1. first order variation of a function measures total oscillation $[0,T]$.

$$F_{V_T}(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|, \quad \Pi: 0 = t_0 < t_1 < t_2 \cdots < t_n = T$$

$$\|\Pi\| = \max(t_{j+1} - t_j)$$

2. $$[f,f](T) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (f(t_{j+1}) - f(t_j))^2$$

Lecture 8

Stochastic Integration and Differentiation

☆ 3. $$[B,B](T) = \lim_{n\to\infty} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = T, \quad t_{j+1} - t_j = \frac{T}{n}$$

of Std Brownian motion

1 • Quadratic variation

2 • Itô integrals

• Wiener integrals ← special case

$\int_0^T f(t,B_t)\,dB_t$ — $f(t,B_t)$: integrand

• Itô multiplication table

3 • Itô's formula $df(t,B_t)$: $df(B_t)$, $df(t,B_t)$, $df(t,X_t)$

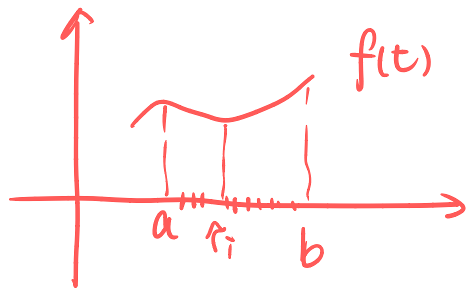
• Computing Itô integrals via Itô's formula

8.1 Introduction

integrand

- Recall that the *Riemann integral* $\int_a^b f(t)dt$ is defined as follows.

  1. Partition the interval $[a, b]$ into n subintervals
  $$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$
  2. Choose a point τ_i in the subinterval $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, n-1$.
  3. Form the *Riemann sum*
  $$S_n = \sum_{i=0}^{n-1} f(\tau_i)(t_{i+1} - t_i)$$



  4. Define the integral as the limit
  $$\int_a^b f(t)dt := \lim_{n\to\infty} S_n \quad = S$$
  where the subintervals are chosen so that $\max_i\{t_{i+1} - t_i\} \to 0$.

- In this lecture, we introduce the following new kinds of integrals (called *Itô integrals*) $dB_t = B_{t_{i+1}} - B_{t_i}$: $dB_t = B(t+\Delta t) - B(t)$.

$B(t)$

$$\int_a^b f(t)\, dB_t \qquad \int_a^b f(t, B_t)\, dB_t \qquad \int_a^b f(t, B_t)\, (dB_t)^2$$

B_t

where B_t (i.e. $B(t)$) is the standard Brownian motion process.

rate of change

- If $B(t)$ were differentiable, then $\frac{dB_t}{dt} = B'(t)$

$$\Rightarrow \int_a^b f(t)\, dB_t \cancel{=} \int_a^b f(t) B'(t)\, dt,$$

$dB_t = B'(t) \cdot dt$

$\frac{B(t+\Delta t) - B(t)}{\Delta t} \sim N(0, \frac{1}{\Delta t})$

which is a classical integral. It is because $B(t)$ is non-differentiable that new calculus rules are sought.

$Var(X) = \Delta t$

- This lecture is to introduce basic calculus rules which will be used in later lectures to stock price, interest rate, and bond price with *stochastic differential equations* (SDE).

$Var(\frac{X}{\Delta t}) = \frac{1}{\Delta t}$

$\Delta t \to 0$, $Var(X) \to \infty$.

8.2 Non-anticipating Processes

integrand.

- Consider the standard Brownian motion B_t (i.e. $B(t)$).
- A process F_t is called a *non-anticipating process* if F_t is independent of any future increment of B_t.

$f(B_t)$ $f(t, B_t)$

- E.g. $F_t = B_t$, e^{B_t}, $B_t^2 + t$ are non-anticipating.
- E.g. $F_t = B_{t+1}$, B_{2t}, $(B_{t+1} - B_t)^2$ are not.
- Itô integrals apply only to non-anticipating processes.

→ adapted to the natural filtration of Brownian motion. $\mathcal{F}_t = \{\sigma(B_s), 0<s<t\}$.

8.3 Increments of Brownian Motions

Proposition 8.1 *Let B_t be the standard Brownian motion. If $s < t$, then we have*

$X = B(t) - B(s)$, $X \sim N(0, t-s)$

$EX^2 = \text{Var}(X) + (EX)^2 = t-s$.

1. $E[(B_t - B_s)^2] = t - s$;
2. $E[(B_t - B_s)^4] = 3(t - s)^2$;
3. $\text{Var}\left((B_t - B_s)^2\right) = 2(t - s)^2$.

