Chapter 5 Systems of First Order Linear Equations

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Table of contents

1 Basic Theory of Systems of First Order Linear Equations

General system of first order ODEs.

$$\begin{aligned} x_1'(t) &= F_1(x_1, x_2, \ldots, x_n, t) \\ x_2'(t) &= F_2(x_1, x_2, \ldots, x_n, t) \\ &\vdots \\ x_n'(t) &= F_n(x_1, x_2, \ldots, x_n, t) \end{aligned}$$

System of first order linear ODEs

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2'(t) &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n'(t) &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

Initial conditions:

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0.$$

Theorem 1.1 **(Existence and Uniqueness of Solution to the IVP)** *If the functions $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval $I: \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system that also satisfies the initial conditions, where t_0 is any point in I, and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.*

The system can be written in matrix form as

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t).$$

The system of equations is called **homogeneous** if $\mathbf{g}(t) = \mathbf{0}$,

Consider homogeneous system

$$\boxed{\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)} \tag{1.1}$$

Theorem 1.2 **(Principle of Superposition)** *If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (1.1), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.*

Definition 1.3 *The* ***Wronskian*** *of* n *solutions* $\mathbf{x}^{(i)}, i = 1, \ldots, n$ *of the system (1.1) is*

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}] = \det[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}].$$

Theorem 1.4 *If the vector functions* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly independent solutions of the system (1.1) for each point in the interval* $\alpha < t < \beta$*, then each solution* $\mathbf{x} = \boldsymbol{\phi}(t)$ *of the system (1.1) can be expressed as a linear combination of* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

$$\boldsymbol{\phi}(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$$

in exactly one way.

Definition 1.5 *If the vector functions* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly independent solutions of the system (1.1) for each point in the interval* $\alpha < t < \beta$*, then we call* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *a* ***fundamental set of solutions****, and*

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$$

the general solution.

Theorem 1.6 *If* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are solutions of the system (1.1) on the interval* $\alpha < t < \beta$*, then in this interval* $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *either is identically zero or else never vanishes.*

Theorem 1.7 *Let*

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

further, let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be the solutions of the system (1.1) that satisfy the initial conditions*

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \quad \ldots, \quad \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)},$$

respectively, where t_0 *is any point in* $\alpha < t < \beta$*. Then* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *form a fundamental set of solutions of the system.*

Theorem 1.8 *Consider the system (1.1), where each element of* $\mathbf{P}$ *is a real-valued continuous function. If* $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ *is a complex-valued solution, then its real part* $\mathbf{u}(t)$ *and its imaginary part* $\mathbf{v}(t)$ *are also solutions of this equation.*

Theorem 1.9 *The general solution of the inhomogeneous system* $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$ *is*

$$\mathbf{x}(t) = c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n + \mathbf{x}_p(t),$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *is a fundamental set of solutions of the homogeneous system* $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$*, and* $\mathbf{x}_p(t)$ *is a particular solution of the inhomogeneous system.*

2 Homogeneous Linear Systems with Constant Coefficients

Example 2.1. (Decoupled system) Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$$

Answer. Write each component one by one,

$$\begin{aligned} x_1'(t) &= 2x_1(t) \\ x_2'(t) &= -3x_2(t) \end{aligned}$$

Solving them one by one,

$$x_1(t)=c_1e^{2t}, \quad x_2(t)=c_2e^{-3t}.$$

Write the solution in vector form,

$$\mathbf{x}(t)=\begin{pmatrix} c_1e^{2t} \\ c_2e^{-3t} \end{pmatrix}=c_1\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}+c_2\begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix}:=c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t),$$

where

$$\mathbf{x}_1(t)=\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}=e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t)=e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now

$$W[\mathbf{x}_1,\mathbf{x}_2](t)=\begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix}=e^{-t}\neq 0.$$

So $\mathbf{x}_1, \mathbf{x}_2$ form a fundamental set of solutions, and the general solution is $c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)$.

Consider the system

$$\mathbf{x}'=\mathbf{A}\mathbf{x}.$$

We look for solutions of the form

$$\mathbf{x}(t)=e^{\lambda t}\boldsymbol{\xi},$$

where $\boldsymbol{\xi}\neq\mathbf{0}$ is a contant vector.

