9 Q.

1+2 20pts

3+4 $^{10\%}$ 20pts

5 30 pts

6 10 pts

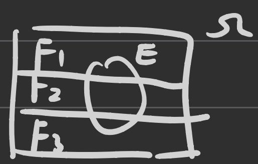
7+8 20-30 pts

1 to int + dif

key concepts and examples

Chap 1. Review of Prob. theory

①



$$P(E) = \sum_{i=1}^{n} P(E \cap F_i)$$

$$= \sum_{i=1}^{n} P(E|F_i) P(F_i)$$

example 1.3

$$p(F_j|E) = \frac{P(E \cap F_j)}{P(E)} = \frac{P(E|F_j) \cdot P(F_j)}{\sum_{i=1}^{n} P(E|F_j) \cdot P(F_j)}$$

② simple symmetric random walk

example 1.19

$$p(\text{up } k \text{ before down } n) = \frac{n}{n+k}$$

Possion distribution

$$p(i) = P\{x=i\} = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \quad i=0,1,2\cdots$$

$$E(X) = Var(X) = \lambda.$$

Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$f(x,y)$ is the joint probability density function of X and Y

$$\begin{cases} f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \\ f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \end{cases}$$

Random walk

for any r.v. X, Y $E(X+Y) = E(X)+E(Y)$

for independent r.v. X, Y
$$\begin{cases} p(x,y) = p_X(x)\,p_Y(y) \\ f(x,y) = f_X(x)\,f_Y(y) \end{cases}$$

$$\boxed{E(XY) = E(X)E(Y)}$$

$$\text{Covariance}(X,Y) = E(X-E(X))E(Y-E(Y)) = E(XY) - E(X)E(Y)$$

chapter 2 Conditional probability, expectation and application

① total expectation formula

Dis: $E(X) = \sum_{y} E(X|Y=y) \cdot P\{Y=y\}$. Ex 2.4.

Conti: $E(X) = \int_{-\infty}^{\infty} E(X|Y=y) \cdot f_Y(y)\, dy$

② Computing probability using total probability formula

$P(E) = \sum P(E|Y=y) \cdot P\{Y=y\}$ Ex: star graph (3.5 + Ass 2-Q.1)

$P(E) = \int_{-\infty}^{\infty} P(E|Y=y) f_Y(y) dy$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

Ex. $f(x,y) = \begin{cases} 6xy(2-x-y), & 0<x<1, \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

compute $E[X|Y=y]$, for $0<y<1$.

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 6xy(2-x-y)\,dx = \int_0^1 12yx - 6yx^2 - 6y^2x\,dx = \left[6yx^2 - 2yx^3 - 3y^2x^2\right]_0^1$$

$$= 6y - 2y - 3y^2 = -3y^2 + 4y = y(4-3y)$$

$$f_{X|Y} = \frac{f(x,y)}{f_Y(y)} = \frac{6x\cancel{y}(2-x-y)}{\cancel{y}(4-3y)} = \frac{6x(2-x-y)}{4-3y}$$

$$E(X|Y=y) = \int_0^1 x \cdot \frac{6x(2-x-y)}{4-3y}\,dx = \frac{1}{4-3y}\int_0^1 (2-y)\cdot 6x^2 - 6x^3\,dx$$

$$= \frac{1}{4-3y}\cdot\left[(2-y)\cdot 2x^3 - \frac{3}{2}x^4\right]_0^1$$

$$= \frac{1}{4-3y}\left(4-2y-\frac{3}{2}\right) = \frac{1}{4-3y}\cdot\frac{5-4y}{2} = \frac{5-4y}{8-6y}$$

$X_1, \cdots X_n$, $E[X_i] = E[X]$, if n is fixed, then $E[\sum X_i] = n \cdot E[X]$

if n isn't fixed, then $E[\sum_{i=1}^{N} X_i] = E_N\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right]$

$$= E_N[N E[X]]$$

$$= E[N] \cdot E[X]$$

Star graph

chapter 3 Possion Process

① memoryless property of exponetial distribution

- Probability $X \sim Exp(\lambda)$ $P\{X > t+s \mid X > t\} = P(X > S)$
- conditional expectation $E(X \mid X > t) = t + \frac{1}{\lambda}$

② Possion Process $\{N(t) \mid t \geq 0\}$

- $N(0) = 0$
- $\{N(t) \mid t \geq 0\}$ independent increament in disjoint time intervals
- ☆ $P\{N(s+t) - N(s) = n\} = \dfrac{e^{-\lambda t} (\lambda t)^n}{n!}$ $(n \geq 0)$

order statistic of uniform distribution

- $E(N(t)) = \lambda \cdot t$

Ex 3.8

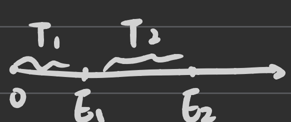
indass ex 2.

