Discrete Mathematics Homework 5

1. Find the recurrence relation of $D_n = \begin{vmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a+b & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_{n\times n}$. Hence calculate D_n.

Solution

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Expand along the first row, we have

$$D_n = \begin{vmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a+b & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_{n\times n}$$

$$= (a+b)\begin{vmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a+b & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_{(n-1)\times(n-1)} - ab\begin{vmatrix} 1 & ab & 0 & \cdots & 0 \\ 0 & a+b & ab & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a+b & ab \\ 0 & \cdots & 0 & 1 & a+b \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= (a+b)D_{n-1} - abD_{n-2}$$

8 explain

We now obtain the relation $D_n - (a+b)D_{n-1} + abD_{n-2} = 0, n \geq 3$ with

$\to$ 10

$$D_1 = a+b, D_2 = \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} = (a+b)^2 - ab = a^2 + ab + b^2.$$

The characteristic equation $r^2 - (a+b)r + ab = 0$ has roots a and b.

Case (i): one of a, $b = 0$

Suppose $b = 0$.

$D_n = aD_{n-1} = a^{n-1}D_1 = a^n$.

3

$a=b=0$ 1

$a=0$ $b\neq 0$ 1

$a\neq 0$ $b=0$ 1

Case (ii): $a \neq b$, $a, b \neq 0$.

$$D_n = c_1 a^n + c_2 b^n$$
$$D_1 = a + b = c_1 a + c_2 b,$$
$$D_2 = a^2 + ab + b^2 = c_1 a^2 + c_2 b^2$$
$$\begin{pmatrix} a & b \\ a^2 & b^2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a+b \\ a^2+ab+b^2 \end{pmatrix}$$

3

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{ab^2 - ba^2}\begin{pmatrix} b^2 & -b \\ -a^2 & a \end{pmatrix}\begin{pmatrix} a+b \\ a^2+ab+b^2 \end{pmatrix}$$

$\rightarrow$ 10

$$= \frac{1}{ab^2 - ba^2}\begin{pmatrix} b^2(a+b) - b(a^2+ab+b^2) \\ -a^2(a+b) + a(a^2+ab+b^2) \end{pmatrix}$$
$$= \frac{1}{ab^2 - ba^2}\begin{pmatrix} -ba^2 \\ ab^2 \end{pmatrix}$$
$$= \frac{1}{b-a}\begin{pmatrix} -a \\ b \end{pmatrix}$$
$$D_n = \frac{b^{n+1} - a^{n+1}}{b-a}$$

Case (iii): $a = b \neq 0$.

4

$$D_n = (c_1 + c_2 n)a^n$$
$$D_1 = 2a = (c_1 + c_2)a,$$
$$D_2 = 3a^2 = (c_1 + 2c_2)a^2$$
$$c_1 = c_2 = 1$$
$$D_n = (1+n)a^n$$

2. Solve each of the following recurrence relations. 20

(a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \quad n \geq 0, \quad a_0 = 0, \quad a_1 = 1.$ 4 explain 2

(b) $a_{n+2} + 4a_{n+1} + 4a_n = 7, \quad n \geq 0, \quad a_0 = 1, \quad a_1 = 2.$ 4 answer 2

(c) $a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n, \quad n \geq 0, \quad a_0 = 1, \quad a_1 = 1.$ 4

(d) $a_{n+1} - 2a_n = 2^n, \quad n \geq 0, \quad a_0 = 1.$ 4

(e) $a_n + 2a_{n-1} + 2a_{n-2} = 0, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 3.$ 4

Solution

(a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$.
With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2 + 3r + 2 = 0 = (r+2)(r+1)$ yields the characteristic roots $r = -1, -2$.
Hence $a_n^{(h)} = A(-1)^n + B(-2)^n$, while $a_n^{(p)} = C(3)^n$.
$C(3)^{n+2} + 3C(3)^{n+1} + 2C(3)^n = 3^n \Longrightarrow 9C + 9C + 2C = 1 \Longrightarrow C = 1/20.$
$a_n = A(-1)^n + B(-2)^n + (1/20)(3)^n$
$0 = a_0 = A + B + (1/20)$
$1 = a_1 = -A - 2B + (3/20)$
Hence $1 = a_0 + a_1 = -B + (4/20)$ and $B = -4/5$. Then $A = -B - (1/20) = 3/4$.
$a_n = (3/4)(-1)^n + (-4/5)(-2)^n + (1/20)(3)^n$, $n \geq 0$
(b) $a_n = (2/9)(-2)^n - (5/6)(n)(-2)^n + (7/9)$, $n \geq 0$

(c) Let $a_n^2 = b_n, n \geq 0$
$b_{n+2} - 5b_{n+1} + 6b_n = 7n$
$b_n^{(h)} = A(3^n) + B(2^n)$, $b_n^{(p)} = Cn + D$
$C(n+2) + D - 5[C(n+1) + D] + 6(Cn + D) = 7n \Longrightarrow C = 7/2,\ D = 21/4$
$b_n = A(3^n) + B(2^n) + (7n/2) + (21/4)$
$b_0 = a_0^2 = 1,\ b_1 = a_1^2 = 1$
$1 = b_0 = A + B + 21/4$
$1 = b_1 = 3A + 2B + 7/2 + 21/4$
$3A + 2B = -31/3$
$2A + 2B = -34/4$
$A = 3/4,\ B = -5$
$a_n = [(3/4)(3)^n - 5(2)^n + (7n/2) + (21/4)]^{1/2}$, $n \geq 0$

(d) $a_n = 2^n + n(2^{n-1})$, $n \geq 0$.

