Ordinary Differential Equations

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1 Motivation

Suppose we deposite u_0 in a bank account, and the annual interest rate is r. After t years,

1. If the interest is compounded annualy, then the balance is

$$u(t) = u_0 (1 + r)^t.$$

2. If the interest is compounded monthly, then the balance is

$$u(t) = u_0 \left(1 + \frac{r}{12}\right)^{12t}$$

3. In general, if the interest is compounded m times a year, then the balance is

$$u(t) = u_0\left(1+\frac{r}{m}\right)^{mt}.$$

4. Taking the limit as $m \to \infty$, then for any fixed t, we have

$$u(t) = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mt} = \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{\frac{m}{r}rt} = e^{rt}.$$

But we can obtain the same result by using a differential equation:

$$u'(t) = u(t)\, r. \tag{1}$$

We can solve it later and obtain the same result.

Definition 1. *A* ***ordinary differential equation*** *(ODE) about a function $u(t)$ is an equation involving $u(t)$ and its derivatives.*

2 Classification of ODE

Any ODE can be written in the abstract form

$$F(u, u', u'', \ldots, u^{(n)}) = 0. \tag{2}$$

For example, Eq. (1) can be written as

$$F(u, u') = r\,u - u' = 0.$$

In Eq. (2), n is the **order** of the equation, i.e. the order is the highest order derivative of u in the equation.

So Eq. (1) is an ODE of order 1.

If F is linear in terms of $u, u', \ldots, u^{(n)}$, then the equation is called **linear**. Otherwise it's called **nonlinear**. The general form of a linear ODE is

$$a_n(t)\,u^{(n)} + a_{n-1}(t)\,u^{(n-1)} + \cdots + a_1(t)\,u + a_0(t) = 0.$$

So Eq. (1) is linear. Examples of nonlinear equations:

$$u' - u^2 = 5, \quad u\, u' + 5x = e^x, \quad u' = \sin u$$

3 Solutions of an ODE

A solution of an ODE

$$F(u, u', u'', \ldots, u^{(n)}) = 0$$

is a function $u = \phi(t)$ satisfying the equation, i.e.

$$F(\phi, \phi', \phi'', \ldots, \phi^{(n)})(t) = 0.$$

Example 2. Can you give solutions of

$$u' = 2\, u$$

possible solution: $u = e^{2t}$, in fact, $u = c\, e^{2t}$ is a solution for any constant c.

Example 3. Can you give solutions of

$$u'' + 4\,u = 0$$

A possible solution is $u = \sin 2\,t$, another solution is $u(t) = \cos 2t$. In fact, any function

$$u = a\sin 2\,t + b\cos 2\,t$$