Assignment 2 **MA-4093**

Complex Numbers.

1. Write the following complex numbers in polar form:

$$z_1 = 3 + 3i, \quad z_2 = 2 - i3\sqrt{2}, \quad z_3 = -3 - \sqrt{3}i.$$

2. Find each of the following numbers and give your answer in the form $z = x + iy$:

$$z_1 = \sqrt{-8i}, \quad z_2 = \sqrt{-11 + 60i}, \quad z_3 = \sqrt[4]{-1}.$$

3. Find all solutions of the equations

$$iz^2 + 2iz + 2 + i = 0, \tag{1}$$
$$z^4 + z^2 + 1 = 0. \tag{2}$$

4. Describe the image $f(D)$ of the set $D \subset \mathbb{C}$ under the mapping f, if:
   (a) $f(z) = \overline{z}$ and $D := \{z \in \mathbb{C} : |z - (2+i)| \leq 1\}$;
   (b) $f(z) = (\mathrm{Re}\,(z))^2 + 2i$ and $D := \{z \in \mathbb{C} : |z| \leq 1\}$;
   (c) $f(z) = z + 6$ and $D := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$;
   (d) $f(z) = 1/z$ and $D := \{z \in \mathbb{C} : 0 < |z| < 1\}$;
   (e) $f(z) = z^2$ and $D := \{z \in \mathbb{C} : 1 \leq \mathrm{Re}\ \leq 2\}$;

5. Determine the following complex numbers:

$$z_1 = 3\log(1 + i\sqrt{3}), \quad z_2 = \log(1 + i\sqrt{3})^3, \quad z_3 = \log(1+i)^{\pi i}.$$

6. Solve each of the equations:

$$e^z = 1 + i, \tag{3}$$
$$\mathrm{Log}\,(1 + z) = i\pi, \tag{4}$$
$$e^{z+1} + 2 = 0. \tag{5}$$

1. $z_1 = 3 + 3i$

$= \sqrt{3^2+3^2}\ \frac{3}{\sqrt{3^2+3^2}} + \frac{3}{\sqrt{3^2+3^2}} i$

$= 3\sqrt{2}\ (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i)$

$= 3\sqrt{2}\left[\cos(\frac{\pi}{4}+2k\pi) + i\sin(\frac{\pi}{4}+2k\pi)\right]$

$(k \in \mathbb{Z})$

$z_2 = 2 - i3\sqrt{2}$

$|z_2| = \sqrt{4+18} = \sqrt{22}$

$\text{Arg } z_2 = \arctan \frac{-3\sqrt{2}}{2}$

$= -1.13$

Hence $z_2 = \sqrt{22}\left[\cos(-1.13+2k\pi) + i\sin(-1.13+2k\pi)\right]$

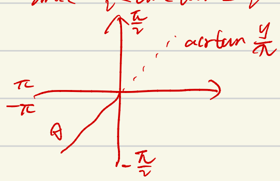
note : $x<0,\ y\geq 0$.

$\text{Arg } z = \arctan(\frac{y}{x}) + \pi$

$x<0,\ y<0$,

$\text{Arg } z = \arctan(\frac{y}{x}) - \pi$

Since $-\frac{\pi}{2} < \arctan < \frac{\pi}{2}$



$z_3 = -3 - \sqrt{3}\ i$

$|z_3| = \sqrt{9+3} = 2\sqrt{3}$

$\text{Arg } z_3 = \arctan \frac{-\sqrt{3}}{-3} - \pi$

$= -\frac{5\pi}{6}$

Hence, $z_3 = 2\sqrt{3}\left[\cos(-\frac{5\pi}{6}+2k\pi) + i\sin(-\frac{5\pi}{6}+2k\pi)\right]$

2. $z_1 = \sqrt{-8i}$

$z_1^2 = -8i$

$z_1^2 = 8\ \cos(-\frac{\pi}{2}+2k\pi) + i\sin(-\frac{\pi}{2}+2k\pi)$, $k \in \mathbb{Z}$

$z_1 = 2\sqrt{2}\ \cos(-\frac{\pi}{4}+k\pi) + i\sin(-\frac{\pi}{4}+k\pi)$

where $n=2$, then $k=0$ or $k=1$

Hence $z_1 = 2\sqrt{2}\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

or $z_1 = 2\sqrt{2}\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$z_2 = \sqrt{-11+60i}$

$z_2^2 = -11 + 60i$

$|-11+60i| = \sqrt{11^2+60^2} = 61$

$\text{Arg}\ (-11+60i) = \arctan(\frac{60}{-11}) + \pi$

$\doteq 1.75$

$z_2^2 = 61\left[\cos(1.75+2k\pi) + i\sin(1.75+2k\pi)\right]$ $k \in \mathbb{Z}$

$z_2 = \sqrt{61}\left[\cos(0.875+k\pi) + i\sin(0.875+k\pi)\right]$

$n=2$, $k=0$ or $k=1$

$z_2 = \sqrt{61}\ (\cos 0.875 + i\sin 0.875)$ or

$z_2 = \sqrt{61}\left[\cos(0.875+\pi) + i\sin(0.875+\pi)\right]$

let $z_2 = x + iy$

$z_2^2 = x^2 - y^2 + 2xyi$

$\begin{cases} x^2-y^2 = -11 \\ 2xy = 60 \end{cases} \Rightarrow \begin{cases} x=5 \\ y=6 \end{cases}$ or $\begin{cases} x=-5 \\ y=-6 \end{cases}$

