Lecture 7

Pricing Stock Options

- The Arbitrage Theorem
- The Black-Scholes option pricing formula
- Stock price model with Poisson process
- Pricing an arbitrary payoff

7.1 The Arbitrage Theorem

Arbitrage

Consider the following game:

- A (possibly unfair) coin is to be flipped.
- Two kinds of betting (wager) are available:
  1. Pay \$100 to enter the game. Payoff is \$200 if head and \$50 if tail.
  2. Pay \$20 to enter the game. Payoff is \$50 if head and \$0 if tail.
- A player can bet
  (a) x units on Game 1, and
  (b) y units on Game 2

  for any value of x and y, even negative (i.e., accepting bets from other players).

- E.g.
  - If the player buys 1 unit of Game 1 and sells 3 units of Game 2, then he pays \$40 net to enter the games.
  - If head, he receives $\$(200 - 3 \times 50) = \$50$ payoff. If tail, he receives $\$(50 - 3 \times 0) = \$50$ payoff.
  - In any case, he has a net profit (return) of \$10.
- This kind of sure-win opportunity is called an *arbitrage*.
- The existence of an arbitrage stems from the mispricing of the games. For example, if Game 2 charges $\$\frac{50}{3}$, then arbitrage is impossible.
- We are interested to see how games like this should be priced. The idea extends to option pricing.

The Arbitrage Theorem

- Consider an experiment with possible outcomes $\Omega = \{1, 2, \ldots, m\}$.
- Suppose that n wagers are available.
- The (net) return of the ith wager is $r_i(j)$ per unit bet if the outcome of the experiment is j.
- A *betting scheme* is a vector $\mathbf{x} = (x_1, \ldots, x_n)$ with the interpretation that x_i units are bet on wager i.
- Each x_i can be positive, negative, or zero.
- The total return of a betting scheme is

$$R(j) = \sum_{i=1}^{n} x_i \cdot r_i(j),$$

when the outcome is j.

- If one can find a betting scheme $\mathbf{x}$ such that the total return $R(j) > 0$ for any outcome j, then it is an arbitrage.
- We are interested to know whether an arbitrage exists in each given setting.
- The Arbitrage Theorem gives a simple characterization of the existence of arbitrage opportunities.
- Remark: Another version of arbitrage is $R(j) \geq 0$ for all j and $R(j) > 0$ for at least one j. The Arbitrage Theorem will also be slightly different.
- Let X be the random variable which is equal to the outcome. Denote $E^{\mathbf{q}}$ be the expectation under a probability distribution $\mathbf{q}$.

Theorem 7.1 (The Arbitrage Theorem) Exactly one of the following is true: Either

(i) there exists a probability vector $\mathbf{q} = (q_1, \ldots, q_m)$ for which

$$E^{\mathbf{q}}[r_i(X)] = \sum_{j=1}^{m} q_j \cdot r_i(j) = 0 \qquad \text{for all } i = 1, 2, \ldots, n$$

or

(ii) there exists a betting scheme $\mathbf{x} = (x_1, \ldots, x_n)$ for which

$$\sum_{i=1}^{n} x_i \cdot r_i(j) > 0 \qquad \text{for all } j = 1, 2, \ldots, m.$$

- The probability vector $\mathbf{q}$ does not depend on the probabilities of the outcomes. In finance terminologies, $\mathbf{q}$ is the vector of *risk-neutral probabilities*, not *real-world probabilities*. They are fictitious probabilities determined by the returns $r_i(j)$ for all i, j.

- We can arrange the returns into a matrix:

$$\begin{bmatrix} r_1(1) & r_1(2) & \cdots & r_1(m) \\ r_2(1) & r_2(2) & \cdots & r_2(m) \\ \vdots & \vdots & \cdots & \vdots \\ r_n(1) & r_n(2) & \cdots & r_n(m) \end{bmatrix}.$$

- The Arbitrage Theorem says that if there is a distribution $\mathbf{q}$ s.t.

$$\begin{bmatrix} r_1(1) & r_1(2) & \cdots & r_1(m) \\ r_2(1) & r_2(2) & \cdots & r_2(m) \\ \vdots & \vdots & \cdots & \vdots \\ r_n(1) & r_n(2) & \cdots & r_n(m) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

then no arbitrage exists. Otherwise, there will be a betting scheme $\mathbf{x}$ s.t.

