Ordinary Differential Equations

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Chapter 4: Series Solutions of Second Order Linear Equations

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1 Review of Power Series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

- We say the series converges at a point $x=x_1$ if the series $\sum_{n=0}^{\infty} a_n (x_1-x_0)^n$ converges.
- We say the series converges absolutely at a point $x=x_1$ if the series $\sum_{n=0}^{\infty} a_n (x_1-x_0)^n$ converges absolutely, i.e. $\sum_{n=0}^{\infty} |a_n (x_1-x_0)^n|$ converges.
- ratio test:

$$r_n=\frac{|a_{n+1}(x-x_0)^{n+1}|}{|a_n(x-x_0)^n|}=\left|\frac{a_{n+1}}{a_n}\right| |x-x_0| \xrightarrow{n\to\infty} L\,|x-x_0|.$$

1. If $|x-x_0|<\dfrac{1}{L}$, then the series converges absolutely.

2. If $|x - x_0| > \frac{1}{L}$, then the series diverges.
3. If $|x - x_0| = \frac{1}{L}$, the test is inconclusive. Other tests are needed.

Example 1.1. For which values of x does the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$$

converge?

Answer: Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \to 1,$$

1. If $1 \times |x-2| < 1$, i.e. $1 < x < 3$, then the series converges absolutely.
2. If $x < 1$ or $x > 3$, then the series diverges.
3. If $x = 1$, the series becomes $\sum_{n=1}^{\infty} -n$, which is divergent.
4. If $x = 3$, the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1} n$, which is divergent.

So the series converges for $x \in (1, 3)$. □

- If the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at $x = x_1$, then the series converges absolutely if $|x - x_0| < |x_1 - x_0|$. If the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ diverges at $x = x_1$, then the series diverges if $|x - x_0| > |x_1 - x_0|$.
- There are three possibilities for the convergence of a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$:
  1. The series converges only for $x = x_0$.
  2. The series converges only for any $x \in \mathbb{R}$.
  3. There exists a number $\rho > 0$ such that the series converges absolutely for $x \in (x_0 - \rho, x_0 + \rho)$. We call ρ the **radius of convergence**. We say $\rho = 0$ in case 1 and $\rho = \infty$ in case 2.

  The interval I such that the series converges for any $x \in I$ is called the **interval of convergence**. If the radius of convergence is ρ, then the interval of convergence may be one of these

$$(x_0 - \rho, x_0 + \rho), \quad (x_0 - \rho, x_0 + \rho], \quad [x_0 - \rho, x_0 + \rho), \quad [x_0 - \rho, x_0 + \rho].$$

Example 1.2. Determine the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 2^n}$$

Answer:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1) 2^{n+1}}}{\frac{1}{n 2^n}} = \frac{n 2^n}{(n+1) 2^{n+1}} = \frac{n}{2(n+1)} \to \frac{1}{2}.$$

So the series converges absolutely for $|x+1| < \frac{1}{1/2} = 2$. So the radius of convergence is $\rho = 2$. The series converges absolutely in

$$-2 < x+1 < 2 \quad \Rightarrow \quad -3 < x < 1.$$

If $x = -3$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is convergent.
If $x = 1$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent.

So the interval of convergence is $[-3, 1)$.

Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x - x_0| < \rho$, $\rho > 0$.

- The sum of the series is a new series whose term is obtained termwise.
$$\sum_{n=0}^{\infty} a_n(x-x_0)^n + \sum_{n=0}^{\infty} b_n(x-x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n.$$
The new series converges at least for $|x - x_0| < \rho$.
- The difference of the series is a new series whose term is obtained termwise.
$$\sum_{n=0}^{\infty} a_n(x-x_0)^n - \sum_{n=0}^{\infty} b_n(x-x_0)^n = \sum_{n=0}^{\infty} (a_n - b_n)(x-x_0)^n.$$
The new series converges at least for $|x - x_0| < \rho$.
- The product of the series
$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \sum_{n=0}^{\infty} b_n(x-x_0)^n = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$
where
$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$
The new series converges at least for $|x - x_0| < \rho$.
- Assuming $b_0 \neq 0$. The quotient of the series is
$$\frac{\sum_{n=0}^{\infty} a_n(x-x_0)^n}{\sum_{n=0}^{\infty} b_n(x-x_0)^n} = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$
where c_n can be found by multiplying both sides by $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ and comparing the coefficients.

- The derivative of the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is
$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}.$$
The new series has the same radius of convergence as the original series.
- The value of a_n is given by
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$
- $$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n \quad \Rightarrow \quad a_n = b_n \forall n$$
$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0 \quad \Rightarrow \quad a_n = 0 \forall n$$

2 Series Solutions Near an Ordinary Point

Consider a 2nd order linear ODE
$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = 0.$$

2.1 Polynomial coefficients

Suppose P, Q, R are polynomials without common factors of the form $x - c$.

Definition 2.1 **(Ordinary point)** *A point x_0 is called an **ordinary point** if $P(x_0) \neq 0$.*

If x_0 is an ordinary point, then $P(x) \neq 0, x \in (c, d)$ for some open interval (c, d) containing x_0. So we can divide the equation by $P(x)$ to obtain
$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0,$$
which has a unique solution in (c, d) satisfying any initial conditiions $y(x_0) = y_0$, $y'(x_0) = y_0'$ by the existence and uniqueness theorem.

Definition 2.2 **(Series solution)** *A power series solution is a solution of the form*
$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad x \in (c, d).$$

Let's find series solutions centered at ordinary points.

Example 2.3. Find a series solution of the equation
$$y'' + y = 0, \quad -\infty < x < \infty.$$

Answer: First of all, $P(x) = 1$. So any number x is an ordinary point. Let $x_0 = 0$.
$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$
Then
$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Plugging into the equation,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now we shift the index so that the form inside the summation is the same for x.

