Lecture 6

Brownian Motion

- Brownian Motion
- Hitting times, maximum variable, and the Gambler's Ruin Problem
- Variations on Brownian motion

Definition

Definition 6.1 *A stochastic process* $\{X(t),\ t \geq 0\}$ *is said to be a* Brownian motion *process if*

(i) $X(0) = 0$*;*

(ii) it has stationary and independent increments;

$\uparrow$: $X(t) - X(0) = X(t)$

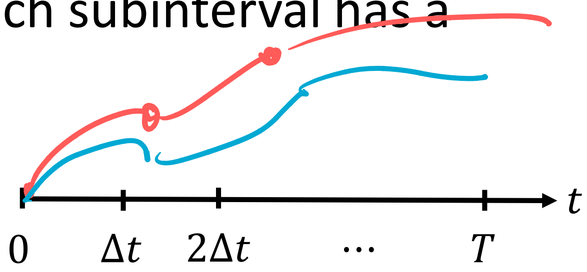
(iii) for every $t > 0$, $X(t) \sim \mathcal{N}(0, \sigma^2 t)$*;* $X(t) - X(s) \sim N(0, \sigma^2(t-s))$.

(iv) the path $t \mapsto X(t)$ *is continuous with probability* 1.

- Brownian motion is also called *Wiener process.*
- When $\sigma = 1$, it is called *standard Brownian motion*, denoted $B(t)$.
- Robert Brown discovered motion of particles in liquid or gas in 1827
- Albert Einstein explained it with bombardment by molecules in 1905
- Norbert Wiener developed the mathematics in 1918

To simulate sample paths of standard Brownian motion

- Discrete-time version (Monte Carlo simulation):
  - Divide the time interval $[0, T]$ into N subintervals, each subinterval has a length of Δt, i.e. $\Delta t = \frac{T}{N}$.
  - Denote $z(i \times \Delta t)$ by z_i for $i = 0,1,2, \ldots, N$.



- Example: Simulate sample path of Brownian motion from 0 to T, where $T = 4$, $N = 16$, $\Delta t = \frac{T}{N} = 0.25$, $z(0) = 0$.

To simulate sample paths of standard Brownian motion

- The increment of Brownian motion follows normal distribution.
- Draw a random number Δz_1 from $\mathcal{N}(0,0.25)$. Let $z_1 = z_0 + \Delta z_1$.
- Draw another random number Δz_2 from $\mathcal{N}(0,0.25)$. Let $z_2 = z_1 + \Delta z_2$…
- Draw another random number Δz_N from $\mathcal{N}(0,0.25)$. Let $z_N = z_{N-1} + \Delta z_N$.

Simulation 1

i	Δz_i	z_i
1	0.47	0.47
2	−0.59	−0.12
3	0.59	0.46
4	−0.12	0.35
5	0.65	1.00

……

Simulation 1000

i	Δz_i	z_i
1	−0.37	−0.37
2	−0.83	−1.20
3	−0.18	−1.38
4	0.22	−1.16
5	−0.19	−1.35

To simulate sample paths of standard Brownian motion

Simulation 1

i	Δz_i	z_i
1	0.47	0.47
2	−0.59	−0.12
3	0.59	0.46
4	−0.12	0.35
5	0.65	1.00
6	0.54	1.54
7	−0.23	1.31
8	0.92	2.23
9	0.58	2.80
10	0.14	2.95
11	−0.21	2.74
12	0.16	2.90
13	0.82	3.72
14	−0.39	3.32
15	−0.46	2.86
16	0.17	3.03