Example 2.2. Find the general solution of the system

$$\mathbf{x}'=\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}\Rightarrow\begin{cases} x_1'=x_1+x_2 \\ x_2'=4x_1+x_2 \end{cases}$$

Then plot a **direction field** and draw a **phase portrait**.

Answer. Let $\mathbf{x}(t)=e^{\lambda t}\boldsymbol{\xi}$, then

$$\mathbf{x}'=\lambda e^{\lambda t}\boldsymbol{\xi}=\mathbf{A}\mathbf{x}=\mathbf{A}e^{\lambda t}\boldsymbol{\xi} \quad\Leftrightarrow\quad \mathbf{A}\boldsymbol{\xi}=\lambda\boldsymbol{\xi}.$$

So λ is an eigenvalue and $\boldsymbol{\xi}$ is an associated eigenvector of $\mathbf{A}$.

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}=(1-\lambda)^2-4=0 \quad\Rightarrow\quad 1-\lambda=\pm 2 \quad\Rightarrow\quad \lambda_1=-1, \lambda_2=3.$$

For $\lambda_1=-1$,

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\Rightarrow\boldsymbol{\xi}_1=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2=3$,

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}=\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}=\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\Rightarrow\boldsymbol{\xi}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So we have two solutions

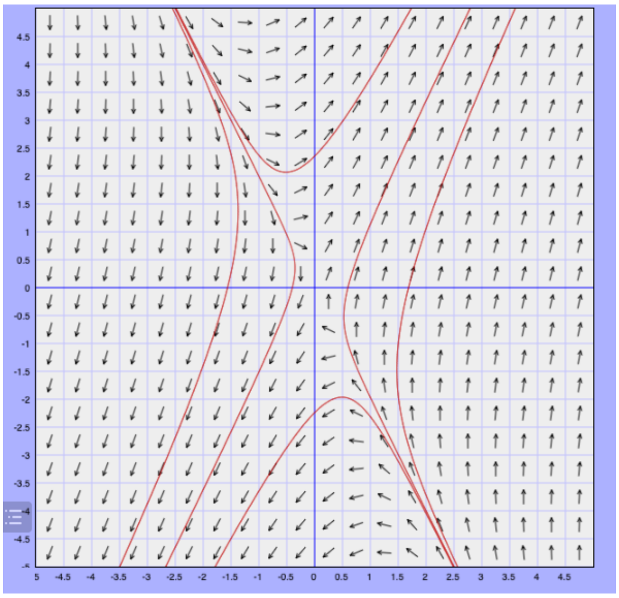
$$\mathbf{x}_1(t)=e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2(t)=e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and

$$W[\mathbf{x}_1,\mathbf{x}_2](t)=e^{2t}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix}=4e^{2t}\neq 0.$$

So $\mathbf{x}_1,\mathbf{x}_2$ form a fundamental set of solutions, and the general solution is

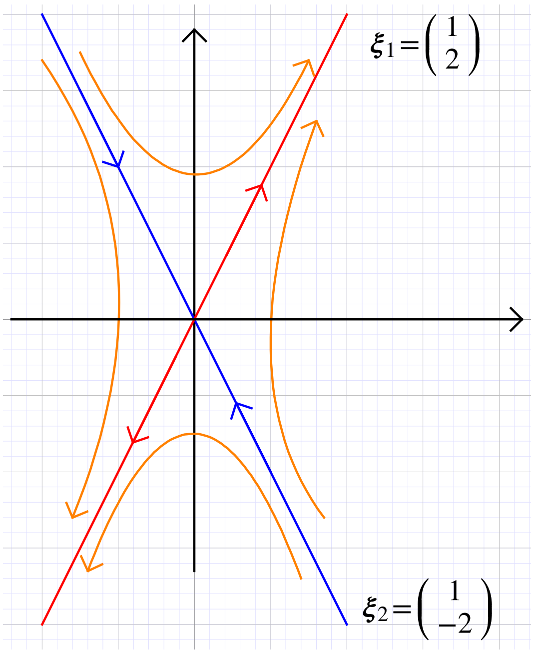
$$\mathbf{x}(t)=c_1e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}+c_2e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$