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0. \\
\Rightarrow\ &\sum_{n=0}^{\infty} [a_n + (n+2)(n+1)a_{n+2}]x^n = 0 \\
\Rightarrow\ &a_n + (n+2)(n+1)a_{n+2} = 0
\end{aligned}$$

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}a_n$$

$$\begin{aligned}
a_2 &= -\frac{1}{1\cdot 2}a_0 = -\frac{1}{2!}a_0 \\
a_4 &= -\frac{1}{3\cdot 4}a_2 = \frac{1}{4!}a_0 \\
a_6 &= -\frac{1}{5\cdot 6}a_4 = -\frac{1}{6!}a_0
\end{aligned}$$

In general,

$$a_{2k} = \frac{(-1)^k}{(2k)!}a_0, \quad k = 1, 2, \ldots$$

We can prove this by mathematical induction.

1. $k=1$, it's clear.
2. Suppose it's true for some $k \geqslant 1$. Then

   $$a_{2(k+1)} = a_{2k+2} = -\frac{1}{(2k+1)(2k+2)}a_{2k} = -\frac{1}{(2k+1)(2k+2)}\frac{(-1)^k}{(2k)!}a_0 = \frac{(-1)^{k+1}}{(2(k+1))!}a_0.$$

   So it's true for $k+1$.

Similarly,

$$\begin{aligned}
a_3 &= -\frac{1}{2\cdot 3}a_1 = -\frac{1}{3!}a_1 \\
a_5 &= -\frac{1}{4\cdot 5}a_3 = \frac{1}{5!}a_1 \\
a_6 &= -\frac{1}{6\cdot 7}a_5 = -\frac{1}{7!}a_1
\end{aligned}$$

In general,

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1,$$

which can also be proved by mathematical induction.
Note that a_0 and a_1 can be arbitrary numbers. So

$$\begin{aligned}
y(x) &= a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} + a_1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\
&:= a_0 C(x) + a_1 S(x),
\end{aligned}$$

where

$$C(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}, \quad S(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

By the ratio test, both series converges for any $x\in\mathbb{R}$. Moreover,

$$C'(x) = \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}2kx^{2k-1}=\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)!}x^{2k-1}=-\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}=-S(x),$$

Similarly we have $S'(x)=C(x)$. The Wronskian is

$$W[C,S](0)=\begin{vmatrix} C(0) & S(0) \\ C'(0) & S'(0)\end{vmatrix}=\begin{vmatrix} 1 & 0 \\ 0 & 1\end{vmatrix}=1.$$

So $C(x)$ and $S(x)$ form a fundamental set of solutions, and the general solution is

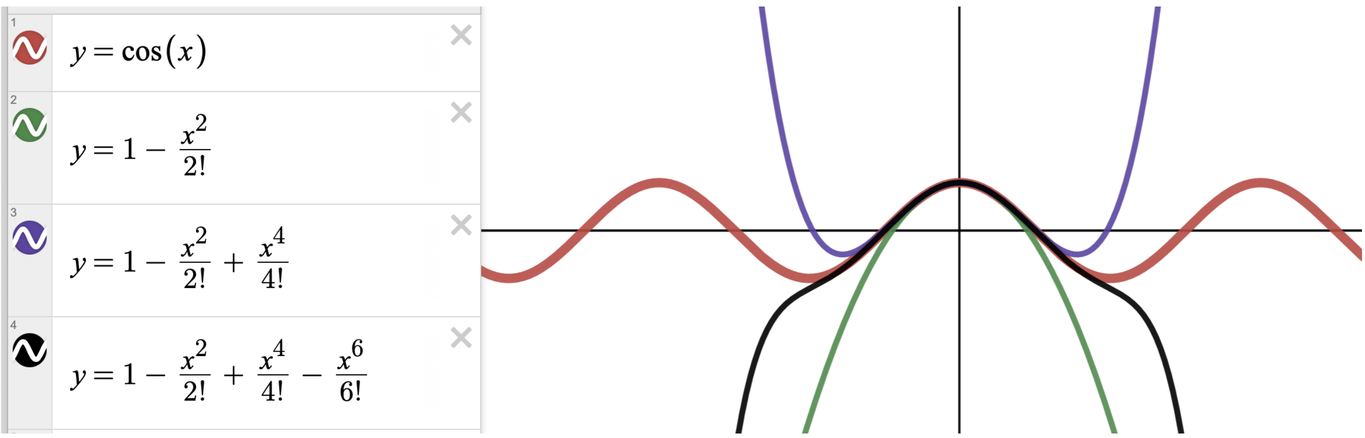
$$y=c_1C(x)+c_2S(x).$$

In fact $C(x)=\cos(x), S(x)=\sin(x)$. So we can consider $\cos(x)$ as the unique solution of the IVP

$$y''+y=0, \quad y(0)=1, \quad y'(0)=0.$$

and $\sin(x)$ as the unique solution of the IVP

$$y''+y=0, \quad y(0)=0, \quad y'(0)=1.$$



Example 2.4. Find a series solution in powers of x of Airy's equation

$$y''-xy=0, \quad -\infty<x<\infty.$$

Answer: We have $P(x)=1$, so any number x is an ordinary point. Let $x_0=0$. Consider the series solution

$$y=\sum_{n=0}^{\infty}a_nx^n.$$

Then

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}, \quad y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

