or no

yes → 2 Discrete Mathematics Homework 8

Cycle or path → 3

1. Find a Hamilton cycle, if one exists, for each of the graphs or multigraphs in the below figure. If the graph has no Hamilton cycle, determine whether it has a Hamilton path.

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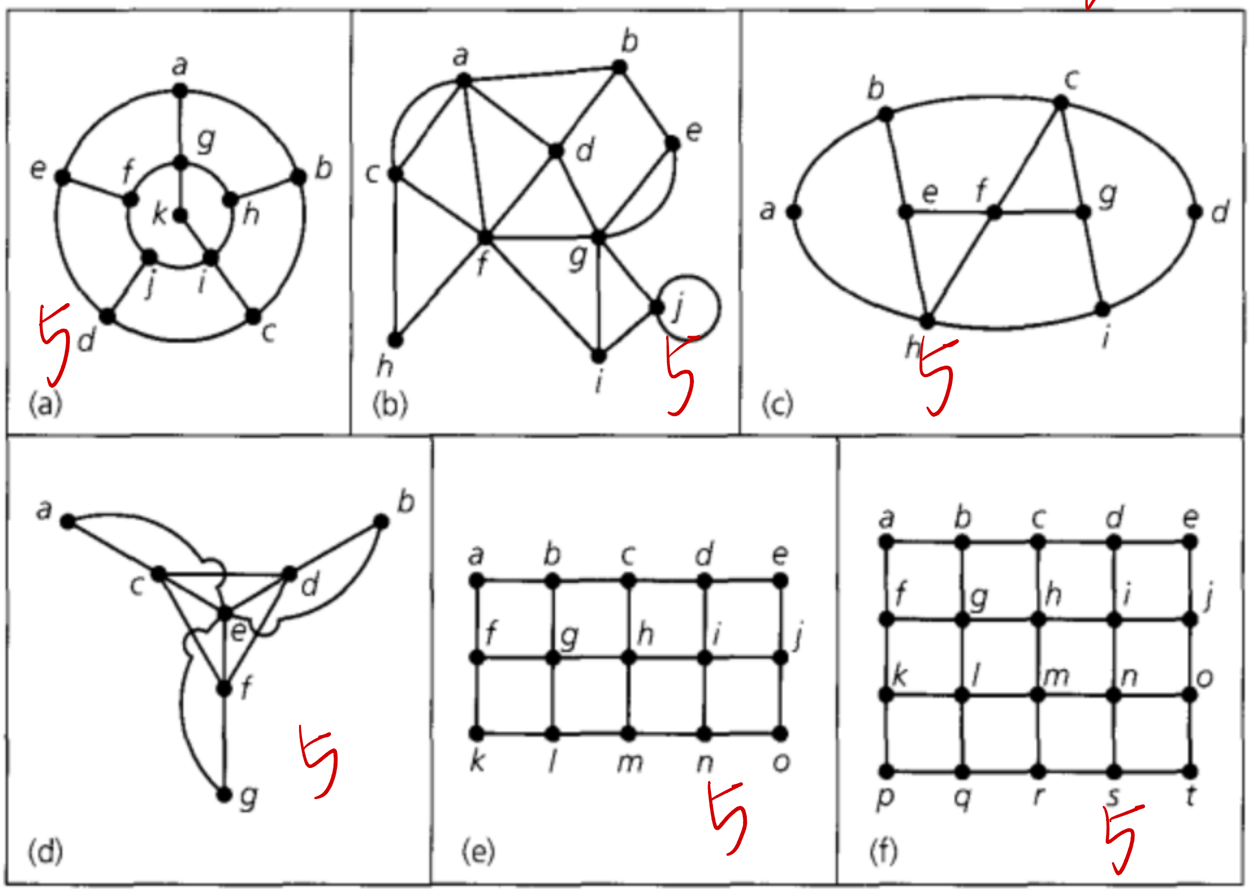

