# Ordinary Differential Equations

BY YULIANG WANG

## Chapter 2: First Order Differential Equations

General form

$$F(u, u') = 0$$

**Example 1. (falling object in air, general solution, integral curves, initial value problem)**

The motion of the object is governed by the Newton's law. Let $v(t)$ be the velocity of the object at time $t$.

$$mv' = mg - \gamma v,$$

where $m$ is the mass of the object, $g$ is the gravitational constant, and $\gamma$ is the coefficient of air resistant force.

This is a first order linear ODE.

# 1 Method of integrating factors

**Example 2.** Find the general solutions of

$$u' + u = 2$$

idea: combine $u' + u$ into the derivative of another function.

Consider the product rule of differentiation:

$$(fg)' = f'g + fg'.$$

Let $f = u(t), g = e^t$, then

$$[u(t)e^t]' = u'e^t + ue^t = e^t(u' + u).$$

Now, we multiply the original equation by $e^t$:

$$(u' + u)e^t = 2e^t \Longrightarrow [u(t)e^t]' = 2e^t \Longrightarrow u(t)e^t = \int 2e^t \mathrm{d}t = 2e^t + c$$

Divide by $e^t$:

$$u(t) = e^{-t}[2e^t + c] = 2 + ce^{-t}.$$

**Method of integrating factors:**

Consider the 1st order linear ODE (standard form):

$$u'(t) + p(t)u(t) = q(t).$$

Multiply left side by $\mu(t)$:

$$[u'(t) + p(t)u(t)]\mu(t) = u'\mu + pu\mu.$$

We want

$$\mu = g, p\mu = g'$$

i.e.

$$\mu'(t) = p(t)\mu(t) \Longrightarrow \mu(t) = e^{\int p(t)\mathrm{d}t}.$$

Check (by the chain rule and fundamental theorem of calculus)

$$\mu' = e^{\int p(t)\mathrm{d}t}p(t) = \mu(t)p(t).$$

Derivation directly:

$$\mu'(t) = p(t)\mu(t) \Longrightarrow \frac{\mu'(t)}{\mu(t)} = p(t) \Longrightarrow [\ln \mu(t)]' = p(t)$$

$$\Longrightarrow \ln \mu(t) = \int p(t)\mathrm{d}t \Longrightarrow \mu(t) = e^{\int p(t)\mathrm{d}t.}$$

So the original ODE becomes

$$[\mu(t)u(t)]' = \mu(t)q(t) \Longrightarrow \mu(t)u(t) = \int \mu(t)q(t)\mathrm{d}t + c$$

$$\Longrightarrow u(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\mathrm{d}t + c\right],$$

where

$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

is called the **integrating factor**.

**Example 3.** Find the **general solution** of

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

**Answer:** Rewrite it in the standard form

$$y' + \frac{2t}{4+t^2}y = \frac{4t}{4+t^2}.$$

Here $p(t) = \frac{2t}{4+t^2}$. Then find the integrating factor $\mu(t)$:

$$\mu(t) = e^{\int p(t)\mathrm{d}t} = e^{\ln(4+t^2)} = 4+t^2.$$

So the solution is

$$y(t) = \frac{1}{4+t^2}\left(\int (4+t^2)\frac{4t}{4+t^2}\mathrm{d}t + c\right) = \frac{1}{4+t^2}(2t^2+c)$$

**Example 4.** Solve the **initial value problem**

$$ty' + 2y = 4t^2$$
$$\text{initial condition: } y(1) = 2$$

**Answer:** First rewrite the equation into the standard form:

$$y' + \frac{2}{t}y = 4t.$$

Find the integrating factor:

$$\mu = e^{\int \frac{2}{t}\mathrm{d}t} = e^{2\ln|t|} = e^{\ln t^2} = t^2.$$

The general solution is

$$y = \frac{1}{t^2}\left[\int t^2 4t\mathrm{d}t + c\right] = \frac{1}{t^2}[t^4 + c] = t^2 + \frac{c}{t^2}.$$