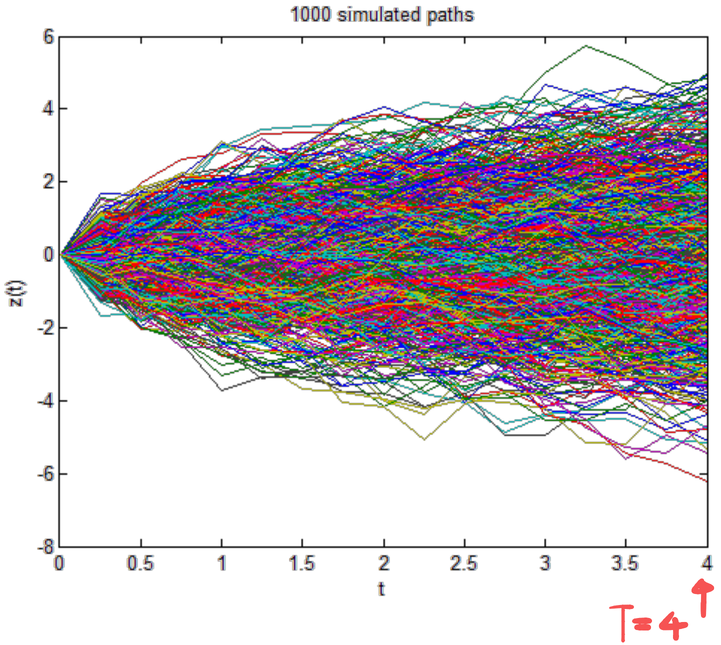
……

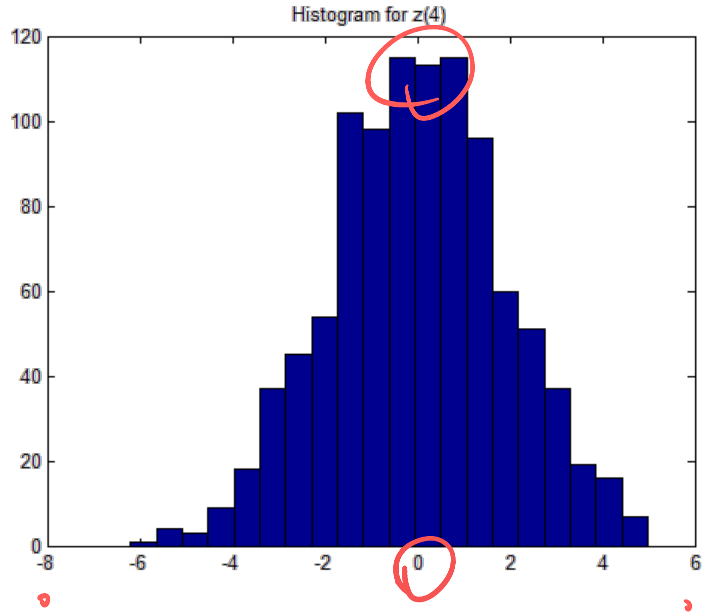
Simulation 1000

i	Δz_i	z_i
1	−0.37	−0.37
2	−0.83	−1.20
3	−0.18	−1.38
4	0.22	−1.16
5	−0.19	−1.35
6	−0.47	−1.81
7	0.54	−1.28
8	1.19	−0.09
9	0.77	0.68
10	−0.01	0.67
11	0.56	1.24
12	0.08	1.31
13	0.70	2.02
14	−0.35	1.67
15	0.27	1.94
16	0.49	2.43

Sample paths of standard Brownian motion

- Each simulated path requires drawing 16 random numbers in $\mathcal{N}(0, 0.25)$.
- Repeat the simulation for 1000 times to obtain 1000 paths.





Further Properties

1. Paths of Brownian motion are nowhere differentiable.

   Intuition:
   $$\frac{X(t+h)-X(t)}{h} \sim \mathcal{N}(0, \sigma^2/h)$$
   and therefore it has mean 0 and variance ∞ as $h \to 0$.

2. Since $X(t) \sim \mathcal{N}(0, \sigma^2 t)$, its density is
   $$f_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{x^2}{2\sigma^2 t}}.$$

Sum of normally distributed random variables

- Independent random variables $X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2), Z = X + Y$.

Then, $Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

- The random variables X and Y are **jointly** normally distributed. Then $X + Y$ is still normally distributed and the mean is $\mu_X + \mu_Y$. However, the variances are not additive due to the correlation.

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y},$$

where ρ is the correlation.

Example 6.2 Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. What is the distribution of $B(1) + B(4)$?

$B(1) \sim N(0,1)$

$B(4) \sim N(0,4)$

$$B(1) + B(4) = \underbrace{B(4) - B(1)}_{\text{increase from } T=1 \text{ to } T=4} + \underline{B(1) + B(1)}$$

$E(B(4) - B(1)) = E(B(4)) - E(B(1)) = 0.$

$\underbrace{Var(B(4) - B(1))}_{\sim N(0,3)} = 3$

$Var(2B(1)) = 4\,Var(B(1)) = 4.$

$B(1) + B(4) \sim N(0,7).$

Example 6.3 Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. What is $E[B(1)B(2)]$?