- If $c_1=1, c_2=0$, then $\mathbf{x}=\mathbf{x}_1=e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$,whose graph is a straight line parallel to the vector $(1, -2)^T$. Moreover, $x_1(t)\to 0, x_2(t)\to 0$ as $t\to\infty$.
- If $c_1=0, c_2=1$, then $\mathbf{x}=\mathbf{x}_2=e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, whose graph is a straight line parallel to the vector $(1, 2)^T$. Moreover, $x_1(t)\to\infty, x_2(t)\to\infty$ as $t\to\infty$.
- If $c_1\neq 0, c_2\neq 0$. Then all solutions converges to $\mathbf{x}_2$ asymptotically as $t\to\infty$.
- The equilibrium solution $\mathbf{x}=\mathbf{0}$ is called a **saddle point**. A saddle point is always unstable.

Online tool for direction field and phase portrait: https://aeb019.hosted.uark.edu/pplane.html

How to draw a phase portrait for $\mathbf{x}(t)=c_1e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}+c_2e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



phase portrait (saddle point)

Example 2.3. Find the general solution of the system

$$\mathbf{x}'=\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathbf{x}.$$

Then plot a **direction field** and draw a **phase portrait**.

Answer. First, find eigenvalues,

$$\begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda)-2 = \lambda^2+5\lambda+4 \Rightarrow \lambda_1=-1, \lambda_2=-4.$$

Find the eigenvectors.
For $\lambda_1=-1$

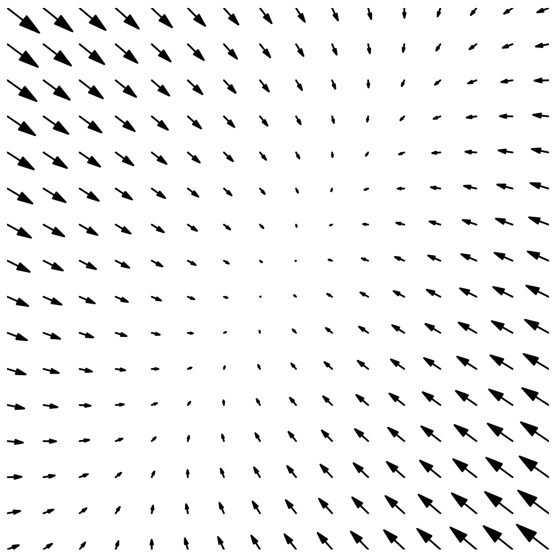
$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \to \begin{pmatrix} -2 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \to \xi_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

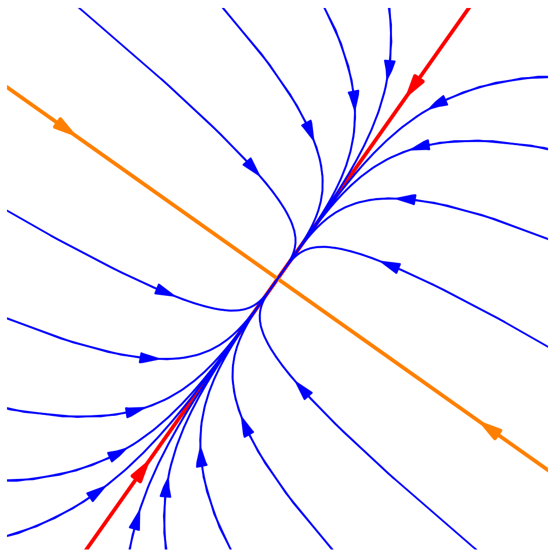
For $\lambda_2=-4$

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \to \xi_2 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$

A fundamental set of solutions are

$$y_1 = e^{-t}\xi_1 = e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad y_2 = e^{-4t}\xi_2 = e^{-4t}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$


direction field


phase portrait (node)

In this case, the equilibrium solution $\mathbf{x}=\mathbf{0}$ is called a **node**. If both eigenvalues are negative, as in this example, the node is **asymptotically stable**. If both eigenvalues are positive, then we still have a node, but it's **unstable**.

For a general $n\times n$ system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

If we have n linearly independent eigenvectors, then a fundamental set of solution is

$$y_i = e^{\lambda_i t}\xi_i, \quad i=1,\ldots,n,$$

even though some eigenvalues are repeated.

An $n\times n$ symmetric matrices always have n linearly indepdent eigenvectors (even better, the eigenvectors for distinct eigenvalues are orthogonal).

