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A generalized volume dimension of complex networks

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Abstract. The fractal and self-similarity properties are investigated in many real complex networks. The volume dimension method is an effective tool to measure the fractal property of complex networks. In this paper, a new volume dimension measure is proposed based on the node degree of complex networks. We apply the proposed method to calculate the fractal dimension of some real networks and Newman–Watts (NW) small-world. The results show that the proposed method is effective when dealing with the fractal dimension problem of complex networks. In addition, we find that the fractal dimension is mainly influenced by the probability of 'adding edges' and the average length of the small-world network.

Keywords: random graphs, networks

Contents

1. Introduction

Complex networks have been studied in various fields, including computer science, physics, management science and biology [1–8]. Some fundamental properties of real complex networks have been revealed such as the 'small-word property' [9] and the scale-free property [10]. In 2005, the fractal and self-similarity properties of complex networks were investigated by Song *et al* [11]. They proposed that the fractal dimension d_B can be given by $N(l) \sim l^{-d_B}$, where $N(l)$ is the minimum number of boxes needed to tile a given network, and box size l corresponds to the box diameter. The fractal dimensions of some real complex networks, including the World Wide Web (WWW), metabolic networks, and protein interaction networks are calculated by applying a greedy box-covering algorithm [12]. The fractal scaling and self-similarity properties have been reviewed and investigated by many scholars, and some algorithms for fractal dimensions of complex networks have been proposed [13–17] such as the random sequential box-covering algorithms [18]. Gao *et al* modified the random sequential box-covering algorithm based on computational efficiency [19]. Recently, Wei *et al* modified the box-covering algorithm to deal with the fractal dimension of weighted complex networks [20]. Although the box-covering algorithm has become a classic method for determining the fractal properties of complex networks, the greedy box-covering algorithm belongs to a family of NP problems. The minimum number of boxes tiling a given network is an approximate solution in the box-covering algorithm [21]. However, the above mentioned difficulty of calculating the dimension of complex networks can be circumvented by calculating its correlation dimension with the time complexity of $O(N^{2.376}\log N)$ [22, 23].

The volume dimension method is another effective tool to measure the fractal property of complex networks [24–26]. The definition of volume is extended from regular lattices to complex networks. Applying this method, Guo *et al* calculated the volume dimension of some real networks [27]. Most recently, Li *et al* [28] improved the volume dimension method to calculate the dimension of spatially embedded networks in which the mass V scales with r as $V \sim r^{d_f}$, where V is the number of nodes within a hypersphere of radius r. With these methods, all nodes of complex networks are undifferentiated. The node degree is a fundamental indicator in the study of complex networks. It is a very important measure to identify the importance of a node [29, 30]. In general, the larger the degree value, the more important the node. Thus, the degrees of nodes cannot be ignored. In this paper, volume dimension is modified by considering the node degree. It is interesting that volume based on node degree emerges as a power law phenomenon with the distances between nodes. Further, we focus on studying the factors that influence the fractal dimension in a model of Newman–Watts small-world. The results show that the probability of 'adding edges' and the average length of the small-world network are the key factors in determining the fractal property.

2. Preliminaries

For a given unweighted complex network $G = (N, M)$, N is the number of nodes and M is the number of edges. The distance between two nodes, i and j, in complex networks is equal to the total number of edges that connect them through the shortest linkages, denoted by d_{ij}, which is different from the Euclidean distance. To investigate the fractal and self-similarity properties of complex networks, the volume dimension method is a successful analytical technique. $V(r)$ is calculated as a function of r to obtain the following scaling [29]

$$V(r) \sim r^{d_{\mathrm{f}}} \tag{1}$$

where r is a constant ranging from one to the diameter of complex networks, the volume $V(r)$ is the number of nodes within a distance r averaged over all nodes, and d_{f} is the volume dimension of complex networks. The volume dimensions of some real networks are calculated and the ratio of the number of nodes in all the boxes to number N of the boxes with size r is defined as average density, denoted as $\langle \rho(r) \rangle$ [27]. $\langle \rho(r) \rangle$ is a function of r with $\langle \rho(r) \rangle \simeq kr^{d_{\mathrm{f}}}$, where d_{f} represents the dimension of the network and k is a geometric constant in [27].

3. Volume dimension of complex networks based on degree

The volume value is represented as the number of nodes in the classical volume dimension method [24, 25]. There are no differences among the nodes in the classical volume dimension method. However, different nodes have different properties; for example, some nodes are called 'hub' nodes. There are different degrees in two volumes, even if they include the same number of nodes. According to the conditions above, the node degree is considered in our method. For a given network, $V_{\mathrm{D}}(r)$ is calculated as a function of r to obtain the following scaling

$$V_{\mathrm{D}}(r) \sim r^{d_{\mathrm{d}}}, \tag{2}$$

where the volume $V_{\mathrm{D}}(r)$ is the sum of the degrees of nodes within distance r. The modified volume dimension is denoted by d_{d} and is the exponent that determines the scaling behaviour of the volume with distance.