Plugging the initial condition:

$$y(1)=2 \Longrightarrow 1+\frac{1}{c}=2 \Longrightarrow c=1.$$

The solution of the initial value problem is

$$y=t^2+\frac{1}{t^2}.$$

# 2 Separable equations

**Example 5.** Solve

$$\frac{dy}{dx}=\frac{x^2}{1-y^2}.$$

This equation is nonlinear. We can separate the variables $x$ and $y$ as follows

$$(1-y^2)\mathrm{d}y=x^2\mathrm{d}x \Longrightarrow \int (1-y^2)\mathrm{d}y=\int x^2\mathrm{d}x$$

$$\Longrightarrow y-\frac{y^3}{3}=\frac{x^3}{3}+c$$

This is an example of **implicit solutions**.

**Definition 6**

*An ODE in the form of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

*is called* ***separable.***

We can solve it as follows

$$\int f(x)\mathrm{d}x = \int g(y)\mathrm{d}y$$

**Example 7.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$

**Solution:**

$$\int (3x^2+4x+2)\mathrm{d}x = \int 2(y-1)\mathrm{d}y$$

$$\Rightarrow \quad x^3+2x^2+2x = y^2-2y+c.$$

Plugging the initial condition

$$y(0) = -1 \Rightarrow 0 = 1 + 2 + c \Rightarrow c = -3.$$

The solution to the initial value problem is

$$x^3 + 2x^2 + 2x = y^2 - 2y - 3.$$

**Question 1.** What is the domain and range of the solution?

**Example 8.** Recall the differential equation for continuous compound interests:

$$u' = ru.$$

Note this equation is both linear and separable. As a separable equation, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru \Rightarrow \frac{\mathrm{d}u}{u} = r\mathrm{d}t \Rightarrow \int \frac{\mathrm{d}u}{u} = \int r\mathrm{d}t \Rightarrow \ln|u| = rt + c$$

$$\Rightarrow |u| = ce^{rt} \Rightarrow u = ce^{rt}.$$

If the initial condition is $u(0) = u_0$, then we find $c = u_0$.

**Exercise 1.** Solve the equation by the method of integrating factor.

# 3 Exact Equations

## 3.1 Motivation and definition

Suppose $\psi(x, y) = c$ is an solution of some ODE. Taking $d / \mathrm{d}x$ on both sides of the solution.

$$\frac{d}{\mathrm{d}x}\psi(x, y) = \frac{d}{\mathrm{d}x}c \Rightarrow \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow M(x, y) + N(x, y)y' = 0,$$

where

$$M(x, y) = \partial_x\psi, \quad N(x, y) = \partial_y\psi.$$

**Example 9.** Solve $2x + y^2 + 2xyy' = 0$.

**Answer.** Guess the solution. Let $\psi = x^2 + y^2 x$. Then

$$\psi_x = 2x + y^2, \quad \psi_y = 2xy.$$

So

$$0 = \psi_x + \psi_y y' = \frac{d}{\mathrm{d}x}\psi(x, y)$$

So the solution is

$$\psi(x, y) = c.$$

**Definition**

*An ODE of the form*

$$M(x, y) + N(x, y)y' = 0 \ \text{ or } \ M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

*is called **exact** if there exists $\psi(x, y)$ such that*

$$\psi_x = M, \quad \psi_y = N.$$

*The solution of the equation is*

$$\psi(x, y) = c,$$

*where $c$ is an arbitrary constant.*

## 3.2 Theorem and method

**Theorem 10**

*Suppose an ODE can be written in the form*

$$M(x,y)+N(x,y)y'=0 \ \ \textit{or} \ \ M(x,y)\mathrm{d}x+N(x,y)\mathrm{d}y=0 \tag{1}$$