$X \sim N(0, \sigma^2)$

$E(X^k) = 0$ if k is add

$E(X^{2n}) = \sigma^{2n} \cdot (2n-1)!!$

$(2n-1)(2n-3) \cdots 3 \cdots 1$

if k is even

$k = 2n$

Proof:

$$E[(B_t - B_s)^2] = E[B_{t-s}^2] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{x^2}{2(t-s)}} dx = t - s$$

$$E[(B_t - B_s)^4] = E[B_{t-s}^4] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} x^4 \cdot e^{-\frac{x^2}{2(t-s)}} dx = 3(t - s)^2$$

$$\text{Var}\left((B_t - B_s)^2\right) = \text{Var}\left(B_{t-s}^2\right) = E[B_{t-s}^4] - (E[B_{t-s}^2])^2 = 2(t-s)^2.$$

□

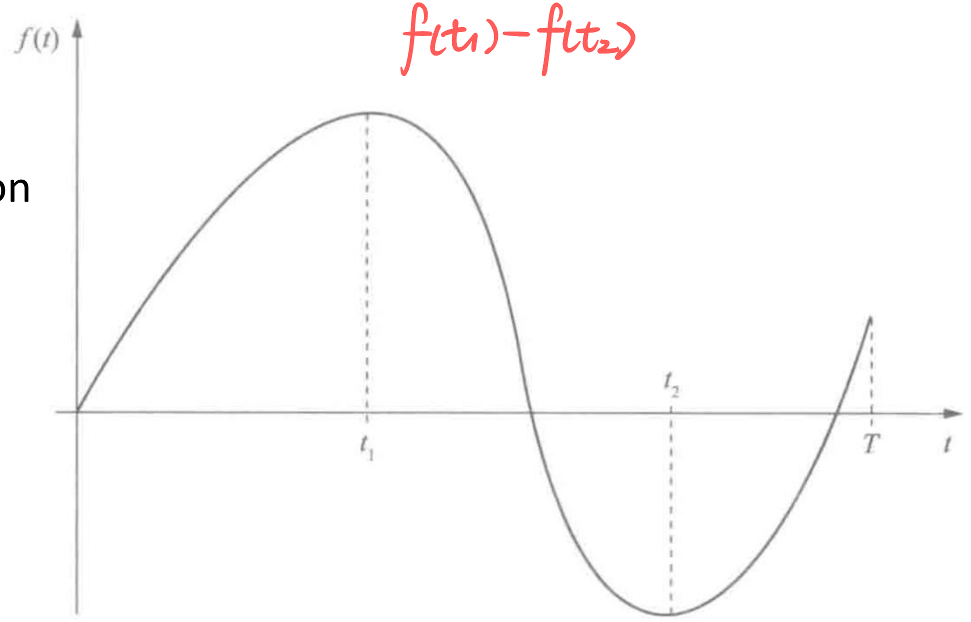
$\int_0^{t_1} f'(t)dt = f(t)\Big|_0^{t_1} = f(t_1) - f(0)$

First-order variation

- To compute the **total oscillation** undergone by a function f between times 0 and T.

$$FV_T(f) = [f(t_1) - f(0)] - [f(t_2) - f(t_1)] + [f(T) - f(t_2)]$$
$$= \int_0^{t_1} f'(t)dt + \int_{t_1}^{t_2}(-f'(t))dt + \int_{t_2}^{T} f'(t)dt$$
$$= \int_0^T |f'(t)|dt$$

$f(t_1) - f(t_2)$



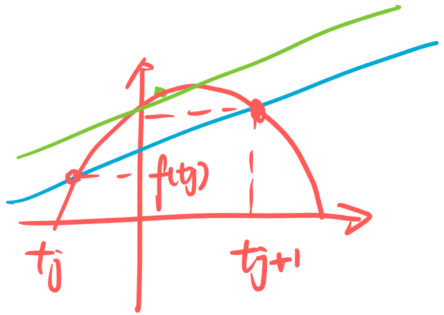
- To compute the first-order variation of a function up to time T, we first choose a partition $\Pi = \{t_0, t_1, \ldots t_n\}$ of $[0, T]$, $0 = t_0 < t_1 < \cdots t_n = T$.
- The maximum step size of the partition will be denoted $||\Pi|| = max_{j=0,\ldots n-1}(t_{j+1} - t_j)$.

mesh size

- We define

$$FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| \quad ②$$

First-order and quadratic variation



- The two expression of $FV_T(f)$ are equivalent, because based on the Mean Value Theorem, which applies to any function $f(t)$ whose derivative $f'(t)$ is defined everywhere, there exist a point t_j^* in the subinterval $[t_j, t_{j+1}]$ such that

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = f'(t_j^*)$$

slope of tangent line

So $FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)| = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)| \, (t_{j+1} - t_j) = \int_0^T |f'(t)| dt$

dt

- Similarly, we could define the **quadratic variation of *f*** up to time T

$$[f, f](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2,$$ where $\Pi = \{t_0, t_1, \ldots t_n\}$ and $0 = t_0 < t_1 < \cdots t_n = T$.

Quadratic variation

- If the function f has a continuous derivative, we could prove that its quadratic variation is 0.

$$\sum_{j=0}^{n-1}[f(t_{j+1}) - f(t_j)]^2 = \sum_{j=0}^{n-1}\left|f'(t_j^*)\right|^2 (t_{j+1} - t_j)^2 \leq ||\Pi|| \sum_{j=0}^{n-1}\left|f'(t_j^*)\right|^2 (t_{j+1} - t_j)$$

$$[f,f](T) \leq \lim_{||\Pi||\to 0} ||\Pi|| \sum_{j=0}^{n-1}\left|f'(t_j^*)\right|^2 (t_{j+1} - t_j) \quad = \lim_{||\Pi||\to 0} ||\Pi|| \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1}\left|f'(t_j^*)\right|^2 (t_{j+1} - t_j)$$

$$= \lim_{||\Pi||\to 0} ||\Pi|| \int_0^T |f'(t)|^2 dt = 0$$

- The path of Brownian motion cannot be differentiated with respect to the time variable, so the mean value theorem does not apply and the above conclusion does not hold.
- The quadratic variation of a Brownian motion quantifies its cumulative **oscillations** over time interval from 0 to T and can be regarded as a measure of variability.

