Lecture 9

Stochastic Differential Equations

- Exact SDE
- Basic rules of differentials
- Linear SDE
- Integrating factors

9.1 Introduction

- Let X_t be a continuous stochastic process and B_t be the standard Brownian motion.
- If small changes in X_t can be written as a linear combination of small changes in t and small increments of B_t, we may write

$$dX_t = a(t, B_t, X_t)\, dt + b(t, B_t, X_t)\, dB_t$$

  and call it a *stochastic differential equation* (SDE).
- The above differential form is indeed a shorthand for

$$X_t = X_0 + \int_0^t a(s, B_s, X_s)\, ds + \int_0^t b(s, B_s, X_s)\, dB_s.$$

- The unknown X_t appears on both sides of the equation. Our aim is to solve this equation to obtain an explicit expression for X_t. $f(t, B_t)$
- Note the difference between an Itô process and a SDE.

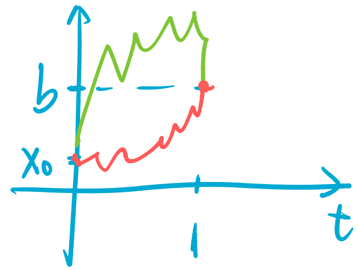
$dX_t = (\alpha X_t + \beta)\,dt + b\cdot dB_t$

Example 9.1 (The Brownian bridge) Let $b \in \mathbb{R}$. The stochastic differential equation

$\int \frac{1}{x}\,dx = \ln|x| + C$

$$dX_t = \frac{b - 1\cdot X_t}{1-t}\cdot dt + 1\,dB_t, \qquad 0 \le t < 1$$

$t \to 1$ $\quad X_t \to b$



has a solution

$$\Rightarrow X_t = X_0(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dB_s, \qquad 0 \le t < 1.$$

$\square$

$\alpha(t) = -\frac{1}{1-t} = \frac{1}{t-1}$, $\beta(t) = \frac{b}{1-t}$, $b(B_t, t) = 1$.

$A(t) = \int_0^t \frac{1}{s-1}\,ds$ $\qquad 0 \le t < 1$

$0 < s \le t < 1$.

$= \ln(1-t)$

IF $e^{-A(t)} = \frac{1}{1-t}$

$d\left(\frac{X_t}{1-t}\right) = \frac{1}{1-t}\cdot\left(\frac{b}{1-t}\,dt + 1\cdot dB_t\right)$

$= \frac{b}{(1-t)^2}\,dt + \frac{1}{1-t}\,dB_t$

Take integral to both sides

$\frac{X_t}{1-t} - \frac{X_0}{1-0} = \underbrace{b\cdot\int_0^t \frac{1}{(1-s)^2}\,ds}_{①} + \int_0^t \frac{1}{1-t}\,dB_s$

① $\int_0^t \frac{1}{(1-s)^2}\,ds = \frac{1}{1-s}\Big|_0^t = \frac{1}{1-t} - 1 = \frac{t}{1-t}$

$\therefore \frac{X_t}{1-t} = X_0 + b\frac{t}{1-t} + \int_0^t \frac{1}{1-s}\,dB_s$

$X_t = X_0(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dB_s$

9.2 Exact Stochastic Differential Equations

- The SDE

$$dX_t = a(t, B_t)dt + b(t, B_t)dB_t$$

is called *exact* if there is a differentiable function $f(t, x)$ such that $X_t = f(t, B_t)$.

Example 9.2

$\int_0^t dX_s = \int_0^t 1\, ds + \int_0^t B_s dB_s$, $X_t - X_0 = t + \frac{1}{2}B_t^2 - \frac{1}{2}t$

1. The SDE $dX_t = dt + B_t\, dB_t$ is exact because the solution

$$X_t = X_0 + \frac{1}{2}(B_t^2 + t)$$

can be expressed as $X_t = f(t, B_t)$.

2. The SDE $dX_t = \cos t\; dt - \sin t\; dB_t$ is not exact because the solution

$$X_t = X_0 + \sin t \cdot (1 - B_t) + \boxed{\int_0^t \cos s \cdot B_s\, ds}$$

cannot be expressed as $X_t = f(t, B_t)$. □

B_t $\{B(s),\ 0<s<t\}$.