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} r_1(1) & r_1(2) & \cdots & r_1(m) \\ r_2(1) & r_2(2) & \cdots & r_2(m) \\ \vdots & \vdots & \cdots & \vdots \\ r_n(1) & r_n(2) & \cdots & r_n(m) \end{bmatrix}$$

is a positive vector.

Example 7.2 Suppose that the wager i is to bet that i is the outcome of the experiment, for $i = 1, 2, \ldots, m$. Let

$$r_i(j) = \begin{cases} o_i, & \text{if } i = j \\ -1, & \text{otherwise.} \end{cases}$$

We shall see how the odds $o_1, \ldots, o_m$ should be specified. In order to satisfy

$$0 = \sum_{j=1}^{m} q_j \cdot r_i(j) = q_i o_i - (1 - q_i),$$

we must have

$$q_i = \frac{1}{1 + o_i}.$$

Next, in order for $\mathbf{q} = (q_1, \ldots, q_m)$ to qualify as a probability vector, we must have

$$q_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{m} q_i = 1.$$

Thus, by the Arbitrage Theorem, the conditions for no arbitrage are

$$o_i > -1 \qquad \text{and} \qquad \sum_{i=1}^{m} \frac{1}{1+o_i} = 1.$$

For example, when $m = 3$, letting $o_1 = 1$, $o_2 = 2$, and $o_3 = 5$ results in no arbitrage. Letting $o_1 = 1$, $o_2 = 2$, and $o_3 = 3$ results in an arbitrage (can you find a sure-win betting scheme?) $\square$

7.2 The Black-Scholes Option Pricing Formula

Introduction to Options

- A *stock option* (or simply *option*) is a financial product that gives an investor the right, but not the obligation, to buy or sell a stock at an agreed upon price and date.
- Two types of options: *call option* and *put option*
  - A call option gives an investor the right to buy a stock.
  - A put option gives an investor the right to sell a stock.
- E.g. A call option is like a supermarket coupon. A holder of the coupon can use it to buy a specific product at a specific price and on a specific date(s). The holder can choose to use it or not. The supermarket (issuer of the coupon) must honour it. We are interested to determine of the value (price) of such a coupon.

Black-Scholes Model

The Black-Scholes model for option pricing (for European options only) comprised of the following.

- Model the stock price with the geometric Brownian motion process.
- Determine the option price by risk-neutral valuation (i.e. Arbitrage Theorem).
- The result is a pricing formula, called the *Black-Scholes formula*.

The Stock

- Consider a stock with price $X(t)$ for $t \geq 0$.
- Let $r > 0$ be the interest rate (assumed constant).
- If we purchase one share at time t, then the *present value* of the **cost** is $X(t)e^{-rt}$. The present value of the share is $X(0)$.

Call Options

- Suppose that the price of a stock is given by $X(t)$ for $t \geq 0$. A *call option* on the stock with maturity T (years) and strike price K operates as follows:
  - At time 0, the investor pays \$$c$ per share to buy an option on one share of the stock.
  - At time T, the payoff to the investor is

$$\text{payoff} = \begin{cases} X(T) - K, & \text{if } X(T) > K \\ 0, & \text{otherwise} \end{cases} = (X(T) - K)^+.$$

- For example, consider a 6-month call option with strike price \$80. The payoff will be \$5 if the stock price is \$85 after 6 months. The payoff will be \$0 if the stock price drops below \$80 after 6 months.
- **In this course, simply treat an option as a function of** $X(T)$. Different options mean different functions of $X(T)$.

Wagers

Assume that two wagers are available in an investment horizon $[0, T]$.

1. Buy (or sell) one share at time s after observing the price at time s. Then, sell (or buy) the share at time t, where $0 \leq s < t \leq T$. The present value of the amount paid (or received) at time s is $X(s)e^{-rs}$. The present value of the amount received (or paid) at time t is $X(t)e^{-rt}$.