The quadratic variation of Brownian motion

- $E\left[\left(B_{t_{j+1}} - B_{t_j}\right)^2\right] = t_{j+1} - t_j$ $\qquad Var\left[\left(B_{t_{j+1}} - B_{t_j}\right)^2\right] = 2(t_{j+1} - t_j)^2$
- $\frac{B_{t_{j+1}} - B_{t_j}}{\sqrt{t_{j+1} - t_j}} = Z_{j+1} \sim N(0,1),\ \frac{\left(B_{t_{j+1}} - B_{t_j}\right)^2}{t_{j+1} - t_j} = Z_{j+1}^2$
- Suppose $t_j = \frac{jT}{n}$, then $\left(B_{t_{j+1}} - B_{t_j}\right)^2 = Z_{j+1}^2 \cdot \frac{T}{n}$

When $n \to \infty, \sum_{j=0}^{n-1} Z_{j+1}^2 \cdot \frac{1}{n} \to E\left(Z_{j+1}^2\right) = 1$ due to the strong law of large numbers. Then,

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(B_{t_{j+1}} - B_{t_j}\right)^2 = \lim_{n\to\infty} Z_{j+1}^2 \cdot \frac{T}{n} = T$$

Each term $\left(B_{t_{j+1}} - B_{t_j}\right)^2$ in this sum can be quite different from its mean $t_{j+1} - t_j = \frac{T}{n}$, but when we sum many terms like this, the deviations cancel in aggregate and the sum converges T.

- We write informally $dB_t\ dB_t = dt$. On an interval $[0, T]$, Brownian motion accumulates T units of quadratic variation.

The quadratic variation of Brownian motion

- Let $\Pi = \{t_0, t_1, \dots t_{\mathrm{n}}\}$ be a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots t_{\mathrm{n}} = T$.

$$[B, B](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \left(B_{t_{j+1}} - B_{t_j}\right)^2 = T$$

- We could also compute the cross variation of B(t) with t and the quadratic variation of t with itself.

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})(t_{j+1} - t_j) = 0$$

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \left(t_{j+1} - t_j\right)^2 = 0$$

- We write informally $dB_t dt = 0, dtdt = 0$

Mean Square Convergence

- A random variable is a function of the outcome $\omega \in \Omega$. Thus, a sequence of random variables $S_1, S_2, \ldots$ is a sequence of functions.
- Convergence of a sequence of functions can be studied in various modes, e.g. pointwise, uniform, and mean square.
- For sequence of random variables, we use mean square convergence, weighted by a probability density.

Definition 8.2 A sequence of random variables $S_1, S_2, \ldots$ is said to converge to a random variable S in *mean square sense* if

$$\lim_{n\to\infty} E\left[(S_n - S)^2\right] = 0.$$

The limit S is denoted by

$$S = \operatorname*{ms-lim}_{n\to\infty} S_n.$$

Quadratic Variation

Proposition 8.3 *Let $T > 0$ and consider the equidistant partition $t_i = \frac{iT}{n}$, for $i = 0, 1, \ldots, n$, of the interval $[0, T]$. Then*

$$\underset{n\to\infty}{\text{ms-lim}} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 = T. \tag{8.1}$$

The expression on the left is called the quadratic variation *of B_t.*

Proof: Consider the random variable

$$S_n = \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2.$$

By Proposition 8.1,

$$E[S_n] = \sum_{i=0}^{n-1} E\left[(B_{t_{i+1}} - B_{t_i})^2\right] = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = T.$$

Since the increments of $\{B_t,\ t \geq 0\}$ are independent, we have

$$\begin{aligned} \mathrm{Var}(S_n) &= \sum_{i=0}^{n-1} \mathrm{Var}\left[(B_{t_{i+1}} - B_{t_i})^2\right] = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2 \\ &= n \cdot 2\left(\frac{T}{n}\right)^2 = \frac{2T^2}{n}. \end{aligned}$$

The second equality above is due to Proposition 8.1 and the fact that $t_{i+1} - t_i = \frac{T}{n}$ for all i.

Next, note that

$$E[(S_n - T)^2] \; = \; E[(S_n - E[S_n])^2] = \mathrm{Var}(S_n) = \frac{2T^2}{n} \to 0$$

as $n \to \infty$. Thus,

$$\underset{n\to\infty}{\text{ms-lim}} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 = T.$$

$\square$

8.4 The Itô Integral

- Let $F_t = f(t, B_t)$ be a non-anticipating process.
- Without loss of generality, we assume the subintervals equidistant, i.e. $t_{i+1} - t_i = \frac{b-a}{n}$.
- Consider the partial sum

$$S_n = \sum_{i=0}^{n-1} F_{t_i}(B_{t_{i+1}} - B_{t_i}).$$

Here, the intermediate points τ_i are chosen to be the <u>left</u> endpoint t_i.

- The choice $\tau_i = t_i$ makes F_{t_i} and $B_{t_{i+1}} - B_{t_i}$ independent.
- The *Itô integral* is the mean square limit of the partial sums S_n

$$\int_a^b F_t \, dB_t := \operatorname*{ms-lim}_{n\to\infty} S_n.$$

8.5 Examples of Itô Integrals

- We provide two examples of computing Itô integrals from first principles.
- However, just like Riemann integrals, we seldom compute them from scratch in practice.
- Later, we shall deduce some rules that allow us to compute Itô integrals more efficiently.
- More precisely, we will introduce a chain rule for stochastic processes (called *Itô's formula*) which can be applied to compute Itô integrals.