*where the functions* $M, N, M_y$ *and* $N_x$ *are all continuous in the rectangular region* $R=[a, b]\times[c,d]$. *Then Eq. (1) is an exact differential equation* ***if and only if***

$$M_y(x,y)=N_x(x,y), \forall(x,y)\in R.$$

**Proof.** $''\Longrightarrow''$. Suppose Eq. (1) is exact. Then there exists a $\psi(x,y)$ such that

$$\psi_x=M, \quad \psi_y=N.$$

Then

$$M_y=\psi_{xy}, \quad N_x=\psi_{yx}.$$

Since $M_y, N_x$ are continuous, we have $\psi_{xy}$ and $\psi_{yx}$ are continuous. So

$$\psi_{xy}=\psi_{yx}.$$

i.e.

$$M_y=N_x.$$

"$\Longleftarrow$" Suppose $M_y = N_x$. We want to find a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. Let

$$\psi = \int M(x, y)\mathrm{d}x + h(y).$$

Then $\psi_x = M$, and

$$\psi_y = \partial_y \int M(x, y)\mathrm{d}x + h'(y).$$

We want $\psi_y = N$, that is

$$h'(y) = N(x, y) - \partial_y \int M(x, y)\mathrm{d}x.$$

We need the RHS to be independent of $x$. That is

$$\frac{\partial}{\partial x}\left[N(x, y) - \partial_y \int M(x, y)\mathrm{d}x\right] = 0.$$

Let's check:

$$\frac{\partial}{\partial x}\left[N(x, y) - \partial_y \int M(x, y)\mathrm{d}x\right] \;=\; N_x - \partial_y \partial_x \int M\mathrm{d}x = N_x - M_y = 0.$$

$\square$

**Example 11.** Solve the ODE

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0.$$

**Answer:**

$$\begin{aligned} M_y &= \cos x + 2xe^y \\ N_x &= \cos x + 2xe^y \end{aligned}$$

So $M_y = N_x$, and the equation is exact.

Next, let

$$\psi = \int M dx = \int y(\cos x) + 2xe^y \, dx = y(\sin x) + x^2 e^y + h(y).$$

Then

$$\psi_y = \sin x + x^2 e^y + h'(y) = N = \sin x + x^2 e^y - 1$$

$$\Longrightarrow \quad h'(y) = -1 \quad \Rightarrow \quad h(y) = -y.$$

So the solution is

$$\psi = y(\sin x) + x^2 e^y - y = c.$$

**Exercise.** Solve the above equation, but using $\psi = \int N dy + h(x)$ first.

**Question.** What is the relationship between separable and exact equations?

## 3.3 Integrating factors

Sometimes we can multiply a function to a non-exact equation to make it exact. Take a function $\mu(x, y) \neq 0$,

$$\begin{aligned} M(x, y)\,dx + N(x, y)\,dy &= 0 \\ \mu(x, y)[M(x, y)\,dx + N(x, y)\,dy] &= 0 \\ \mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy &= 0 \\ \tilde{M}(x, y)\,dx + \tilde{N}(x, y)\,dy &= 0 \end{aligned}$$

where $\tilde{M}(x, y) = \mu(x, y)M(x, y), \tilde{N}(x, y) = \mu(x, y)N(x, y)$. Then let

$$\tilde{M}_y = \mu_y M + \mu M_y, \quad \tilde{N}_x = \mu_x N + \mu N_x.$$

We want $\tilde{M}_y = \tilde{N}_x$, i.e.

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Let's choose $\mu$ such that $\mu_y = 0$. Then the above equation reduces to

$$\mu M_y = \mu_x N + \mu N_x \quad \Leftrightarrow \mu_x = \frac{M_y - N_x}{N}\mu.$$

If the function $(M_y - N_x)/N$ is a function of $x$ only, then we can solve $\mu$ as a separable equation.

Here $\mu$ is called an integrating factor.