(e) $a_n + 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.
$r^2 + 2r + 2 = 0$, $r = -1 \pm i$
$(-1+i) = \sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$
$(-1-i) = \sqrt{2}(\cos(5\pi/4) + i\sin(5\pi/4)) =$
$\sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4)) = \sqrt{2}(\cos(3\pi/4) - i\sin(3\pi/4))$
$a_n = (\sqrt{2})^n[A\cos(3\pi n/4) + B\sin(3\pi n/4)]$
$1 = a_0 = A$
$3 = a_1 = \sqrt{2}[\cos(3\pi/4) + B\sin(3\pi/4)] =$
$\sqrt{2}[(-1/\sqrt{2}) + B(1/\sqrt{2})]$, so $3 = -1 + B$, $B = 4$
$a_n = (\sqrt{2})^n[\cos(3\pi n/4) + 4\sin(3\pi n/4)]$, $n \geq 0$

$\frac{1-4i}{2}(-1+i)^n + \frac{1+4i}{2}(-1-i)^n$

3. Let a_n be the number of expression (not answer) of $\underbrace{1-1-\cdots-1-1}_{n\ \text{"}-\text{"}}$ together with ( ) for each subtraction. For instance,

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\begin{aligned}
a_0 &= 1: && 1 \\
a_1 &= 1: && 1-1 \\
a_2 &= 2: && (1-1)-1, \quad 1-(1-1) \\
a_3 &= 5: && ((1-1)-1)-1, \quad (1-(1-1))-1, \quad (1-1)-(1-1), \quad 1-((1-1)-1), \quad 1-(1-(1-1))
\end{aligned}
$$

(a) List all possible cases for $n = 4$. 7 (0.5 each)

(b) Find the recurrence relation for a_n. Explain your relation in detail. 6

(c) Hence, solve the a_n. 7

Solution

(a) $a_4 = 14$

$1-(((1-1)-1)-1), \quad 1-((1-(1-1))-1), \quad 1-((1-1)-(1-1)), \quad 1-(1-((1-1)-1)), \quad 1-(1-(1-(1-1)))$

$(1-1)-((1-1)-1), \quad (1-1)-(1-(1-1))$

$((1-1)-1)-(1-1), \quad (1-(1-1))-(1-1)$

$(((1-1)-1)-1)-1, \quad ((1-(1-1))-1)-1, \quad ((1-1)-(1-1))-1, \quad (1-((1-1)-1))-1, \quad (1-(1-(1-1)))-1$

(b) $\underbrace{1-1-\cdots-1-1}_{n\ \text{"}-\text{"}} = \underbrace{(\ )}_{k\ \text{"}-\text{"}} \underbrace{-}_{\text{main subtraction}} \underbrace{(\ )}_{n-k-1\ \text{"}-\text{"}}, \quad 0 \le k \le n-1$

$\underbrace{(\ )}_{k\ \text{"}-\text{"}}$ has a_k choices, $\underbrace{(\ )}_{n-k-1\ \text{"}-\text{"}}$ has a_{n-k-1} choices.

$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0, \quad n \ge 1.$

(c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating functions of a_n. Multiplying x^n to above equation, we obtain

$$
\begin{aligned}
a_n x^n &= x\left(a_0 a_{n-1} x^{n-1} + a_1 x a_{n-2} x^{n-2} + \cdots + a_{n-1} x^{n-1} a_0\right), \quad n \ge 1. \\
\sum_{n=1}^{\infty} a_n x^n &= x\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 \\
f(x) - a_0 &= x f(x)^2 \\
f(x) &= 1 + x f(x)^2 \\
x f(x)^2 - f(x) + 1 &= 0
\end{aligned}
$$

Recall $\binom{\frac{1}{2}}{n} = \frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}}{2^n}\cdot\frac{1\cdot 3\cdots(2n-3)}{n!} = (-1)^{n-1}\left(\frac{1}{2}\right)^{2n-1}\frac{(2n-2)!}{n!(n-1)!}, \quad n \ge 1.$