$Cov(B(s), B(t))$

$B(1) \sim N(0,1)$

$= E(B(s) \cdot B(t)) - E(B(s)) E(B(t))$

$B(2) \sim N(0,2)$

for standard $B(t)$, $s<t$.

$E(B(1)B(2)) = E[B(1)(B(2)-B(1)) + B(1)^2]$

$= E[B(s)(B(t)-B(s)) + B(s)^2]$

$= E[\underline{B_1}(\underbrace{B(2)-B(1)}_{\text{increament}})] + E[B(1)^2]$

$= Var(s)$

$= s$. for general case

it's $\min(s,t)$

$= E(B_1) E[(B_2) - (B_1)] + Var(B_1) + E(B_1)^2$

$= 0 + 1 + 0 = 1.$

Example 6.5 Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. Determine $E[e^{B(10)-B(9)}]$.

$B(10)-B(9) \sim N(0,1)$, $E(e^X)$ $X \sim (0,1)$

$$E(e^X) = \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2-2x}{2}} \, dx$$

$$= e^{\frac{1}{2}} \cdot \int_{-\infty}^{\infty} \boxed{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-1)^2}{2}}} \, dx$$

$$= e^{\frac{1}{2}}$$

density function of $N(1,1)$.

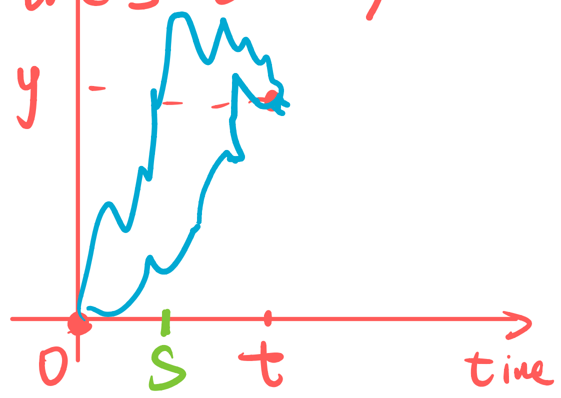
Joint and Conditional Densities

- Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion.
- As $B(t) \sim \mathcal{N}(0, t)$, its density function is $f_t(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$.
- Joint density of $(B(t_1), B(t_2))$ for $t_1 < t_2$: $(B(t_1), B(t_2)) = (x_1, x_2)$ is equivalent to $(B(t_1), B(t_2) - B(t_1)) = (x_1, x_2 - x_1)$.
- By the independent increment assumption, $B(t_1)$ and $B(t_2) - B(t_1)$ are independent.
- By stationary increment assumption, $B(t_2) - B(t_1) \sim \mathcal{N}(0, t_2 - t_1)$.
- Hence, the joint density is

$$
\begin{aligned}
f(x_1, x_2) &= f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \\
&= \frac{1}{\sqrt{(2\pi)^2 t_1 (t_2 - t_1)}} \cdot \exp\left[-\frac{x_1^2}{2t_1} - \frac{(x_2 - x_1)^2}{2(t_2 - t_1)} \right].
\end{aligned}
$$

Browian Bridge : know value for time t
find for time s (s<t)

- If $B(t) = y$, what is the distribution of $B(s)$ for $s < t$?



- The conditional density is

$$
\begin{aligned}
f_{s|t}(x|y) &= \frac{f_s(x) f_{t-s}(y-x)}{f_t(y)} \\
&= K_1 \cdot \exp\left[-\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)}\right] \\
&= K_2 \cdot \exp\left[-x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)}\right) + \frac{yx}{t-s}\right] \\
&= K_2 \cdot \exp\left[-\frac{t}{2s(t-s)}\left(x^2 - \frac{2ys}{t}x\right)\right] \\
&= K_3 \cdot \exp\left[-\frac{(x - ys/t)^2}{2s(t-s)/t}\right], \qquad (6.1)
\end{aligned}
$$

where K_1, K_2, and K_3 do not depend on x.