**Example 12.** Solve the ODE

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

**Answer:** It's first order, nonlinear, and not separable. Check if it's exact:

$$M_y = 3x + 2y, \quad N_x = 2x + y.$$

Not exact!. Next, try integrating factors.

$$\frac{M_y - N_x}{N} = \frac{x + y}{x^2 + xy} = \frac{1}{x}$$

is a function of $x$ only! Let

$$\mu'(x) = \frac{1}{x}\mu \quad \Rightarrow \quad \mu(x) = x.$$

Then multiply $\mu$ to the original equation:

$$x(3xy + y^2) + x(x^2 + xy)y' = 0$$

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

Double check the new equation is exact! Then solve it as usual (exercise).

Similarly, if $(N_x - M_y)/M$ is a function of $y$ only, then we can use the integrating factor $\mu(y)$ solving

$$\mu' = \frac{N_x - M_y}{M}\mu.$$

## 4 Direction fields

Consider the first order ODE:

$$y' = f(t, y)$$

Draw small arrows as a vector $(1, f(t, y))$ at many points $(t, y)$

Online plotter:

https://aeb019.hosted.uark.edu/dfield.html

**Example.** Consider

$$y' = \frac{y \cos x}{1 + 3y^3}$$

# 5 The Existence and Uniqueness Theorem

## 5.1 Linear equations

**Theorem**

*Consider the initial value problem*

$$y' + p(t)y = q(t), \quad y(t_0) = y_0.$$

*If $p, q$ are continuous on an interval $I = [a, b]$ containing $t_0$, then the IVP has a unique solution on $I$.*

**Example.** Consider

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

Solve it by integrating factors,

$$y' + \frac{2}{t}y = 4t \quad \Rightarrow \quad \mu = \mathrm{e}\hat{}\left[\int \frac{2}{t}\,dt\right] = t^2.$$

$$y = \frac{1}{t^2}\left[\int 4t^3\,dt + c\right] = \frac{1}{t^2}[t^4 + c] = t^2 + \frac{c}{t^2}.$$

Plugging $y(1) = 2$, we obtain $c = 1$. The solution is

$$y = t^2 + \frac{1}{t^2}.$$

Now, $p(t) = \frac{2}{t}$, $q(t) = 4t$. So $p, q$ are continuous in $(-\infty, 0) \cup (0, \infty)$. But $1 \in (0, \infty)$ only, so we know from the theorem the IVP has a unique solution in $(0, \infty)$, which is

$$y = t^2 + \frac{1}{t^2}, \quad t \in (0, \infty).$$

If the initial condition is changed to $y(-1) = 2$, then the solution is

$$y = t^2 + \frac{1}{t^2}, \quad t \in (-\infty, 0).$$

## 5.2 Nonlinear equations

**Theorem**

*Consider the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0.$$

*If $f$ and $\partial_y f$ are continuous on a rectangular domain $R = [a, b] \times [c, d]$ containing the point $(t_0, y_0)$. Then the IVP has a unique solution in some interval $I$ containing $t_0$.*

**Example.** Consider the IVP.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

It is separable. Let's solve it first,

$$2(y-1)dy = (3x^2+4x+2)dx \quad \Rightarrow \quad y^2 - 2y = x^3 + 2x^2 + 2x + c$$

$$y(0) = -1 \quad \Rightarrow \quad c = 3.$$

The solution is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

$$y = \frac{2 \pm \sqrt{4 + 4(x^3+2x^2+2x+3)}}{2} = 1 \pm \sqrt{x^3+2x^2+2x+4} = 1 - \sqrt{x^3+2x^2+2x+4}.$$

Here $f$ and $\partial_y f$ are continuous everywhere except $y = 1$.

