Lecture 4

Martingales

- What is a martingale?
- How to verify that a stochastic process is a martingale?
- What is a stopped process?
- Martingale Stopping Theorem and its applications
- Analysis of the martingale gambling strategy

4.1 Martingales

- A stochastic process $\{Z_n | n \geq 1\}$ is said to be a *martingale* process if
$$E[|Z_n|] < \infty \qquad \text{for all } n \qquad \text{(integrability)}$$
and
$$E[Z_{n+1}|Z_1, Z_2, \ldots, Z_n] = Z_n. \tag{4.1}$$

- A martingale is a generalized version of a fair game.

- If we interpret Z_n as a gambler's fortune after the nth game, then Eq. (4.1) states that his expected fortune after the $(n+1)$st gamble is equal to his fortune after the nth gamble no matter what may have previously occurred, i.e. no gain or loss.

- Key properties: 1. Knowledge of previous events does not improve our position. 2. No gain or loss is expected.

- Martingales have a constant (unconditional) expectation. Taking expectations of Eq. (4.1) gives

$$\begin{aligned} E[Z_{n+1}|Z_1, Z_2, \ldots, Z_n] &= Z_n \\ E[E[Z_{n+1}|Z_1, Z_2, \ldots, Z_n]] &= E[Z_n] \\ E[Z_{n+1}] &= E[Z_n]. \qquad \text{(total expectation)} \end{aligned}$$

  Thus,

$$E[Z_n] = E[Z_1] \qquad n = 1, 2, \ldots$$

- E.g. If Z_n represents a gambler's total fortune, then, before any game is played, we expect that the total fortune does not change over time (here "time" refers to the number of games played). But after each game is played, the expected fortune is updated to the most recent value.

  Conditional

- Martingale (probability theory) vs martingale strategy (gambling strategy)

- In some applications, we may have some set of observable variables $Y_1, \ldots, Y_n$ that determines $Z_1, \ldots, Z_n$.
- E.g. Y may represent a stock price and Z may represent an option price, or Y may represent the outcome of roulette and Z may represent the payoff.
- In this case, it is often easier to verify a martingale by conditioning on the more informative variables $Y_1, \ldots, Y_n$:

$$E[Z_{n+1}|Y_1, \ldots, Y_n] = Z_n. \tag{4.2}$$

- If $Y_1, \ldots, Y_n$ is given, then $Z_1, \ldots, Z_n$ is also given, but not vice versa.
- It is easy to show that (4.2) implies (4.1) by using (4.3) below.

Example 4.1 Below are some examples of martingales. The integrability requirement has been taken for granted.

1. Let $X_1, X_2, \ldots$ be independent random variables with 0 mean; and let $Z_n = \sum_{i=1}^{n} X_i$. Then $\{Z_n | n \geq 1\}$ is a martingale since

$$\begin{aligned}
& E[Z_{n+1}|Z_1, Z_2, \ldots, Z_n] \\
= \; & E[Z_n + X_{n+1}|Z_1, Z_2, \ldots, Z_n] \\
= \; & E[Z_n|Z_1, Z_2, \ldots, Z_n] + E[X_{n+1}|\boxed{Z_1, Z_2, \ldots, Z_n}] \\
= \; & Z_n + E[X_{n+1}] \\
= \; & Z_n.
\end{aligned}$$

$Z_1 = X_1$

$Z_2 = X_2 + X_1$

$\vdots$

$Z_n = X_n + X_{n-1} + \cdots + X_1$

$\to X_1, X_2 \cdots X_n$

independent of X_i

$Z_{n+1} = X_{n+1} + \underbrace{X_n + \cdots + X_1}_{Z_n}$

$X_i = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$ random walk

Remark: In this example, think of X_n as the winnings of the nth game and Z_n the total winnings.

$E(XY) = E(X) \cdot E(Y)$

2. Let $X_1, X_2, \ldots$ be independent random variables with mean 1; and let $Z_n = \prod_{i=1}^{n} X_i$. Then $\{Z_n | n \geq 1\}$ is a martingale since

$$\begin{aligned} E[Z_{n+1}|Z_1, Z_2, \ldots, Z_n] &= E[\overset{\prod_{i=1}^{n} X_i}{Z_n} \cdot X_{n+1}|Z_1, Z_2, \ldots, Z_n] \\ &= Z_n \cdot \underline{E[X_{n+1}|Z_1, Z_2, \ldots, Z_n]} \\ &= \underline{Z_n \cdot E[X_{n+1}]} \\ &= \underline{Z_n}. \end{aligned}$$

