Q1 (a): CDF $F_m(m) = P(M \le m)$.

for $m \le 0$, $P(M \le m) = 0$

for $m \ge 1$, $P(M \le m) = 1$.

for $m \in (0,1)$. $P(\max(X_1, X_2 \cdots X_n) \le m) = \prod_{i=1}^{n} P(X_i \le m)$

$P(X_i)$ is uniform distritution, So $P(X_i \le m) = m$.

So $P(M \le m) = \prod_{i=1}^{n} m = m^n$

So $$\text{CDF} = \begin{cases} 0 & m \le 0 \\ m^n & m \in (0,1) \\ 1 & m \ge 1 \end{cases}$$

(b). $$f_m(m) = \frac{d}{dm} F_m(m) = \begin{cases} 0 & m \notin (0,1) \\ n\,m^{n-1} & m \in (0,1) \end{cases}$$

Q2: for a Exponetial distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0. \end{cases}$$

So $$E(x) = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} \, dx = \int_0^{\infty} \lambda x \cdot e^{-\lambda x} \, dx$$

let $-\lambda x = y$ $\quad dy = -\lambda dx$

So $E(x) = \int_0^{-\infty} \cancel{-}y \cdot e^{y} \cdot \cancel{-}\frac{1}{\lambda}\, dy$

$= \frac{1}{\lambda}\int_0^{\infty} y \cdot e^{y}\, dy$ $\qquad (y-1)e^{y}$

$= \frac{1}{\lambda}\left[(y-1)e^{y}\right]_0^{-\infty}$

$= \frac{1}{\lambda} \cdot (0-(-1)) = \frac{1}{\lambda}$

3. $f_Y(y) = \int_0^{y} f(x,y)\, dx = \int_0^{y} \frac{e^{-y}}{y}\, dx = \frac{e^{-y}}{y} \cdot y = e^{-y}$

$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{1}{y} \quad 0 < x < y.$

So $f(x|y)$ is a uniform distribution.

$Var(f(x|y)) = \frac{y^2}{12} = E(x^2|y) - E(x|y)^2$

So $E(x^2|y) = \frac{y^2}{12} + \left(\frac{y}{2}\right)^2 = \frac{y^2}{3}$

4. suppose X to be the number of that the heads first come up, then there are $(X-1)$. tails

$$P(X=n) = (1-p)^{n-1} \cdot p$$

$$E(X) = \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1} \cdot p = p \cdot \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1}$$

$n \cdot (1-p)^{n-1} = f'(p)$. $f(p) = (1-p)^n$, since the sum is linear

So $\sum f'(p) = (\sum f(p))' = \left(\frac{1-p}{1-(1-p)}\right)$

$$= \left(\frac{1-p}{p}\right)' = \frac{-p+1-p}{p^2} = \frac{1}{p^2}$$

So $E(X) = p \cdot \frac{1}{p^2} = \frac{1}{p}$. this means it takes $\frac{1}{p}$ times to succeed, because the every throw has a p rate of success.

Q5: (1) according to Q4. the question can be view as a coin whos head has a probability of $\frac{1}{6}$.

So $E(X) = \frac{1}{p} = 6$.

(2). Since $Y=1$, So the first roll must be 5.

$$P(X=n+1 \mid Y=1) = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} = \frac{5^{n-1}}{6^n}$$

So $E[X=n+1 \mid Y=1] = \sum_{n=1}^{\infty} (n+1) \cdot \frac{5^{n-1}}{6^n}$

$$= \frac{1}{5} \sum_{n=1}^{\infty} (n+1) \cdot \left(\frac{5}{6}\right)^n = 7.$$

6. (a). before miner choose first door. the total time is $\sum_{i=1}^{N} T_i$, and miner needs another 2 hour to get out, so $X = \sum_{i=1}^{N} T_i + 2$.

(b) $N+1$ has a geometric distribution with $E[N+1] = \frac{1}{P} = 3$. So $E[N] = 2$.

(c). T_N means miner chooses either second or third door

So $E[T_N] = \frac{1}{2} \times 3 + \frac{1}{2} \times 5 = 4$

(d). $E[\sum_{i=1}^{N} T_n \mid N=n] = E[N] \cdot E[T_N)$

$= 4n$.

(e) $E[X] = E[\sum_{i=1}^{N} T_n + 2] = E[N] \cdot E[T_N] + 2$

$= 4 \times 2 + 2 = 10$.