J. Stat. Mech. (2014) P10039

Table 1. General characteristics of several real networks. For each network, we indicate the number of nodes, the average degree $\langle k \rangle$, the average path length l, the clustering coefficient C, and the degree distribution $p(k)$.

Network	Size	$\langle k \rangle$	l	C	$p(k)$
Power grid	4941	2.67	18.7	0.08	$e^{-0.59k}$
CNCG	7343	1.62	3.92	0.103	$k^{-2.17}$
Yeast	2361	2.82	4.62	0.04	$k^{-2.11}$

The definition here is different from the box-covering dimension [21] and the classical volume dimension [27]. In the box-covering dimension, the minimum $N(l)$ value must be identified for any given l, which belongs to a family of NP-hard problems. $V_{\mathrm{D}}(r)$ is an exact solution in our method; however, $N(l)$ is an approximate solution. In the classical volume dimension, heterogeneity of degrees is not considered. In our method, we use the modified volume measure that accounts for not only the node degrees but also the number of the nodes. To analyze the dimension property of complex networks, we set all the nodes as the seeds in turn. Then, the node degree is calculated based on all the nodes in the circle with a radius of r. $\frac{\sum_i^N V_{\mathrm{D}}(r)}{N}$ represents the average volume of complex networks with given r, which is denoted as $\langle V(r) \rangle$.

We apply the proposed method to investigate some real complex networks, e.g. technological networks such as the electrial power grid of the western United Sates [9], the collaboration network in computational geometry (CNCG) [31], and the protein–protein interaction network (PIN) in budding yeast [32]. These are spare networks with the small-word effect. Some of their properties are shown in table 1 [27]. In table 1, for each network, we indicate the number of nodes, the average degree $\langle k \rangle$, the average path length l, the diameter d, the clustering coefficient C, and the degree distribution $p(k)$.

Figure 1 shows the degree volume dimensions of these real complex networks: the US power grid with $d_{\mathrm{d}} = 2.287$, the CNCG with $d_{\mathrm{d}} = 2.182$, and the PIN in budding yeast with $d_{\mathrm{d}} = 2.887$. Here, the d_{d} values are somewhat similar to the d_{f} value obtained by the volume dimension method [22], which is shown in table 2. From table 2, the fractal properties are somewhat same although different physical quantities are considered in the fractal definitions. From figure 1, the values of $\langle V(r) \rangle$ hardly change when r reaches a certain value, which is denoted by r' for these real networks. The graph shows that $[l] \leqslant r' \leqslant [\frac{d}{2}]$ for these real networks, where $[x]$ is integral function and d is the diameter of the network. This indicates that the fractal property of these real complex networks can be revealed when the value of r increases near to the value of l.

Many scholars have analyzed how fractality and other properties coexist for complex networks [33–37]. Using the renormalization group approach (RG), the box-covering fractal dimension is revealed in certain scales for some real networks. These networks' structures belong to the unstable phase. For example, the WWW network is fractal up to a given length scale and then crosses over to small-world behavior at large scales [33]. The relationship between the classical volume dimension and the shortcuts in a small-world network is revealed in [27]. For a given complex network size, the larger the shortcut density value of a NW small-world, the larger the volume dimension value [27]. Thus, we detail how the fractal dimension d_{d} depends on the complex network structure such as