$z_2 = 5 + i6$ or

$z_2 = -5 - i6$

$z_3 = \sqrt[4]{-1}$

$z_3^4 = -1$

$z_3^4 = \cos(\pi+2k\pi) + i\sin(\pi+2k\pi)$. $(k \in \mathbb{N})$

$z_3 = \cos(\frac{\pi}{4}+\frac{k}{2}\pi) + i\sin(\frac{\pi}{4}+\frac{k}{2}\pi)$

when $n=4$, $k=0,1,2,3$.

Hence $z_3 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ or

$z_3 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ or

$z_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ or

$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$.

3. (1) $iz^2+2iz+2+i=0$

$-z^2-2z+2i-1=0$

$z^2+2z-2i+1=0$

$(z+1)^2=2i$

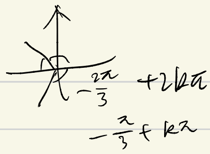
$(z+1)^2=2e^{i(\frac{\pi}{2}+2k\pi)}$

$z+1=\sqrt{2}\,e^{i(\frac{\pi}{4}+k\pi)}$

$=\sqrt{2}\,e^{i\frac{\pi}{4}}\ /\ e^{i\frac{5\pi}{4}}$

$=1+i\ /\ -1-i$

$z=2+i\ /\ -i$



(2) $z^4+z^2+1=0$

$z^4+z^2+\frac{1}{4}+\frac{3}{4}=0$

$(z^2+\frac{1}{2})^2=-\frac{3}{4}$

$z^2+\frac{1}{2}=\frac{\sqrt{3}}{2}i$

$z^2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$z^2=\cos(\frac{2\pi}{3}+2k\pi)+i\sin(\frac{2\pi}{3}+2k\pi)$, $(k\in\mathbb{N})$

$z=\cos(\frac{\pi}{3}+k\pi)+i\sin(\frac{\pi}{3}+k\pi)$

where $n=2$, $k=0$ or $k=1$

Hence $z=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ or

$z=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

or $z^2+\frac{1}{2}=-\frac{\sqrt{3}}{2}i$

$z=\cos-\frac{\pi}{3}+i\sin-\frac{\pi}{3}$ or

$z=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

4. (a) $|z-(2+i)|\le 1$

let $z=x+iy$,

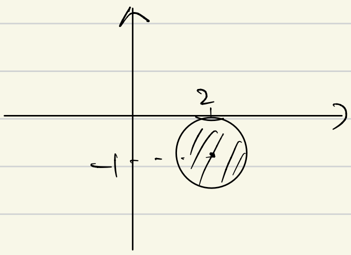
$|x+iy-(2+i)|\le 1$

$|x-2+i(y-1)|\le 1$

$\sqrt{(x-2)^2+(y-1)^2}\le 1$

$(x-2)^2+(y-1)^2\le 1$

For $f(z)=\bar{z}=x-iy$:



(b) $D:=|z|\le 1$ $z\in\mathbb{C}$

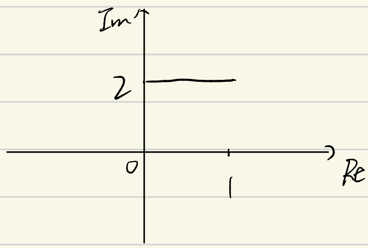
Let $z=x+iy$

Then $x^2+y^2\le 1$

$0<x^2\le 1-y^2\le 1$

$f(z)=(\mathrm{Re}(z))^2+2i$

$=x^2+2i$:



