MATH4205/MATH7620 Probability Theory and Stochastic Processes 2025-2026, Semester 1

Assignment 3

Name	Student ID	Marks

Instructions:

- Due date: 11:59 PM 7 Nov 2025 (Fri)
- 20 marks will be deducted for every 24 hours (rounded up) of late submission
- Submit a soft copy (in PDF) to Moodle
- Layout the intermediate steps systematically

1. (20 marks) Let $\{X_i \mid i \geq 1\}$ be $i.i.d.$ random variables with $P\{X_i = 1\} = P\{X_i = -1\} = 1/2$. The martingale gambling strategy is: start with stake $B_1 = 1$. After a loss, double the next stake. Formally, $B_{i+1} = 2B_i$ if $X_i = -1$. The game continuous until the first win, then the game stops. Define the wealth process $W_0 = 0$ and, for $n \geq 1$, $W_n = \sum_{i=1}^{n} B_i X_i$. Let T be the stopping time, i.e., the "first win." Note that under this strategy, if the first n outcomes are all losses, then $W_n = -(1 + 2 + \cdots + 2^{n-1}) = -(2^n - 1)$. If $X_{n+1} = 1$, then $W_{n+1} = 2^n - (2^n - 1) = 1$.
   a) Show that $\{W_n \mid n \geq 0\}$ is a martingale with respect to $\{X_i \mid i \geq 1\}$.
   b) Following a), the martingale stopping theorem should apply to $\{W_n \mid n \geq 0\}$. Compute $E(W_T)$. Is $E(W_T) = \mathrm{E}(W_0)$? Briefly explain why this conclusion does NOT contradict the Martingale Stopping Theorem. You may refer to lecture notes section 4.3.

2. (20 marks) Consider a random walk on a line which at each step either goes right 1 with probability p or left 1 with probability $q = 1 - p$. Let S_n be the position after n steps and let $S_0 = 0$ be the initial position. Show that $(\frac{q}{p})^{S_n}, n \geq 1$, is a martingale.

3. (20 marks) Let $\{B(t) \mid t \geq 0\}$ be the standard Brownian motion. Let $\{X(t) \mid t \geq 0\}$ be a Brownian motion with drift 0.1 and variance parameter 0.09. Give the answer in terms of the CDF $\Phi(\mathrm{x})$ of the standard normal random variable Z when appropriate.

   a) Determine $P\{T_{-1} < T_1 < T_{-2}\}$.
   b) Determine $P\,\{max_{0\leq s\leq 2}\, B(s) < 0.5\}$.
   c) Determine the conditional distribution of $X(1)$ given that $X(2) = 1$.
   d) Determine $E[e^{X(2)-X(1)}]$.

4. (20 marks) Suppose that the price of a stock changes according to the following geometric Brownian motion:

$$X(t) = 50e^{0.2B(t)+0.1t}.$$

   a) What is the probability that the price will exceed \$55 when $t = 1$?
   b) Find $E[X(2)|X(1) = 52]$.

5. (20 marks) Let $\{B(t) \mid t \geq 0\}$ be the standard Brownian motion.

   a) Show that $\{Y(t) \mid t \geq 0\}$ is a martingale with respect to $\{B(t) \mid t \geq 0\}$ when $Y(t) = B(t)^2 - t$.
   b) Let T to be the first time that B(t) either reaches $-A$ or B. What is $E(T)$?

Assignment 3.

1. a) To show $\{W_n\}$ is a martingale, we need to verify W_n is $\mathcal{F}_n$-measurable

$E[W_n] < \infty$ and $E[W_{n+1} | \mathcal{F}_n] = W_n$.

$W_0 = 0$, and for $n \geq 1$, $W_n = \sum_{i=1}^{n} B_i X_i$ where $B_1 = 1$, $B_{i+1} = 2B_i$

if $X_i = -1$ (lose), then $B_{i+1} = 2B_i$

if $X_i = 1$ (win), Thus, B_{n+1} is $\mathcal{F}_n$-measurable, and W_n is $\mathcal{F}_n$-measurable

as a sum of measurable terms.

for fixed n, $|W_n| \leq \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$. so $E|W_n| < \infty$.

