

# Complex Analysis

Asgn. 1

Solutions

Q. 1.

$$z + z^{-1} = 2 \cos \theta$$

$$z^2 - (2 \cos \theta)z + 1 = 0$$

$$z^2 - 2z \cos \theta + \cos^2 \theta + \sin^2 \theta = 0$$

$$(z - \cos \theta)^2 = -\sin^2 \theta$$

$$z - \cos \theta = \pm \sin \theta i$$

$$z = \cos \theta \pm i \sin \theta$$

Use the ~~moivre~~ moivre formula

$$z^m = (\cos \theta \pm i \sin \theta)^m = \cos(m\theta) \pm i \sin(m\theta)$$

$$z^{-m} = (z^{-1})^m = (\cos \theta \mp i \sin \theta)^m = \cos(m\theta) \mp i \sin(m\theta)$$

$$z^m + z^{-m} = 2 \cos(m\theta).$$

□

Q 2.

Consider two complex numbers

$$\underline{z_1 = 2 + 3i}, \quad \underline{z_2 = 5 + 7i}.$$

$$|z_1|^2 = 2^2 + 3^2 = 13, \quad |z_2|^2 = 5^2 + 7^2 = 74$$

Then

$$13 \times 74 = |z_1|^2 |z_2|^2 = |z_1 z_2|^2$$

$$\begin{aligned} \text{Find } z_1 z_2 &= (2 + 3i)(5 + 7i) = (10 - 21) + (15 + 14)i \\ &= -11 + 29i \end{aligned}$$

Therefore,

$$13 \times 74 = |z_1 z_2|^2 = (-11)^2 + (29)^2 = 11^2 + 29^2$$

You get other solutions if change signs in real and imag. parts of  $z_1, z_2$ .

Q.3

$$\left| \frac{z-9}{z-1} \right| = 3$$

2

$$|z-9|^2 = 3^2 |z-1|^2$$

$$(z-9)\overline{(z-9)} = 9(z-1)\overline{(z-1)}$$

$$(z-9)(\bar{z}-9) = 9(z-1)(\bar{z}-1)$$

$$z\bar{z} - 9z - 9\bar{z} + 81 = 9(z\bar{z} - z - \bar{z} + 1)$$

$$|z|^2 - 9z - 9\bar{z} + 81 = 9(|z|^2 - z - \bar{z} + 1)$$

$$8|z|^2 = 7z + 7\bar{z}$$

$$|z|^2 = 9$$

$$|z| = 3$$

□

Q4.

$$|z|^2 = |z+1|^2, \quad z = x+iy$$

$$|z \neq -1|$$

$$z\bar{z} = (z+1)(\bar{z}+1)$$

$$z\bar{z} = z\bar{z} + z + \bar{z} + 1$$

$$z + \bar{z} = -1 \Leftrightarrow \operatorname{Re} z = -\frac{1}{2}, \quad z = -\frac{1}{2} + iy$$

Other condition

$$z = i\bar{z}, \quad \bar{z} = -\frac{1}{2} - iy$$

$$\text{or } -\frac{1}{2} + iy = -\frac{1}{2}i + y \Leftrightarrow y = -\frac{1}{2}$$

$$\text{Answer: } z = -\frac{1}{2} - \frac{1}{2}i //$$

Q5. (a)

$$\left| \frac{z-a}{z-b} \right| = 1$$

3

$$|z-a|^2 = |z-b|^2$$

$$(z-a)(\bar{z}-\bar{a}) = (z-b)(\bar{z}-\bar{b})$$

$$\cancel{z\bar{z}} - z\bar{a} - a\bar{z} + a\bar{a} = \cancel{z\bar{z}} - z\bar{b} - b\bar{z} + b\bar{b}$$

$$(\bar{b}-\bar{a})z + (b-a)\bar{z} = |b|^2 - |a|^2$$

$$2 \operatorname{Re}((b-a)\bar{z}) = |b|^2 - |a|^2$$

$$b-a = \operatorname{Re}(b-a) + i \operatorname{Im}(b-a)$$

$$z = x + iy$$

$$\operatorname{Re}((b-a)\bar{z}) = \operatorname{Re}(b-a)x - \operatorname{Im}(b-a)y$$

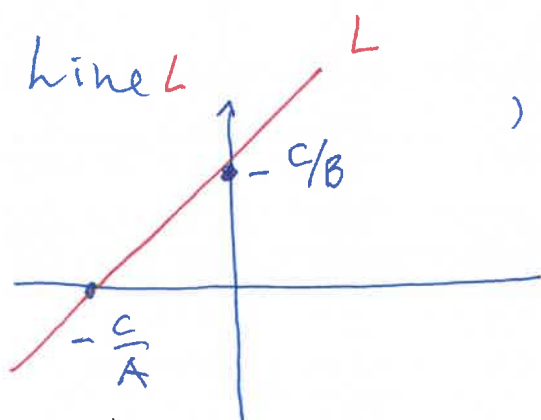
$$A = \{ (x, y) : Ax + By = C \}$$

where

$$A = \operatorname{Re}(b-a)$$

$$B = -\operatorname{Im}(b-a)$$

$$C = \frac{|b|^2 - |a|^2}{2}$$



, Case:  $A \neq 0, B \neq 0$ .

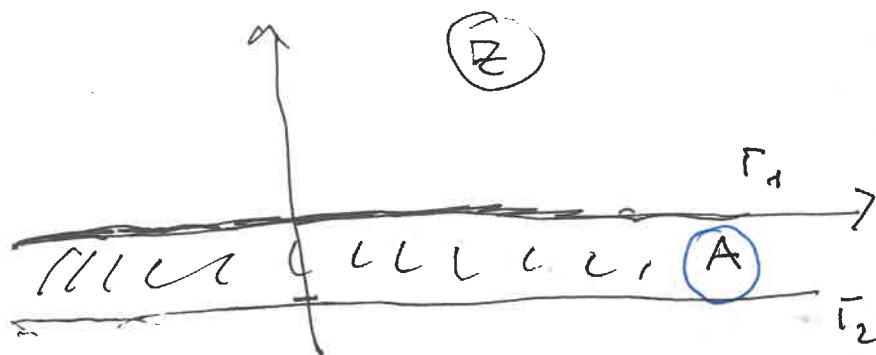
5(b)

$$z = x + iy$$

$$iz = -y + ix$$

$$0 \leq \operatorname{Re}(iz) < 1 \Leftrightarrow 0 \leq -y < 1$$

$$-1 < y \leq 0, x \in \mathbb{R}$$



A is the strip, upper bound  $\Gamma_1$  is included

5(c).  $z = |z| + \operatorname{Re}(z) \leq 1$ ,  $z = x + y$

$$|z| = \sqrt{x^2 + y^2}, \quad \operatorname{Re} z = x$$

$$\sqrt{x^2 + y^2} + x \leq 1 \Leftrightarrow \sqrt{x^2 + y^2} \leq 1 - x \quad (1)$$

1) Cond.  $1 - x \geq 0 \Leftrightarrow x \leq 1$ .

square (1)

$$x^2 + y^2 \leq 1 - 2x + x^2$$

$$y^2 + 2x - 1 \leq 0$$

$$x \leq \frac{1 - y^2}{2}$$

set A is bounded by parabola.

Boundary included!