Plugging into the equation,

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0 \\
&= 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0 \\
2a_2 &= 0 \\
a_{n+2} &= \frac{1}{(n+1)(n+2)}a_{n-1}, \quad n=1,2,\ldots,\infty \\
a_3 &= \frac{1}{2\times 3}a_0 \\
a_6 &= \frac{1}{5\times 6}a_3 = \frac{1}{5\times 6}\frac{1}{2\times 3}a_0 \\
a_9 &= \frac{1}{8\times 9}\frac{1}{5\times 6}\frac{1}{2\times 3}a_0 \\
&\vdots \\
a_2 &= 0 \\
a_5 &= 0 \\
a_8 &= 0 \\
&\vdots \\
a_4 &= \frac{1}{3\times 4}a_1 \\
a_7 &= \frac{1}{6\times 7}\frac{1}{3\times 4}a_1 \\
a_{10} &= \frac{1}{9\times 10}\frac{1}{6\times 7}\frac{1}{3\times 4}a_1 \\
&\vdots \\
y = \sum_{n=0}^{\infty} a_n x^n &= a_0\left(1 + \frac{x^3}{2\times 3} + \frac{x^6}{2\times 3\times 5\times 6} + \cdots\right) \\
&+ a_1\left(x + \frac{x^4}{3\times 4} + \frac{x^7}{3\times 4\times 6\times 7}\right) \\
&= a_0 y_1(x) + a_1 y_2(x),
\end{aligned}
$$

where a_0, a_1 are arbitrary constants, and y_1, y_2 are power series.
Using the ratio test, we can show both y_1, y_2 converge for all x. The Wronskian

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

So y_1, y_2 form a fundamental set of solutions for the ODE. So $a_0 y_1(x) + a_1 y_2(x)$ is the general solution.

Example 2.5. Find a solution of Airy's equation in powers of $x - 1$.

Answer: Let
$$y = \sum_{n=0}^{\infty} a_n (x-1)^n.$$
Then
$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$
Plugging into the equation,
$$\begin{aligned}
y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1+1) \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= (2a_2 - a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} - a_n\right] (x-1)^n = 0 \\
2a_2 - a_0 &= 0 \\
a_{n+2} &= \frac{1}{(n+1)(n+2)} (a_{n-1} + a_n) \\
a_2 &= \frac{1}{2} a_0 \\
a_3 &= \frac{1}{6}(a_0 + a_1) = \frac{1}{6} a_0 + \frac{1}{6} a_1 \\
a_4 &= \frac{1}{12}(a_1 + a_2) = \frac{1}{24} a_0 + \frac{1}{12} a_1 \\
a_5 &= \frac{1}{20}(a_2 + a_3) = \frac{1}{20}\left(\frac{1}{2} a_0 + \frac{1}{6} a_0 + \frac{1}{6} a_1\right) \\
&= \frac{1}{20}\left(\frac{2}{3} a_0 + \frac{1}{6} a_1\right) = \frac{1}{30} a_0 + \frac{1}{120} a_1 \\
y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + \cdots \\
&= a_0 + a_1 (x-1) + \frac{1}{2} a_0 (x-1)^2 + \left(\frac{1}{6} a_0 + \frac{1}{6} a_1\right)(x-1)^3 + \cdots \\
&= a_0 \left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots\right] + a_1 \left[(x-1) + \frac{1}{6}(x-1)^3 + \cdots\right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

No general formula for a_n and not easy to check the convergence. But we can find as many coefficients as we want using a computer program.

2.2 General coefficients

Consider the ODE

$$P(x)\,y''(x)+Q(x)\,y'(x)+R(x)\,y(x)=0,$$

where P,Q,R are continuous functions. Suppose $P(x_0)\neq 0$. We seek a power series solution

$$y=\sum_{n=0}^{\infty} a_n\,(x-x_0)^n.$$

In a neighborhood of x_0 where $P(x)\neq 0$,

$$y''(x)+p(x)\,y'(x)+q(x)\,y(x)=0.$$

Then

$$y(x_0)=a_0 \quad\Rightarrow\quad a_0=y(x_0)$$

$$y'(x)=\sum_{n=1}^{\infty} a_n n\,(x-x_0)^{n-1}=\sum_{n=0}^{\infty} a_{n+1}(n+1)\,(x-x_0)^n$$

$$y'(x_0)=a_1 \quad\Rightarrow\quad a_1=y'(x_0)$$

$$y''(x)=\sum_{n=2}^{\infty} a_n n\,(n-1)\,(x-x_0)^{n-2}=\sum_{n=0}^{\infty} a_{n+2}(n+2)\,(n+1)\,(x-x_0)^n$$

$$\begin{aligned} y''(x_0) &= 2a_2 \quad\Rightarrow\\ a_2 &= \frac{1}{2}y''(x_0)=\frac{1}{2}(-p(x_0)\,y'(x_0)-q(x_0)\,y(x_0))\\ &= -\frac{1}{2}(q(x_0)\,a_0+p(x_0)\,a_1) \end{aligned}$$

Similarly, we can find $a_3,a_4,\ldots$ So the series solution can be written as

$$y=a_0\,y_1(x)+a_1\,y_2(x),\qquad y_1(x)=\sum_{n=0}^{\infty} b_n\,(x-x_0)^n,\qquad y_2(x)=\sum_{n=0}^{\infty} c_n\,(x-x_0)^n.$$

In general, we can not guarantee the convergence or find the radius of convergence of the series.

Definition 2.6 **(Analytic)** *A function $f(x)$ is called* ***analytic*** *at a point x_0 if it's Taylor series about x_0 converges to $f(x)$ in a neighborhood of x_0.*

Example 2.7. Consider

$$f(x)=\begin{cases} e^{-\frac{1}{x}}, & x>0,\\ 0, & x\leqslant 0.\end{cases}$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x}=\lim_{u\to\infty}\frac{e^{-u}}{\frac{1}{u}}=\lim_{u\to\infty}\frac{u}{e^u}=0 \quad\Rightarrow\quad f'(0)=0.$$

In fact, $f\in C^{\infty}(\mathbb{R})$, and $f^{(n)}(0)=0$. So its Taylor series is 0, which does not converge to $f(x)$ in any neighborhood of 0. So f is nonanalytic at $x=0$.