The case $F_t = 1$ (i.e. $\int_0^T dB_t$)

Example 8.4 Let $F_t = 1$ and $[a, b] = [0, T]$. Then,

$$S_n = \sum_{i=0}^{n-1} F_{t_i}(B_{t_{i+1}} - B_{t_i}) = \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i}) = B_T - B_0 = B_T.$$

It follows that $\text{ms-lim}_{n\to\infty} S_n = B_T$. Thus, we have the result

$$\int_0^T dB_t = B_T.$$

□

The case $F_t = B_t$ (i.e. $\int_0^T B_t \, dB_t$)

Example 8.5 Let $F_t = B_t$ and $t_i = \frac{iT}{n}$ for $i = 0, 1, \ldots, n$. Then,

$$S_n = \sum_{i=0}^{n-1} F_{t_i}(B_{t_{i+1}} - B_{t_i}) = \sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}).$$

Since

$$xy = \frac{1}{2}\left[(x+y)^2 - x^2 - y^2\right],$$

letting $x = B_{t_i}$ and $y = B_{t_{i+1}} - B_{t_i}$ yields

$$B_{t_i}(B_{t_{i+1}} - B_{t_i}) = \frac{1}{2}B_{t_{i+1}}^2 - \frac{1}{2}B_{t_i}^2 - \frac{1}{2}(B_{t_{i+1}} - B_{t_i})^2.$$

By summing over i, we have

$$S_n = \frac{1}{2}\sum_{i=0}^{n-1} B_{t_{i+1}}^2 - \frac{1}{2}\sum_{i=0}^{n-1} B_{t_i}^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2.$$

Then, by telescoping, the sum becomes

$$S_n \;=\; \frac{1}{2}B_T^2 - \frac{1}{2}B_0^2 - \frac{1}{2}\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2.$$

By Eq. (8.1), we obtain

$$\underset{n\to\infty}{\text{ms-lim}}\, S_n = \frac{1}{2}B_T^2 - \frac{1}{2}\cdot \underset{n\to\infty}{\text{ms-lim}} \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 = \frac{1}{2}B_T^2 - \frac{1}{2}T.$$

Thus, we have the result

$$\int_0^T B_t\, dB_t = \frac{B_T^2}{2} - \frac{T}{2}.$$

□

Remark: For Riemann integrals, we have $\int_0^T f(x)\, df(x) = \frac{f(T)^2}{2} - \frac{f(0)^2}{2}$, provided that f differentiable. The additional term $-\frac{T}{2}$ appeared in the Itô integral is an instance of *Itô correction*. It is due to the non-differentiability of $B(t)$.

The construction of Itô integral

$f(t, B_t)$

- We fix a positive number of T and try to make sense of $\int_0^T \Delta(t) dB_t$, where B_t, $t \geq 0$, is a Brownian motion together with a filtration $\mathcal{F}(t), t \geq 0$.
- The integrand $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t), t \geq 0$. i.e., the information available up to time t is sufficient to evaluate $\Delta(t)$ at time t.
- We can regard $\Delta(t)$ as the **position** we take in an asset at time t, and this depends on the price path of the asset **up to** time t. i.e., $\Delta(t)$ should not depend on the value of Brownian motion after time t.

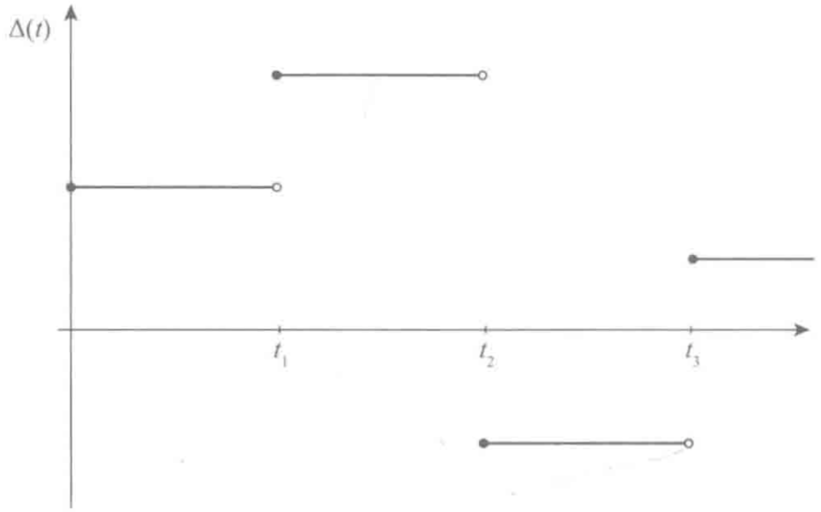
The construction of Itô integral

- We first define the Itô integral form **simple integrands** and then extend it to nonsimple integrands as a limit of the integral of simple integrands.
- Let $\Pi = \{t_0, t_1, \dots t_\mathrm{n}\}$ be a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots t_\mathrm{n} = T$. Assume $\Delta(t)$ is constant in t on each $[t_j, t_{j+1})$.