2. Buy (or sell) one call option at time 0. Then, receive (or pay) a payoff at time T. The present value of the amount paid (or received) at time 0 is c. The present value of the amount received (or paid) at time T is

$$(X(T) - K)^+ e^{-rT}.$$

Pricing the Option

- Let $\mathcal{F}_s = \{X(u),\ 0 \le u \le s\}$ be the stock price until time s.
- Given the present stock price $X(0)$, investors and option writers are interested in the theoretical no-arbitrage price of the option.
- By the Arbitrage Theorem (a suitably generalized version to continuous case), if there exists a $\mathbf{Q}$ on $\{X(t),\ 0 \le t \le T\}$ such that

$$E^{\mathbf{Q}}[X(t)e^{-rt}|\mathcal{F}_s] = X(s)e^{-rs} \quad \text{for } 0 \le s < t \le T \tag{7.1}$$

$$E^{\mathbf{Q}}[(X(T) - K)^+ e^{-rT}] = c, \tag{7.2}$$

  then no arbitrage is possible.

- But our goal is to determine a suitable price c for the option. We will first find a $\mathbf{Q}$ that satisfies Eq. (7.1), then use such a $\mathbf{Q}$ to calculate c via (7.2), so that the option is priced to eliminate arbitrage. This method of computing option price is called *risk-neutral valuation*.

Geometric Brownian Motion Stock Price Model

- To satisfy Eq. (7.1), we let
$$X(t) = X(0)e^{Z(t)},$$
where $Z(t) = (\mu - \frac{\sigma^2}{2})t + \sigma B(t)$ is a Brownian motion with drift.

- By Example 6.10,
$$E[X(t)|\mathcal{F}_s] \;=\; X(s) \cdot e^{\mu(t-s)}.$$

- If we replace μ with r (or a change of probability measure), then
$$\begin{aligned} E[X(t)|\mathcal{F}_s] \;&=\; X(s) \cdot e^{r(t-s)} \\ E[e^{-rt}X(t)|\mathcal{F}_s] \;&=\; X(s) \cdot e^{-rs}. \end{aligned}$$

- Hence, Eq. (7.1) is satisfied. The probability measure $\mathbf{Q}$ is simply the one governing the process $\{Z(t),\ 0 \le t \le T\}$, which is the Brownian motion with drift $r - \frac{\sigma^2}{2}$ and variance rate σ^2.

Black-Scholes Formula

- Having found a suitable $\mathbf{Q}$, now we apply it to calculate the European call option price using risk-neutral valuation.
- Recall that $Z(T) \sim \mathcal{N}((r - \frac{\sigma^2}{2})T, \sigma^2 T)$ under $\mathbf{Q}$.
- Hence, by Eq. (7.2),

$$
\begin{aligned}
c &= E^{\mathbf{Q}}[(X(T) - K)^+ e^{-rT}] \\
&= e^{-rT} E^{\mathbf{Q}}[(X(0)e^{Z(T)} - K)^+] \\
&= e^{-rT} \int_{-\infty}^{\infty} (X(0)e^z - K)^+ \cdot \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z-(r-\sigma^2/2)T)^2}{2\sigma^2 T}} \, dz \\
&= e^{-rT} \int_{\ln(K/X(0))}^{\infty} (X(0)e^z - K) \cdot \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z-(r-\sigma^2/2)T)^2}{2\sigma^2 T}} \, dz \\
&= \mathrm{I} - \mathrm{II}.
\end{aligned}
$$

- Let

$$a = \frac{\ln(K/X(0)) - (r + \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}} \qquad b = \frac{\ln(K/X(0)) - (r - \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}}.$$

-

$$\begin{aligned}
\mathrm{I} &= e^{-rT} X(0) \int_{\ln(K/X(0))}^{\infty} e^z \cdot \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z-(r-\sigma^2/2)T)^2}{2\sigma^2 T}}\, dz \\
&\vdots \\
&= X(0) \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \\
&= X(0) \cdot \Phi(-a)
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &= K e^{-rT} \int_{\ln(K/X(0))}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z-(r-\sigma^2/2)T)^2}{2\sigma^2 T}}\, dz \\
&= K e^{-rT} \int_b^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw \\
&= K e^{-rT} \Phi(-b).
\end{aligned}$$

- Combining I and II, we have

$$c = X(0) \cdot \Phi\left(\frac{\ln \frac{X(0)}{K} + (r + \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}}\right) - Ke^{-rT} \cdot \Phi\left(\frac{\ln \frac{X(0)}{K} + (r - \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}}\right).$$

- This formula is known as the *Black-Scholes Formula* (for European call options).