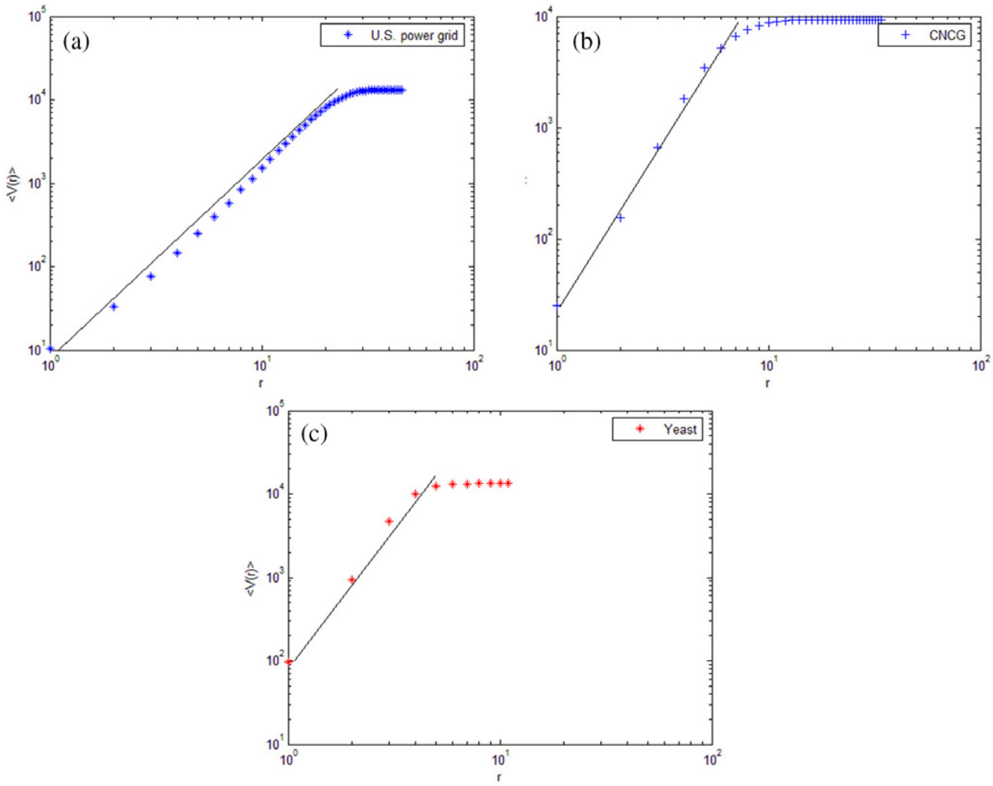

Figure 1. Fractal scaling analysis of real networks based on degree volume. (*a*) Power grid network, the reference line has slope 2.287. (*b*) CNCG network, the reference line has slope 2.182 and (*c*) yeast network, the reference line has slope 2.887.

Table 2. Dimensions based on different volume methods.

Network	Power grid	CNCG	Yeast
d_{f}	2.286	1.842	2.349
d_{d}	2.287	2.182	2.887

the shortcuts in small-world networks. To do so, we carried out empirical analysis of a small-world network. A model for building a small-world network was given by Newman and Watts [32] and is denoted by NW networks. The origin network is a lattice consisting of N nodes arranged in a ring, where each node is connected to its $2z$ neighbors. For each pair of originally unconnected nodes, with probability p, we add an edge to connect them. In this process, between any pair of nodes, there will be no multiple edges and no node will have self-loops. We employed this model to construct a small-world network with different adding edges probability p, z and network size z. After this preparation, the fractal dimensions were calculated by the proposed method. The relations between d_{d} and p, z and n are shown in figure 2.

Figures 2(*a*) and (*b*) show that the value of d_{d} increases as the probability p increases with the same value z. However, the dimension values are almost the same with increase of network size for a given probability. In figures 2(*a*) and (*c*), it can be seen that the