- Regard $B(t)$ as the asset price at time t. Think of $t_0, t_1, \dots t_{\mathrm{n}-1}$ as the trading dates in the assets, and think of $\Delta(t_0)$, $\Delta(t_1)\dots\ \Delta(t_{n-1})$ as the position (number of shares) taken in the asset at each trading date and held to the next trading date.
- The gain from trading at each time t is

$$I(t) = \Delta(t_0)[B(t) - B(t_0)] = \Delta(0)B(t),\ 0 \le t \le t_1$$

$$I(t) = \Delta(0)B(t_1) + \Delta(t_1)[B(t) - B(t_1)],\ t_1 \le t \le t_2$$



A path of a simple process

The construction of Itô integral

$I(t) = \Delta(0)B(t_1) + \Delta(t_1)[B(t_2) - B(t_1)] + \Delta(t_2)[B(t) - B(t_2)],\ t_2 \leq t \leq t_3.$

- In general, if $t_k \leq t \leq t_{k+1}$,

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[B(t_{j+1}) - B(t_j)] + \Delta(t_k)[B(t) - B(t_k)] \qquad (*)$$

- This process $I(t)$ is the Itô integral of the simple process $\Delta(t)$.
- In particular, if we take $t = t_\mathrm{n} = T, I(T)$ provides a definition for $\int_0^T \Delta(t) dB_t$.

Properties of the Integral

The Itô integral defined by (*)

- Is a martingale (proposition 8.8)
- satisfies $E[I^2(t)] = E\left[\int_0^t \Delta^2(u)du\right]$ (Isometry, P205)

The RHS is an ordinary Lebesgue integral where the integrand is a stochastic process.

- The quadratic variation accumulated up to time t is

$$[I,I](t) = \int_0^t \Delta^2(u)\, du$$

Summary:

- The result of quadratic variation can depend on path and the size of the positions we take.
- The variance of $I(t)$ is an average over all possible paths of the quadratic variation.

Differential form of Itô integral

- Recall $dB_t\ dB_t = dt$ which can be interpreted as "Brownian motion accumulates quadratic variation at rate one per unit time."
- $I(t) = \int_0^t \Delta(t) dB_t$ can be written in differential form as $dI(t) = \Delta(t)\, dB_t$. Then
$$dI(t)dI(t) = \Delta^2(t)dB_t\, dB_t = \Delta^2(t)dt$$
- This equation says the Itô integral accumulates quadratic variation at rate $\Delta^2(t)$ per unit time.

Itô integral for general integrands

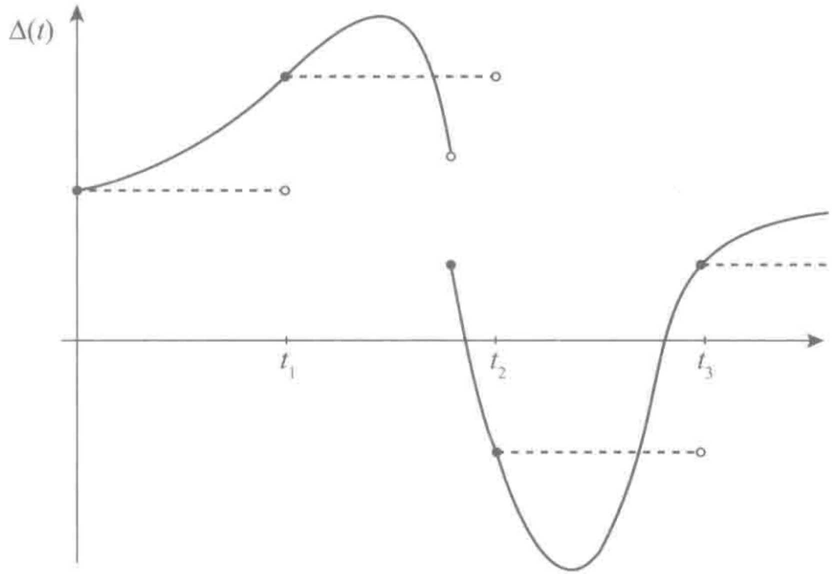
- In this section, we define $\Delta(t)$, $t \geq 0$, is adapted to the filtration $\mathcal{F}(t), t \geq 0$. We also assume the square-integrability condition $E[\int_0^T \Delta^2(t)dt] < \infty$.
- In general, it is possible to choose a sequence $\Delta_n(t)$ of simple processes such that as $n \to \infty$, these processes converge to the continuously varying $\Delta(t)$.

$$\lim_{n\to\infty} E\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt < \infty.$$

- Then we define

$$\int_0^t \Delta(t)dB_t = \lim_{n\to\infty} \int_0^t \Delta_n(t)dB_t, 0 \leq t \leq T.$$

- This integral inherits the properties of Itô integral of simple processes.



Approximating a continuously varying integrand

8.6 Properties of the Itô Integral

Proposition 8.6 *Let $F_t = f(t, B_t)$ and $G_t = g(t, B_t)$ be two non-anticipating processes and $c \in \mathbb{R}$. Then we have*

1. *Additivity:*
$$\int_a^b (F_t + G_t)\, dB_t = \int_a^b F_t\, dB_t + \int_a^b G_t\, dB_t.$$
2. *Homogeneity:*
$$\int_a^b cF_t\, dB_t = c \int_a^b F_t\, dB_t.$$
3. *Partition property:*
$$\int_a^b F_t\, dB_t = \int_a^c F_t\, dB_t + \int_c^b F_t\, dB_t$$
for any $a < c < b$.