Definition 2.8 **(Ordinary point, Singualr Point)** *If $\dfrac{Q(x)}{P(x)}$ and $\dfrac{R(x)}{P(x)}$ are analytic at x_0, then x_0 is called an* ***ordinary point*** *of the equation*

$$P(x)\,y''(x)+Q(x)\,y'(x)+R(x)\,y(x)=0.$$

Otherwise, x_0 is called a ***singular point*** *of the equation.*

Theorem 2.9 *Suppose x_0 is an ordinary point of the equation*

$$P(x)\,y''(x)+Q(x)\,y'(x)+R(x)\,y(x)=0,$$

i.e. the functions $p(x)=Q(x)\,/\,P(x)$ and $q(x)=R(x)\,/\,P(x)$ are analytic at x_0. Then the general solution of the equation is given by

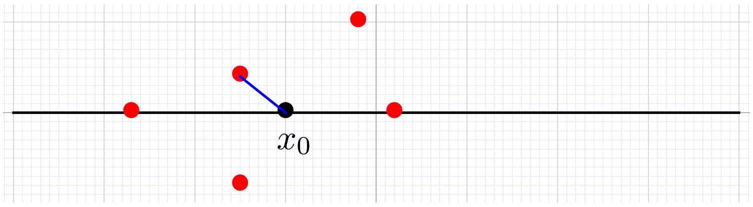
$$y=a_0\,y_1(x)+a_1\,y_2(x),$$

where

$$y_1=\sum_{n=0}^{\infty} b_n\,(x-x_0)^n,\quad y_2=\sum_{n=0}^{\infty} c_n\,(x-x_0)^n,$$

and their radius of convergence are at least the minimum of the radius of convergence for the Taylor series of p and q about x_0.

Remark 2.10. If P, Q are polynomials, and $P(x_0)\neq 0$, then $p(x)=Q(x)\,/\,P(x)$ is analytic at x_0. Moreover, the radius of convergence of the Taylor series about x_0 is the distance from x_0 to the nearest (real and complex) root of P.



Example 2.11. What is the radius of convergence of the Taylor series for $(1+x^2)^{-1}$about $x=0$?

Answer: The roots of $1+x^2$ is $\pm i$. The distance from 0 to i is 1, and the distance from 0 to $-i$ is 1. This implies the radius of convergence is 1. Check the answer by finding the Taylor series.

Example 2.12. What is the radius of convergence of the Taylor series for $(x^2-2x+2)^{-1}$ about $x=0$? about $x=1$?

Answer:

$$x^2-2x+2=(x-1)^2+1=0\quad\Rightarrow\quad x=1\pm i.$$

The minimal distance from 0 to the roots are $\sqrt{2}$, so the radius of convergence is $\sqrt{2}$.
The minimal distance from 1 to the roots are 1, so the radius of convergence is 1.

Example 2.13. Determine a lower bound for the radius of convergence of series solutions about $x=0$ for the Legendre equation

$$(1-x^2)\,y''-2x\,y'+\alpha\,(\alpha+1)\,y=0,$$

where α is a constant.

Answer: $P(0)\neq 0$. The roots of P are ± 1. The minimal distance from 0 to ± 1 is 1. So the radius of convergence is 1.

Example 2.14. Can we determine a series solution about $x=0$ for the differential equation

$$y''+(\sin x)\,y'+(1+x^2)\,y=0,$$

and if so, what is the radius of convergence?

Answer: We have $p = \sin x$, $q = 1 + x^2$. They are analytic everywhere. The Taylor series for p about 0 has an infinite radius of convergence. The Taylor series for q about $x = 0$ also has an infinite radius of convergence. So both y_1 and y_2 in the series solution has an infinite radius of convergence.

3 Euler Equations; Regular Singular Points

Definition 3.1 **(Euler's Equations)**

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = 0,$$

where α, β are given constants.

Now $P(x) = x^2$, so $x = 0$ is a singular point. Let's look at solutions for $x > 0$.
Let's try to compute a solution using an ansatz

$$y = x^r.$$

Then the equation reduces to

$$[r(r-1) + \alpha r + \beta] x^r = 0,$$

which implies the **characteristic equation**

$$r^2 + (\alpha - 1) r + \beta = 0.$$

The roots are

$$r_{1,2} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}.$$

3.1 Real and distinct roots

Suppose $r_1 \neq r_2$ are both real. Then we have two solutions

$$y_1 = x^{r_1}, \quad y_2 = x^{r_2}.$$

Check the Wronskian

$$W[y_1, y_2] = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1 - 1} & r_2 x^{r_2 - 1} \end{vmatrix} = (r_2 - r_1) x^{r_1 + r_2 - 1} \neq 0.$$

So the general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_2}.$$

We are concerned with the properties of the solution as $x \to 0^+$.

- If $r_1 > 0$, then $x^{r_1} \to 0$.
- If $r_1 < 0$, then $x^{r_1} \to \infty$.
- If $r_1 = 0$, then $x^{r_1} \equiv 1$.

