Chapter 4 : Linear Transformation

4.1 Def / examples

① Def

A mapping L from a vector space V into a vector space W is said to be a **linear transformation** if

so $L(0_V) = 0_W$

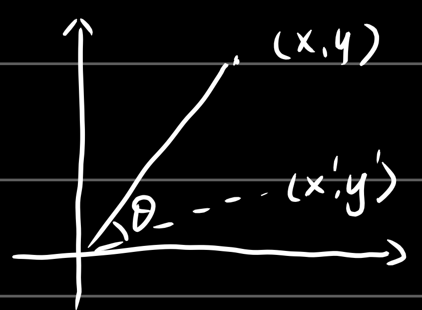
$$L(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2) \qquad (1)$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and for all scalars α and β.

② linear operator

A linear transformation $L: V \to V$ is called linear operator

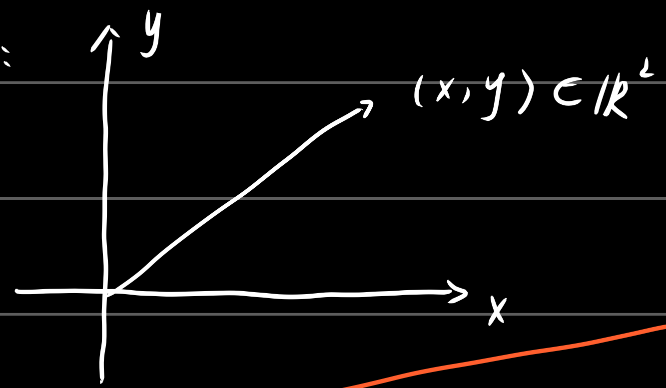
Ex Rotation



$L: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\begin{cases} x' = \cos\theta\, x + \sin\theta\, y \\ y' = -\sin\theta\, x + \cos\theta\, y \end{cases}$$

Ex :



Projection :

$\boxed{L: \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \to \begin{pmatrix} x \\ 0 \end{pmatrix}}$

$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$L(\alpha_1 V_1 + \alpha_2 V_2) = L\begin{pmatrix} \alpha_1 x_1 + \alpha_2 x_2 \\ \alpha_1 y_1 + \alpha_2 y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 x_1 + \alpha_2 x_2 \\ 0 \end{pmatrix}$

$\alpha_1 L(V_1) + \alpha_2 L(V_2) = \alpha_1 L\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \alpha_2 L\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \alpha_1 x_1 + \alpha_2 x_2 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$: the matrix representation for linear trans

Ex: let L be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$

satisfying $L(e_1) = e_1$, $L(e_2) = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

Find $L\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $L\begin{pmatrix} x \\ y \end{pmatrix}$

For any $v \in \mathbb{R}^2 = \text{span}\{e_1, e_2\}$

$L(v) = L(v_1 e_1 + v_2 e_2) = v_1 L(e_1) + v_2 L(e_2)$

③ kernel and range of linear transformation

The Image and Kernel

Let $L\colon V \to W$ be a linear transformation. We close this section by considering the effect that L has on subspaces of V. Of particular importance is the set of vectors in V that get mapped into the zero vector of W.

Definition Let $L\colon V \to W$ be a linear transformation. The **kernel** of L, denoted $\ker(L)$, is defined by

$$\ker(L) = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}_W\}$$

Definition Let $L\colon V \to W$ be a linear transformation and let S be a subspace of V. The **image** of S, denoted $L(S)$, is defined by

$$L(S) = \{\mathbf{w} \in W \mid \mathbf{w} = L(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in S\}$$

The image of the entire vector space, $L(V)$, is called the **range** of L.

understand in matrix A: $\ker(L)$ is null (A)

$L(S)$ is the Column space of A

Ex: $L = \frac{d}{dx}$ $P_3 \to P_2$

$P_3 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

$v \in V$ $v = a + bx + cx^2$

$L(v) = \frac{d}{dx} v = b + 2cx$

Question: Find $\ker(L)$ and $L(V)$

Find $v \in V$ Such that

$L(v) = 0$

$\Rightarrow b + 2cx = 0$

$\Rightarrow \begin{cases} b=0 \\ c=0 \end{cases}$

$v = a$

$\ker(L) = \{a \mid a \in \mathbb{R}\}$

$L(v) = \{b + 2cx\}$

$= \{\alpha + \beta x \mid \alpha, \beta \in \mathbb{R}\} = P_2$

③ Theorem

Theorem 4.1.1 *If $L: V \to W$ is a linear transformation and S is a subspace of V, then*