- As a random variable, Itô integrals have the following properties.

Proposition 8.7 *Let $F_t = f(t, B_t)$ and $G_t = g(t, B_t)$ be two non-anticipating processes. We have*

1. *Zero mean:*
$$E\left[\int_a^b F_t\, dB_t\right] = 0.$$

2. *Isometry (variance):*
$$E\left[\left(\int_a^b F_t\, dB_t\right)^2\right] = E\left[\int_a^b F_t^2\, dt\right].$$

3. *Covariance:*
$$E\left[\int_a^b F_t\, dB_t \cdot \int_a^b G_t\, dB_t\right] = E\left[\int_a^b F_t G_t\, dt\right].$$

Main Ideas Behind Proposition 8.7

- Zero mean: $\int_a^b F_t \, dB_t \approx S_n = \sum_{i=0}^{n-1} F(t_i)(B(t_{i+1}) - B(t_i))$. Note that $E[S_n] = \sum_{i=0}^{n-1} E[F(t_i)]E[B(t_{i+1}) - B(t_i)] = 0$ for all n. The limit $E[S]$ also has a zero mean. Of course, some rigorous arguments are needed to show that $E[S] = E[\lim_n S_n] = \lim_n E[S_n]$, but we skip these details.
- Isometry: $\left(\int_a^b F_t \, dB_t\right)^2 \approx S_n^2 = \left[\sum_{i=0}^{n-1} F(t_i)(B(t_{i+1}) - B(t_i))\right]^2$. Hence, for each n, we have

$$\begin{aligned}
E[S_n^2] &= 2\sum_{i<j} E[F(t_i)F(t_j)(B(t_{i+1}) - B(t_i))]E[B(t_{j+1}) - B(t_j)] \\
&\quad + \sum_{i=0}^{n-1} E[F(t_i)^2]E[(B(t_{j+1}) - B(t_j))^2] \\
&= \sum_{i=0}^{n-1} E[F(t_i)^2]E[(B(t_{j+1}) - B(t_j))^2] \\
&= \sum_{i=0}^{n-1} E[F(t_i)^2](t_{i+1} - t_i) = E\left[\sum_{i=0}^{n-1} F(t_i)^2(t_{i+1} - t_i)\right] \approx E\left[\int_a^b F(t)^2 dt\right].
\end{aligned}$$

- By considering the upper integration limit as the time variable t, the integral $I_t = \int_0^t f(u, B_u)\, dB_u$ is a continuous-time process.
- Denote by $\mathcal{F}_s = \{B_u,\ 0 \le u \le s\}$ the historical values of the Brownian motion available at time s.

Proposition 8.8 *The Itô integral* $I_t = \int_0^t F_u\ dB_u$*, where* $F_u = f(u, B_u)$*, is a martingale. That is, for any* $s < t$*, we have*

$$E[I_t|\mathcal{F}_s] = I_s.$$

Proof:

$$\begin{aligned} E[I_t|\mathcal{F}_s] &= E\left[\int_0^s F_u\, dB_u \middle| \mathcal{F}_s\right] + E\left[\int_s^t F_u\, dB_u \middle| \mathcal{F}_s\right] \\ &= \int_0^s F_u\, dB_u + 0 = I_s. \end{aligned}$$

Here, $E\left[\int_s^t F_u\ dB_u|\mathcal{F}_s\right] = 0$ by a reason similar to the zero mean property in Prop. 8.7. $\square$

8.7 The Wiener Integral

- The *Wiener integral* is a special case of the Itô integral in which $f(t, B_t) = f(t)$, i.e. a deterministic function.
- E.g. $\int t\, dB_t$ and $\int e^t\, dB_t$.
- Being an Itô integral, a Wiener integral inherits all properties of the Itô integral.
- But it has an additional property that it is normally distributed.

Proposition 8.9 *The Wiener integral* $I(f) = \int_a^b f(t)\, dB_t$ *is normally distributed with mean* 0 *and variance*

$$\mathrm{Var}(I(f)) = \int_a^b f(t)^2 dt.$$

Proof: The mean and variance are basic properties of Itô integrals, followed directly from Proposition 8.7. To see that $I(f)$ is normally distributed, note that

$$S_n = \sum_{i=0}^{n-1} f(t_i)(B_{t_{i+1}} - B_{t_i}).$$

Since the increments $B_{t_{i+1}} - B_{t_i} \sim \mathcal{N}(0, t_{i+1} - t_i)$, it follows that

$$S_n \sim \mathcal{N}\left(0, \sum_{i=0}^{n-1} f(t_i)^2(t_{i+1} - t_i)\right).$$

We claim without giving details that the mean square limit is also normally distributed (under some suitable assumptions on f). □

Example 8.10 The following are examples of Wiener integrals.

1. The random variable $I = \int_1^T \frac{1}{\sqrt{t}}\, dB_t$ is normally distributed with mean 0 and variance $\ln T$.