- Hence, the conditional density is normal with mean and variance given by

slope

$$
\begin{aligned}
E[B(s)|B(t) = y] &= \frac{s}{t} \cdot y \quad = \frac{y}{t} \cdot s \\
\text{Var}[B(s)|B(t) = y] &= \frac{s(t-s)}{t}.
\end{aligned}
$$

- We see that the expected value $E[B(s)|B(t) = y]$ lies on the straight line joining $B(0) = 0$ and $B(t) = y$.
- The variance $\text{Var}[B(s)|B(t) = y]$ is zero at $s = 0$ and $s = t$ and is maximum at $s = t/2$.
- Needing no computation, we know that K_3 must be given by

$$
K_3 = \frac{1}{\sqrt{2\pi s(t-s)/t}}.
$$

$X = (X_1, X_2)$

$X_1 \mid X_2 = x_2 \sim N(\quad , \quad)$

$$X \sim N\left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}^2 \end{pmatrix} \right)$$

$$E(X_1 \mid X_2 = x_2) = \mu_1 + \frac{\sigma_{12}(x_2 - \mu_2)}{\sigma_{22}^2}$$

$\mu_1 = \mu_2 = 0, \quad \sigma_{11}^2 = s, \quad \sigma_{22}^2 = t, \quad \sigma_{12} = s.$

$$E(X_1 \mid X_2 = y) = \frac{s}{t} \cdot y$$

$$\Sigma_{1|2} = \sigma_{11}^2 - \frac{\sigma_{12}^2}{\sigma_{22}^2} = \frac{s(t-s)}{t}$$

Example 6.6 Find $P\{B(1) > 0 | B(2) = 2\}$.

$s=1, \quad y=2$

$t=2$

X

$B(1) \mid B(2) = 2 \sim N(1, \frac{1}{2})$.

$E(B(1) \mid B(2)=2) = \frac{s}{t} \cdot y = \frac{1}{2} \cdot 2 = 1$.

$Var(B(1) \mid B(2)=2) = \frac{s(t-s)}{t} = \frac{1 \cdot 1}{2} = \frac{1}{2}$

$X \sim N(1, \frac{1}{2})$, $P(X>0) = P\{\frac{X-1}{\sqrt{\frac{1}{2}}} > \frac{0-1}{\sqrt{\frac{1}{2}}}\}$

$= P\{Z > -\sqrt{2}\}$.

$= \Phi(\sqrt{2})$

$\square$

6.2 Hitting Times, Maximum Variable, and the Gambler's Ruin Problem

Hitting Times

- Let T_a denote the first time the standard Brownian motion hits a.
- When $a > 0$, we are interested to compute $P\{T_a \leq t\}$, the probability that the process will reach a by time t.
- Conditioning $P\{B(t) \geq a\}$ on $\{T_a \leq t\}$:

multiple sample

$$\underline{P\{B(t) \geq a\}} = P\{B(t) \geq a \mid \underline{T_a \leq t}\} \cdot P\{T_a \leq t\} + P\{B(t) \geq a \mid \underline{T_a > t}\} \cdot P\{T_a > t\}.$$

- By symmetry, $P\{B(t) \geq a \mid T_a \leq t\} = \frac{1}{2}$. ?
- The term $P\{B(t) \geq a \mid T_a > t\}$ is 0 because it is given that the process does not reach a before t.

when T=t, B(t) hit (a) first
So it's impossible for B(t) to hit a before t.

$P(B(t) \geq a)$
$= P(B(t) \geq a \mid T(a) \leq t) \cdot P(T_a \leq t)$
$= \frac{1}{2} \cdot P(T_a \leq t)$
So $P(T_a \leq t) = 2P(B(t) \geq a)$

- Let Z be the standard normal random variable. Then,

$$\begin{aligned} P\{T_a \leq t\} &= 2 \cdot P\{B(t) \geq a\} \\ &= 2 \cdot P\left\{\frac{B(t)}{\sqrt{t}} \geq \frac{a}{\sqrt{t}}\right\} \\ &= 2 \cdot P\left\{Z \geq \frac{a}{\sqrt{t}}\right\} \\ &= 2 \cdot \Phi(-a/\sqrt{t}). \end{aligned}$$

- If $a < 0$, by symmetry, $P\{T_a \leq t\} = P\{T_{|a|} \leq t\}$, so that

$$P\{T_a \leq t\} = P\{T_{|a|} \leq t\} = 2 \cdot \Phi(-|a|/\sqrt{t}).$$

- If $a = 0$, what is $P\{T_0 \leq t\}$? $=1.$

- Later, we will build a stock price model based on Brownian motion. Then, $\boxed{P\{T_a \leq t\}}$ is related to the probability that a callable bull/bear contract (barrier options) will be knocked out.