Example 3.2. Solve

$$2x^2 y'' + 3xy' - y = 0, \quad x > 0.$$

Answer: Let $y = x^r$,

$$2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1), \quad r_1 = \frac{1}{2}, \quad r_2 = -1.$$
$$y_1 = x^{\frac{1}{2}}, \quad y_2 = x^{-1}.$$

3.2 Real and equal roots

Suppose $r_1 = r_2$ are real. We have a solution

$$y_1 = x^{r_1}.$$

The second solution can be found by the method of reduction of order (Exercise). Here we use another method. Let

$$L[y] = x^2 y''(x) + \alpha x y'(x) + \beta y(x).$$

Then

$$L[x^r] = [r(r-1) + \alpha r + \beta] x^r = (r - r_1)^2 x^r.$$

So

$$\frac{\partial}{\partial r} L[x^r] = 2(r - r_1) x^r + (r - r_1) r x^{r-1} \quad \Rightarrow \quad \frac{\partial}{\partial r} L[x^r](r_1) = 0.$$

Then under appropriate assumption,

$$\frac{\partial}{\partial r} L[x^r] = L\left[\frac{\partial x^r}{\partial r}\right] = L[x^r \ln x](r_1) = 0.$$

So

$$y_2 = x^r \ln x$$

is also a solution. Then

$$W[y_1, y_2] = \begin{vmatrix} x^r & x^r \ln x \\ r x^{r-1} & r x^{r-1} \ln x + x^{r-1} \end{vmatrix} = r x^{2r-1} \ln x + x^{2r-1} - r x^{2r-1} \ln x = x^{2r-1} \neq 0.$$

So the general solution is

$$y = c_1 x^r + c_2 x^r \ln x.$$

As $x \to 0^+$,

- If $r > 0$, then $x^r \to 0$ and $x^r \ln x \to 0$.
- If $r < 0$, then $x^r \to \infty$ and $x^r \ln x = -\infty$.
- If $r = 0$, then $x^r \equiv 1$ and $x^r \ln x \to -\infty$.

Example 3.3. Solve

$$x^2 y'' + 5 x y' + 4 y = 0, \quad x > 0.$$

Answer:

$$r(r-1) + 5r + 4 = r^2 + 4r + 4, \quad r_{1,2} = -2$$
$$y_1 = x^{-2}, \quad y_2 = x^{-2} \ln x$$

3.3 Complex roots

Suppose the roots are

$$r_{1,2} = \lambda \pm i\mu.$$

Then

$$x^r = e^{r \ln x}.$$

$$x^{\lambda + i\mu} = e^{(\lambda + i\mu)\ln x} = e^{\lambda \ln x + i\mu \ln x} = e^{\lambda \ln x}[\cos \mu \ln x + i \sin \mu \ln x] = x^{\lambda}[\cos \mu \ln x + i \sin \mu \ln x].$$

So we have two complex valued solutions

$$x^{\lambda + i\mu}, \quad x^{\lambda - i\mu}.$$

and they are linearly independent. Taking the real and imaginary parts, we obtain two real valued solutions

$$y_1 = x^\lambda \cos \mu \ln x, \quad y_2 = x^\lambda \sin \mu \ln x,$$

and they are also linearly independent. So the general solution is

$$y = x^\lambda [c_1 \cos \mu \ln x + c_2 \sin \mu \ln x].$$

As $x \to 0^+$,

- If $\lambda > 0$, then $y \to 0$ and oscillates increasingly fast.
- If $\lambda < 0$, then $y \to \infty$ and oscillates increasingly fast.
- If $\lambda = 0$, then y oscillates increasingly fast.

Example 3.4. Solve

$$x^2 y'' + x y' + y = 0.$$

Answer:

$$r(r-1) + r + 1 = r^2 + 1, \quad r_{1,2} = \pm i.$$

$$y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x)$$

3.4 Extend to $x < 0$

Let $\xi = -x$, $\tilde{y}(\xi) = y(x)$

$$\frac{dy}{dx} = \frac{d\tilde{y}}{d\xi}\frac{d\xi}{dx} = -\frac{d\tilde{y}}{d\xi}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{d\tilde{y}}{d\xi}\right) = \frac{d}{d\xi}\left(-\frac{d\tilde{y}}{d\xi}\right)\frac{d\xi}{dx} = \frac{d^2\tilde{y}}{d\xi^2}$$

$$x^2 y''(x) + \alpha x y'(x) + \beta y(x) = \xi^2 \tilde{y}''(\xi) + \alpha \xi \tilde{y}'(\xi) + \beta \tilde{y}(\xi) = 0.$$

So the equation for $\tilde{y}(\xi)$ is the same as the equation for $y(x)$. So the solutions are the same. For example, if r_1, r_2 are real and distinct, then

$$\tilde{y}_1(\xi) = \xi^{r_1} = (-x)^{r_1} = y_1(x).$$

Similarly, we obtain solutions as before, with x replaced by $-x$.
In the domain $(-\infty, 0) \cup (0, \infty)$, the solution can be written as before, with x replaced by $|x|$.

3.5 Extend to $x - x_0$

$$(x - x_0)^2 y'' + \alpha (x - x_0) y' + \beta y = 0.$$

Using the change of variable $\xi = x - x_0$, we obtain the same solutions, with x replaced by $x - x_0$.

3.6 Regular singular points

Definition 3.5 **(Regular Singular Point)** *Consider the equation*

$$y'' + p(x) y' + q(x) y = 0.$$

*A point x_0 is called a **regular singular point** if x_0 is a singular point, and*

$$(x - x_0) p(x), \quad (x - x_0)^2 q(x)$$

*are analytic at x_0. A singular point is called **irregular** if it is not regular.*

Example 3.6. Determine the singular points of the Legendre equation

$$(1-x^2)\,y''-2xy'+\alpha(\alpha+1)\,y=0$$

and determine whether they are regular or irregular.

Answer:

$$y''-\frac{2x}{1-x^2}\,y'+\frac{\alpha(\alpha+1)}{1-x^2}\,y=0.$$

So $x_0=\pm1$ are singular points.