**Example.** Consider

$$y' = y^{1/3}, \quad y(0) = 0 \quad (t \geq 0)$$

First, let's solve it as a separable equation.

$$y^{-1/3}\, dy = dt \quad \Rightarrow \quad \frac{3}{2} y^{2/3} = t + c$$

Plugging $y(0) = 0$ yields $c = 0$. So

$$y = \pm \left(\frac{2}{3} t\right)^{3/2}$$

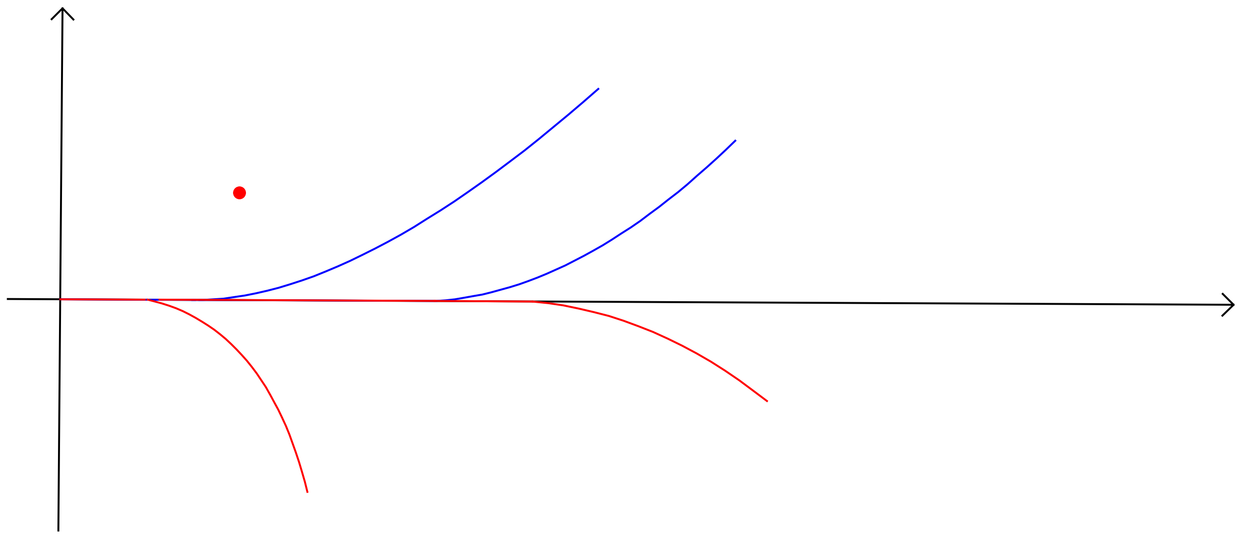
are two solutions. In addition

$$y = 0$$

is also a solution. In fact, we have infinitely many solutions defined as

$$y = \begin{cases} 0, & t < t_0 \\ \left[\frac{2}{3}(t - t_0)\right]^{3/2}, & t \geqslant t_0 \end{cases}, \text{ or } y = \begin{cases} 0, & t < t_0 \\ -\left[\frac{2}{3}(t - t_0)\right]^{3/2}, & t \geqslant t_0 \end{cases}$$

for any $t_0 > 0$. (**Exercise:** check $y$ is continuous and differentiable at $t = t_0$.)



In fact,

$$f = y^{1/3}, \quad \partial_y f = \frac{1}{3} y^{-2/3}.$$

So $\partial_y f$ is discontinuous near $(0,0)$. So there exists no rectangle $R$ containing $(0,0)$ such that $f, \partial_y f$ are both continuous in $R$. So we can't gaurantee the existence and uniqueness of solution for the IVP.

**Note 13.** One may not be able to find all solutions to nonlinear equations using one method.

**Example.** Consider

$$\frac{dy}{dt} = y^2, \quad y(0) = y_0 \neq 0$$

We can first solve it as a separable eqn.

$$y^{-2} dy = dt \quad \Rightarrow \quad -y^{-1} = t + c \quad \Rightarrow \quad y = -\frac{1}{t+c} \quad \Rightarrow \quad y = -\frac{1}{t - \frac{1}{y_0}}.$$

Now $f = y^2, \partial_y f = 2y$ are continuous everywhere. However, the solution is not defined for every $t$. For example, if $y_0 > 0,$ then the solution is defined only in $\left(-\infty, \frac{1}{y_0}\right)$.