Maximum Variable

- Some financial products have payoffs depending the maximum / minimum stock price observed during the life of the product, e.g. look-back options.
- Interestingly, the distribution of maximum boils down to the distribution of hitting time.
- For $a > 0$,

$$\begin{aligned} P\left\{\max_{0\le s\le t} B(s) \ge a\right\} &= P\{T_a \le t\} \qquad \text{(by continuity of } B(t)\text{)} \\ &= 2\cdot\Phi(-a/\sqrt{t}). \end{aligned}$$

The Gambler's Ruin Problem

- Let $A > 0$ and $B > 0$. We are interested to know the probability that the Brownian motion hits A before $-B$.
- We calculate it by taking limit of a symmetric random walk.
- Recall that when the step size is ± 1, we have

$$P(\text{up } k \text{ before down } n) = \frac{n}{n+k}.$$

- Suppose that $A = k\Delta x$ and $B = n\Delta x$. Then, when the step size is $\pm\Delta x$, we have (for symmetric random walk)

$$P(\text{up } A \text{ before down } B) = \frac{n}{n+k} = \frac{B/\Delta x}{A/\Delta x + B/\Delta x} = \frac{B}{A+B}.$$

- Let $\Delta x \to 0$, we have, for Brownian motion,

$$P(T_A < T_{-B}) = P(\text{up } A \text{ before down } B) = \frac{B}{A+B}.$$

Example 6.7 Determine $P\{T_1 < T_{-1} < T_2\}$.

Solution:

$$\begin{aligned}
& P\{T_1 < T_{-1} < T_2\} \\
= \; & P\{T_1 < T_{-1},\ T_{-1} < T_2\} \\
= \; & P\{T_1 < T_{-1}\} \cdot P\{T_{-1} < T_2 \mid T_1 < T_{-1}\} \\
= \; & P\{\text{up 1 before down 1}\} \cdot P\{\text{down 2 before up 1}\} \\
= \; & \frac{1}{2} \cdot \frac{1}{3}.
\end{aligned}$$

$\square$

6.3 Variations on Brownian Motion

6.3.1 Brownian Motion with Drift

- To model processes with an upward or downward trend, we introduce a *drift*.
- Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. Define

$$X(t) = \sigma B(t) + \mu t.$$

- Then, $X(t)$ is a Brownian motion with *drift coefficient* μ and *variance parameter* σ^2.
- $X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$.

Example 6.8 Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift 1 and variance parameter 1.

(a) What is the conditional distribution of $X(3)$ given that $X(1) = -1$?

Solution:

$$\begin{aligned} X(3) &= X(3) - X(1) + X(1) \\ &= X(3) - X(1) - 1 \\ &\sim X(2) - 1 \\ &\sim \mathcal{N}(2,2) - 1 \\ &\sim \mathcal{N}(1,2) \end{aligned}$$

Here, the notation $X \sim Y$ denotes that X and Y have the same distribution. $\square$

(b) What is the conditional distribution of $X(1)$ given that $X(3) = -1$?

6.3.2 Geometric Brownian Motion

- The plain Brownian motion (with drift) is unsuitable for modeling stock price for the following two reasons:

  1. $X(t)$ can become negative;
  2. the increment $X(t+s) - X(t)$ is normally distributed with a fixed mean μs regardless of the value of $X(t)$. But we expect the percentage change to have a fixed mean instead.

- These problems can be easily resolved with a transformation.

- Consider the process

$$X(t) = X(0)e^{Z(t)},$$

  where $Z(t)$ is a Brownian motion with drift. The process $X(t)$ is called *geometric Brownian motion* (GBM).

- Let $Z(t) = \sigma B(t) + (\mu - \frac{\sigma^2}{2})t$, where $B(t)$ is standard Brownian motion. We have
$$X(t) = X(0)e^{\sigma B(t)+(\mu-\frac{\sigma^2}{2})t}.$$

- Since
$$\begin{aligned}\ln X(t) &= \ln X(0) + \sigma B(t) + (\mu - \tfrac{\sigma^2}{2})t \\ &\sim \mathcal{N}\left(\ln X(0) + (\mu - \tfrac{\sigma^2}{2})t, \sigma^2 t\right),\end{aligned}$$
we see that $X(t)$ follows a *log-normal* distribution.

- The *log-return* over an interval $[t, t+s]$ is defined as $\boxed{\ln\left(\frac{X(t+s)}{X(t)}\right)}$.