2. The random variable $I = \int_1^T \sqrt{t}\, dB_t$ is normally distributed with mean 0 and variance $(T^2 - 1)/2$.

3. The random variable $I = \int_0^T e^{T-t}\, dB_t$ is normally distributed with mean 0 and variance $(e^{2T} - 1)/2$.

$\square$

8.8 Fundamental Relations of Differentials

- By Eq. (8.1),

$$\underset{n\to\infty}{\text{ms-lim}} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 = T$$

- The right hand side can be regarded as a Riemann integral $\int_0^T dt$.
- The left hand side can be regarded as a stochastic integral $\int_0^T (dB_t)^2$.
- Therefore, we have the integral equation

$$\int_0^T (dB_t)^2 = \int_0^T dt.$$

- The above equation written in differential form is $(dB_t)^2 = dt$. The differential form is a shorthand for the integral equation.

- Likewise, we can show that

$$\operatorname*{ms-lim}_{n\to\infty} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})(t_{i+1} - t_i) = \operatorname*{ms-lim}_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = 0.$$

- This yields the integral equations

$$\int_0^T dB_t\, dt = 0 \qquad \text{and} \qquad \int_0^T (dt)^2 = 0,$$

whose differential forms are $dB_t\, dt = 0$ and $(dt)^2 = 0$.

- The following are fundamental relations in stochastic calculus:

  1. $(dB_t)^2 = dt$;
  2. $dB_t\, dt = 0$;
  3. $(dt)^2 = 0$.

  They are also known as the *Itô multiplication table.*

Itô formula for Brownian motion

- We want to find a "differentiate" expression of the form $f(B_t)$, where $f(x)$ is a differentiable function and B_t is a Brownian motion.
- If B_t is also differentiable,

$\frac{d}{dt} f(B_t) = f'(B_t)B_t{}'$, or $df(B_t) = f'(B_t)dB_t$

- But because B_t has nonzero quadratic variation, the correct formula has an extra term

$$df(B_t) = f'(B_t)dB_t + \boxed{\frac{1}{2} f''(B_t)dt,}$$

Which is the Itô formula in differential form. Integrating this, we obtain the integral form:

$$f(B_t) - f(0) = \int_0^t f'(B_u)dB_u + \frac{1}{2}\int_0^t f''(B_u)du$$

Itô formula for Brownian motion

- Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x), f_x(t, x)$, and $f_{xx}(t, x)$ are defined and continuous. Let B_t be a Brownian motion. Then, for every $T \geq 0$,

$$f(T, B_T) = f(0, B_0) + \int_0^T f_t(t, B_t)dt + \int_0^T f_x(t, B_t)dB_t + \frac{1}{2}\int_0^T f_{xx}(t, B_t)dt$$

8.9 Itô's Formula

Itô Processes

- A process X_t is called an *Itô diffusion* (a.k.a. *Itô process*) if

$$X_t = X_0 + \int_0^t a(s, B_s)\, ds + \int_0^t b(s, B_s)\, dB_s$$

  for some functions $a(t, x)$ and $b(t, x)$.

- This equation can also be expressed in differential form

$$dX_t = a(t, B_t)\, dt + b(t, B_t)\, dB_t.$$

  This form is again a shorthand for the integral form above.

- Examples of Itô processes:

$$\begin{aligned} dX_t &= \mu\, dt + \sigma\, dB_t \qquad \text{(Brownian motion with drift)} \\ dS_t &= \mu S_t\, dt + \sigma S_t\, dB_t \qquad \text{(GBM model of stock prices)} \end{aligned}$$

Differential of $F_t = f(t, X_t)$

- Consider a process $F_t = f(t, X_t)$ where X_t is an Itô process. We would like to devise the chain rule for evaluating dF_t.

- By Taylor expansion of $f(t, x)$

$$df(t,x) = \frac{\partial f(t,x)}{\partial t}\,dt + \frac{\partial f(t,x)}{\partial x}\,dx + \frac{1}{2}\cdot\frac{\partial^2 f(t,x)}{\partial x^2}(dx)^2 + \cdots .$$

- Substituting $x = X_t$ (an Itô diffusion) and yields

$$df(t,X_t) = \frac{\partial f(t,X_t)}{\partial t}\,dt + \frac{\partial f(t,X_t)}{\partial x}\,dX_t + \frac{1}{2}\cdot\frac{\partial^2 f(t,X_t)}{\partial x^2}(dX_t)^2.$$

- All higher terms, e.g. $(dt)^2$, $dt\ dX_t$, and $(dX_t)^3$, are 0 because of the Itô multiplication table.

- By using $dX_t = a(t, B_t)dt + b(t, B_t)dB_t$ and the Itô multiplication table, we have

$$(dX_t)^2 = a^2(dt)^2 + 2ab\ dt\ dB_t + b^2(dB_t)^2 = b^2 dt.$$

- Thus, we arrive at the *Itô's formula*

$$\begin{aligned} dF_t \ &= \left[\frac{\partial f(t, X_t)}{\partial t} + a(t, B_t)\frac{\partial f(t, X_t)}{\partial x} + \frac{b(t, B_t)^2}{2} \cdot \frac{\partial^2 f(t, X_t)}{\partial x^2}\right] dt \\ &\quad + b(t, B_t)\frac{\partial f(t, X_t)}{\partial x}\ dB_t. \end{aligned} \tag{8.2}$$

Remark: Note that the traditional chain rule for smooth functions is

$$df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t}dt + \frac{\partial f(t, X_t)}{\partial x}dX_t.$$

Thus, the Itô formula has an additional term, called *Itô correction*,

$$\frac{b(t, B_t)^2}{2} \cdot \frac{\partial^2 f(t, X_t)}{\partial x^2}dt.$$

Example 8.11 Determine $d(tB_t^2)$.