(i) $\ker(L)$ *is a subspace of V.*

(ii) $L(S)$ *is a subspace of W.*

4.2 matrix representation of linear transformation

① $L: \mathbb{R}^n \to \mathbb{R}^m$

$L: v \in \mathbb{R}^n \to w \to \mathbb{R}^m$

Q: can we find the corresponding $A_{m \times n}$ such that

$Av = L(v)$

Theorem 4.2.1 *If L is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that*

$$L(\mathbf{x}) = A\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^n$. In fact, the jth column vector of A is given by

$$\mathbf{a}_j = L(\mathbf{e}_j) \qquad j = 1, 2, \dots, n$$

$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ jth

Proof For $j = 1, \dots, n$, define

$$\mathbf{a}_j = L(\mathbf{e}_j)$$

and let

$$A = (a_{ij}) = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$$

If

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

is an arbitrary element of $\mathbb{R}^n$, then

$$\begin{aligned} L(\mathbf{x}) &= x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) + \cdots + x_n L(\mathbf{e}_n) \\ &= x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n \\ &= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\mathbf{x} \end{aligned}$$

Theorem 4.2.2 Matrix Representation Theorem

If $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are ordered bases for vector spaces V and W, respectively, then, corresponding to each linear transformation $L\colon V \to W$, there is an $m \times n$ matrix A such that

$$[L(\mathbf{v})]_F = A[\mathbf{v}]_E \qquad \text{for each } \mathbf{v} \in V$$

A is the matrix representing L relative to the ordered bases E and F. In fact,

$$\mathbf{a}_j = \left[L(\mathbf{v}_j)\right]_F \qquad j = 1, 2, \ldots, n$$

$L: V \to W$ finite dimensional space

ordered base

Theorem: V: $E = \{v_1, v_2 \cdots V_n\}$ base

W: $F = \{w_1, w_2 \cdots w_m\}$ base

For each $v \in V \Rightarrow [v]_E$

$w = L(v) \Rightarrow [w]_F = [L(v)]_F$

There exists a matrix $A_{m \times n}$ such that

$[w]_F = [L(v)]_F = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

$[L(v)]_F = A[v]_E$

$[w]_F = A[v]_E$

representing matrix, where $a_j = [L(v_j)]_F$

proof: $v = x_1 v_1 + \cdots x_n v_n$

$= \sum x_j v_j$

$[v]_E = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$w = L(v) = y_1 w_1 + y_2 w_2 + \cdots y_m w_m = \sum y_j w_j$

$[w]_F = [L(v)]_F = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

$w \to L(v_j) = \underset{a_{1j}}{z_{1j}} w_1 + \cdots \underset{a_{mj}}{z_{mj}} w_m = \sum a_{ij} w_i$

Let $a_j = [L(v_j)]_F$

$L(v) = L(x_1 v_1 + x_2 v_2 + \cdots x_n v_n)$

$= x_1 L(v_1) + x_2 L(v_2) + \cdots x_n L(v_n)$

$= \sum x_j L(v_j)$

$= \sum_j x_j \sum_i a_{ij} w_i$

$L(v) = \sum_i w_i \sum_{j=1}^{n} a_{ij} x_j = \sum y_i w_i$

$\sum a_{ij} x_j = y_i$

or $Ax = y$ $\quad A[v]_E = [L(v)]_E$

Ex: $L = \frac{d}{dx} : P_3 \to P_3$

$v \in P_3$, $v = a + bx + cx^2$

$L(v) = \frac{dv}{dx} = b + 2cx$

$E = \{1, x, x^2\}$, $F = \{1, x, x^2\}$

$L(1) = 0 = 0 + 0x + 0x^2 \Rightarrow [L(1)]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$L(x) = 1 = 1 + 0x + 0x^2 \Rightarrow [L(x)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$L(x^2) = 2x = 0 + 2x + 0x^2 \Rightarrow [L(x^2)]_E = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