Solution

(a) Hamilton cycle: $a \to g \to k \to i \to h \to b \to c \to d \to j \to f \to e \to a$
(b) Hamilton cycle: $a \to d \to b \to e \to g \to j \to i \to f \to h \to c \to a$
(c) Hamilton cycle: $a \to h \to e \to f \to g \to i \to d \to c \to b \to a$
(d) The edges $\{a,c\}$, $\{c,d\}$, $\{d,b\}$, $\{b,e\}$, $\{e,f\}$, $\{f,g\}$ provide a Hamilton path for the given graph. However, there is no Hamilton cycle, for such a cycle would have to include the edges $\{b,d\}$, $\{b,e\}$, $\{a,c\}$, $\{a,e\}$, $\{g,f\}$, and $\{g,e\}$ – and, consequently, the vertex e will have degree greater than 2.
(e) The path $a \to b \to c \to d \to e \to j \to i \to h \to g \to f \to k \to l \to m \to n \to o$ is one possible Hamilton path for this graph. Another possibility is the path $a \to b \to c \to d \to i \to h \to g \to f \to k \to l \to m \to n \to o \to j \to e$. However, there is no Hamilton cycle. For if we try to construct a Hamilton cycle we must include the edges $\{a,b\}$, $\{a,f\}$, $\{f,k\}$, $\{k,l\}$, $\{d,e\}$, $\{e,j\}$, $\{j,o\}$ and $\{n,o\}$. This then forces us to eliminate the edges $\{f,g\}$ and $\{i,j\}$ from further consideration. Now consider the vertex i. If we use edges $\{d,i\}$ and $\{i,n\}$, then we have a cycle on the vertices d,e,j,o,n and i – and we cannot get a Hamilton cycle for the given graph. Hence we must use only one of the edges $\{d,i\}$ and $\{i,n\}$. Because of the symmetry in this graph let us select edge $\{d,i\}$ – and then edge $\{h,i\}$ so that vertex i will have degree 2 in the Hamilton cycle we are trying to construct. Since edges $\{d,i\}$ and $\{d,e\}$ are now being used, we eliminate edge $\{c,d\}$ and this then forces us to include edges $\{b,c\}$ and $\{c,h\}$ in our construction. Also we must include the edge $\{m,n\}$ since we eliminated edge $\{i,n\}$ from consideration. Next we eliminate edges $\{h,m\}$, $\{h,g\}$ and $\{b,g\}$. Finally we must include edge $\{m,l\}$ and then eliminate edge $\{l,g\}$. But now we have eliminated the four edges $\{b,g\}$, $\{f,g\}$, $\{h,g\}$ and $\{l,g\}$ and g is consequently isolated.
(f) For this graph we find the Hamilton cycle $a \to b \to c \to d \to e \to j \to i \to h \to g \to l \to m \to n \to o \to t \to s \to r \to q \to p \to k \to f \to a$.

2 (a) Show that the Petersen graph [Fig. 7.9(a)] has no Hamilton cycle but that it has a Hamilton path.

10 (explain no cycle 5, path 5)

(b) Show that if any vertex (and the edges incident to it) is removed from the Petersen graph, then the resulting subgraph has a Hamilton cycle.

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Solution

(a) Consider the graph as shown in Fig. 7.9(a). We demonstrate one case. Start at vertex a and consider the partial path $a \to f \to i \to d$. These choices require the removal of edges $\{f, h\}$ and $\{g, i\}$ from further consideration since each vertex of the graph will be incident with exactly two edges in the Hamilton cycle. At vertex d we can go to either vertex c or vertex e. (i) If we go to vertex c we eliminate edge $\{e, d\}$ from consideration, but we must now include edges $\{e, j\}$ and $\{e, a\}$, and this forces the elimination of edge $\{a, b\}$. Now we must consider vertex b, for by eliminating edge $\{a, b\}$ we are now required to include edges $\{b, g\}$ and $\{b, c\}$ in the cycle. This forces us to remove edge $\{c, h\}$ from further consideration. But we have now removed edges $\{f, h\}$ and $\{c, h\}$ and there is only one other edge that is incident with h, so no Hamilton cycle can be obtained. (ii) Selecting vertex e after d, we remove edge $\{d, c\}$ and include $\{c, h\}$ and $\{b, c\}$. Having removed $\{g, i\}$ we must include $\{g, b\}$ and $\{g, j\}$. This forces the elimination of $\{a, b\}$, the inclusion of $\{a, e\}$ (and the elimination of $\{e, j\}$). We now have a cycle containing a, f, i, d, e, hence this method has also failed.

However, this graph does have a Hamilton path: $a \to b \to c \to d \to e \to j \to h \to f \to i \to g$.

(b) For example, remove vertex j and the edges $\{e, j\}, \{g, j\}, \{h, j\}$. Then $e \to a \to f \to h \to c \to b \to g \to i \to d \to e$ provides a Hamilton cycle for this subgraph.

3. Give an example of a loop-free connected undirected multi-graph $G = (V, E)$ such that $|V| = n$ and $\deg(x) + \deg(y) \geq n - 1$ for all $x, y \in V$, but G has no Hamilton path.

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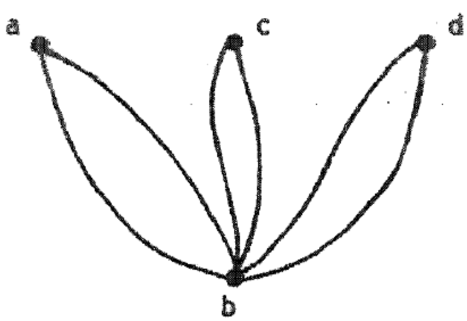
Solution



For the multigraph in the given figure, $|V| = 4$ and $\deg(a) = \deg(c) = \deg(d) = 2$ and $\deg(b) = 6$. Hence $\deg(x) + \deg(y) \geq 4 > 3 = 4 - 1$ for all nonadjacent $x, y \in V$, but the multigraph has no Hamilton path.

