Discrete Mathematics Homework 4

1. Show that the number of partitions of a positive integer n where no summand appears more than twice equals the number of partitions of n where no summand is divisible by 3.

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Solution

Let $f(x)$ be the generating funcion for the number of partitions of n where no summand appears more than twice. Let $g(x)$ be the generating function for the number of partitions of n where no summand is divisible by 3.

$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7} \cdots$

$f(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots$

$= \frac{1-x^3}{1-x} \cdot \frac{1-x^6}{1-x^2} \cdot \frac{1-x^9}{1-x^3} \cdot \frac{1-x^{12}}{1-x^4} \cdots = g(x).$

$\to 5$

$\to 5$

2. (a) Find the exponential generating function for the number of ways to arrange n letters, $n \geq 0$, selected from each of the following words.

(i) HAWAII

(ii) MISSISSIPPI

(iii) ISOMORPHISM

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(Pxt)

(b) For (ii) of part (a), what is the exponential generating function if the arrangement must contain at least two I's?

Solution

(a) (i) $(1+x)^2(1+x+(x^2/2!))^2$

(ii) $(1+x)(1+x+(x^2/2!))(1+x+(x^2/2!)+(x^3/3!)+(x^4/4!))^2$

(iii) $(1+x)^3(1+x+(x^2/2!))^4$

(b) $(1+x)\cdot(1+x+(x^2/2!))\cdot(1+x+(x^2/2!)+(x^3/3!)+(x^4/4!))\cdot((x^2/2!)+(x^3/3!)+(x^4/4!))$.

3. In how many ways can we select seven nonconsecutive integers from $\{1, 2, 3, \ldots, 50\}$?

Solution

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$1 \le x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 \le 50$

Using the ideas developed in Example 3.8, we consider one such subset: $1 \le 1 < 3 < 6 < 10 < 15 < 30 < 42 \le 50$. This subset determines the differences 0,2,3,4,5,15,12,8, which sum to 49.

A second such subset is $1 \le 7 < 9 < 15 < 21 < 32 < 43 < 50 \le 50$, which provides the differences 6,2,6,6,11,11,7, 0, which also sum to 49.

These observations suggest a one-to-one correspondence between the subsets and the integer solutions of $c_1 + c_2 + c_3 + \ldots + c_8 = 49$ where $c_1, c_8 \ge 0$ and $c_i \ge 2$ for $2 \le i \le 7$. The number of these solutions is the coefficient of x^{49} in the generating function $(1+x+x^2+\ldots)(x^2+x^3+\ldots)^6(1+x+x^2\ldots) = [1/(1-x)^2][x^{12}/(1-x)^6] = x^{12}/(1-x)^8$.

The answer then is the coefficient of x^{37} in $(1-x)^{-8}$ and this is $\binom{-8}{37}(-1)^{37} = (-1)^{37}\binom{8+37-1}{37}(-1)^{37} = \binom{44}{37}$.

4. How many 20-digit quaternary (0, 1, 2, 3) sequences are there where:

(a) There is at least one 2 and an odd number of 0's?

(b) No symbol occurs exactly twice?

(c) No symbol occurs exactly three times?

(d) There are exactly two 3's or none at all?

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(4x7)

Solution

$\frac{e^x - e^{-x}}{2}$

$(e^x - 1)$

(a) $f(x) = (x + (x^3/3!) + (x^5/5!) + \ldots) \cdot (x + (x^2/2!) + (x^3/3!) + \ldots) \cdot (e^x)(e^x) = (1/2)(e^x - 1)^2(e^{2x}) = (1/2)(e^x - 1)(e^{3x} - e^x) = (1/2)(e^{4x} - e^{3x} - e^{2x} + e^x)$.
The answer is the coefficient of $x^{20}/(20!)$ in $f(x)$ which is $(1/2)[4^{20} - 3^{20} - 2^{20} + 1]$.

(b) $g(x) = (1 + x + (x^3/3!) + (x^4/4!) + \ldots)^4 = (e^x - (x^2/2))^4 = e^{4x} - \binom{4}{1}e^{3x}(x^2/2) + \binom{4}{2}e^{2x}(x^2/2)^2 - \binom{4}{3}e^x(x^2/2)^3 + (x^2/2)^4$. The coefficient of $x^{20}/(20!)$ in $g(x)$ is $4^{20} - \binom{4}{1}(1/2)(3^{18})(20)(19) + \binom{4}{2}(1/4)(2^{16})(20)(19)(18)(17) - \binom{4}{3}(1/8)(1^{14})(20)(19)(18)(17)(16)(15)$

$x^3/6$

(c) $h(x) = (1 + x + (x^3/3!) + (x^4/4!) + \ldots)^4 = (e^x - (x^2/2))^4 = e^{4x} - \binom{4}{1}e^{3x}(x^3/6) + \binom{4}{2}e^{2x}(x^3/6)^2 - \binom{4}{3}e^x(x^3/6)^3 + (x^3/6)^4$. The coefficient of $x^{20}/(20!)$ in $h(x)$ is $4^{20} - \binom{4}{1}(1/6)(3^{17})(20)(19)(18) + \binom{4}{2}(1/6)^2(2^{14})(20)(19)(18)(17)(16)(15) - \binom{4}{3}(1/6)^3[(20!)/(11!)]$.

(d) The coefficient of $x^{20}/(20!)$ in $(e^x)^3(1 + (x^2/2!)) = e^{3x} + e^{3x}(x^2/2!)$ is $3^{20} + (1/2)(3^{18})(20)(19)$.