- We have
$$\begin{aligned}\ln\left(\frac{X(t+s)}{X(t)}\right) &= \sigma[B(t+s) - B(t)] + (\mu - \tfrac{\sigma^2}{2})s \\ &\sim \mathcal{N}\left((\mu - \tfrac{\sigma^2}{2})s, \sigma^2 s\right).\end{aligned}$$

Aim: if the stock price is driven by GBM, the percentage change of stock price will have fixed mean.

log-return: $\boxed{r = \ln\left(\frac{X(t+\Delta t)}{X(t)}\right)} \sim \underline{N\left((\mu - \frac{\sigma^2}{2})\cdot\Delta t,\ \sigma^2\cdot\Delta t\right)}$

percentage change:

$$R = \frac{X(t+\Delta t) - X(t)}{X(t)} = \frac{X(t+\Delta t)}{X(t)} - 1, \quad = \boxed{e^r} - 1$$

$$E(R) = E(e^r - 1) = E(e^r) - 1$$

$$= e^{(\mu - \frac{\sigma^2}{2})\cdot\Delta t + \frac{\sigma^2\cdot\Delta t}{2}}$$

$$= e^{\mu\cdot\Delta t} - 1$$

$r \sim N(m, v)$

$E(e^r) = e^{m + \frac{v}{2}}$

Example 6.9 Suppose that the price of a stock changes according to the following geometric Brownian motion:

$$X(t) = 100e^{0.2B(t)+0.1t}.$$

What is the probability that the price will exceed \$110 when $t = 2$?

Solution: Let Z be the standard normal random variable.

$$\begin{aligned}
P\{X(2) > 110\} &= P\left\{100e^{0.2B(2)+0.2} > 110\right\} = P\left\{e^{0.2B(2)+0.2} > 1.1\right\} \\
&= P\{0.2B(2) + 0.2 > \ln(1.1)\} \\
&= P\left\{B(2) > \frac{\ln(1.1) - 0.2}{0.2}\right\} \\
&= P\left\{\frac{B(2)}{\sqrt{2}} > \frac{\ln(1.1) - 0.2}{0.2\sqrt{2}}\right\} \\
&= P\left\{Z > \frac{\ln(1.1) - 0.2}{0.2\sqrt{2}}\right\} \\
&= \Phi\left(-\frac{\ln(1.1) - 0.2}{0.2\sqrt{2}}\right) \approx \Phi(0.3701) \approx 0.6443
\end{aligned}$$

$B(2) \sim N(0,2)$

$Z \sim N(0,1)$

□

Example 6.10 Let $\{X(t), t \geq 1\}$ be a geometric Brownian motion with $X(0) = 1$.

$\sigma Y \sim N(0, \sigma^2 t)$

(a) Determine $E[X(t)]$. $= e^{(\mu - \frac{\sigma^2}{2})t}$ $\quad E(e^{\sigma Y}) = e^{\frac{\sigma^2 t}{2}}$

Solution: Let $X(t) = e^{\sigma B(t) + (\mu - \frac{\sigma^2}{2})t}$ where $B(t)$ is the standard Brownian motion. Since $B(t) \sim \mathcal{N}(0, t)$, we have

1. the expected value of stock price grows exponentially at the rate μ

2. $\mu = \frac{\sigma^2}{2}$ special case

$X = e^{B(t)}$

$$
\begin{aligned}
E[X(t)] &= E[e^{\sigma Y + (\mu - \frac{\sigma^2}{2})t}] \qquad (\text{where } Y \sim \mathcal{N}(0, t)) \\
&= \int_{-\infty}^{\infty} e^{\sigma y + (\mu - \frac{\sigma^2}{2})t} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \, dy \\
&= e^{(\mu - \frac{\sigma^2}{2})t} \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2\sigma t y}{2t}} \, dy \\
&= e^{(\mu - \frac{\sigma^2}{2})t} \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-\sigma t)^2 - \sigma^2 t^2}{2t}} \, dy \\
&= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-\sigma t)^2}{2t}} \, dy = e^{\mu t}. \qquad \square
\end{aligned}
$$

(b) Given the history $\mathcal{F}_s = \{X(u), 0 \le u \le s\}$ up to a time s, determine the expectation of $X(t)$ at a later time t, i.e. $t > s$.