Example 2.4. Find the general solution of

$$\mathbf{x}'=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\mathbf{x}.$$

Answer. First, find eigenvalues

$$\begin{aligned}\begin{vmatrix}-\lambda&1&1\\1&-\lambda&1\\1&1&-\lambda\end{vmatrix} &= -\lambda(\lambda^2-1)-(-\lambda-1)+(1+\lambda)\\ &= -\lambda(\lambda^2-1)+2(\lambda+1)=(\lambda+1)(-\lambda^2+\lambda+2)=(\lambda+1)^2(-\lambda+2)\end{aligned}$$

$$\Rightarrow\quad \lambda_{1,2}=-1,\quad \lambda_3=2.$$

Next, find eigenvectors.
For $\lambda_{1,2}=-1$,

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}\to x_1+x_2+x_3=0\to\xi=\begin{pmatrix}-s-t\\s\\t\end{pmatrix}=s\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\xi_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad \xi_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

For $\lambda_3=2$,

$$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\to\begin{pmatrix}0&3&-3\\0&-3&3\\1&1&-2\end{pmatrix}\to\begin{pmatrix}1&1&-2\\0&3&-3\\0&-3&3\end{pmatrix}\to\begin{pmatrix}1&1&-2\\0&1&-1\\0&0&0\end{pmatrix}\to\xi_3=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

So a fundamental set of solutions are

$$y_1=e^{-t}\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad y_2=e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix},\quad y_3=e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

3 Complex Eigenvalues

If the matrix $\mathbf{A}$ is real-valued, then the complex eigenvalues appear in conjugate pairs. Let's consider the complex eigenvalues

$$\lambda\pm i\mu.$$

with eigenvectors

$$\mathbf{u}\pm i\mathbf{v}.$$

We have complex-valued solutions

$$\begin{aligned}e^{(\lambda+i\mu)t}(\mathbf{u}+i\mathbf{v}) &= e^{\lambda t}(\cos\mu t+i\sin\mu t)(\mathbf{u}+i\mathbf{v})\\ &= e^{\lambda t}[(\mathbf{u}\cos\mu t-\mathbf{v}\sin\mu t)+i(\mathbf{v}\cos\mu t+\mathbf{u}\sin\mu t)].\end{aligned}$$

So fundamental set of real-valued solutions are

$$y_1 = e^{\lambda t}(\mathbf{u}\cos\mu t - \mathbf{v}\sin\mu t), \quad y_2 = e^{\lambda t}(\mathbf{v}\cos\mu t + \mathbf{u}\sin\mu t).$$

(You should check the Wronskian).

Example 3.1. Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\mathbf{x}.$$

Plot a phase portrait and graphs of components of typical solutions.

Answer. First, find eigenvalues,

$$\begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{vmatrix} = \left(-\frac{1}{2}-\lambda\right)^2 + 1 \quad \Rightarrow \quad \lambda_{1,2} = -\frac{1}{2} \pm i.$$

Then find eigenvectors. For $\lambda_2 = -\frac{1}{2} - i$,

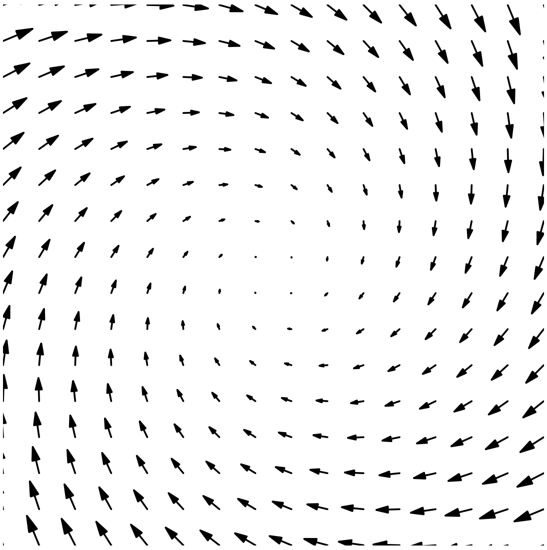
$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \to \xi_2 = \begin{pmatrix} i \\ 1 \end{pmatrix} \to \xi_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