$dX_t = a\,dt + b\,dB_t$

① $df(B_t)$
② $df(B_t, t)$
③ $df(X_t, t)$

- If the equation is exact, then by Itô's formula (with $a = 0$, $b = 1$)

$$
\begin{aligned}
dX_t &= d(f(t, B_t)) \\
&= \left(\frac{\partial f(t, B_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, B_t)}{\partial x^2} \right) dt + \frac{\partial f(t, B_t)}{\partial x} dB_t,
\end{aligned}
$$

so that we must have $(f_t + \frac{1}{2} f_{xx})\,dt + f_x\,dB_t$

$$
\boxed{\left\{
\begin{aligned}
a(t, x) &= \frac{\partial f(t, x)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, x)}{\partial x^2} \\
b(t, x) &= \frac{\partial f(t, x)}{\partial x}.
\end{aligned}
\right.}
$$

- Since $a(t, x)$ and $b(t, x)$ are given, we can solve the above equations to obtain $f(t, x)$. Then, the solution to the SDE is given by $X_t = f(t, B_t)$.

- If the above equations have no solutions, then the SDE is not exact.

- Equations for $f(t, x)$:

$$
\begin{aligned}
a(t,x) &= \frac{\partial f(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial x^2} \qquad (1) \\
b(t,x) &= \frac{\partial f(t,x)}{\partial x}. \qquad (2)
\end{aligned}
$$

- Step 1: Integrate $b(t, x)$ to yield $f(t, x)$ up to an additive function $g(t)$

$$
f(t,x) = \int b(t,x)dx + g(t).
$$

- Step 2: Substitute $f(t, x)$ into $a(t, x)$ and determine $g(t)$

$$
\begin{aligned}
a(t,x) &= \int \frac{\partial b(t,x)}{\partial t}\, dx + g'(t) + \frac{1}{2}\frac{\partial b(t,x)}{\partial x} \\
g(t) &= \int a(t,x)dt - \int\int \frac{\partial b(t,x)}{\partial t}\, dx\, dt - \frac{1}{2}\int \frac{\partial b(t,x)}{\partial x}\, dt + c.
\end{aligned}
$$

- Step 3: $X_t = f(t, B_t)$
- Step 4: Use the initial condition $X(0) = X_0$ to determine the constant c in terms of X_0.

Example 9.3 Solve the SDE as an exact equation

$$dX_t = e^t(1 + B_t^2)\,dt + (1 + 2e^t B_t)\,dB_t, \qquad X_0 = 0.$$

In this case $a(t,x) = e^t(1 + x^2)$ and $b(t,x) = 1 + 2e^t x$. The associated system is

f_x

$$e^t(1 + x^2) = \boxed{\frac{\partial f(t,x)}{\partial t}} + \boxed{\frac{1}{2}\frac{\partial^2 f(t,x)}{\partial x^2}} \quad \text{and} \quad 1 + 2e^t x = \frac{\partial f(t,x)}{\partial x}. \text{ (2)}$$

Integrating in x in the second equation yields

$$f(t,x) = \int (1 + 2e^t x)dx = x + e^t x^2 + \boxed{g(t)}.$$

Substituting this into the first equation yields

$$e^t(1 + x^2) = \boxed{e^t x^2 + g'(t)} + \boxed{e^t}. = e^t(1+x^2) + g'(t)$$

This implies $g'(t) \equiv 0$, i.e. $g(t) \equiv c$. Hence,

$$X_t = f(t, B_t) = B_t + e^t B_t^2 + c.$$

$X(t) = B_t + e^t B_t^2$

The initial condition $X_0 = 0$ yields $c = 0$. $\square$

Example 9.4 What if we attempt to solve a non-exact equation as if it were exact? Consider the SDE

$$dX_t = (1 + B_t^2)\, dt + (t^4 + B_t^2)\, dB_t.$$

The coefficient functions are $a(t,x) = 1 + x^2$ and $b(t,x) = t^4 + x^2$. Integrating $b(t,x)$ w.r.t. (with respect to) x yields

$$f(x,t) = \int (t^4 + x^2)dx = t^4 x + \frac{x^3}{3} + g(t).$$

$f_x = t^4 + x^2$

$f_t + \frac{1}{2} f_{xx}$

$f_{xx} = 2x$

Then,

$$\frac{\partial f(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2} = 4t^3 x + g'(t) + x.$$

The equation for $g(t)$, i.e. $\partial_t f + (1/2)\partial_{xx} f = a$, reads

$$4t^3 x + g'(t) + x = 1 + x^2,$$

which has no solution. Note that the solution is not of the exact form:

$$X_t = X_0 + t^4 B_t + \frac{B_t^3}{3} + t + \int_0^t \left[B_s^2 - (4s^3 + 1)B_s\right] ds. \qquad \square$$