# 6 Applications

## 6.1 Falling object in the air

$$mv' = mg - \gamma v,$$

where $v$ is the velocity, $m, g, \gamma$ are constants.

- Analyze the solutions using direction field.

- Solve it by integrating factors.

$$v' + \frac{\gamma}{m} v = g$$

integrating factor

$$\mu = e^{\int \gamma/m} = e^{\frac{\gamma}{m}t}$$

$$v(t) = e^{-\frac{\gamma}{m}t}\left[\int g e^{\frac{\gamma}{m}t} dt + c\right] = e^{-\frac{\gamma}{m}t}\left[\frac{gm}{\gamma} e^{\frac{\gamma}{m}t} + c\right] = \frac{gm}{\gamma} + c e^{-\frac{\gamma}{m}t}.$$

If the initial condition is $v(0) = v_0$. Then $c = v_0 - gm/\gamma$. So the solution of the IVP is

$$v(t) = \frac{gm}{\gamma} + \left[v_0 - \frac{gm}{\gamma}\right] e^{-\frac{\gamma}{m}t}.$$

So

$$\lim_{t\to\infty} v(t) = \frac{gm}{\gamma}.$$

All other solutions converge to the **equilibrium solution** $v = gm/\gamma$ as $t \to \infty$. This equilibrium solution is a **stable** one.

## 6.2 Compound interest with deposits/withdrawals

Assume the annual interest rate is $r$. The continuous rate of deposit/withdrawal is $k$. Then the ODE model for the total balance $u(t)$ is

$$u' = ru + k.$$

integrating factor

$$\mu = e^{-rt}$$

$$u = e^{rt}\left[\int ke^{-rt} + c\right] = e^{rt}\left[-\frac{k}{r}e^{-rt} + c\right] = -\frac{k}{r} + ce^{rt}.$$

If the initial condition is $u(0) = u_0$, then $c = u_0 + k/r$. So the solution of the IVP is

$$u = -\frac{k}{r} + \left(u_0 + \frac{k}{r}\right)e^{rt}.$$

The equilibrium solution is $u = -\frac{k}{r}$, and it is an **unstable** one since all other solutions diverge from it as $t \to \infty$.

## 6.3 Population dynamics

### 6.3.1 Exponential growth

$$y' = ry,$$

The solution is

$$y = y_0 e^{rt}$$

where $y_0 = y(0)$.

- If $r > 0$, we have exponential growth
- If $r < 0$, we have exponential decay, such as radioactive decay.

### 6.3.2 Logistic growth

$$y' = (r - ay)y.$$

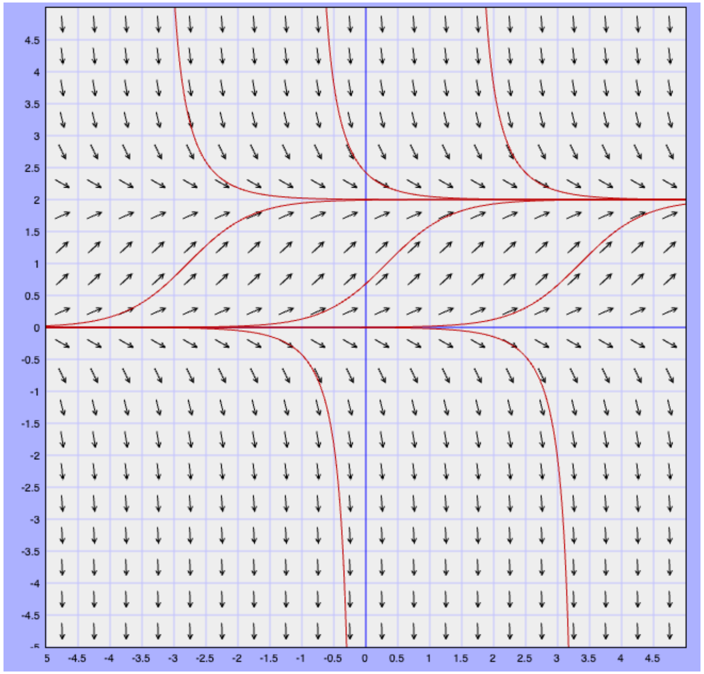
Note that the right-hand-side depends on $y$ only. In general, ODE of the form