Solution: Let $F_t = f(t, X_t) = tB_t^2$. Therefore, we must have $a = 0$, $b = 1$, and $f(t, x) = tx^2$. Then,

$$\frac{\partial f}{\partial t} = x^2, \quad \frac{\partial f}{\partial x} = 2tx, \quad \frac{\partial^2 f}{\partial x^2} = 2t.$$

By Itô's formula (8.2) (with $a = 0$, $b = 1$),

$$\begin{aligned} d(tB_t^2) &= \left(B_t^2 + \frac{1}{2} \cdot 2t \right) dt + 2tB_t \, dB_t \\ &= (B_t^2 + t)dt + 2tB_t \, dB_t. \end{aligned}$$

$\square$

Example 8.12 If X_t is a process such that $dX_t = \mu dt + \sigma dB_t$ where μ and σ are constants, determine $d(e^{-t}X_t)$.

Solution: Let $F_t = f(t, X_t) = e^{-t}X_t$. Therefore, we have $f(t,x) = e^{-t}x$. Then,

$$\frac{\partial f}{\partial t} = -e^{-t}x, \quad \frac{\partial f}{\partial x} = e^{-t}, \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

By Itô's formula (8.2) (with $a = \mu$, $b = \sigma$)

$$\begin{aligned} d(e^{-t}X_t) &= \left(-e^{-t}X_t + \mu e^{-t}\right)\, dt + \sigma e^{-t}\, dB_t \\ &= e^{-t}(\mu - X_t)\, dt + \sigma e^{-t}\, dB_t. \end{aligned}$$

$\square$

Example 8.13 Consider the stock price S_t generated by the GBM model $dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$ where μ and σ are constants, determine $d(\ln S_t)$.

Solution: Let $F_t = f(t, S_t) = \ln S_t$. Therefore, we have $f(t, x) = \ln x$. Then,

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}.$$

By Itô's formula (8.2) (with $a = \mu S_t$ and $b = \sigma S_t$)

$$\begin{aligned} d(\ln S_t) &= \left(0 + \mu S_t \cdot \frac{1}{S_t} - \frac{1}{2} \cdot (\sigma S_t)^2 \cdot \frac{1}{S_t^2}\right) dt + \sigma S_t \cdot \frac{1}{S_t}\, dB_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dB_t. \end{aligned}$$

□

This example shows that the log price is a Brownian motion with a drift.

8.10 Stochastic Integration Techniques: Itô's Formula

- Let $X_t = B_t$, i.e. an Itô process with $a = 0$, $b = 1$. By Itô's formula,

$$df(t, B_t) = \left[\frac{\partial f(t, B_t)}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 f(t, B_t)}{\partial x^2}\right] dt + \frac{\partial f(t, B_t)}{\partial x} \, dB_t.$$

- Integrating both sides, we have

$$\int_0^t \frac{\partial f(s,B_s)}{\partial x} \, dB_s = f(t, B_t) - f(0, 0) - \int_0^t \left[\frac{\partial f(s,B_s)}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 f(s,B_s)}{\partial x^2}\right] ds$$

- Suppose that we'd like to determine an Itô integral $\int_0^t g(s, B_s) \, dB_s$.
- We can thus determine $\int_0^t g(s, B_s) \, dB_s$ by finding an $f(t, x)$ such that $\frac{\partial f(t,x)}{\partial x} = g(t, x)$ and applying the above formula.

Example 8.14 Determine $\int_0^t B_s \, dB_s$.

Solution: Let $\frac{\partial f}{\partial x}(t, X_t) = B_t$. Therefore, we have $a = 0$, $b = 1$, and $f(t,x) = \frac{x^2}{2}$. Let $F_t = f(t, X_t) = \frac{B_t^2}{2}$. By Itô's formula (with $a = 0$, $b = 1$)

$$d\left(\frac{B_t^2}{2}\right) = \frac{1}{2}dt + B_t \, dB_t.$$

Hence,

$$\frac{B_t^2}{2} = \frac{B_0^2}{2} + \frac{t}{2} + \int_0^t B_s \, dB_s,$$

so that

$$\int_0^t B_s \, dB_s = \frac{B_t^2}{2} - \frac{t}{2}.$$

$\square$

Example 8.15 Determine $\int_0^t sB_s\, dB_s$.

Solution: Let $\frac{\partial f}{\partial x}(t, X_t) = tB_t$. Therefore, we have $a = 0$, $b = 1$, and $f(t, x) = \frac{tx^2}{2}$. Let $F_t = f(t, X_t) = \frac{tB_t^2}{2}$. By Itô's formula (with $a = 0$, $b = 1$)

$$d\left(\frac{tB_t^2}{2}\right) = \left(\frac{B_t^2}{2} + \frac{t}{2}\right) dt + tB_t\, dB_t.$$

Hence,

$$\frac{tB_t^2}{2} = \frac{0 \cdot B_0^2}{2} + \int_0^t \left(\frac{B_s^2}{2} + \frac{s}{2}\right) ds + \int_0^t sB_s\, dB_s,$$

so that

$$\int_0^t sB_s\, dB_s = \frac{tB_t^2}{2} - \frac{1}{2}\int_0^t B_s^2\, ds - \frac{t^2}{4}.$$

$\square$

Remark: $\int_0^t B_s^2\, ds$ is a Riemann integral; it cannot be further simplified.