- In practice, the current stock price $X(0)$ and interest rate r are known. The maturity date T and the strike price K are specified in the option contract. The only unknown is σ, called *volatility*.

- We can estimate σ using historical stock prices because the log-return follows

$$\ln(X(t+s)/X(t)) \sim \mathcal{N}((r - \tfrac{\sigma^2}{2})s, \sigma^2 s).$$

  Calculate the daily log-returns $\ln \frac{X_2}{X_1}, \ln \frac{X_3}{X_2}, \ldots, \ln \frac{X_{n+1}}{X_n}$. They correspond to $s = 1/252$ years. The sample variance gives an estimate of $\sigma^2/252$.

Example 7.3 The current price of a stock is \$100. Suppose that the price of the stock changes according to a geometric Brownian motion process with drift coefficient $\mu = 2$ and variance parameter $\sigma^2 = 1$. Give the Black-Scholes cost of a call option with maturity $T = 10$ (years) and strike price $K = 100$. The continuously compounded interest rate is 5% per annum.

Solution:

$$
\begin{aligned}
d_1 &= \frac{\ln(X(0)/K) + rT + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(100/100) + 0.05 \times 10 + 10/2}{\sqrt{10}} = 1.7393 \\
d_2 &= \frac{\ln(X(0)/K) + rT - \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(100/100) + 0.05 \times 10 - 10/2}{\sqrt{10}} = -1.4230 \\
c &= X(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \\
&= 100\Phi(1.7393) - 100e^{-0.05\times 10}\Phi(-1.4230) = 91.2081.
\end{aligned}
$$

$\square$

7.3 Other Payoff Functions

For an arbitrary payoff $g(X(T))$ at time T, the current price is

$$e^{-rT}E[g(X(T))] = e^{-rT}\int_{-\infty}^{\infty} g(X(0)e^z)f_{Z(T)}(z)dz,$$

where $f_{Z(T)}$ is the pdf for $\mathcal{N}((r-\frac{\sigma^2}{2})T, \sigma^2 T)$.

Example 7.4 Suppose that the payoff of an investment at time T is $X(T)-K$, i.e. a futures contract. Determine the current price f of the investment, assuming a geometric Brownian motion model for $X(t)$.

Solution: By the risk-neutral valuation method,

$$\begin{aligned} f &= E[e^{-rT}(X(T)-K)] \\ &= E[e^{-rT}X(T)] - Ke^{-rT} \\ &= e^{-rT}X(0)E[e^{Z(T)}] - Ke^{-rT} = X(0) - Ke^{-rT}. \end{aligned}$$

The price is independent of μ and σ. □

7.3.1 Futures Options

Recall that a *futures contract* is an agreement between two parties to buy/sell a pre-determined number of stocks at a pre-determined price K on a pre-determined date T. The party who will pay money and receive stocks at time T is called the *long party.* It has been shown that when the stock price follows a geometric Brownian motion, the present value of a futures contract (for the long party and per share of stock) is

$$f(0;T) = S(0) - Ke^{-rT}.$$

If we denote the current time by t instead of 0, then the value of the contract is

$$f(t;T) = S(t) - Ke^{-r(T-t)}.$$

(In fact, this formula is independent of the stock price model. We can prove the same formula by arbitrage arguments without assuming any specific stock price model.)

