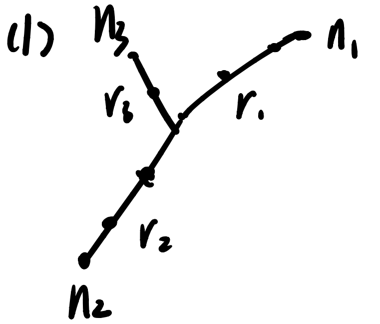
(1)



let L denote the first leaf visited
R denote the first Ray

$$P\{L=i\} = \sum_{j=1}^{3} p_j \cdot P\{L=i \mid R=j\}$$

if the particle moves from node 0 to ray i. then the probability that it will reach the leaf before leaving the ray is $\frac{1}{n_i}$

Consider whether the particle will reach a leaf or start over

$$P\{L=i \mid R=i\} = \frac{1}{n_i} + \left(1-\frac{1}{n_i}\right) P\{L=i\}$$

$$P\{L=i \mid R=j\} = \left(1-\frac{1}{n_j}\right) P\{L=i\} \quad i \neq j$$

so

$$P\{L=i\} = p_i \cdot P\{L=i \mid R=i\} + \sum_{j \neq i} p_j \, P\{L=i \mid R=j\}$$

$$= p_i \left[\frac{1}{n_i} + \left(1-\frac{1}{n_i}\right) P\{L=i\}\right]$$

$$+ \sum_{i \neq j} p_j \left[\left(1-\frac{1}{n_j}\right) P\{L=i\}\right]$$

$$P\{L=i\} \left\{1 - \left[p_i\left(1-\frac{1}{n_i}\right) + \sum_{j \neq i} p \left(1-\frac{1}{n_j}\right)\right]\right\} = \frac{p_i}{n_i}$$

so $$P\{L=i\} = \frac{\frac{p_i}{n_i}}{\sum_{j=1}^{3} \frac{p_j}{n_j}}$$

(2) $Z = X + Y$, $Y = Z - X$

$$P(Z=k) = P(X+Y=k) = \sum_{i=0}^{k} P(X=i)P(Y=k-i)$$

$$= \sum_{i=0}^{k} \left(e^{-\lambda_1} \frac{\lambda_1^i}{i!}\right) \left(e^{-\lambda_2} \cdot \frac{\lambda_2^{k-i}}{(k-i)!}\right)$$

$$= \sum_{i=0}^{k} e^{-(\lambda_1+\lambda_2)} \cdot \frac{\lambda_1^i \cdot \lambda_2^{k-i}}{i!\,(k-i)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \sum_{i=0}^{k} \cdot \frac{\lambda_1^i \cdot \lambda_2^{k-i}}{i!\,(k-i)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{k!} \cdot \sum_{i=0}^{k} \frac{k!}{i\,(k-i)!} \cdot \lambda_1^i \cdot \lambda_2^{k-i}$$

by binomial theorem. we get that

$$P(Z=k) = e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^k}{k!}$$

So $Z \sim \text{Possion}(\lambda_1 + \lambda_2)$, which indicate that two independent possion distributions' sum is also possion

(3) let X denote the number of customers entering the store,

Y denote the number of customers buy the stuff.

is a binomial distributition.

$$P(Y=k) = \sum_{n=k}^{\infty} P(Y=k \mid X=n) \cdot P(X=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$= p^k \cdot e^{-\lambda} \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} (1-p)^{n-k} \cdot \frac{\lambda^n}{\cancel{n!}}$$

$$= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{[(1-p)\lambda]^{n-k}}{(n-k)!}.$$

let $m = n-k$,

$$\sum_{m=0}^{\infty} \frac{[(1-p)\lambda]^m}{m!} = e^{(1-p)\lambda}$$

So $P(S=k) = e^{-p\lambda} \frac{(p\lambda)^k}{k!}$, therefore $S \sim \text{Possion}(p\lambda)$

which indicate that the joint distributition of possion and bionomial is still possion.

(4) The possion process has interval times $\{T_i\}$, which are i.i.d $\text{Exp}(\lambda)$, fix $r>0$ $P(R>r \mid N(t)=k)$, k is the number of arrivals by time t, let S_k be the k-th arrival

then $\{R>r\}$ is equal to no arrival in $(t, t+r]$

by memoryless property, we get $P(R>r \mid S_k = s) = P(T>r) = e^{-\lambda r}$

which holds for all $s \le t$, so $P(R>r) = e^{-\lambda r}$ $r \ge 0$.

$F_R(r) = P(R \le r) = 1 - e^{-\lambda r}$, which is the CDF of $Exp(\lambda)$

so $f_R(r) = \lambda e^{-\lambda r}$ $r \ge 0$.

which again implies the memoryless properties of Possion distribution.

(5) (a) $P\{N(s)=a, N(t)=b\} = P\{N(s)=a\} \cdot P\{N(t)-N(s)=b-a\}$

$$= e^{-\lambda s} \cdot \frac{(\lambda s)^a}{a!} \cdot e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^{b-a}}{(b-a)!}$$

$$= e^{-\lambda t} \cdot \lambda^b \frac{s^a (t-s)^{b-a}}{a!(b-a)!}$$

(b) $P\{S_a < s \mid N(t) = a\} = \frac{P\{S_a < s\} \cap P\{N(t)=a\}}{P\{N(t)=a\}}$.

if $N(s) = a$, then $S_1, S_2, \cdots S_a < S$.

$$P\{N(t)=a\} = e^{-\lambda t} \cdot \frac{(\lambda t)^a}{a!}$$

$P\{N(s)=a, N(t)=a\} = P\{N(s)=a\} \cdot P\{N(t)-N(s)=0\}$

$$= e^{-\lambda s} \frac{(\lambda s)^a}{a!} \cdot e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^0}{0!}$$

$$= e^{-\lambda t} \cdot \frac{(\lambda s)^a}{a!}$$

so $P\{S_a < s \mid N(t) = a) = \left(\frac{s}{t}\right)^a$

(6) (a) in possion process, the arrival time of n-th events

follows a Gamma distribution

So $E[S_4] = \frac{n}{\lambda} = \frac{4}{\lambda}$

(b) Given $N(1)=2$, So 2 events has occured by time $t=1$

So the remaing 2 events happens after $t=1$

since the possion process is memoryless, So

$$E[\text{Remaining 2 events}] = \frac{n}{\lambda} = \frac{2}{\lambda}$$

So $E[S_4 \mid N(1)=2] = 1 + E(R) = 1 + \frac{2}{\lambda}$