Ass 2 Qs. 6.

③ Distribution of interval time and arrival time. $S_1 = T_1$

T_i iid $exp(\lambda)$

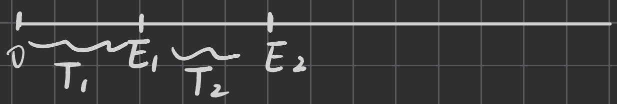
$S_2 = T_1 + T_2$



$S_n \sim Gamma(n\lambda)$ Conditional distribution

$$Var(X) = E[Var(X|Y)] + Var(E(X|Y))$$

Consider a possion process



all interval time are i.i.d exponentially distributed random variable with mean $\frac{1}{\lambda}$

The arrival time until nth event is $S_n = \sum_{i=1}^{n} T_i, \ n \geq 1$

N_t (events) is count, total time

S_a (time) is the time point ath time

$$F_{S_n}(t) = P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{i=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$$

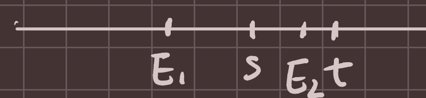
$$f_{S_n(t)} = F'_{S_n}(t) = \sum_{i=n}^{\infty} (-\lambda) e^{-\lambda t} \cdot \frac{(\lambda t)^i}{i!} + \sum_{i=n}^{\infty} e^{-\lambda t} \cdot \lambda \frac{(\lambda t)^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Gamma with n, λ

$$E(S_n) = \frac{n}{\lambda}$$

$$Var(S_n) = \frac{n}{\lambda^2}$$



$P\{N(s)=1, N(t)=2\}$ for $s<t$.

$P\{N(t)=2 \mid N(s)=1\} \cdot P\{N(s)=1\}$

$= P\{N(t)-N(s)=1\} \cdot P\{N(s)=1\}$

$= P\{N(t-s)=1\} \cdot P\{N(s)=1\}$

$= e^{-\lambda(t-s)} \cdot \lambda(t-s) \cdot e^{-\lambda s} \cdot \lambda s$

$= \lambda^2 s(t-s) e^{-\lambda t}$

key trick: To transforme the distribution of time to the distribution of n.o. of events.

④ compound possion process

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad E(X(t)) = \lambda t \cdot E(Y)$$

$$Var(X(t)) = \lambda t \cdot E(Y^2)$$

Ex: 5.12

☆ 背过不会

Chapter 4. martingales. defination

Ex: 4.1. Ass 3 - Q1, Q2, Q5
inclass 4.

Definition: $\{Z_n \mid n \geq 1\}$, $E[|Z_n| < \infty] < \infty$ for all n (intergrability)

$$E[Z_{n+1} \mid Z_1, Z_2, \cdots Z_n] = Z_n$$

$$\begin{aligned} E[Z_{n+1} \mid Z_1, Z_2 \cdots Z_n] &= E[Z_n \cdot X_{n+1} \mid Z_1, \cdots Z_n] \\ &= Z_n \cdot E[X_{n+1} \mid Z_1 \cdots Z_n] \\ &= Z_n \cdot E[X_{n+1}] \\ &= Z_n \end{aligned}$$

Gamble's ruin

$P\{T_{-1} < T_1\}$ $P\{T_1 < T_{-2} \mid T_1\}$

$$\frac{1}{1+1} \cdot \frac{2}{3} = \frac{1}{3}$$

Chapter 5. Brownian motions

1. Definition   Ex 6.3, in-class 3.

$X(t) - X(s) \sim N(0, \sigma^2(t-s))$. $\sigma^2 = 1$. Standard Brownian motion

2. Joint conditional distribution $(B(t_1), B(t_2))$.

Joint distribution: bi-variate normal $(s<t)$

Conditional distribution: $B(t)=y$, conditional distribution of $B(s)$

normal distribution

$$\boxed{\begin{aligned} &E(B(s) \mid B(t)=y) = \frac{s}{t} \cdot y \\ &Var(B(s) \mid B(t)=y) = \frac{s(t-s)}{t}. \end{aligned}}$$

Ex 6.6.