J. Stat. Mech. (2014) P10039

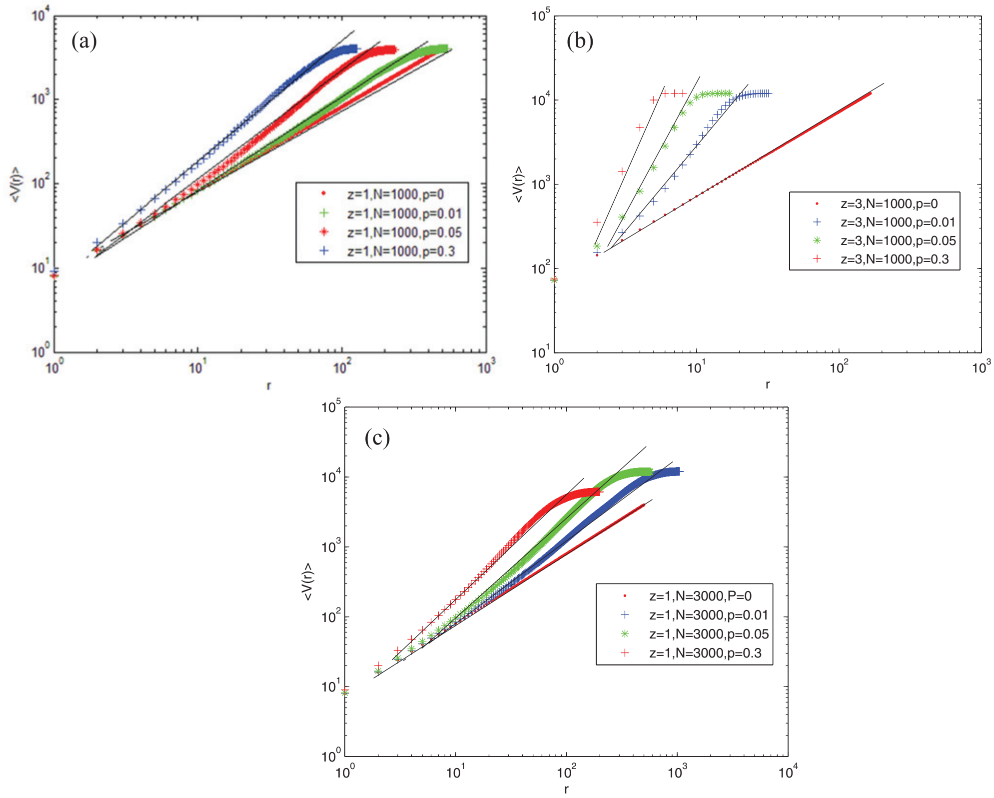

Figure 2. The volume of $\langle V(r) \rangle$ as a function of the boxes within size r in an NW small-world network with various probability p. (a) $z = 1, N = 1000$ (b) $z = 1, N = 3000$ and (c) $z = 3, N = 1000$.

Table 3. Dimensions of NW networks with varying probability.

NW network	$p = 0$	$p = 0.01$	$p = 0.05$	$p = 0.3$
$z = 1, N = 1000$	1	1.005	1.221	1.309
$z = 1, N = 3000$	1	1.076	1.201	1.231
$z = 3, N = 1000$	1	1.814	2.256	3.012

Table 4. Average shortest paths of NW networks with varying probability.

NW network	$p = 0.01$	$p = 0.05$	$p = 0.3$
$z = 1, N = 1000$	187.2793	100.2390	53.37951
$z = 1, N = 3000$	351.9427	164.9937	66.05098
$z = 3, N = 1000$	13.7833	7.9233	4.6509

dimension values are strongly affected by the value of fixed range z for the same adding edges probability and size. These dimension values are shown in table 3.

We also calculate the average shortest paths, which are shown in table 4.

Figures 2(a) and (b) show the evolution of the average volume $\langle V(r) \rangle$ as the box size r with the same complex network size N, different FNP z, and shortcut density p. Similarly,

J. Stat. Mech. (2014) P10039

figures 2(a) and (c) represent the evolution of the average volume $\langle V(r)\rangle$ as the box size r with the same FNP z, different complex network size N, and shortcut density p. We found that $d_{\mathrm{d}} = 1$ for $p = 0$ and $d_{\mathrm{d}} \succ 1$ for $p \succ 0$ in all cases. d_{d} increases as the shortcut density p increases, which is consistent with [27]. The classical volume dimension d_{f} is independent of the FNP z in an NW small-world network. However, in our method, we found that the fractal dimension d_{d} increases as the FNP z increases. The reason for this is that the degree of nodes is considered in the proposed method. Hence, the degree of disorder of complex networks is reflected by the dimension d_{d}.

4. Conclusions

Recently, it was shown that some real networks exhibit fractal scaling. With the volume dimension method, one can obtain $V(r_{ij}) \sim r^{d_{\mathrm{f}}}$, where the $V(r_{ij})$ is number of nodes j within a distance r_{ij} of node i, averaged over i. In this paper, a modified version of the volume dimension method is introduced, $V_{\mathrm{D}}(r_{ij}) \sim r^{d_{\mathrm{d}}}$, where the volume $V_{\mathrm{D}}(r_{ij})$ is the sum degree of nodes j within a distance r_{ij} of node i, averaged over i. The numerical examples show that the proposed approach can well reveal the fractal property for a complex network. To discuss the factors that mainly determine the fractal dimension, we launch a study by altering the typical qualities in an NW small-world network. We find that the adding edges probability, fixed range of the orginal ring, and average shortest path play important roles in determining the dimension, but the dimension values do not depend on the size of a network.

Our results are different from the d_{B} value obtained from the box-covering method such as the C elegans with $d_{\mathrm{d}} = 2.262$ and $d_{\mathrm{B}} = 3.5$ [19]. In addition, the power grid network is classified as a fractal network in our method and the volume dimension method [22]. However, it is undetermined if the power grid network is classified as a fractal network by applying the box-covering method [30]. On the other hand, the box-covering method is an approximation algorithm since obtaining the minimum number of boxes is an NP problem. However, the volume in the boxes with size r is an exact solution in our method and [27] and is easy to apply on networks. Compared to the classical volume dimension method, the proposed method takes the node degree into consideration when calculating the volume, which is natural and necessary from the view of the topology because the functions of complex networks are usually dominated by the 'hub' nodes.

Acknowledgments

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J. Stat. Mech. (2014) P10039