9.3 Basic Rules of Differentials

- We introduced $(dB_t)^2 = dt$ as a shorthand of $\int (dB_t)^2 = \int dt$.
- We also introduced $dX_t = a(t, X_t, B_t)\ dt + b(t, X_t, B_t)\ dB_t$ as a shorthand of its integral form.
- The differential dX_t can also be **manipulated** as follows

$$\underline{dX_t} = X_{t+dt} - X_t.$$

$dt = t + \Delta t - t$

- The constant multiple rule:

$$\underline{d(cX_t)} = cX_{t+dt} - cX_t = c(X_{t+dt} - X_t) = c\, dX_t.$$

- The sum/difference rule:

$$\begin{aligned} d(\boxed{X_t \pm Y_t}) &= (X_{t+dt} \pm Y_{t+dt}) - (X_t \pm Y_t) \\ &= (X_{t+dt} - X_t) \pm (Y_{t+dt} - Y_t) \\ &= \boxed{dX_t} \pm \boxed{dY_t}. \end{aligned}$$

- The product rule:

$(f \cdot g)' = f'g + fg'$

$d(f \cdot g) = df \cdot g + f \cdot dg$

$$
\begin{aligned}
d(X_t Y_t) &= X_{t+dt} Y_{t+dt} - X_t Y_t \\
&= X_t (Y_{t+dt} - Y_t) + Y_t (X_{t+dt} - X_t) \\
&\quad + (X_{t+dt} - X_t)(Y_{t+dt} - Y_t) \\
\Rightarrow d(X_t Y_t) &= \underbrace{X_t \, dY_t}_{①} + \underbrace{Y_t \, dX_t}_{②} + \underbrace{dX_t \, dY_t}_{③}.
\end{aligned}
$$

- If X_t and Y_t are Itô processes

$$
\begin{aligned}
dX_t &= a(t, B_t) \, dt + b(t, B_t) \, dB_t \\
dY_t &= c(t, B_t) \, dt + d(t, B_t) \, dB_t,
\end{aligned}
$$

then by the Itô multiplication table,

$$
d(\boxed{X_t Y_t}) = \underbrace{X_t \, dY_t}_{①} + \underbrace{Y_t \, dX_t}_{②} + \underbrace{b(t, B_t) d(t, B_t)}_{③} \, dt.
$$

$x \, dX_t \, dY_t$

$d(X_t Y_t)$ ③ $(a dt + b dB_t)(c dt + d \, dB_t)$

$= ac \underbrace{(dt)^2}_{0} + ad \underbrace{dt \, dB_t}_{0} +$

$bc \underbrace{dB_t \, dt}_{0} + bd \underbrace{(dB_t)^2}_{dt}$

$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2}f''(B_t)\,dt$

$f(x)=x^2,\ f'(x)=2x,\ f''(x)=2$

Example 9.5 Determine $d(B_t^2)$.

$d(B_t)^{2\prime} = 2B_t + \frac{1}{2}\cdot 2\cdot dt$

Solution: By the product rule and $(dB_t)^2 = dt$,

$X_t \quad Y_t$

$$
\begin{aligned}
d(B_t^2) &= d(B_t B_t) \\
&= B_t\,dB_t + B_t\,dB_t + (dB_t)^2 \\
&= 2B_t\,dB_t + dt.
\end{aligned}
$$

$\square$

$d(f(t,x)) = f_t\,dt + f_x\,dx + \frac{1}{2}f_{xx}\,dt$

Example 9.6 Determine $d(tB_t)$.

$f(t,x) = t\cdot x,\ f_t = x,\ f_x = t,\ f_{xx} = 0$

Solution: By the product rule and $dt\,dB_t = 0$,

$d(tB_t) = B_t\cdot dt + t\cdot dB_t + 0$

$$
\begin{aligned}
d(tB_t) &= B_t\,dt + t\,dB_t + dt\,dB_t \\
&= B_t\,dt + t\,dB_t.
\end{aligned}
$$

$dX_t = 1\cdot dt + 0\cdot dB_t$

$d(X_tY_t) = t\cdot dB_t + B_t\,dt + 0$

$\square$

$dY_t = 0\cdot dt + 1\cdot dB_t$

$dX_t\cdot dY_t = (dt)(dB_t) = 0$

9.4 Linear Stochastic Differential Equations

- Consider the SDE with drift term linear in X_t

$\alpha \cdot X_t + \beta$

$$\Rightarrow \quad dX_t = [\alpha(t)X_t + \beta(t)]\, dt + b(t, B_t)\, dB_t.$$

- Let $\boxed{A(t)} = \int_0^t \alpha(s)\, ds$. Multiplying the integrating factor $e^{-A(t)}$, we have