4. Let $n \in N$ with $n \geq 4$, and let the vertex set V' for the complete graph K_{n-1} be $\{v_1, v_2, v_3, \cdots, v_{n-1}\}$. Now construct the loop-free undirected graph $G_n = (V, E)$ from K_{n-1} as follows: $V = V' \cup \{v\}$, and E consists of all the edges in K_{n-1} except for the edge $\{v_1, v_2\}$, which is replaced by the pair of edges $\{v_1, v\}$ and $\{v, v_2\}$.

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(a) Determine deg(x) + deg(y) for all nonadjacent vertices x and y in V. 5 (explain: 3)

(b) Does G_n have a Hamilton cycle? 5 (explain: 3)

(c) How large is the edge set E? 5 (explain: 3)

(d) Do the results in parts (b) and (c) contradict Corollary 8.6? 5 (explain: 3)

Solution

(a) If $x \neq v$ and $y \neq v$, then $\deg(x) = \deg(y) = n-2$, and $\deg(x) + \deg(y) = 2n - 4 \geq n$, for $n \geq 4$.
If one of x, y is v, say x, then $\deg(x) = 2$ and $\deg(y) = n - 2$, and $\deg(x) + \deg(y) = n$.
(b) From part (a) it follows that $\deg(x) + \deg(y) \geq n$ for all nonadjacent x, y in V. Therefore G_n has a Hamilton cycle — by virtue of Theorem 8.5.
(c) Here $|E| = \binom{n-1}{2} - 1 + 2$, where we subtract 1 for the edge $\{v_1, v_2\}$, and add 2 for the pair of edges $\{v_1, v\}$ and $\{v, v_2\}$. Consequently, $|E| = \binom{n-1}{2} + 1$.
(d) The results in parts (b) and (c) do not contradict Corollary 8.6. They show that the converse of this corollary is false — as is its inverse.

5. If $G = (V, E)$ is an undirected graph, a subset I of V is called **independent** if no two vertices in I are adjacent.

Let $G = (V, E)$ be an undirected graph with subset I of V an independent set. For each $a \in I$ and each Hamilton cycle C for G, there will be $\deg(a) - 2$ edges in E that are incident with a and not in C. Therefore there are at least $\sum_{a \in I}[\deg(a) - 2] = \sum_{a \in I} \deg(a) - 2|I|$ edges in E that do not appear in C.

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(a) Why are these $\sum_{a \in I} \deg(a) - 2|I|$ edges distinct? 5

(b) Let $v = |V|$, $e = |E|$. Prove that if

$$e - \sum_{a \in I} \deg(a) + 2|I| < v,$$ 5

then G has no Hamilton cycle. 5

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(c) Select a suitable independent set / and use part (b) to show that the graph in the below figure (known as the Herschel graph) has no Hamilton cycle.

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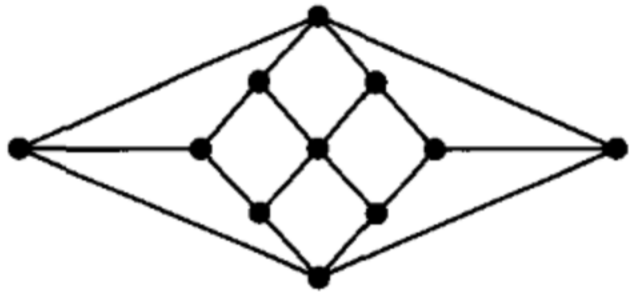
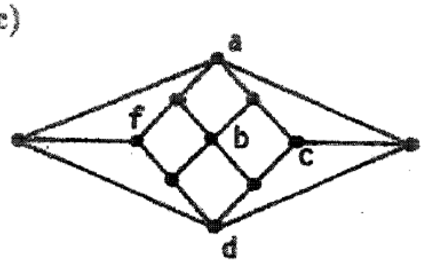


Solution

(a) If not, there is an edge $\{a, b\}$ in E where $a, b \in I$. This contradicts the independence of I.

(b) A Hamilton cycle on v vertices must have v edges.

(c)



Let $I = \{a, b, c, d, f\}$, as shown in the figure. Here $v = 11, e = 18$, and $e - \sum_{v \in I} \deg(v) + 2|I| = 18 - (4 + 4 + 3 + 4 + 3) + 2(5) = 10 < 11$, so by part (b), the Herschel graph has no Hamilton cycle.