Chapter 1
Matrix

① Def.

An array of number in the field Q

② operation

Equal

$A = (a_{ij}) = B = (b_{ij})$

if A and B have the same demension

and $a_{ij} = b_{ij}$

$i = 1, 2, 3 \dots m$

$j = 1, 2, 3 \dots n$ $\quad A_{m \times n} = (a_{ij})$

scalar multiplication $k \cdot A_{m \times n} = (k a_{ij})_{m \times n}$

Addition

$A = (a_{ij})_{m \times n}$

$B = (b_{ij})_{m \times n}$

A and B must have the same demension $m \times n$

$C = A + B$ with $c_{ij} = a_{ij} + b_{ij}$

$A - A = (0)$ matrix, not a number

$AB = C$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$i = 1, 2, \dots m$

$j = 1, 2, \dots n$

-matrix multiplication

If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times r}$

AB is a matrix denoted by C

be careful of demension

of the demension $m \times r$ and

demension restrection

$$C_{m\times r} = \underbrace{A_{m\times n} \times B_{n\times r}}$$

only in this case, we have the definition of multiplication

Square matrix

$$A_{mm} = \begin{Bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mm} \end{Bmatrix} \to \text{diagonal}$$

$$A_{m\times n} B_{n\times r} = A(b_1\ b_2 \cdots b_r)$$
$$= (Ab_1, Ab_2 \cdots Ab_r)$$

$$A_{m\times n}\ b_{n\times r}$$

$$\begin{Bmatrix} 1 & 2 \\ 0 & -1 \end{Bmatrix} \quad \begin{Bmatrix} -2 & 0 & 1 \\ 3 & 1 & 1 \end{Bmatrix} \quad \underline{2\times 3}$$

2×2 $\qquad 2\times 3$

$1\times -2 + 2\times 3 = 4$ (6) $\qquad 1\times 0 + 2\times 1 = 2 \qquad 1\times 1 + 2\times 1 = 3$

$0\times -2 + -1\times 3 = -3 \qquad 0\times 0 + -1\times 1 = -1 \qquad 0\times 1 + -1\times 1 = -1$

$$\begin{Bmatrix} 4 & 2 & 3 \\ -3 & -1 & -1 \end{Bmatrix}$$

Ex: $A = \begin{Bmatrix} 1 & 2 \\ 0 & -1 \end{Bmatrix}$ $\quad B = \begin{Bmatrix} -2 & 0 & 1 \\ 3 & 1 & 1 \end{Bmatrix}$

$$C_{2\times 3} = \underbrace{A_{2\times 2}\ B_{2\times 3}} = \begin{Bmatrix} 1\cdot(-2) + 2\times 3 \\ \\ 0\cdot(-2) + (-1)\times 3 \end{Bmatrix}$$

Ex $\underset{2\times 3}{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ $\underset{3\times 1}{B} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\underset{3\times 3}{C} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ $\underset{1\times 3}{D} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$

$\underset{2\times 1}{AB} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ $\underset{3\times 3}{BD} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ -2 & -1 & 0 \end{bmatrix}$

$\underset{2\times 3}{AC} = \begin{bmatrix} 1 & 3 & 1 \\ 8 & 2 & 1 \end{bmatrix}$ $\underset{1\times 1}{DB} = \begin{bmatrix} 4 \end{bmatrix}$

DCB $\quad \underset{1\times 3}{(DC)} \times \underset{3\times 1}{B} = \underset{1\times 1}{\begin{bmatrix} 4 \end{bmatrix}}$

$\underset{3\times 1}{CB} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

- transpose

Given $A = (a_{ij})_{m\times n}$

transpose of A: $A^T = (\tilde{a}_{ij})_{n\times m}$

$$\boxed{\tilde{a}_{ij} = a_{ji}}$$

$(AB)^T = B^T A^T$

Proof: entry (i,j) of $B^T A^T = \sum_{k=1} \tilde{b}_{ik} \tilde{a}_{kj}$

$$= \sum_{k=1} b_{ki} a_{jk}$$

$$= \sum_{k=1} a_{jk} b_{ki}$$

$=$ entry (j,i) of AB

$=$ entry (i,j) of $(AB)^T$

Symmetric $A_{n\times n}$ (square)

$a_{ij}=a_{ji}$ (i.e $A^T=A$)

Ex: $A_{m\times n}$ prove that A^TA, AA^T are symmetric

$(A^TA)^T=(A^T(A^T)^T=A^TA$

$(AA^T)^T=(A^T)^TA^T=AA^T$

Ex: Let A and B to be $\begin{cases}\text{upper}\\ \text{lower}\end{cases}$ trangiral matrix

prove AB is also $\begin{cases}\text{upper}\\ \text{lower}\end{cases}$ trangoal matrix

$a_{ij}=0 \quad i<j \qquad b_{ij}=0 \quad i<j$

$AB=C \qquad C_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$

Since $\begin{cases} a_{ik}=0 & \text{for } i<k \\ b_{kj}=0 & \text{for } k<j \end{cases}$

so $C_{ij}=0$ for $i<j$

Uniqueness: Assume that B_1 and B_2 are inverse of A

$\begin{cases} AB_1=B_1A=I \\ AB_2=B_2A=I \end{cases}$ $\qquad B_1=B_1I=B_1AB_2=IB_2=B_2$

So inverse is unique

Ex: A, B, C are matrix. if $AB=AC$, is $B=C$?