1. If $x_0=1$, then

$$(x-x_0)\,p(x)=-(x-1)\frac{2x}{1-x^2}=\frac{2x}{1+x},\quad (x-x_0)^2\,q(x)=\alpha(\alpha+1)\frac{1-x}{1+x}$$

   are both analytic at x_0. So 1 is a regular singular point.
2. Similarly, we verify -1 is a regular singular point.

Example 3.7. Determine the singular points of the differential equation

$$2x(x-2)^2y''+3xy'+(x-2)\,y=0$$

and classify them as regular or irregular.

Answer:

$$y''+\frac{3}{2(x-2)^2}\,y'+\frac{1}{2x(x-2)}\,y=0.$$

So the singular points are $0,2$.

1. If $x_0=0$, then

$$(x-x_0)\,p(x)=\frac{3x}{2(x-2)^2},\quad (x-x_0)^2\,q(x)=\frac{x}{2(x-2)}$$

   are both analytic at 0. So 0 is a regular singular point.
2. If $x_0=2$, then

$$(x-x_0)\,p(x)=\frac{3}{2(x-2)},\quad (x-x_0)^2\,q(x)=\frac{x-2}{2x}.$$

   So $(x-x_0)\,p(x)$ is nonanalytic at 2. So 2 is an irregular singular point.

Example 3.8. Determine the singular points of the differential equation

$$x^2y''+(\sin x)\,y'+2\,y=0$$

and classify them as regular or irregular.

Answer:

$$y''+\frac{\sin x}{x^2}\,y'+\frac{2}{x^2}\,y=0.$$

So $x_0 = 0$ is the singular point.

$$(x - x_0)\, p(x) = \frac{\sin x}{x}, \quad (x - x_0)^2 q(x) = 2$$

are both analytic at x_0 (use $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ as the function value). So 0 is a regular singular point.

4 Series Solutions Near a Regular Singular Point

WLOG, assume $x_0 = 0$ is a regular singular point of the equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Multiplying the equation by x^2 to obtain

$$\begin{aligned} & x^2 y'' + x^2 p(x)\, y' + x^2 q(x)\, y = 0 \\ \Rightarrow \quad & x^2 y'' + x\,[x\,p(x)]\,y' + [x^2 q(x)]\, y \end{aligned}$$

Consider the ansatz of the solution

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

The property of y as $x \to 0$ is determined by r.

Example 4.1. Solve the differential equation

$$2x^2 y'' - x\,y' + (1 + x)\, y = 0.$$

Answer: One can verify $x = 0$ is a regular singular point. Seek solution in the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0.$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}.$$

Plugging into the equation,

$$2x^2 \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} + (1 + x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Rearranging terms,

$$\sum_{n=0}^{\infty} 2\,(n + r)\,(n + r - 1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

$$\sum_{n=0}^{\infty} 2\,(n + r)\,(n + r - 1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0.$$

$$\sum_{n=0}^{\infty} 2\,(n + r)\,(n + r - 1)\, a_n x^n - \sum_{n=0}^{\infty} (n + r)\, a_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

For $n=0$, we have

$$[2r(r-1)-r+1]\,a_0=0 \quad \Rightarrow \quad 2r(r-1)-r+1=0$$

which is the same equation as derived for the corresponding Euler's equation.

$$2r^2-3r+1=0 \quad \Rightarrow \quad r_1=1, \quad r_2=\frac{1}{2}.$$

For $n \geqslant 1$, we have

$$[2(n+r)(n+r-1)-(n+r)+1]\,a_n+a_{n-1}=0$$

$$\begin{aligned}\Rightarrow \quad a_n &= -\frac{1}{2(n+r)(n+r-1)-(n+r)+1}\,a_{n-1} \\ &= -\frac{1}{[2(n+r)-1][n+r-1]}\,a_{n-1}\end{aligned}$$

$\boxed{\text{For } r_1=1}$, we have

$$a_n=-\frac{1}{(2n+1)n}\,a_{n-1},$$

$$a_1=-\frac{1}{3\cdot 1}\,a_0, \quad a_2=-\frac{1}{5\cdot 2}\,a_1=\frac{1}{(5\cdot 2)\cdot(3\cdot 1)}\,a_0, \quad a_3=-\frac{1}{(7\cdot 3)\cdot(5\cdot 2)\cdot(3\cdot 1)}\,a_0, \ldots$$

In general,

$$a_n=\frac{(-1)^n}{n!\,(3\cdot 5\cdot 7\cdots(2n+1))}\,a_0=\frac{(-1)^n(2\cdot 4\cdot 6\cdots 2n)}{n!\,(2n+1)!}\,a_0=\frac{(-1)^n 2^n n!}{n!\,(2n+1)!}\,a_0=\boxed{\frac{(-1)^n 2^n}{(2n+1)!}\,a_0}$$

So the first solution is

$$y_1=x\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(2n+1)!}x^n=x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{(2n+1)!}x^n\right]$$

By the ratio test, y_1 has a radius of convergence of ∞.

$\boxed{\text{For } r_2=\frac{1}{2}}$, we have a second solution

$$y_2=x^{\frac{1}{2}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{(2n)!}x^n\right].$$

Now let us consider the general problem of determining a solution of the equation

$$L[y]=x^2y''+x[xp(x)]\,y'+[x^2q(x)]\,y=0,$$

Since $xp(x)$ and $x^2q(x)$ are analytic at 0, they are represented by their Taylor series near 0:

$$xp(x)=\sum_{n=0}^{\infty}p_nx^n, \quad x^2q(x)=\sum_{n=0}^{\infty}q_nx^n$$

Seek solutions of the form

$$y = \phi(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad a_0 = 1.$$

Remark 4.2. The essential property of $\phi(r,x)$ as $x \to 0$ is determined by r.