3. hitting time, maxium value

T_a: the first time the SB motion hits a

$$\boxed{\begin{aligned} &P\{T_a < t\} = 2 \cdot \Phi(-|a|/\sqrt{t}) \\ &P\{\max_{0 \le s \le t} B(s) \ge a\} = 2\Phi(-|a|/\sqrt{t}) \end{aligned}}$$

Ass 3 - Q3

The maxium value of standard Brownian motion within $[0,t]$ exceeds a <=>
The first hit of a happens within $[0,t]$.

4. Gambler's ruin problem

$P\{T_A < T_{-B}\} = P\{\text{up } A \text{ before down } B\} = \frac{B}{A+B}$

Ex. 6.7
Ass 3 Q3

5. $X(t) = \sigma B(t) + \mu t$ $E = \mu t$, $Var = \sigma^2 t$ Ex 6.8 Ass 3 Q3

- conditioning on a past value ⟶ independent, Stationary properity
- conditioning on a future value → part 2 formula

6. Geometric Brownian motion

$X(t) = X(0) \cdot e^{Z(t)}$, where $Z(t) = \sigma B(t) + (\mu - \frac{\sigma^2}{2})t$

$E(X(t)) = e^{\mu t}$, $E(e^{\sigma B(t)}) = e^{\frac{\sigma^2 t}{2}}$ Ex 6.9 Ass 3 Q4

X, Y are jointly normally distributed

$$\mu_{X+Y} = \mu_X + \mu_Y$$

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\,\sigma_X\sigma_Y = \sigma_X^2 + \sigma_Y^2 + 2\,\frac{E(XY)-E(X)E(Y)}{\sigma_X\,\sigma_Y}$$

$$E(B(1)B(2)) = E[\underbrace{(B(2)-B(1))}_{0}\cdot \underbrace{B(1)}_{0} + B(1)^2] = E(B^2(1)) = Var(B(1)) + E(B(1))^2$$
$$= 1 + 0 = 1.$$

$$E[B(s)\cdot B(t)] = \min(s,t)$$

$r \sim N(m, V)$ $\quad E(e^r) = e^{m+\frac{V}{2}}$

$$Var(e^r) = e^{2m+2V} - e^{2m+V}$$

$\{B(t),\ t \geq 0\}$ std BM, $X(t) = \underbrace{\sigma B(t)}_{\text{R.V.}} + \mu t$

$E(X(t)) = \mu t$, $\sigma^2(X(t)) = \sigma^2 t$

chapter 6 option pricing

portfolio
- Stock
- investment ← price?

① set the $\mu = r$ to calculate the (risk neutral) prob of Stock price goes up and down

② price of the investment is the expected payoff where the discount rate is risk free r Ex. P20

Compound n times a year after T years: $FV = (1+\frac{r}{n})^{nT}$

Continous compounding: $FV = e^{rT}$

Ito Integral:

B_t: Std Brownian motion, if $s<t$, then

1. $E[(B_t-B_s)^2]=t-s$
2. $E[(B_t-B_s)^4]=3(t-s)^2$
3. $Var((B_t-B_s)^2)=2(t-s)^2$

First-order variation (爬一座山上下山海拔路程之和)

$$FV_T(f)=\int_0^T |f'(t)|\,dt \quad ①$$

$$=\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f(t_{j+1})-f(t_j)| \quad ②$$

$dB_t\,dt=0$ why?

$$\begin{cases} 0=t_0<t_1<\cdots t_n=T \\ \|\Pi\|=\max_{j=0,\ldots,n-1}(t_{j+1}-t_j) \end{cases}$$

$$\begin{cases} (dB_t)^2=dt \\ (dt)^2=0 \end{cases} \qquad dB_t\,dt=0$$

$F_t=f(t,X_t)$, $\quad df(t,X_t)=\dfrac{\partial f(t,X_t)}{\partial t}dt+\dfrac{\partial f(t,X_t)}{\partial x}dX_t+\dfrac{1}{2}\dfrac{\partial^2 f(t,X_t)}{\partial x^2}(dX_t)^2$

$$dX_t=a\,dt+b\,dB_t$$

$$df(t,X_t)=(f_t+a\cdot f_x+\tfrac{1}{2}b^2 f_{xx})\,dt+b\cdot f_x\,dB_t$$