Discrete Mathematics Homework 9

27

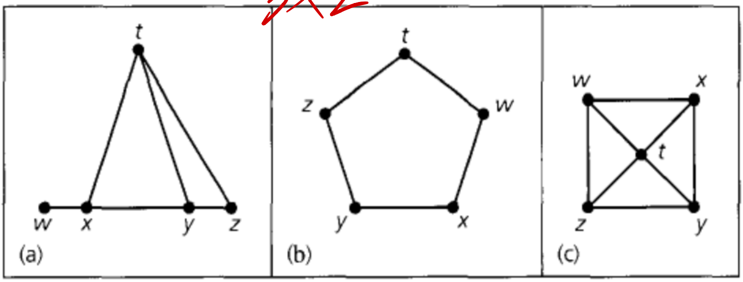
1 (a) Determine the chromatic polynomials for the graphs in below figure.

4×3
(explain 2)

(b) Find $\chi(G)$ for each graph. 3×3

(c) If five colors are available, in how many ways can the vertices of each graph be properly colored?

3×2



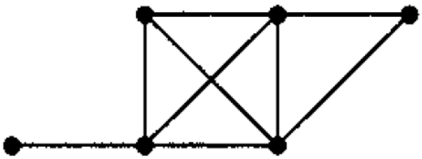
Solution

(a) (1) $\lambda(\lambda-1)^2(\lambda-2)^2$; (2) $\lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$;
(3) $\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$

(b) (1) 3: (2) 3; (3) 3

(c) (1) 720; (2) 1020; (3) 420

2. Find the chromatic polynomial of the following graph by simplicial elimination ordering.



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Solution

6

In the graph below, $v_6, \ldots, v_1$ is a simplicial elimination ordering. When we form the graph in the order $v_1, \ldots, v_6$, the values $d(1), \ldots, d(6)$ are 0, 1, 1, 2, 3, 2, and the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$. 6

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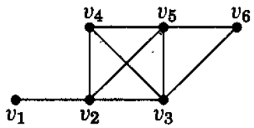


3. For $n \ge 3$, let $G_n = (V, E)$ be the undirected graph obtained from the complete graph K_n upon deletion of one edge.

Determine $P(G_n, \lambda)$ and $\chi(G_n)$. 5 15

Solution

Let e be the deleted edge. Then $G_n = K_n - e$ and $(K_n)'_e = K_{n-1}$.

$$P(G_n,\lambda) = P(K_n,\lambda) + P(K_{n-1},\lambda)$$
$$= \lambda(\lambda-1)\boxed{?}(\lambda-n+2)(\lambda-n+1) + \lambda(\lambda-1)\quad \boxed{?}-n+2)$$
$$= \lambda(\lambda-1)\boxed{?}(\lambda-n+3)(\lambda-n+2)^2$$ 5

$\chi(G_n) = n-1$

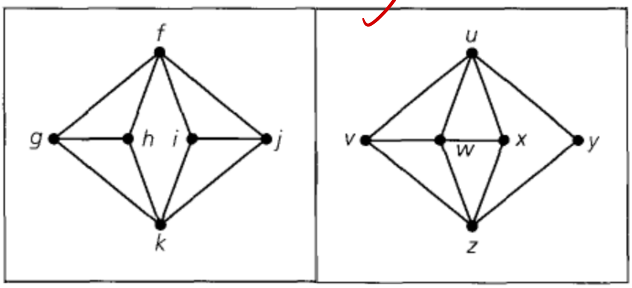
5

(explain: 5)

4 (a) Determine whether the graphs in below figure are isomorphic. 10

(b) Find $P(G,\lambda)$ for each graph. 10 (explain: 5) 25

(c) Comment on the results found in parts (a) and (b). 5



Solution

(a) These graphs are not isomorphic. The first graph has two vertices of degree 4 – namely, f and k. The second graph has three vertices of degree 4 – namely u,w,z.

(b) For the first graph there are two cases to consider.

Case (i): Vertices f and k have the same color: Here there are $\lambda(\lambda-1)^2(\lambda-2)^2$ ways to properly color the vertices.

Case (ii): Vertices f and k are colored with different colors: Here the vertices can be properly colored in $\lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^2$ ways.

By the rule of sum, $P(G,\lambda) = \lambda(\lambda-1)^2(\lambda-2)^2 + \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^2 =$
$\lambda(\lambda-1)^{\cancel{2}}(\lambda-2)^2(\lambda^2-5\lambda+8)$. $\rightarrow$ $\lambda(\lambda-1)(\lambda-2)^2(\lambda^2-5\lambda+8)$

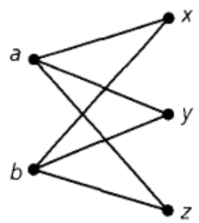
Using the same type of argument, with the two cases for vertices u and z, the chromatic polynomial for the second graph is also found to be $\lambda(\lambda-1)^2(\lambda-2)^2(\lambda^2-5\lambda+8)$.

(c) If G_1, G_2 are two graphs with $P(G_1,\lambda) = P(G_2,\lambda)$, it need not be the case that G_1 and G_2 are isomorphic.

$\lambda(\lambda-1)(\lambda-2)^2(\lambda^2-5\lambda+8)$

5. Find the chromatic polynomial of the following graph.

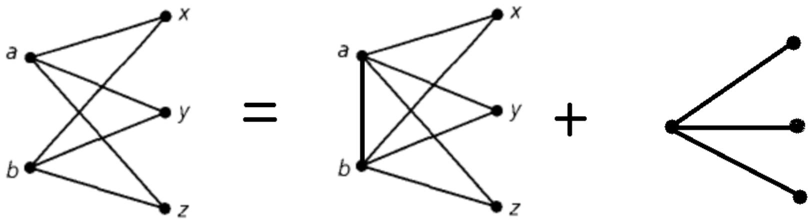
15 (explain 10)



Solution

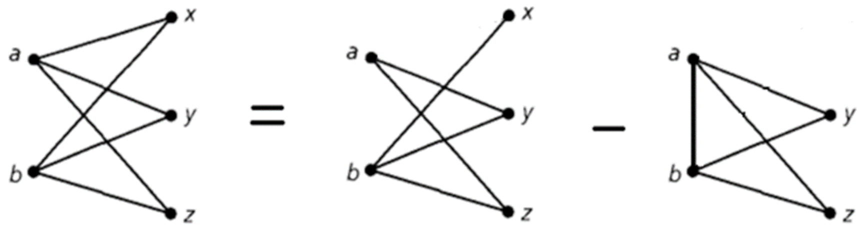
Method 1:

Join a and b, we have



$P(K_{2,3},\lambda) = \lambda(\lambda-1)(\lambda-2)^3 + \lambda(\lambda-1)^3 = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)$

Method 2:



$$P(K_{2,3}, \lambda) = \lambda(\lambda-1)^2(\lambda^2 - 3\lambda + 3) - \lambda(\lambda-1)(\lambda-2)^2 = \lambda(\lambda-1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)$$