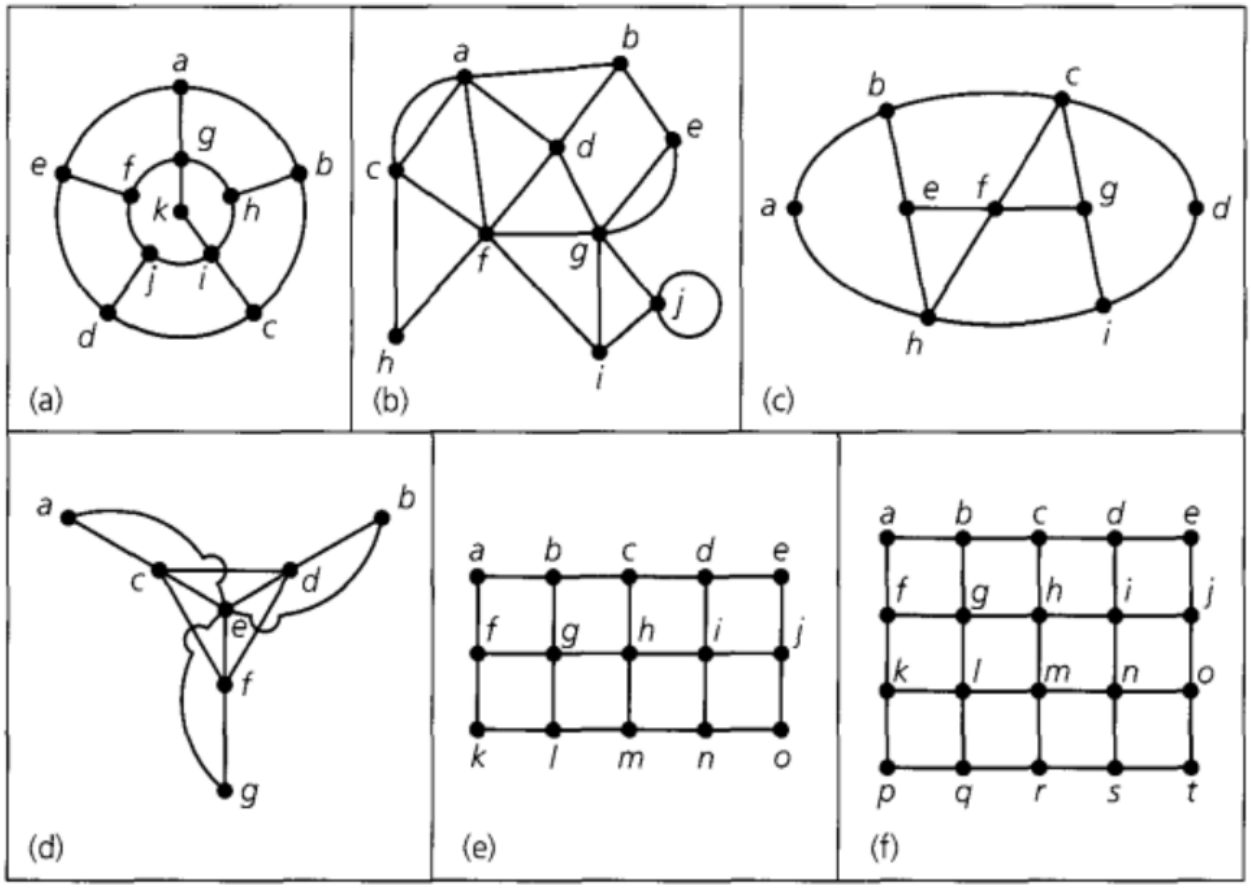
Discrete Mathematics Homework 8

1. Find a Hamilton cycle, if one exists, for each of the graphs or multigraphs in the below figure. If the graph has no Hamilton cycle, determine whether it has a Hamilton path.



2 (a) Show that the Petersen graph [Fig. 7.9(a)] has no Hamilton cycle but that it has a Hamilton path.

(b) Show that if any vertex (and the edges incident to it) is removed from the Petersen graph, then the resulting subgraph has a Hamilton cycle.

3. Give an example of a loop-free connected undirected multi-graph $G = (V, E)$ such that $|V| = n$ and $\deg(x) + \deg(y) \geq n - 1$ for all $x, y \in V$, but G has no Hamilton path.

4. Let $n \in N$ with $n \geq 4$, and let the vertex set V′ for the complete graph K_{n-1} be $\{v_1, v_2, v_3, \cdots, v_{n-1}\}$. Now construct the loop-free undirected graph $G_n = (V, E)$ from K_{n-1} as follows: $V = V' \cup \{v\}$, and E consists of all the edges in K_{n-1} except for the edge $\{v_1, v_2\}$, which is replaced by the pair of edges $\{v_1, v\}$ and $\{v, v_2\}$.

(a) Determine $\deg(x) + \deg(y)$ for all nonadjacent vertices x and y in V.

(b) Does G_n have a Hamilton cycle?

(c) How large is the edge set E?

(d) Do the results in parts (b) and (c) contradict Corollary 8.6?

5. If $G = (V, E)$ is an undirected graph, a subset I of V is called **independent** if no two vertices in I are adjacent.

Let $G = (V, E)$ be an undirected graph with subset I of V an independent set. For each $a \in I$ and each Hamilton cycle C for G, there will be $\deg(a) - 2$ edges in E that are incident with a and not in C. Therefore there are at least $\sum_{a \in I} [\deg(a) - 2] = \sum_{a \in I} \deg(a) - 2|I|$ edges in E that do not appear in C.

(a) Why are these $\sum_{a \in I} \deg(a) - 2|I|$ edges distinct?

(b) Let $v = |V|$, $e = |E|$. Prove that if

$$e - \sum_{a \in I} \deg(a) + 2|I| < v,$$

then G has no Hamilton cycle.

(c) Select a suitable independent set I and use part (b) to show that the graph in the below figure (known as the Herschel graph) has no Hamilton cycle.