$W_{n+1} = W_n + B_{n+1} X_{n+1}$, Then

$$E[W_{n+1} | \mathcal{F}_n] = W_n + B_{n+1} E[X_{n+1} | \mathcal{F}_n] = W_n + B_{n+1} \cdot 0 = W_n.$$

Thus, $\{W_n\}$ is a martingale.

b). T is the stopping time, so $X_T = 1$, $X_i = -1$ for $i < T$.

Also, $B_i = 2^{i-1}$ for $i = 1, 2, \cdots T$, Thus

$$W_T = \sum_{i=1}^{T-1} 2^{i-1}(-1) + 2^{T-1} \cdot 1 = 1. \quad \text{so } E(W_T) = 1.$$

but $E(W_0) = 0$. so $E(W_0) \neq E(W_T)$

in this case, T isn't bounded, $P(T > n) = (\frac{1}{2})^n > 0$ for all n

so it doesn't contradict with the theorem.

2. $S_{n+1} = S_n + X_{n+1}$. where X_{n+1} is the increament at step $n+1$, independent of $\{S_0, S_1 \cdots S_n\}$ with $X_{n+1} = \begin{cases} 1 & p \\ 0 & 1-p=q \end{cases}$

$$E[(\frac{q}{p})^{S_{n+1}} \mid S_0, S_1, \cdots S_n] = E[(\frac{q}{p})^{S_n + X_{n+1}} \mid S_n] = (\frac{q}{p})^{S_n} \cdot E[(\frac{q}{p})^{X_{n+1}} \mid S_n].$$

$$E[(\frac{q}{p})^{X_{n+1}}] = P(X_{n+1} = 1) \cdot (\frac{q}{p})^1 + p(X_{n+1} = -1) \cdot (\frac{q}{p})^{-1}$$

$$= p \cdot \frac{q}{p} + q \cdot \frac{p}{q} = q + p = 1 - p + p = 1.$$

So $E[(\frac{q}{p})^{S_{n+1}} \mid S_0, \cdots S_n] = (\frac{q}{p})^{S_n} \cdot 1 = (\frac{q}{p})^{S_n}$

thus, $(\frac{q}{p})^{S_n}$, $n \geq 1$ is a martingales.

3. a) The probability of a standard Brown emotion hitting a boundary "a" before a boundary "b" is $\frac{|b|}{|a|+|b|}$, so $P(T_{-1} < T_1) = \frac{|1|}{|-1|+|1|} = \frac{1}{2}$

$$P(T_1 < T_{-2} \mid X_0 = -1) = \frac{|-2|}{|-2|+|1|} = \frac{2}{3}, \text{ Thus } P\{T_{-1} < T_1 < T_{-2}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

b) let $M_t = \max_{0 \leq s \leq t} B(s)$, $P(M_t \leq a) = 2\Phi(\frac{a}{\sqrt{t}} - 1) = 2\Phi(\frac{0.5}{2}) - 1 = 2\Phi(\frac{\sqrt{2}}{4}) - 1.$

c). $X(t)$ is Brown emotion with $\mu = 0.1$ and $\sigma = 0.3$. the increament

$$X(2) - X(1) \sim N(\mu \cdot 1, \sigma^2 \cdot 1) = N(0.1, 0.09).$$

$X(1) \mid X(2) = 1$ is normal with μ_1 and σ_1

$$\mu_1 = X(0) + \frac{t}{T}(X(T) - X(0)) = \frac{1}{2} \cdot 1 = 0.5$$

$$\sigma_1^2 = \sigma^2 t (1 - \frac{t}{T}) = 0.09 \cdot 1 \cdot (1 - \frac{1}{2}) = 0.045.$$