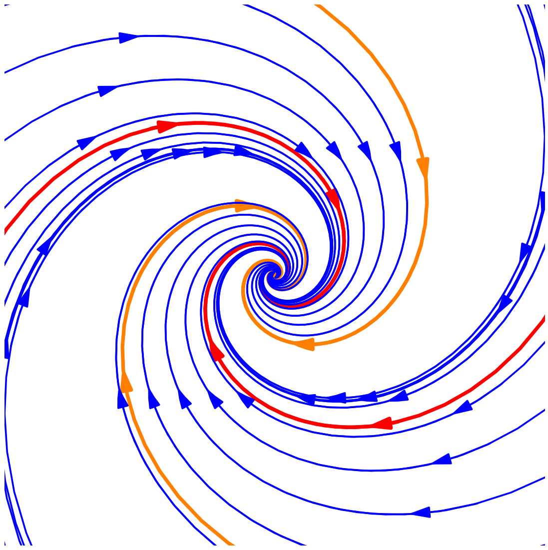
So the complex solutions are

$$\begin{aligned} y &= e^{(-\frac{1}{2}+i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix} = e^{-\frac{1}{2}t}(\cos t + i\sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= e^{-\frac{1}{2}t}\begin{pmatrix} \sin t - i\cos t \\ \cos t + i\sin t \end{pmatrix} = e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + ie^{-\frac{1}{2}t}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}. \end{aligned}$$

A fundamental set of real solutions are

$$y_1 = e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad y_2 = e^{-\frac{1}{2}t}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$





phase portrait (spiral)

In this case, the equilibrium solution $\mathbf{x}=\mathbf{0}$ is called **spiral**. A spiral point is asyptotically stable if the real part of the eigenvalue is negative, and unstable if the real part is positive.

Question 1. The system

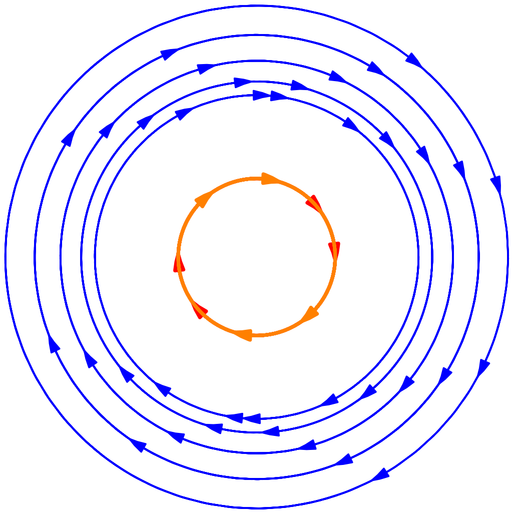
$$\mathbf{x}'=\begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix}\mathbf{x}$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behavior of the trajectories in the phase plane changes markedly.

Answer. Find the eigenvalues,

$$\begin{vmatrix} \alpha-\lambda & 2 \\ -2 & -\lambda \end{vmatrix}=\lambda^2-\alpha\lambda+4=0 \quad\Rightarrow\quad \lambda_{1,2}=\frac{\alpha\pm\sqrt{\alpha^2-16}}{2}.$$

1. If $|\alpha|>4$, then $\lambda_{1,2}$ are both real and distinct.
   a. If $\alpha>4$, then both eigenvalues are positive. So we have an unstable node.
   b. If $\alpha<-4$, then both eigenvalues are negative. So we have a stable node.
2. If $|\alpha|<4$, then both eigenvalues are complex.
   a. If $0<\alpha<4$, then the real part is positive. So have an unstable spiral.
   b. If $-4<\alpha<0$, then the real part is negative. So have a stable spiral.
   c. If $\alpha=0$, then the real part is zero. The solutions are ellipses, and we have a **center**. A center is stable (but not asymptotically stable).



phase portrait (center)

3. If $\alpha=4$, then both eigenvalues are positive and equal, and we have an unstable **improper node**.
4. If $\alpha=-4$, then both eigenvalues are positive and equal, and we have an asymptotically stable **improper node**.

