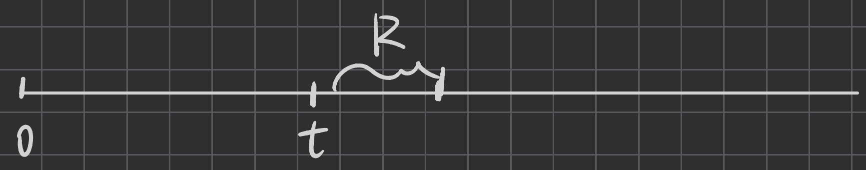
4. (15 marks) Let $\{N(t)|t \geq 0\}$ be a Poisson process with rate λ. At a fixed time $t > 0$, let R be the duration until the next arrival. This is called the "residual time" because it is only part of the interarrival time that t falls into. What's the distribution of R? Interpret the result using your own words.



$$N(t+r) - N(t) = 0$$

$$P(R>r) = e^{-\lambda r} \cdot \frac{(\lambda r)^0}{0!} = e^{-\lambda r}$$

$$P(R \leq r) = 1 - e^{-\lambda r}$$

$$f_R(r) = \lambda e^{-\lambda r}$$

$P\{N(s)=a,\ N(t)=b\} = P\{N(s)=a,\ N(t)-N(s)=b-a\} = P\{N(s)=a,\ N(t-s)=b-a\}$

$$= e^{-\lambda s}\cdot\frac{(\lambda s)^a}{a!}\cdot e^{-\lambda(t-s)}\cdot\frac{(\lambda(t-s))^{b-a}}{(b-a)!}$$

$$= e^{-\lambda t}\,\frac{\lambda^b s^a (t-s)^{b-a}}{a!\,(b-a)!}$$

(b) $P\{S_a<s \mid N(t)=a\} = \dfrac{P\{S_a<s\}\cap P\{N(t)=a\}}{P\{N(t)=a\}} = \dfrac{P\{N(s)=a\}\cdot P\{N(t)-N(s)=0\}}{P\{N(t)=a\}}.$

a events

$Y(t+1) = B(t+1)^2-(t+1)$

$Y(t) = B(t)^2 - t$

$E(B(t)^2 \mid \mathcal{F}_s)$

$=$

$E[W_n] < \infty$

W_n is a function of $X_1, \dots X_n$

$E[W_{n+1} \mid W_1, W_2 \dots W_n] = W_n$

$= E[$

$W_n + B_{i+1}X_{i+1}$

if $X_i=-1$

$B_{i+1}=2B_i$

$2^n - (2^n - 1)$

$$E\left(\left(\frac{q}{p}\right)^{S_{n+1}} \middle| \left(\frac{q}{p}\right)^{S_1} \dots \left(\frac{q}{p}\right)^{S_n}\right) = \left(\frac{q}{p}\right)^{S_n}$$

$$\left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] = \left(\frac{q}{p}\right)\cdot \frac{q}{p}\cdot p + q\cdot\frac{p}{q}$$

$$= \left(\frac{q}{p}\right)^{S_n}$$