$\square$

$Z_1 = X_1$

$Z_2 = X_1 \cdot X_2$

$\vdots$

$Z_n = X_1 \cdot X_2 \cdots X_n$

$Z_{n+1} = \underbrace{X_1 \cdots X_n}_{Z_n} X_{n+1}$

3. First, we note that the total expectation formula $E[X] = E[E[X|Y]]$ extends to conditional expectation:

$$E[X|U] = E\,[\,E[X|U,Y]\,\,|U]\,, \tag{4.3}$$

and also multiple random variables:

$$E[X|U_1,\ldots,U_m] = E[E[X|U_1,\ldots,U_m,Y_1,\ldots,Y_n]|U_1,\ldots,U_m].$$

That is,

$$E[X|\mathbf{U}] = E[E[X|\mathbf{U},\mathbf{Y}]|\mathbf{U}],$$

where $\mathbf{U} = (U_1,\ldots,U_m)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$.

Next, let $X, Y_1, Y_2, \ldots$ be some arbitrary random variables such that $E[|X|] < \infty$, and let

$Z_1 = E(X|Y_1)$
$Z_2 = E(X|Y_1, Y_2)$
$\vdots$
$Z_n = E(X|Y_1 Y_2 \ldots Y_n)$

$$Z_n = E[X|Y_1, \ldots, Y_n].$$

(E.g. In Blackjack, X may represent the winnings of a player at the end of the game, Y_i may represent the ith card known to the player.) Note that $Z_1, \ldots, Z_n$ is determined by $Y_1, \ldots, Y_n$. It then follows that $\{Z_n | n \geq 1\}$ is a martingale:

$Z_{n+1} = E(X|Y_1, Y_2 \ldots Y_n, Y_{n+1})$.

$$\begin{aligned} Z_n = \boxed{E[Z_{n+1}|Y_1, \ldots, Y_n]} &= E[E[X|Y_1, \ldots, Y_n, Y_{n+1}]|Y_1, \ldots, Y_n] \\ &= E[X|Y_1, \ldots, Y_n] \qquad \text{(by (4.3))} \\ &= Z_n. \end{aligned}$$

$\Rightarrow U: Y_1, Y_2 \ldots Y_n$

$Y: Y_{n+1}$ $\qquad X: X$

This martingale is called a *Doob-type martingale*. $\square$

$E(E(X|U, Y)|U) = E(X|U)$

$\Rightarrow E(E(X|Y_1, Y_2 \ldots Y_n, Y_{n+1})|Y_1 Y_2 Y_3 \ldots Y_n) =$

$E(X|Y_1 Y_2 \ldots Y_n)$

4. For any random variables $X_1, X_2, \ldots$, the random variable

$$X_i - E[X_i|X_1, \ldots, X_{i-1}], \qquad i \geq 1,$$

have mean 0 (but non-necessarily independent). Let

$$Z_n = \sum_{i=1}^{n} (X_i - E[X_i|X_1, \ldots, X_{i-1}])\,.$$

Then, $\{Z_n|n \geq 1\}$ is a martingale. To verify this, note that

$$Z_{n+1} \;=\; Z_n + X_{n+1} - E[X_{n+1}|X_1, \ldots, X_n].$$

Since $X_1, \ldots, X_n$ determines $Z_1, \ldots, Z_n$, we will condition on the more informative variables.

$Z_1 = X_1 - E(X_1|X_0)$

$Z_2 = X_1 - E(X_1|X_0) + X_2 - E(X_2|X_0, X_1)$

$\vdots$

$Z_n =$

Recall that

$$Z_{n+1} = Z_n + X_{n+1} - E[X_{n+1}|X_1, \ldots, X_n].$$

Conditioning on $X_1, \ldots, X_n$ yields that

$E(E(X|Y)|Y) = E(X|Y)$

$$\begin{aligned}
E[Z_{n+1}|X_1, \ldots, X_n] &= Z_n \\
= E[Z_n|X_1, \ldots, X_n] &+ E[X_{n+1}|X_1, \ldots, X_n] \\
&- E[E[X_{n+1}|X_1, \ldots, X_n]|X_1, \ldots, X_n] \\
= \; Z_n &+ E[X_{n+1}|X_1, \ldots, X_n] - E[X_{n+1}|X_1, \ldots, X_n] \\
= \; Z_n&.
\end{aligned}$$