$AB=0$, A and B can be both not 0 matrix

$A^{-1}\cdot AB=A^{-1}\cdot AC \qquad IB=IC \qquad B=C$

(i) $(\alpha A)^{-1}=\frac{1}{\alpha}A^{-1}$ if $\alpha\neq 0$

(ii) $(A^T)^{-1}=(A^{-1})^T$

Since A is invertible $A\cdot A^{-1}=I \quad A^{-1}\cdot A=I$

$(AA^{-1})^T=I^T=I=(A^{-1})^T\cdot A^T$

$(A^{-1}A)^T = I^T = I = A^T \cdot (A^{-1})^T$

Ex: A is symmetric and invertible prove A^{-1} is symmetric

$A^T = A$ $\quad A \cdot A^{-1} = I$

$(A \cdot A^{-1})^T = I^T = I \quad (A^{-1})^T \cdot A^T = I$

$(A^{-1})^T \cdot A = I = A^{-1} \cdot A$

$(A^{-1})^T = (A^{-1})$ So (A^{-1}) is symmetric

$AB = I \overset{?}{\Rightarrow} BA = I$

$$\begin{cases} x_1 + 4x_2 - 2x_3 = 4. \\ 2x_1 - 2x_2 + x_3 = 1 \\ 3x_1 + 2x_2 + 2x_3 = 11 \end{cases}$$

matrix formula

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & -2 & 1 \\ 3 & 2 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 1 \\ 11 \end{bmatrix}$$

扩充

Augemented matrix

$$[A\ b] = \left(\begin{array}{ccc:c} 1 & 4 & -2 & 4 \\ 2 & -2 & 1 & 1 \\ 3 & 2 & 2 & 11 \end{array}\right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 & -2 & 4 \\ 0 & -10 & 5 & -7 \\ 0 & -10 & 8 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 = R_3 - R_2]{} \begin{pmatrix} 1 & 4 & -2 & 4 \\ 0 & -10 & 5 & -7 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

back substitution $3x_3 = 6 \Rightarrow x_3 = 2$

structure of solution

- consistency : a system has a certain solution
- inconsistency : a system doesn't have a certain solution

equivalent : two systems have the same solution.

parametric form (have infinite solution)

$$\begin{cases} x_1 = 3-3\alpha \\ x_2 = 1-\alpha \\ x_3 = \alpha \quad (\alpha \in \mathbb{R}) \end{cases}$$

Elementary Row Operations

I. Interchange two rows.

II. Multiply a row by a nonzero real number.

III. Replace a row by its sum with a multiple of another row.

Row1 = Row1 + k · Row 2

Augmented matrix

Ex: $[A, b] = \begin{bmatrix} 2 & -3 & -1 & 2 & -2 \\ 1 & 0 & 3 & 1 & 6 \\ 2 & -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & -2 & 4 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - \frac{1}{2}R_2} \begin{bmatrix} 2 & -3 & -1 & 2 & -2 \\ 0 & \frac{3}{2} & \frac{7}{2} & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ no solution

1.2 Row echelon form

① Def

A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each nonzero row is 1.

(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

leading zero

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & \cdots \end{array} \\ \begin{bmatrix} 1 & x & x & x & x & \cdots \\ 0 & 0 & 1 & x & x & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \; 1 \; x \; x \\ 0 & \cdots & & & & 0 \end{bmatrix} \begin{array}{l} \\ \text{row } k \\ \text{row } k+1 \\ \\ \end{array} \end{array}$$

For linear system (x_1, x_3) leading varible

(x_2) others: free varible

$$\begin{cases} x_1 = \boxed{1\alpha} \\ x_2 = \alpha \\ x_3 = \boxed{1\alpha} \end{cases}$$

② Gaussian elimination

$Ax = b$

if there are solutions

$[A, b] \xrightarrow{\text{row operation}}$ [echelon form] $\xrightarrow{\text{back substitution}}$ Sol $\begin{cases} \text{one} \\ \text{infinite} \end{cases}$

if no sol say no

③ Gaussian - Jordan reduction

$$[A, b] \xrightarrow{\text{Row operation}} \begin{bmatrix} 1 & 1 & -3 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 = R_2 + 4R_3]{R_1 = R_1 + 3R_3} \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \text{reduced echelon form}$$