(c) $D:=z\in\mathbb{C}, \mathrm{Re}\,z>0$

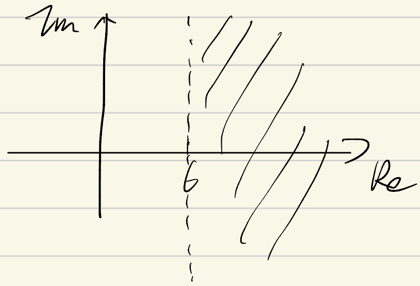
Let $z=x+iy$,

then $x>0$

$f(z)=z+6$

$=x+6+iy$

$x+6>6$, y is arbitrary



(d) $D:=0<|z|<1$, $z\in\mathbb{C}$

let $z=x+iy$, then

$0<x^2+y^2<1$

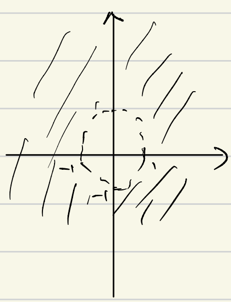
$f(z)=1/z$

$=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$

$=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$

$|f(z)|=(\frac{x}{x^2+y^2})^2+(\frac{y}{x^2+y^2})^2$

$=\frac{1}{x^2+y^2}>1$



(e) $D:=\{z\in\mathbb{C}:1\le\mathrm{Re}\le 2\}$

let $z=x+iy$, then

Boundary Condition: $1\le x\le 2$

$f(z)=z^2=(x+iy)^2$

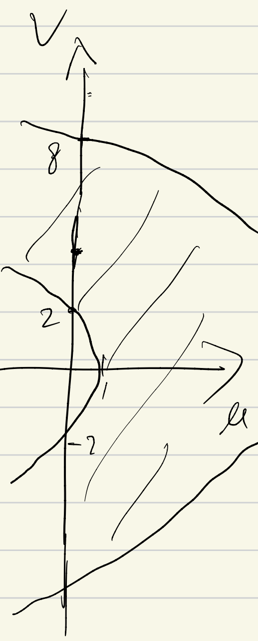
$=x^2-y^2+i2xy$

$x^2-y^2\le 4$, $2xy$ is arbitrary

$1\le x^2\le 4$

$u=x^2-y^2\Rightarrow u=x^2-\frac{v^2}{4x^2}$

$v=2xy\Rightarrow y=\frac{v}{2x}$



5. $z_1 = 3\ln(1+i\sqrt{3})$

let $w = 1+i\sqrt{3}$

$|w| = \sqrt{1+3} = 2$

$\operatorname{Arg} w = \arctan\sqrt{3} = \frac{\pi}{3}$

$w = 2e^{i(\frac{\pi}{3}+2k\pi)}$ $k \in \mathbb{Z}$

$z_1 = 3\ln w = 3[\ln 2 + i(\frac{\pi}{3}+2k\pi)]$

$= 3\ln 2 + i(\pi + 6k\pi)$, $k \in \mathbb{Z}$

$z_2 = \ln(1+i\sqrt{3})^3$

$w = 1+i\sqrt{3}$

from z_1, $w = 2e^{i(\frac{\pi}{3}+2k\pi)}$

then, $z_2 = \ln w^3$

$= \ln 2^3 e^{3i(\frac{\pi}{3}+2k\pi)}$

$= 3\ln 2 + i(\pi + 6k\pi)$, $k \in \mathbb{Z}$

$z_3 = \ln(1+i)^{\pi i}$

$z_3 = \ln\left[\sqrt{2}e^{i(\frac{\pi}{4}+2k\pi)}\right]^{\pi i}$, $k \in \mathbb{Z}$

$= \ln e^{[1/2\ln 2 + i(\frac{\pi}{4}+2k\pi)]\pi i}$

$= \ln e^{(-\frac{\pi^2}{4}-2k\pi^2)+i\frac{\pi}{2}\ln 2}$

$= -\frac{\pi^2}{4} - 2k\pi^2 + i(\frac{\pi}{2}\ln 2 + 2j\pi)$, $j, k \in \mathbb{Z}$

6. (3) $e^z = 1+i$

$z = \ln(1+i)$

$= \ln\sqrt{2}e^{i(\frac{\pi}{4}+2k\pi)}$, $k \in \mathbb{Z}$

$= \ln\sqrt{2} + i(\frac{\pi}{4}+2k\pi)$

$e^z = 1+i$

let $z = x+iy$

$e^{x+iy} = 1+i$

$e^x e^{iy} = \sqrt{2}e^{i(\frac{\pi}{4}+2k\pi)}$, $k \in \mathbb{Z}$

$\begin{cases} e^x = \sqrt{2} \\ y = \frac{\pi}{4} + 2k\pi \end{cases}$

$x = 1/2\ln 2$

Hence $z = 1/2\ln 2 + i(\frac{\pi}{4}+2k\pi)$, $k \in \mathbb{Z}$

(4) $\operatorname{Log}(1+z) = i\pi$

$1+z = re^{i\theta}$ $(-\pi < \theta \le \pi$

$\operatorname{Log}(1+z) = \ln r + i\theta = i\pi$

$r = 1$, $\theta = \pi$

$1+z = e^{i\pi}$

$z = e^{i\pi} - 1$

$= -2$

(5) $e^{z+1} + 2 = 0$

let $z = x+iy$

$z+1 = x+1+iy$

$e^{x+1+iy} + 2 = 0$

$e^{x+1}e^{iy} = -2$

$\begin{cases} e^{x+1} = 2 \\ e^{iy} = -1 \end{cases} \Rightarrow \begin{cases} x = \ln 2 - 1 \\ y = \pi + 2k\pi \end{cases}$, $k \in \mathbb{Z}$

Hence $z = \ln 2 - 1 + i(\pi + 2k\pi)$, $k \in \mathbb{Z}$