5. For a positive integer n, we partition n into summands of 1, 2 and 3 with **ordering**. For example, 3 can be partitioned into

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$$3 = 1 + 1 + 1 = 1 + 2 = 2 + 1$$

First two of these partitions have an odd number of parts, and the last two have an even number of parts.

(4×8)

(a) Let $p(n)$ be the number of partitions (with ordering) of n. Find a recurrence relation of $p(n)$, $n \geq 4$. Explain your relation in detail. Calculate $p(7)$.

(b) Find the generating function of partitions (with ordering) of n that have an odd number of parts. Find the generating function of partitions (with ordering) of n that have an even number of parts.

(c) Show that the number of partitions (with ordering) of $4n - 1$ that have an odd number of parts is equal to the number of partitions (with ordering) of $4n - 1$ that have an even number of parts.

(d) List all partitions (with ordering) of the integer 7 into odd number and even number of parts into summands of 1, 2 and 3.

Solution

(a) $p(n) = p(n-1) + p(n-2) + p(n-3), \quad n \geq 4.$ $\to$ 4

$$\left.\begin{array}{l}[\text{partition (with ordering) of } n-1]+1 \\ [\text{partition (with ordering) of } n-2]+2 \\ [\text{partition (with ordering) of } n-3]+3\end{array}\right\} = \text{partition (with ordering) of } n$$

$p(1) = 1: \quad 1$

$p(2) = 2: \quad 2 = 1 + 1$

$p(3) = 4: \quad 3 = 1+1+1 = 2+1 = 1+2$ $\to$ 4

$$p(4) = 7: \quad \underbrace{3+1 = 1+1+1+1 = 2+1+1 = 1+2+1}_{p(3)\text{ (ending with +1)}} = \underbrace{2+2 = 1+1+2}_{p(2)\text{ (ending with +2)}} = \underbrace{1+3}_{p(1)\text{ (ending with +3)}}$$

$p(5) = p(2) + p(3) + p(4) = 13$

$p(6) = p(3) + p(4) + p(5) = 24$

$p(7) = p(4) + p(5) + p(6) = 44$

(b) $3 \to (x + x^2 + \underset{\uparrow}{x^3})$

$1+1+1 \to (\underset{\uparrow}{x} + x^2 + x^3)(\underset{\uparrow}{x} + x^2 + x^3)(\underset{\uparrow}{x} + x^2 + x^3)$

$1+2 \to (\underset{\uparrow}{x} + x^2 + x^3)(x + \underset{\uparrow}{x^2} + x^3)$

$2+1 \to (x + \underset{\uparrow}{x^2} + x^3)(\underset{\uparrow}{x} + x^2 + x^3)$ → 4

Consider $n = \underset{1,2 \text{ or } 3}{x_1} + \underset{1,2 \text{ or } 3}{x_2} + \cdots + \underset{1,2 \text{ or } 3}{x_r}$

The number of ways to partition n into r summand of 1, 2 or 3 with ordering is the coefficient of x^n in $(x + x^2 + x^3)^r$.

The generating function of partitions (with ordering) of n that have an odd number of parts is

$$f(x) = (x + x^2 + x^3) + (x + x^2 + x^3)^3 + (x + x^2 + x^3)^5 + \cdots = \frac{x + x^2 + x^3}{1 - (x + x^2 + x^3)^2}.$$ → 2

The generating function of partitions (with ordering) of n that have an even number of parts is

$$g(x) = (x + x^2 + x^3)^2 + (x + x^2 + x^3)^4 + \cdots = \frac{(x + x^2 + x^3)^2}{1 - (x + x^2 + x^3)^2}.$$ → 2

(c) Consider $f(x) - g(x)$

$$\begin{aligned}
f(x) - g(x) &= \frac{x + x^2 + x^3}{1 - (x + x^2 + x^3)^2} - \frac{(x + x^2 + x^3)^2}{1 - (x + x^2 + x^3)^2} \\
&= (x + x^2 + x^3)\frac{1 - (x + x^2 + x^3)}{1 - (x + x^2 + x^3)^2} \\
&= \frac{(x + x^2 + x^3)}{1 + x + x^2 + x^3} \\
&= 1 - \frac{1}{1 + x + x^2 + x^3} \\
&= 1 - \frac{1 - x}{1 - x^4} \\
&= 1 - \left(1 - x + x^4 - x^5 + \cdots - 0x^{4n-1} + x^{4n} - x^{4n+1} + \cdots\right) \\
&= x - x^4 + x^5 + \cdots + 0x^{4n-1} - x^{4n} + x^{4n+1} + \cdots
\end{aligned}$$ → 4

(the number of partitions (with ordering) of $4n - 1$ that have an odd number of parts) −

(the number of partitions (with ordering) of $4n - 1$ that have an even number of parts)

$= $ the coefficient of x^{4n-1} in $f(x) - g(x) = 0$ $\rightarrow$ 4

Therefore, the number of partitions (with ordering) of $4n - 1$ that have an odd number of parts is equal to the number of partitions (with ordering) of $4n - 1$ that have an even number of parts.

(d) Odd number of parts:

$3 + 3 + 1$ and its permutation (3 ways)

$3 + 2 + 2$ and its permutation (3 ways)

$3 + 1 + 1 + 1 + 1$ and its permutation (5 ways)

$2 + 2 + 1 + 1 + 1$ and its permutation ($C(5, 2) = 10$ ways)

$1 + 1 + 1 + 1 + 1 + 1 + 1$ (1 way)

Even number of parts:

$2 + 2 + 2 + 1$ and its permutation (4 ways)

$3 + 2 + 1 + 1$ and its permutation $\left(\dfrac{4!}{2!1!1!} = 12 \text{ ways}\right)$

$2 + 1 + 1 + 1 + 1 + 1$ and its permutation (6 ways)