$E(E(X|Y)|Y) = E(X|Y)$

$\square$

4.2 Stopping Times

- Let Z_n be the total winnings of a gambler after he played n games. Suppose that the following sequence of total winnings is observed:

$$(Z_1, Z_2, \dots) = (1, 2, 5, 6, 3, 1, -2, \dots)$$

- Suppose a second gambler makes exactly the same bets, except that he quits when he wins \$5 or more. Thus, the quitting time is

$$N = \min\{n \mid Z_n \geq 5\}.$$

- We have $N = 3$ in this case. The sequence of total winnings is

$$(\overline{Z}_1, \overline{Z}_2, \dots) = (1, 2, 5, 5, 5, 5, 5, \dots)$$

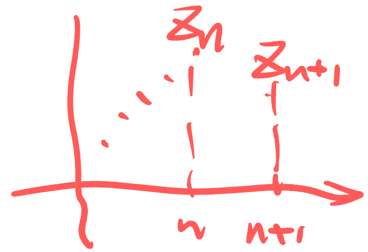
- After Z_1 is observed, we know $\{N = 1\}$ does not occur. But the occurrence of $\{N = 2\}$ is unknown. Likewise, after $Z_1, \dots, Z_n$ are observed, the occurrence of $\{N = n\}$ is known, but not $\{N = k\}$ for any $k > n$. Obviously, $Z_1, \dots, Z_n$ determines $\overline{Z}_1, \dots, \overline{Z}_n$.

$\{N=n\} \in \sigma\{Z_1, Z_2, \cdots Z_n\}$

- A positive integer-valued, possibly infinite, random variable N is said to be a *random time* for the process $\{Z_n | n \geq 1\}$ if the event $\{N = n\}$ is determined by the random variables $Z_1, \ldots, Z_n$.
- The variable N depicts the time at which something special happens.
- A random time N is said to be a *stopping time* if $P\{N = \infty\} = 0$, i.e. the special event must occur. We also write $P\{N < \infty\} = 1$.
- Let N be a random time and let

time for the first peak

$$\overline{Z}_n = \begin{cases} Z_n, & \text{if } n \leq N \\ Z_N, & \text{if } n > N. \end{cases}$$



Then, $\{\overline{Z}_n |\ n \geq 1\}$ is called the *stopped process*.

- Since $N \geq 1$, $\overline{Z}_1 = Z_1$.
- If $\omega \in \Omega$ denotes the whole sequence of outcomes, then $Z_n = Z_n(\omega)$ for each n, $\overline{Z}_n = \overline{Z}_n(\omega)$ for each n, and $N = N(\omega)$.

Example 4.2 *Let Z_n be a martingale. Let $N = \min\{n \mid Z_n \geq 5\}$ be the stopping time. Suppose the experiments are repeated three times with the outcomes depicted in the table below.*

outcome	Z_1	Z_2	Z_3	Z_4	Z_5	$\cdots$
ω_1	1	3	5	2	4	$\cdots$
ω_2	4	7	3	1	6	$\cdots$
ω_3	2	4	1	8	2	$\cdots$

We have:

- $N(\underline{\omega_1}) = 3,\ N(\omega_2) = 2,\ N(\omega_3) = 4;$ stopping time
- $Z_N(\omega_1) = 5,\ Z_N(\omega_2) = 7,\ Z_N(\omega_3) = 8;$ total winning of outcome sequence w_i at stopping time N
- $Z_5(\omega_1) = 4,\ Z_5(\omega_2) = 6,\ Z_5(\omega_3) = 2;$
- $\overline{Z}_5(\omega_1) = 5,\ \overline{Z}_5(\omega_2) = 7,\ \overline{Z}_5(\omega_3) = 8;$
- $\overline{Z}_3(\omega_1) = 5,\ \overline{Z}_3(\omega_2) = 7,\ \overline{Z}_3(\omega_3) = 1.$ □

Proposition 4.3 *If N is a random time for the martingale $\{Z_n\}$, then the stopped process $\{\overline{Z}_n\}$ is also a martingale.*

- Since the stopped process is also a martingale,

$$E[\overline{Z}_n] = E[\overline{Z}_1] = E[Z_1] \quad \text{for all } n.$$

- Suppose that $P\{N < \infty\} = 1$, i.e. N is a stopping time. Since

$$\overline{Z}_n = \begin{cases} Z_n, & \text{if } n \leq N \\ Z_N, & \text{if } n > N \end{cases}$$

  and $N < \infty$ with probability 1, it follows that, for each $\omega \in \Omega$, $\overline{Z}_n = Z_N$ when n is sufficiently large.

- Hence,

$$\overline{Z}_n \to Z_N \quad \text{as } n \to \infty, \text{ with probability 1.}$$

- Is it also true that $E[\overline{Z}_n] \to E[Z_N]$? If yes, then $E[Z_N] = E[Z_1]$.