Then

$$(1-x)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}(-x) + \cdots + \binom{\frac{1}{2}}{n}(-x)^n + \cdots = 1 - \frac{1}{2}x - \cdots - \left(\frac{1}{2}\right)^{2n-1}\frac{(2n-2)!}{n!(n-1)!}x^n - \cdots.$$

$$\begin{aligned}
f(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \frac{1}{2x}\left(1 - \left(1 - \frac{1}{2}(4x) - \cdots - \left(\frac{1}{2}\right)^{2n-1}\frac{(2n-2)!}{n!(n-1)!}(4x)^n - \cdots\right)\right) \\
&= \frac{1}{2x}\left(\frac{1}{2}(4x) + \cdots + \left(\frac{1}{2}\right)^{2n-1}\frac{(2n-2)!}{n!(n-1)!}2^{2n}x^n + \cdots\right) \\
&= \frac{1}{2x}\left(\frac{1}{2}(4x) + \cdots + \frac{(2n-2)!}{n!(n-1)!}2x^n + \cdots\right) \\
&= 1 + \cdots + \frac{(2n-2)!}{n!(n-1)!}x^{n-1} + \cdots
\end{aligned}$$

explain : f

$$a_n = \frac{(2n)!}{n!(n+1)!}$$

4. Solve the systems of recurrence relation

$$\begin{aligned}
a_{n+1} &= -2a_n - 4b_n \\
b_{n+1} &= 4a_n + 6b_n \\
n \ge 0,\ a_0 &= 1,\ b_0 = 0
\end{aligned}$$

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Solution

$a_0 = 1$

$a_1 = -2$

$$\begin{aligned}
a_{n+2} &= -2a_{n+1} - 4b_{n+1} \\
&= 4a_n + 8b_n - 16a_n - 24b_n \\
&= -12a_n - 16b_n
\end{aligned}$$

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$$a_{n+2} - 4a_{n+1} = 4a_n$$

$$c_1 2^n + c_2 n 2^n$$

$a_{n+1} = -2a_n - 4b_n$
$b_{n+1} = 4a_n + 6b_n$
$n \geq 0,\ a_0 = 1,\ b_0 = 0.$
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n,\ g(x) = \sum_{n=0}^{\infty} b_n x^n.$
$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = -2\sum_{n=0}^{\infty} a_n x^{n+1} - 4\sum_{n=0}^{\infty} b_n x^{n+1}$

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$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = 4\sum_{n=0}^{\infty} a_n x^{n+1} + 6\sum_{n=0}^{\infty} b_n x^{n+1}$
$f(x) - a_0 = -2xf(x) - 4xg(x)$
$g(x) - b_0 = 4xf(x) + 6xg(x)$
$f(x)(1+2x) + 4xg(x) = 1$
$f(x)(-4x) + (1-6x)g(x) = 0$

$$f(x) = \frac{\begin{vmatrix} 1 & 4 \\ 0 & (1-6x) \end{vmatrix}}{\begin{vmatrix} (1+2x) & 4x \\ -4x & (1-6x) \end{vmatrix}} = (1-6x)/(1-2x)^2 =$$

$(1-6x)(1-2x)^{-2} = (1-6x)[\binom{-2}{0} + \binom{-2}{1}(-2x) + \binom{-2}{2}(-2x)^2 + \ldots]$
$a_n = \binom{-2}{n}(-2)^n - 6\binom{-2}{n-1}(-2)^{n-1} = 2^n(1-2n),\ n \geq 0$ 5
$f(x)(-4x) + (1-6x)g(x) = 0 \implies g(x) = (4x)f(x)(1-6x)^{-1} \implies g(x) =$
$4x(1-2x)^{-2}$ and $b_n = 4\binom{-2}{n-1}(-2)^{n-1} = n(2^{n+1}),\ n \geq 0.$

5

3. Find the recurrence relation of $D_n = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \vdots & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{n\times n}$. Hence calculate D_n.

Hint: $C_1 - C_n \to C_1$.

Solution

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \vdots & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{n\times n} = \begin{vmatrix} -1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{n\times n} \qquad [C_1 - C_n \to C_1]$$

$$= \begin{vmatrix} -2 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{n\times n} \qquad [R_1 - R_n \to R_1]$$

$$= -2\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{(n-1)\times(n-1)} + (-1)^{1+n}\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= -2\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{(n-1)\times(n-1)} + (-1)^{1+n}(-1)^{1+n-1}\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}_{(n-2)\times(n-2)}$$

$D_n + 2D_{n-1} + D_{n-2} = 0$ → 8 explain

$D_n = (-1)^n (A + Bn)$

$(-1)(A+B) = D_1 = 0, \quad (A+2B) = D_2 = -1$

$B = -1, \quad A = 1$ → 1 → 1

$D_n = (-1)^n (1-n)$

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$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix}_{n\times n} \underline{\underline{C_1 = C_1 - C_n}} \begin{vmatrix} -1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix}_{n\times n}$$

$$= (-1)^{1+1}(-1) \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$+ (-1)^{n+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & & 0 & 1 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= -D_{n-1} + (-1)^{n+1} \underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & & 0 & 1 \end{vmatrix}_{(n-1)\times(n-1)}}$$

$$D_n = -D_{n-1} + (-1)^{n+1}$$

$$R_2 = R_2 - R_1$$

$$\vdots$$

$$R_{n-1} = R_{n-1} - R_1 \quad \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{vmatrix}$$

$$= (-1)^n (-1)^{n-2}$$

$$= 1$$