So $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

③ $L: \mathbb{R}^n \to \mathbb{R}^m$

$E = \{v_1, v_2 \cdots v_n\} \in \mathbb{R}^n$

$F = \{w_1, w_2 \cdots w_m\} \in \mathbb{R}^m$

By the theorem

$a_j = [L(v_j)]_F = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$

$$L(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots a_{mj}w_m$$
$$= (w_1, w_2 \cdots w_m)\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$$
$$= B a_j$$

$$[B \quad C] = \boxed{\overset{Y}{E_k \cdots E_1}}\,[I \quad X]$$
$$\Downarrow$$
$$A = B^{-1}C$$

$B = Y$	$a_j = [L(v_j)]_F$
$C = YX$	$a_j = B^{-1}L(v_j)$
$X = Y^{-1}C$	$A = (a_1 \cdots a_n) = B^{-1}(C_1, C_2 \cdots C_n)$
$X = B^{-1}C$	$A = B^{-1}C$

4.3 similarity

$L : V \to W$

Finite Finite

$E = \langle v_1 \cdots v_n \rangle$ $F = \langle w_1, w_2 \cdots w_m \rangle$ ordered base

$v \in E = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ $[v]_E = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

there exist $A[v]_E = [L(v)]_F$, for any $v \in V$

if $V = \mathbb{R}^n$ $W = \mathbb{R}^m$

$$A = B^{-1}(L(v_1) \cdots L(v_n))$$

A: the matrix representing L with respect to E and F

Linear operator : $L : V \to V$
E Y

L: linear operator on V

S: transition matrix : $E = (v_1 \cdots v_n)$

$F = (w_1 \cdots w_n)$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$

$= \beta_1 w_1 + \cdots \beta_n w_n$

$[v]_E = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$, $[v]_F = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

$S[v]_F = [v]_E$ from F to E

$$\begin{tikzcd}
\text{base } E & V \arrow[r, "L"', "A"] & V \\
\text{base } F & V \arrow[u, "S"', "\text{Coordinate}"] \arrow[r, "B", "L"'] & V \arrow[u]
\end{tikzcd}$$

$A[v]_E = [L(v)]_E$

$A[v]_F = [L(F)]_F$

what's the relation between A and B

Theorem : $B = S^{-1} A S$ ✡

proof: For any $v \in V$, $v = x_1 v_1 + x_2 v_2 + \cdots v_n v_n$

$= y_1 w_1 + y_2 w_2 + \cdots y_n w_n$

$\begin{cases} [v]_E = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X \\ [v]_F = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y \end{cases} \Rightarrow \begin{cases} Ax = [L(v)]_E \\ By = [L(v)]_F \\ S[L(v)]_F = [L(v)]_E \end{cases}$

$\Rightarrow \begin{cases} SBy = Ax \\ S^{-1}ASy = S^{-1}Ax \end{cases} \Rightarrow \begin{cases} By = S^{-1}Ax \\ S^{-1}Asy = S^{-1}Ax \end{cases} \Rightarrow S^{-1}ASy = By$ for any x

So $S^{-1}AS = B$

Ex : $P = \frac{d}{dx} : P_3 \to P_3$

$\frac{d}{dx}(a + bx + cx^2) = b + 2cx$

(i) $E = (1, x, x^2)$

matrix representing L with respect to V

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

<ii> $F=(1, 2x, 4x^2-2)$

Question : Find the matrix representing L with respect to F

Answer: $B=S^{-1}AS$ by theorem

$v=\alpha+\beta(2x)+\gamma(4x^2-2)=\alpha-2\gamma+2\beta x+4\gamma x^2$

$[v]_F=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$

$[v]_E=\begin{bmatrix}\alpha-2\gamma\\ 2\beta\\ 4\gamma\end{bmatrix}$

$S[v]_F=[v]_E$, $S=\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}$

$L(v)=2\beta+8\gamma x$

$[L(v)]_F=\begin{bmatrix}2\beta\\ 4\gamma\\ 0\end{bmatrix}$

Since $B[v]_F=[L(v)]_F$

$B\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}2\beta\\ 4\gamma\\ 0\end{bmatrix}$

$\Rightarrow B=\begin{bmatrix}0&2&0\\0&0&4\\0&0&0\end{bmatrix}$

② properties

Let A and $B\in\mathbb{R}^{n\times n}$, There exists a nonsingular matrix $B=S^{-1}AS$,

Then it says A is similar to B

$A\sim B\sim C$

<i> $\det(A)=\det(B)$ <ii> A^T is similar to B^T

<iii> $\text{Rank}(A)=\text{rank}(B)$