4 Repeated Eigenvalues

Suppose λ is a repeated eigenvalue of $\mathbf{x}'=\mathbf{A}\mathbf{x}$ with eigenvector $\boldsymbol{\xi}$. So one solution is

$$y_1=e^{\lambda t}\boldsymbol{\xi}.$$

Consider a second solution of the form

$$y_2=te^{\lambda t}\boldsymbol{\xi}+e^{\lambda t}\boldsymbol{\eta}=e^{\lambda t}(t\boldsymbol{\xi}+\boldsymbol{\eta}).$$

What should be $\boldsymbol{\eta}$? Plugging y_2 into the equation,

$$y_2' = \lambda e^{\lambda t}(t\boldsymbol{\xi}+\boldsymbol{\eta}) + e^{\lambda t}\boldsymbol{\xi} = \mathbf{A}y_2 = e^{\lambda t}A(t\boldsymbol{\xi}+\boldsymbol{\eta})$$

$$\Rightarrow \quad A(t\boldsymbol{\xi}+\boldsymbol{\eta}) = \lambda(t\boldsymbol{\xi}+\boldsymbol{\eta})+\boldsymbol{\xi} \quad \Rightarrow \quad \mathbf{A}\boldsymbol{\eta} = \lambda\boldsymbol{\eta}+\boldsymbol{\xi} \quad \Rightarrow \quad \boxed{(\mathbf{A}-\lambda\mathbf{I})\boldsymbol{\eta}=\boldsymbol{\xi}} \quad \Rightarrow \quad \boxed{(\mathbf{A}-\lambda\mathbf{I})^2\boldsymbol{\eta}=\mathbf{0}}$$

We call $\boldsymbol{\eta}$ a **generalized eigenvector**.

Example 4.1. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$$

and draw a phase portrait for this system.

Answer. Eigenvalues,

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 \quad \Rightarrow \quad \lambda_{1,2} = 2.$$

Eigenvectors

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \to \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So one solution is

$$y_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We seek a second solution in the form

$$y_2 = e^{2t}(t\boldsymbol{\xi}+\boldsymbol{\eta}),$$

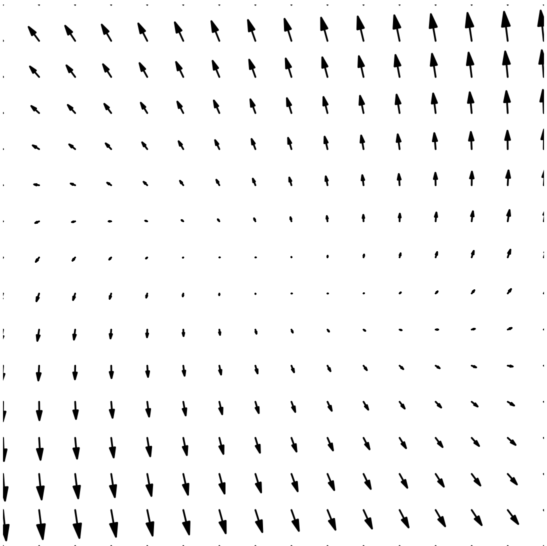
and $\boldsymbol{\eta}$ can be found as follows.

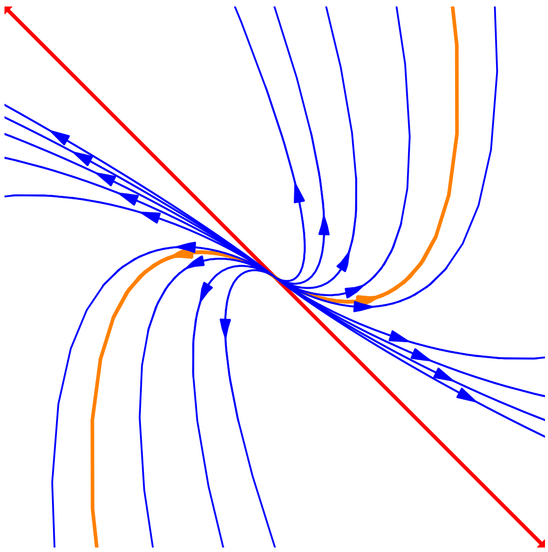
$$(\mathbf{A}-\lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi} \quad \Rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \eta_1 + \eta_2 = -1 \quad \Rightarrow \quad \boldsymbol{\eta} = \begin{pmatrix} -1-s \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Note that we can take $s=0$. So a second is

$$y_2 = e^{2t}\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} t-1 \\ -t \end{pmatrix}.$$





phase portrait (improper node)