Solution: Let $X(t) = e^{\sigma B(t)+(\mu-\frac{\sigma^2}{2})t}$ where $B(t)$ is the standard Brownian motion. We have

$$\begin{aligned}
E[X(t)|\mathcal{F}_s] &= E[e^{\sigma B(t)+(\mu-\frac{\sigma^2}{2})t}|\mathcal{F}_s] \\
&= E[e^{\sigma(B(t)-B(s))+\sigma B(s)+(\mu-\frac{\sigma^2}{2})t}|\mathcal{F}_s] \\
&= e^{\sigma B(s)+(\mu-\frac{\sigma^2}{2})t}\cdot E[e^{\sigma(B(t)-B(s))}|\mathcal{F}_s] && (B(s)\text{ given}) \\
&= e^{\sigma B(s)+(\mu-\frac{\sigma^2}{2})t}\cdot E[e^{\sigma(B(t)-B(s))}] && (\text{indep. inc.}) \\
&= e^{\sigma B(s)+(\mu-\frac{\sigma^2}{2})t}\cdot E[e^{\sigma B(t-s)}] && (\text{stationary inc.}) \\
&= e^{\sigma B(s)+(\mu-\frac{\sigma^2}{2})t}\cdot e^{\frac{\sigma^2(t-s)}{2}} && (\text{by part (a) with } \mu = \tfrac{\sigma^2}{2}) \\
&= e^{\sigma B(s)+(\mu-\frac{\sigma^2}{2})s+\mu(t-s)} = X(s)\cdot e^{\mu(t-s)}. && \square
\end{aligned}$$

Example 6.11 Determine $E[X(3)X(2)|X(1) = 1]$, where $X(t) = e^{B(t)+t}$. Use the fact that $E[e^{\sigma B(t)}] = e^{\frac{\sigma^2 t}{2}}$.

$X(3) = e^{B(3)+3}$ $\quad X(2) = e^{B(2)+2}$ $\quad X(1) = e^{B(1)+1} \Rightarrow B(1) = -1.$

$E(e^{B(3)+3+B(2)+2} | B(1) = -1) = E(e^{B(3)+B(2)+5} | B(1) = -1)$

$= E(e^{B(3)-B(2)+2B(2)+5} | B(1) = -1) = E(e^{\underline{B(3)-B(2)}+2(\underline{B(2)-B(1)})+2B(1)+5} | B(1) = -1)$

ind inc

$= E(e^{B(3)-B(2)+2(B(2)-B(1))+3}) = E(e^{B(3)-B(2)}) \cdot E(e^{2(B(2)-B(1))}) \cdot e^3$

$\sigma=1, t=1$ $\quad \sigma=2, t=1$

$= e^{\frac{1}{2}} \cdot e^2 \cdot e^3 = e^{\frac{11}{2}}$

6.4 Continuous-time Martingales

Definition 6.12 *A continuous-time stochastic process* $\{X(t),\ t \geq 0\}$ *is said to be a* martingale *if, for* $s < t$*,*

$$\Rightarrow E[X(t)|\mathcal{F}_s] = X(s).$$

Here, $\mathcal{F}_s = \{X(u),\ 0 \leq u \leq s\}$ *is the history of* $X(t)$ *up to time* s*.*

Example 6.13 Verify that $X(t) = \underline{e^{cB(t)-c^2t/2}}$ is a martingale, where c is any constant.

$$E(X(t)|\mathcal{F}_s) = e^{-\frac{c^2t}{2}} \cdot E(e^{cB(t)}|\mathcal{F}_s) = e^{-\frac{c^2t}{2}} \cdot E(e^{c(B(t)-B(s))+cB(s)}|\mathcal{F}_s)$$

$$= e^{cB(s)-\frac{c^2t}{2}} E(e^{\overbrace{c(B(t)-B(s))}^{B(t-s)}})$$

$$= e^{cB(s)-\frac{c^2t}{2}} \cdot e^{\frac{c^2(t-s)}{2}} =$$

Filtration for Brownian motion

We need notation for the amount of information available at each time.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $B(t), t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t), t \geq 0$, satisfying:

1. **(Information accumulates)** For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. In other words, there is at least as much information available at the later time $\mathcal{F}(t)$ as there is at the earlier time $\mathcal{F}(s)$.

$\mathcal{F}(s) \subseteq \mathcal{F}(t)$

Filtration for Brownian motion

2. **(Adaptivity)** For each $t \geq 0$, the Brownian motion $B(t)$ at time t is $\mathcal{F}(t)$**-**measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion $B(t)$ at that time.

3. **(Independence of future increments)** For $0 \leq t \leq u$, the increment $B(u) - B(t)$ is independent of $\mathcal{F}(t)$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.