Thus $X(1) \mid X(2) = 1 \sim N(0.5, 0.045)$.

d) let $Y = X(2) - X(1) \sim N(0.1, 0.09)$. for $Y \sim N(0.1, 0.09)$.

$$E[e^Y] = E[e^{X(2) - X(1)}] = e^{0.1 + \frac{0.09}{2}} = e^{0.145}$$

4\. a). $X(1) = 50e^{0.2B(1)+0.1}$, we need $P(X(1)>55)$

$$50e^{0.2B(1)+0.1} > 55 \Rightarrow e^{0.2B(1)+0.1} > 1.1 \Rightarrow 0.2B(1)+0.1 > \ln(1.1)$$

$$B(1) > \frac{\ln(1.1)-0.1}{0.2}$$, since $B(1) \sim N(0,1)$.

so $P(X(1)>55) = P(B(1) > \frac{\ln(1.1)-0.1}{0.2}) = \Phi(\frac{0.1-\ln(1.1)}{0.2})$

b) $X(1)=52$, $50e^{0.2B(1)+0.1} = 52 \Rightarrow B(1) = \frac{\ln(1.04)-0.1}{0.2}$

$B(2) = B(1) + (B(2)-B(1))$, $B(2)-B(1) \sim N(0,1)$

$$E[X(2) \mid X(1)=52] = E[50e^{0.2B(2)+0.2} \mid X(1)=52]$$

$$= E[50e^{0.2[B(1)+(B(2)-B(1))]+0.2} \mid \cancel{B(1)}\, X(1)=52]$$

$$= 50\,E[e^{0.2(B(2)-B(1))}] \cdot E[\cancel{0.2}\, e^{0.2B(1)+0.2} \mid X(1)=52]$$

$$= 50 \cdot e^{0+\frac{1}{2}(0.2)^2 \cdot 1} \; \cancel{e^{0.02}} \cdot 1.04 \cdot e^{0.1}$$

$$= 52 \cdot e^{0.12}$$

5\. a) Since $Y(t)$ is function of $B(t)$, so it's $\mathcal{F}$-measurable

$E[B(t)^2] = t$, so $E[Y(t)] = 0$. $Var(Y(t)) = E[(B(t)^2 - t)^2]$

$= E[B(t)^4] - 2t\,E[B(t)^2] + t^2 = 3t^2 - 2t^2 + t^2 = 2t^2$

so ~~$E|Y(t)|$~~ $E|Y(t)| < \infty$ for each $t \geq 0$.

for $0 \leq s < t$, $E[Y(t) \mid \mathcal{F}_s] = E[B(t)^2 - t \mid \mathcal{F}_s] = E[(B(t)-B(s)+B(s))^2 \mid \mathcal{F}_s]$

$= E[(B(t)-B(s))^2 \mid \mathcal{F}_s] + 2B(s)E[B(t)-B(s) \mid \mathcal{F}_s] + B(s)^2 - t$

since $B(t)-B(s)$ is independent of $\mathcal{F}_s$ with $E(B(t)-B(s)=0)$.

$E[(B(t)-B(s))^2] = t-s$.

$(t-s) + 2B(s) \cdot 0 + B(s)^2 - t = B(s)^2 - s = Y(s)$.

So $\{Y(t)\}$ is a martingale

b) $Y(t)$ is a martingale, apply optional stopping theorem.

$$E[Y(T)] = E[B(T)^2 - T] = E[B(T)^2] - E[T] = 0.$$

So $E[T] = E(B(T)^2]$.

$P(\text{hit } B \text{ first}) = \frac{A}{A+B}$, $\quad P(\text{hit } -A \text{ first}) = \frac{B}{A+B}$.

$$E(B(T)^2) = \frac{A}{A+B} \cdot B^2 + \frac{B}{A+B} \cdot A^2 = \frac{AB^2 + A^2B}{A+B} = AB$$

Thus, $E(T) = AB$.