$A'(t) = \alpha t$

$$e^{-A(t)}\, dX_t - e^{-A(t)}\alpha(t)X_t\, dt = e^{-A(t)}\,[\beta(t)\, dt + b(t, B_t)\, dB_t]\,.$$

- By the product rule $\quad Y_t\,dX_t + X_t\,dY_t + dY_t \cdot dX_t$

$X_t \quad Y_t$

$$d\left(X_t e^{-A(t)}\right) = e^{-A(t)}\, dX_t - X_t A'(t)e^{-A(t)}\, dt - A'(t)e^{-A(t)}\, dt\, dX_t$$

$X_t = X_t \cdot Y_t = e^{-A(t)}$

$$= e^{-A(t)}\, dX_t - e^{-A(t)}\alpha(t)X_t\, dt.$$

$\cdot\, dY_t = de^{-A(t)} = e^{-A(t)} \cdot (-A'(t))\, dt \qquad = e^{-A(t)}\,(dX_t - \alpha X_t\, dt)$

- Thus

☆☆☆
$$\boxed{d\left(X_t e^{-A(t)}\right) = e^{-A(t)}\,[\beta(t)\, dt + b(t, B_t)\, dB_t]\,.}$$

$e^{-A(t)} \quad A(t) = -e \qquad e^t\,[2\,dt + e^t \cdot B_t\, dB_t]$

- Integrating yields

$$X_t e^{-A(t)} = X_0 e^{-A(0)} + \int_0^t e^{-A(s)} \beta(s)\, ds + \int_0^t e^{-A(s)} b(s, B_s)\, dB_s$$

or

$$\begin{aligned} X_t &= X_0 e^{A(t)} + e^{A(t)} \int_0^t e^{-A(s)} \beta(s)\, ds \\ &\quad + e^{A(t)} \int_0^t e^{-A(s)} b(s, B_s)\, dB_s. \end{aligned}$$

$dX_t = (\alpha X_t + \beta)\, dt + b \cdot dB_t$

Example 9.7 Solve the linear SDE

$A(t) = \int_0^t \alpha(s)\, ds = 2t$ $\quad IF = e^{-A(t)} = e^{-2t}$

$$dX_t = (2X_t + 1)\, dt + e^{2t}\, dB_t.$$

$\alpha \quad \beta \quad b$

Solution: Multiply by the integrating factor e^{-2t} to get

$X_t \quad Y_t$ ① ②

$$\rightarrow d\left(X_t e^{-2t}\right) = e^{-2t}\, dt + dB_t.$$

$d(X_t e^{-A}) = e^{-A}(\beta\, dt + b\, dB_t)$

$= e^{-2t}(dt + e^{2t}\, dB_t)$

$= e^{-2t}\, dt + dB_t$

Integrating ① and multiplying ② by e^{2t}, and obtain

$$\begin{aligned} X_t &= X_0 e^{2t} + e^{2t} \int_0^t e^{-2s}\, ds + e^{2t} \int_0^t dB_s \\ &= X_0 e^{2t} + e^{2t} \cdot \frac{1 - e^{-2t}}{2} + e^{2t} B_t \\ &= X_0 e^{2t} + \frac{e^{2t} - 1}{2} + e^{2t} B_t. \end{aligned}$$

① LHS

$X_t e^{-2t} - X_0 e^{0} =$

$\int_0^t e^{-2s}\, ds + \int_0^t 1\, dB_s$

RHS: $-\frac{1}{2} e^{-2s} \Big|_0^t + B_t$

$= -\frac{1}{2} e^{-2t} - \left(-\frac{1}{2} \cdot 1\right) + B_t$

$= \frac{1}{2}(1 - e^{-2t}) + B_t$

$X_t \cdot e^{-2t} - X_0 = \frac{1}{2}(1 - e^{-2t}) + B_t$

$X_t = X_0 \cdot e^{-2t} + \frac{1}{2}(e^{2t} - 1) + e^{2t} \cdot B_t$

□

Example 9.8 Solve the linear SDE $dX_t = (\alpha X_t + \beta)\,dt + b \cdot dB_t$

$$dX_t = (-X_t + 2)\,dt + e^{-t}B_t\,dB_t.$$

$\alpha = -1 \quad \Rightarrow \quad A(t) = \int_0^t \alpha\,dt = -t \qquad I_f = e^{-A(t)} = e^{t}$