If two parties initiate a new contract now (time t) and if they agree to set the delivery price to $K = e^{r(T-t)}S(t)$, then the contract has zero current value. Such a special delivery price is called the *futures price* (a.k.a. forward price), denoted by $F(t)$:

$$F(t) := e^{r(T-t)}S(t).$$

Under a geometric Brownian motion model for the stock price, the futures price is

$$F(t) = F(0)e^{-\frac{\sigma^2 t}{2}+\sigma B(t)},$$

from which we can deduce the distribution of $F(t)$. Hence, we can use it together the risk-neutral valuation method to deduce a pricing formula for options on futures. Suppose the current price of a futures contract with delivery price K_f and maturity T_f is $F(0)$.

Suppose a call option on the futures has a strike price K_o and maturity $T_o < T_f$. The payoff of the option is $(F(T_o) - K_o)^+$. The current option price is therefore

$$E[e^{-rT}(F(T_o) - K_o)^+] = e^{-rT}[F(0)\Phi(d_1) - K_o\Phi(d_2)],$$

where $d_1 = \frac{\ln(F(0)/K_o) + \sigma^2 T_o/2}{\sigma\sqrt{T_o}}$ and $d_2 = \frac{\ln(F(0)/K_o) - \sigma^2 T_o/2}{\sigma\sqrt{T_o}}$.

7.4 A Stock Price Model with Poisson Process

- In the derivation of the Black-Scholes formula, we found a **P** by assuming $X(t) = X(0)e^{Z(t)}$ for some Brownian motion $Z(t)$.
- We can assume other models for $X(t)$. As long as we can find a **P** such that
$$E^{\mathbf{P}}[X(t)e^{-rt}|\mathcal{F}_s] = X(s)e^{-rs},$$
i.e. $X(t)e^{-rt}$ is a continuous-time martingale, then we can use this **P** to compute the corresponding no-arbitrage price for the option.
- We present another stock price model whose paths consist of jumps.
- Let $Y_1, Y_2, \ldots$ be i.i.d. with a common mean $E[Y_i] = \mu$.
- Suppose that this process is independent of $\{N(t),\ t \geq 0\}$, a Poisson process with rate λ.

- Let
$$X(t) = X(0) \prod_{i=1}^{N(t)} Y_i.$$
Thus, we see that $X(t)$ is a step function. It gets multiplied by an additional factor $Y_{N(t)}$ when an event occurs.

- Using the identity
$$X(t) = X(0) \cdot \prod_{i=1}^{N(s)} Y_i \cdot \prod_{j=N(s)+1}^{N(t)} Y_j = X(s) \cdot \prod_{j=N(s)+1}^{N(t)} Y_j$$
and the independent and stationary increment assumptions of the Poisson process, we see that, for $s < t$,
$$E[X(t) \mid \mathcal{F}_s] = X(s) \cdot E\left[\prod_{j=N(s)+1}^{N(t)} Y_j\right] = X(s) \cdot E\left[\prod_{j=1}^{N(t-s)} Y_j\right].$$

- Conditioning on $N(t-s)$ yields

$$\begin{aligned}
E\left[\prod_{j=1}^{N(t-s)} Y_j\right] &= E\left[E\left[\prod_{j=1}^{N(t-s)} Y_j \middle| N(t-s)\right]\right] = E[\mu^{N(t-s)}] \\
&= \sum_{n=0}^{\infty} \mu^n \cdot \frac{e^{-\lambda(t-s)}\lambda^n(t-s)^n}{n!} = e^{-\lambda(t-s)} \cdot \sum_{n=0}^{\infty} \frac{\mu^n\lambda^n(t-s)^n}{n!} \\
&= e^{-\lambda(t-s)} \cdot e^{\mu\lambda(t-s)} = e^{(\mu-1)\lambda(t-s)}.
\end{aligned}$$

- Therefore, if we choose $(\mu-1)\lambda = r$, then

$$E[X(t)e^{-rt}|\mathcal{F}_s] = X(s)e^{-rs}.$$

- Using such a model for $X(t)$, the no-arbitrage option price is

$$c = e^{-rT}E[(X(T)-K)^+|\mathcal{F}_s].$$

- To determine $E[(X(T)-K)^+|\mathcal{F}_s]$, we first need to specify of a distribution for Y_i. But we shall not pursue it.