④ special cases $Ax=b$ $A_{m \times n}$

Over determined systems $m > n$

the number of equations are more than the number of unknowns

under determined systems $m < n$

the number of equations are less than the number of unknowns

$n > m \geq$ no. of leading variable

$n - m =$ no. of free variables

- Homogeneous systems $Ax = 0$
  - $x = 0$ trivial solution
  - non zero sol nontrivial solution (including infinitely many solutions)

If $Ax=0$ has a nonzero sol. $\hat{x}$?
$\hat{x} = \alpha \hat{x}$ $2A\hat{x} = 0$ $A(2\hat{x}) = 0$

Ex: $Ax = 0$

$$A = \begin{bmatrix} 1 & -1 & 0 & -2 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Row operation}} \begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1/5 \end{bmatrix}$$

set $x_4 = \alpha$

$x_3 = -\frac{\alpha}{5}$

Ex: Cosider $Ax = b$

let $Ax=0$ be the corresponding homogeneous system

Assume that $\hat{x}$ that $A\hat{x} = 0$

$A(\tilde{x} + \hat{x}) = A\tilde{x} + A\hat{x} = 0 + b = b$, So $\tilde{x} + \hat{x}$ is a sol of $Ax = b$

— $A_{n\times n}$ is invertible

$Ax=b \quad A^{-1}Ax=A^{-1}b \quad Ix=A^{-1}b \quad x=A^{-1}b \Rightarrow$ unique Sol.

For $m<n$

Theorem: For $m<n$, (i) the system $A_{m\times n}x=0$ always has infinitely many solutions

(ii) if the system $A_{m\times n}x=b$ is consistent, then the system has infinitely many solutions.

1.5 Elementary matrices

① Def

② Ex $\quad Ax=b$, $A=\begin{bmatrix} 1 & 4 & -2 \\ 2 & -2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ $\quad b=\begin{bmatrix} 4 \\ 1 \\ 11 \end{bmatrix}$

$$[A\ b]=\begin{pmatrix} 1 & 4 & -2 & 4 \\ 2 & -2 & 1 & 1 \\ 3 & 2 & 2 & 11 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \begin{pmatrix} 1 & 4 & -2 & 4 \\ 0 & -10 & 5 & -7 \\ 0 & -10 & 8 & -1 \end{pmatrix}$$

② Therom *If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.*

Row equivalent (Def) A matrix B is **row equivalent** to a matrix A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A$$

Theorem 1.5.2 Equivalent Conditions for Nonsingularity

Let A be an $n \times n$ matrix. The following are equivalent:

(a) *A is nonsingular.*
(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution* $\mathbf{0}$.
(c) *A is row equivalent to I.*
(d) $Ax=b$ has a unique solution for any b

1.5 Elementary matrix

Theorem 1.5.2 Equivalent Conditions for Nonsingularity

Let A be an $n \times n$ matrix. The following are equivalent:

(a) *A is nonsingular.*
(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.*
(c) *A is row equivalent to I.*
(d) $Ax=b$ has a unique solution for any b

$c \Rightarrow d$: Since (c), then $A = E_k E_{k-1} \cdots E_1 I$ $\quad E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} A = I$

$$E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} A x = E_k^{-1} \cdots E_1^{-1} b$$

$$x = E_k^{-1} \cdots E_1^{-1} b$$

Let $\hat{x}$ be another solution, $A\hat{x} = b$

$$E_k^{-1} \cdots E_1^{-1} A \hat{x} = E_k^{-1} \cdots E_1^{-1} b$$

$$\hat{x} = E_k^{-1} \cdots E_1^{-1} b$$

$d \Rightarrow a$ $\quad Ax=b$ has a unique solution for any b

If A is singular, by (b) $Ax=0$ has nonezero solution.

Therom: If $AB = I$, then $BA = I$ and $B = A^{-1}$

proof: Since $AB = I$, There exist $E_k \cdots E_1$ such that

$$E_k \cdots E_1 A = U, \quad \underbrace{E_k \cdots E_1 A}\, B = E_k \cdots E_1 I$$

UB

③ Computation of inverse

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix} \qquad AB = I, \text{ to find the } B$$

Gauss-Jordan

$$(A \,|\, I) = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 / -6} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & 1/-6 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & 1/-6 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{7}{6} & 1 \end{array}\right] \xrightarrow{R_3 = R_3 \cdot -2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/3 & -1/6 & 0 \\ 0 & 0 & 1 & 4/3 & 1/2 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & 2/3 & 4/3 & -5 \\ 0 & 1 & 0 & -1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{bmatrix}$$

$$(A, I) \xrightarrow{\text{Row operation}} (I, A^{-1})$$

$$E_k \cdots E_1 [A, I] = [I, B]$$

$$\begin{cases} E_k \cdots E_1 A = I \Rightarrow BA = I \\ E_k \cdots E_1 I = B \end{cases}$$

④ Triangular factorization

1.6 partioned matrix

① Def

② "row" operation

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} A = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

③ some special cases

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \qquad \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}$$