$\beta = 2$

$b(t, B_t) = e^{-t}B_t$

$d(X_t e^t) = e^t [2\,dt + e^{-t} \cdot B_t\,dB_t].$

$d(X_t Y_t) = dX_t \cdot Y_t + dY_t \cdot X_t + dX_t \cdot dY_t$

$= [(-X_t + 2)dt + e^{-t}B_t\,dB_t]e^t + e^t \cdot X_t \cdot dt + 0$

$= 2e^t\,dt + B_t\,dB_t$

$X_t \cdot e^t - X_0 e^0 = 2\int_0^t e^s\,ds + \int_0^t B_s\,dB_s$

$X_t = e^{-t}(X_0 + 2(e^s|_0^t) + \frac{1}{2}(B_t^2 - t))$

$= X_0 e^{-t} + 2e^{-t}(e^t - 1) + \frac{1}{2}e^{-t}(B_t^2 - t)$

$= X_0 \cdot e^{-t} + 2(1 - e^{-t}) + \frac{1}{2}e^{-t}(B_t^2 - t)$

$b(B_t, t)$

9.5 Integrating Factors

$dX_t = (\alpha X_t + \beta) \cdot dt + b\, dB_t$

- The method of integrating factors can be applied to a class of SDE of the type

$$dX_t = a(t, X_t)dt + g(t)X_t dB_t$$

where a and g are given deterministic functions.

- The integration function is given by

$$\rho_t = e^{\frac{1}{2}\int_0^t g^2(s)\, ds - \int_0^t g(s)\, dB_s}.$$

- The equation can be brought into the form

$$d(\rho_t X_t) = \rho_t a(t, X_t)\, dt, \tag{9.1}$$

which may be solved by integration in some cases of $a(t, X_t)$. We will discuss two such cases in finance applications.

- Let's verify Eq. (9.1), $d(\rho_t X_t) = \rho_t a(t, X_t)\, dt$.
- First, we compute $d\rho_t$.
- Let
$$Z_t = \frac{1}{2}\int_0^t g^2(s)\, ds - \int_0^t g(s)\, dB_s,$$
so that $\rho_t = e^{Z_t}$.
- Note that
$$dZ_t = \frac{1}{2}g^2(t)\, dt - g(t)\, dB_t$$
so that Z_t is an Itô process.
- Since Z_t is an Itô process, we can apply Itô's formula
$$\begin{aligned} d\rho_t &= de^{Z_t} \\ &= \left(\frac{1}{2}g^2(t)e^{Z_t} + \frac{1}{2}(-g(t))^2 e^{Z_t}\right) dt + (-g(t))e^{Z_t}\, dB_t \\ &= \rho_t \left(g^2(t)\, dt - g(t)\, dB_t\right). \end{aligned}$$

- By the product rule

$$
\begin{aligned}
d(\rho_t X_t) &= \rho_t\, dX_t + X_t\, d\rho_t + d\rho_t\, dX_t \\
&= \rho_t \left(a\, dt + gX_t\, dB_t\right) + \rho_t X_t \left(g^2\, dt - g\, dB_t\right) \\
&\quad + \rho_t \left(g^2\, dt - g\, dB_t\right)\left(a\, dt + gX_t\, dB_t\right) \\
&= \rho_t a\, dt + \rho_t g X_t dB_t + \rho_t X_t g^2 dt - \rho_t X_t g dB_t - \rho_t g^2 X_t dt \\
&= \rho_t a\, dt.
\end{aligned}
$$

Example 9.9 Solve the SDE

$$dX_t = r\,dt + \alpha X_t\,dB_t,$$

with r and α constants.

Solution: The integration factor is $\rho_t = e^{\frac{1}{2}\alpha^2 t - \alpha B_t}$. Thus, Eq. (9.1) reads

$$d(\rho_t X_t) = r\rho_t\,dt = re^{\frac{1}{2}\alpha^2 t - \alpha B_t}\,dt.$$

Integrating both sides yields

$$\begin{aligned} \rho_t X_t &= \rho_0 X_0 + r\int_0^t e^{\frac{1}{2}\alpha^2 s - \alpha B_s} ds \\ X_t &= \frac{X_0}{\rho_t} + \frac{r}{\rho_t}\int_0^t e^{\frac{1}{2}\alpha^2 s - \alpha B_s} ds \\ &= X_0 e^{-\frac{1}{2}\alpha^2 t + \alpha B_t} + r\int_0^t e^{\frac{1}{2}\alpha^2 (s-t) - \alpha(B_s - B_t)} ds. \end{aligned}$$

$\square$

Example 9.10 Solve the SDE

$$dX_t = X_t \, dt + X_t \, dB_t.$$