Theorem 4.4 (The Martingale Stopping Theorem) *If either*

1. *$\overline{Z}_n$ are uniformly bounded (for all n and for all ω), or,*
2. *N is bounded (for all ω), or,*
3. *$E[N] < \infty$, and there is an $M < \infty$ such that*

$$E[|Z_{n+1} - Z_n| \, |Z_1, \dots, Z_n] \leq M,$$

then $E[\overline{Z}_n] \to E[Z_N]$ so that $E[Z_N] = E[Z_1]$.

- This theorem states that in a fair game even if a gambler uses a stopping time to decide when to quit, his expected final fortune is still equal to his initial fortune.
- Significance: On average, no successful gambling system is possible.
- To apply #3 of the theorem, it is necessary to verify that $P\{N = \infty\} = 0$ and $E[N] = \sum_{n=1}^{\infty} n \cdot P\{N = n\} < \infty$ first. But we handwave it.

Example 4.6 (The Matching Rounds Problem) Consider the problem in Example 2.6. Let R denote the number of rounds until all people have a match. Show that $E[R] = n$.

Solution:

- Let X_i denote the number of matches on the ith round, for $i = 1, 2, \ldots, R$. Define $X_i \equiv 1$ for $i > R$.
- By Example 2.6(a), for $i \le R$, we have

$$E[X_i | X_1, \ldots, X_{i-1}] = 1.$$

  If $X_1, \ldots, X_{i-1}$ signifies $i > R$, then the above holds because $X_i \equiv 1$.
- Let Z_k be

$$Z_k = \sum_{i=1}^{k} (X_i - E[X_i | X_1, \ldots, X_{i-1}]) = \sum_{i=1}^{k} (X_i - 1).$$

  Then, $\{Z_k \mid k \ge 1\}$ is a zero-mean martingale.

- R is a random time for the martingale $\{Z_k \mid k \geq 1\}$; it is the smallest k for which $\sum_{i=1}^{k} X_i = n$.
- Note that

$$|Z_{k+1} - Z_k| = |X_{k+1} - 1| \leq n + 1 \qquad \text{for all } k \geq 1.$$

monoticity of conditional expectation.

Thus, $E\left[|Z_{k+1} - Z_k| \,|X_1, \ldots, X_k\right] \leq n + 1$ for all $k \geq 1$.

- By the Martingale Stopping Theorem, n

$$\begin{aligned} 0 &= E[Z_R] \\ &= E\left[\sum_{i=1}^{R}(X_i - 1)\right] \\ &= n - E[R]. \end{aligned}$$

$z_1 = X_1 - 1$
$E(z_1) = E(X_1) - 1 = 0$

Thus, $E[R] = n$.

□

4.3 The Martingale Gambling Strategy

- Consider tossing a fair coin. For each dollar bet, the payoff is \$2 (i.e. winnings is \$1) so that the game is fair.
- The *martingale strategy* is to double the bet until one wins.
- Let X_n denote the winnings per dollar bet of the nth game:
$$X_n = \begin{cases} 1, & \text{if wins,} \\ -1, & \text{if loses.} \end{cases}$$
- Let
$$N = \min\{n \mid X_n = 1\}.$$
Then, it is a random time for $\{X_n \mid n \geq 1\}$. It is also a stopping time because
$$P\{N = \infty\} < P\{N > n\} = P(\text{losing first } n \text{ games}) = \frac{1}{2^n}$$
for any n, so that $P\{N = \infty\} = 0$.

- The bet size for the nth game is

$$\alpha_n = \begin{cases} 2^{n-1}, & \text{for } N > n-1 \\ 0, & \text{for } N \leq n-1. \end{cases}$$

- Let $Z_n = \sum_{i=1}^{n} \alpha_i X_i$ be the total winnings after the nth game. Then, $\{Z_n \mid n \geq 1\}$ is a martingale with mean 0 because

$$E[Z_{n+1}|X_1,\ldots,X_n] = Z_n + \alpha_{n+1} \cdot E[X_{n+1}|X_1,\ldots,X_n] = Z_n.$$

- Before the player stops, he has lost $-1-2-2^2-\cdots-2^{n-1} = 1-2^n$. Thus,

$$Z_n = \begin{cases} 1-2^n, & \text{for } n < N \\ 1, & \text{for } n \geq N. \end{cases}$$

- The probability of winning \$1 is

$$P\{N < \infty\} = 1 - P\{N = \infty\} = 1.$$

- What is $E[N]$?

$$
\begin{aligned}
E[N] &= \sum_{n=1}^{\infty} n \cdot P\{N = n\} \\
&= \sum_{n=1}^{\infty} \frac{n}{2^n} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{2} \cdot \frac{1}{(1 - \frac{1}{2})^2} = 2.
\end{aligned}
$$

- The expected winnings are

$$
\begin{aligned}
E[Z_1] &= E[Z_2] = \cdots = E[Z_n] = \cdots = 0 \\
E[Z_N] &= 1.
\end{aligned}
$$

- Thus, it is an example where the Martingale Stopping Theorem does not apply. All the three conditions of the theorem are not met.