$$y' = f(y)$$

is called **autonomous**. There are two equilibrium solutions

$$y = 0, \quad y = \frac{r}{a}.$$

From the direction field we can tell the equilibrium solution $y = 0$ is unstable, while the solution $y = \frac{r}{a}$ is stable.



Now let's solve the equation:

$$\frac{dy}{(r-ay)y} = dt \quad \Rightarrow \quad \int \frac{dy}{(r-ay)y} = \int dt \quad \Rightarrow$$

$$\frac{1}{(r-ay)y} = \frac{A}{y} + \frac{B}{r-ay} = \frac{A(r-ay)+By}{y(r-ay)} \quad \Rightarrow \quad 1 = A(r-ay)+By$$

$$y=0 \Rightarrow A = 1/r, \quad y = r/a \Rightarrow B = a/r$$

So

$$\int \frac{dy}{(r-ay)y} = \int \frac{1}{ry} + \frac{a/r}{r-ay} dy = \frac{1}{r}\ln|y| + \frac{a}{r}\left(\frac{1}{-a}\right)\ln|r-ay| = \frac{1}{r}\ln|y| - \frac{1}{r}\ln|r-ay|$$

$$= \frac{1}{r}\ln\frac{|y|}{|r-ay|} = \int dt = t+c$$

$$\Rightarrow \quad \frac{|y|}{|r-ay|} = e^{r(t+c)} = ce^{rt} \quad \Rightarrow \quad \frac{y}{r-ay} = ce^{rt}.$$

$$\Rightarrow \quad y = \frac{rce^{rt}}{1+ace^{rt}} = \frac{rc}{e^{-rt}+ac}.$$

Suppose the initial condition is $y(0) = y_0$, then

$$c = \frac{y_0}{r-ay_0}.$$

1. If $y_0 < 0$, then $c < 0$.

$$y' = \frac{-rc}{(e^{-rt}+ac)^2}(-r)e^{-rt} = \frac{r^2ce^{-rt}}{(e^{-rt}+ac)^2} < 0$$

So the solution is always decreasing.

2. If $0 < y_0 < \frac{r}{a}$, then $c > 0$. So $y' > 0$, and the solution is always increasing.
3. If $y_0 > \frac{r}{a}$, then $c < 0$, So $y' < 0$, and the solution is always decreasing.

By taking $y''$, we can find an inflection point (exercise).

# 7 Euler's method

Consider a general 1st order ODE

$$y' = f(t, y).$$

Take $(t_0, y_0)$, then

$$y'(t_0) = f(t_0, y_0)$$

$$y'(t_0) = \lim_{h \to 0} \frac{y(t_0 + h) - y(t_0)}{h} \approx \frac{y(t_1) - y(t_0)}{t_1 - t_0}$$

if $|t_1 - t_0|$ is small. So

$$y(t_1) \approx y(t_0) + (t_1 - t_0) y'(t_0) = y(t_0) + (t_1 - t_0) f(t_0, y_0)$$

Let

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0).$$

So $y_1 \approx y(t_1)$. Repeat this process, we obtain an algorithm: For a sequence of $t_0, t_1, t_2, \ldots$

$$y_{k+1} = y_k + (t_{k+1} - t_k) f(t_k, y_k)$$

This sequence of $y_0, y_1, y_2, \ldots$ is an approximation of the true values $y(t_0), y(t_1), y(t_2), \ldots$