Now

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}$$

Hence

$$\begin{aligned}
L[\phi](r,x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} \\
&+ \left[\sum_{n=0}^{\infty} (r+n) a_n x^{r+n}\right]\left[\sum_{n=0}^{\infty} p_n x^n\right] + \left[\sum_{n=0}^{\infty} a_n x^{r+n}\right]\left[\sum_{n=0}^{\infty} q_n x^n\right] \\
x^{-r} L[\phi](r,x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^n \\
&+ \left[\sum_{n=0}^{\infty} (r+n) a_n x^n\right]\left[\sum_{n=0}^{\infty} p_n x^n\right] + \left[\sum_{n=0}^{\infty} a_n x^n\right]\left[\sum_{n=0}^{\infty} q_n x^n\right] \\
&= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^n \\
&+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} (r+k) a_k p_{n-k} x^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k q_{n-k} x^n \\
&= \sum_{n=0}^{\infty} \left\{ (r+n)(r+n-1) a_n + \sum_{k=0}^{n} [(r+k) p_{n-k} + q_{n-k}] a_k \right\} x^n \\
&= [r(r-1) + r p_0 + q_0] \\
&+ \sum_{n=1}^{\infty} \left\{ (r+n)(r+n-1) a_n + \sum_{k=0}^{n} [(r+k) p_{n-k} + q_{n-k}] a_k \right\} x^n \\
&= [r(r-1) + r p_0 + q_0] \\
&+ \sum_{n=1}^{\infty} \left\{ [(r+n)(r+n-1) + (r+n) p_0 + q_0] a_n + \sum_{k=0}^{n-1} [(r+k) p_{n-k} + q_{n-k}] a_k \right\} x^n \\
&= F(r) + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} [(r+k) p_{n-k} + q_{n-k}] a_k \right\} x^n
\end{aligned}$$

where

$$\boxed{F(r) = r(r-1) + r p_0 + q_0}$$

Setting coefficients to zero,

$$F(r)=0,$$
$$F(r+n)a_n+\sum_{k=0}^{n-1}[(r+k)p_{n-k}+q_{n-k}]a_k=0.$$

From $F(r)=0$, we obtain roots r_1, r_2. Subsituting $r_{1,2}$ into the 2nd equation, we may obtain recurrence relations for a_n. We need $F(n+r)\neq 0$.

4.1 Real distinct roots, $r_1>r_2$ and $r_1-r_2\notin\mathbb{N}$

In this case, $F(n+r_1)\neq 0$ for $n\geqslant 1$. So at least we have the solution

$$y_1(x)=|x|^{r_1}\left[1+\sum_{n=1}^{\infty}a_n(r_1)x^n\right].$$

Since $r_1-r_2\notin\mathbb{N}$, r_2+n can not be a root of F. So $F(r_2+n)\neq 0$ for any $n\geqslant 1$. So we have a second solution

$$y_2(x)=|x|^{r_2}\left[1+\sum_{n=1}^{\infty}a_n(r_2)x^n\right].$$

The general solution is

$$y=c_1y_1+c_2y_2.$$

Example 4.3. Discuss the nature of the solutions of the equation

$$2x(1+x)y''+(3+x)y'-xy=0$$

near the singular points.

Answer: Rewrite the equation as

$$y''+\frac{3+x}{2x(1+x)}y'-\frac{1}{2(1+x)}y=0.$$

The singular points are $0,-1$.
At 0,

$$xp(x)=\frac{3+x}{2(1+x)},\quad x^2q(x)=-\frac{x^2}{2(1+x)}$$

are analytic at 0. So 0 is a regular singular point.
Seek solution of the form

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r}.$$

Plugging into the equation,

$$2x(1+x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+(3+x)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-x\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}$$
$$+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}=0.$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^n$$
$$+ \sum_{n=0}^{\infty} 3(n+r)a_n x^{n-1} + \sum_{n=0}^{\infty} (n+r)a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

$$\Rightarrow \quad \sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1}x^n + \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^n$$
$$+ \sum_{n=-1}^{\infty} 3(n+r+1)a_{n+1}x^n + \sum_{n=0}^{\infty} (n+r)a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

For $n=-1$,

$$[2r(r-1)+3r]\,a_0 = 0 \quad \Rightarrow \quad 2r^2+r=0 \quad \Rightarrow \quad r_1=0, \quad r_2=-\frac{1}{2}.$$

So we have two solutions

$$y_1 = 1+\sum_{n=0}^{\infty} a_n(r_1)x^n, \quad y_2 = |x|^{-\frac{1}{2}}\left[1+\sum_{n=0}^{\infty} a_n(r_2)x^n\right].$$

So $y_1 \to 1$ and $y_2 \to \infty$ as $x \to 0$

Method 2:

$$F(r) = r(r-1)+p_0 r + q_0.$$

Here

$$p_0 = \lim_{x\to 0} \frac{3+x}{2(1+x)} = \frac{3}{2}, \quad q_0 = \lim_{x\to 0} -\frac{x^2}{2(1+x)} = 0.$$

So

$$F(r) = r(r-1)+\frac{3}{2}r = r^2+\frac{1}{2}r = 0 \quad \Rightarrow \quad r_1=0, \quad r_2=-\frac{1}{2}.$$

4.2 Real equal roots, $r_1 = r_2$

In this case $F(r_1+n) \neq 0$ for any $n \geqslant 1$. So we have a first solution

$$y_1(x) = |x|^{r_1}\left[1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right].$$

Consider a_n as a function of r. For the selection of the coefficients to satisfy the ODE, we have

$$L[\phi](r,x) = x^r F(r).$$

Then

$$\frac{\partial}{\partial r}L[\phi](r,x) = x^r(\ln x)F(r) + x^r F'(r) = 0 \quad \text{if} \quad r=r_1$$

since $F(r) = a_0(r-r_1)^2$. Switching the order of partial derivatives,

$$L\left[\frac{\partial \phi}{\partial r}\right](r_1,x) = 0.$$

So

$$\begin{aligned}
y_2 &= \frac{\partial \phi}{\partial r}(r_1,x) = \frac{\partial y_1}{\partial r}(r_1,x)\\
&= |x|^{r_1}\ln|x|\left[1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right] + |x|^{r_1}\left[\sum_{n=1}^{\infty} a_n'(r_1)x^n\right]\\
&= y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n, \quad b_n(r_1) = a_n'(r_1)
\end{aligned}$$

is also a solution of the ODE.