- We have a winning probability of 1. What is the catch?

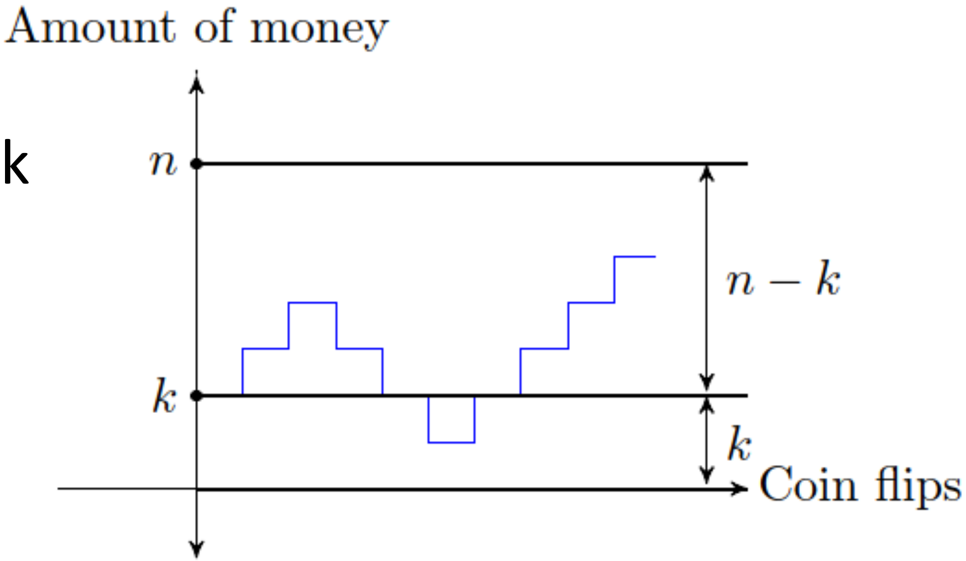
$$
\begin{aligned}
E[Z_{N-1}] = E[E[Z_{N-1}|N]] &= \sum_{n=1}^{\infty} E[Z_{N-1}|N=n]P\{N=n\} \\
&= \sum_{n=1}^{\infty}(1-2^{n-1})\frac{1}{2^n} \\
&= \sum_{n=1}^{\infty}\frac{1}{2^n} - \sum_{n=1}^{\infty}\frac{1}{2} \\
&= 1-\infty \\
&= -\infty.
\end{aligned}
$$

- Thus, to adopt the martingale strategy, an infinite amount of capital is needed.

The Gambler’s ruin - revisit

- The game consists of flipping a fair coin repeatedly. Every time the outcome is heads, the gambler wins \$1. If, however, the outcome is tails, the gambler loses \$1. The gambler starts the game with \$k and stops playing when he loses all his money or when he reaches a total of \$n, where n > k, i.e., the gambler stops if he hits 0 or n.

  The **total** amount of money: random walk



The Gambler’s ruin - revisit

- Random walk: “A random process that describes a path that consists of **a succession of** random steps”

- *What is the probability of hitting n (win) before hitting 0 (lose)?*

X_i denotes the amount of money the gambler makes at the i^{th} round.

$$P(X_i = +\$1) = P(X_i = -\$1) = \frac{1}{2}$$

Let $Z_0 = k, Z_i = \sum_{k=1}^{i} X_k$

The Gambler’s ruin - revisit

- The set of observable variables $X_1, X_2, \dots X_n$ determines $Z_1, Z_2 \dots Z_n$
- Show that $\{Z_n | n \geq 1\}$ is a martingale.

- The expected value of total fortune **over time** is k.

The Gambler's ruin - revisit

- $Z_1, Z_2 \ldots$ is a martingale with respect to $X_1, X_2, \ldots$. Let T be the time the gambler hits 0 or n, T is a stopping time for $X_1, X_2, \ldots X_n$. $|Z_i|\ i = 1,2,\ldots n$ are bounded by n.
- By the martingale stopping theorem

$$E(Z_T) = E(Z_0) = k$$
$$n \times P(win) + 0 \times P(lose) = k$$
$$\therefore P(win) = \frac{k}{n}$$