4.3 Real distinct roots, $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{N}$

$$y_1(x) = |x|^{r_1}\left[1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right], \quad y_2(x) = a\,y_1(x)\ln|x| + |x|^{r_2}\left[1+\sum_{n=1}^{\infty} c_n(r_2)x^n\right]$$

5 Bessel's Equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

$$y'' + \frac{1}{x}y' + \frac{x^2-\nu^2}{x^2}y = 0.$$

So $x=0$ is a singular point.

$$xp(x) = 1, \quad x^2q(x) = x^2 - \nu^2$$

are both analytic at 0. So $x=0$ is a regular singular point.

$$p_0 = 1, \quad q_0 = -\nu^2.$$

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \nu^2 = r^2 - \nu^2$$

$$r_{1,2} = \pm\nu.$$

5.1 $\nu = 0$

$$y_1 = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

$$y_1' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \quad y_1'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

$$\begin{aligned}
& x^2y_1'' + xy_1' + x^2y_1\\
={}& x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x^2\sum_{n=0}^{\infty}a_n x^{n+r}\\
={}& \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2}\\
={}& [r(r-1)a_0 + (r+1)ra_1x + ra_0 + (r+1)a_1x]x^r\\
+{}& \sum_{n=2}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r}\\
={}& [r(r-1)+r]a_0x^r + [(r+1)r + (r+1)]a_1x^{1+r}\\
+{}& \sum_{n=2}^{\infty}\{[(n+r)(n+r-1)+(n+r)]a_n + a_{n-2}\}x^{n+r}.
\end{aligned}$$

So

$$\begin{aligned} [r(r-1)+r]\,a_0 &= 0, \\ [(r+1)r+(r+1)]a_1 &= 0, \\ \{[(n+r)(n+r-1)+(n+r)]a_n+a_{n-2}\} &= 0. \end{aligned}$$

For $r=0$, we obtain

$$a_0 \text{ is arbitrary}, \quad a_1=0, \quad a_n=-\frac{1}{n^2}\,a_{n-2}.$$

$$\Rightarrow \quad a_{2m+1}=0$$

$$a_2=-\frac{1}{2^2}\,a_0, \quad a_4=\frac{1}{4^2}\frac{1}{2^2}\,a_0, \quad a_6=-\frac{1}{6^2\cdot 4^2\cdot 2^2}\,a_0, \quad a_{2m}=\frac{(-1)^m}{(2^m m!)^2}\,a_0.$$

So the first solution is

$$y_1=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2^m m!)^2}x^{2m}=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{(2^m m!)^2}x^{2m}:=J_0(x)$$

which is called **Bessel function of order zero of the first kind.**

The second solution, according to the general result, is given by

$$y_2=y_1(x)\ln x+x^{r_1}\sum_{n=1}^{\infty}b_n(r_1)\,x^n, \quad b_n=a_n'(r_1).$$

To find $b_n(r_1)$.

$$a_1'(r_1)=0.$$

$$[(n+r)(n+r-1)+(n+r)]a_n+a_{n-2}=0 \quad \Rightarrow \quad a_n(r)=-\frac{1}{(n+r)^2}\,a_{n-2}(r)$$

$$a_3(r)=-\frac{1}{(3+r)^2}\,a_1(r) \quad \Rightarrow \quad a_3'(r_1)=0 \quad \Rightarrow \cdots \Rightarrow \quad a_{2m+1}'(r_1)=0.$$

$$a_2(r)=-\frac{1}{(2+r)^2}, \quad a_4(r)=\frac{1}{(4+r)^2\cdot(2+r)^2}, \quad \cdots \quad a_{2m}(r)=\frac{1}{(2m+r)^2\cdots(2+r)^2}$$

$$\begin{aligned} a_{2m}(r) &= \prod_{k=1}^{m}(2k+r)^{-2} \\ a_{2m}'(r) &= -2\sum_{k=1}^{m}\left[(2k+r)^{-3}\prod_{\substack{j=1\\ j\neq k}}^{m}(2j+r)^{-2}\right] \\ &= -2\sum_{k=1}^{m}\left[(2k+r)^{-1}\prod_{j=1}^{m}(2j+r)^{-2}\right]=-2a_{2m}(r)\sum_{k=1}^{m}(2k+r)^{-1} \\ a_{2m}'(r_1) &= -2a_{2m}(r_1)\sum_{k=1}^{m}(2k)^{-1}=-H_m a_{2m}(r_1)=\frac{(-1)^{m+1}H_m}{(2^m m!)^2}\,a_0. \end{aligned}$$

where

$$H_m=\sum_{k=1}^{m}\frac{1}{k}.$$

Hence the 2nd solution is

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{(2^m m!)^2} x^{2m}.$$

Bessel function of order zero of the second kind is defined as (other definitions exist)

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)\, J_0(x)\right],$$

where

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) \cong 0.5772$$

is known as the Euler-Máscheroni constant.