MATH4205/MATH7620 Probability Theory and Stochastic Processes 2025-2026, Semester 1

In-class Exercise 2

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Instructions:

1. You should finish and hand-in the in-class exercise **in class**.
2. If you have any questions about the questions, you can ask the instructor during the class time.
3. Late submission of in-class exercise will NOT be entertained and no marks will be given for the in-class exercises.
4. Please provide a difficulty level (DL) score between 0 and 10 to indicate how difficult you found this exercise after completing it. 0: extremely easy. 10: extremely difficult.

Questions:

1. Suppose that X_1 and X_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$; What is $P\{X_1 < X_2\}$? (Hint: By conditioning on X_1, use the total probability formula in Section 2.4).

$$P(X_1 < X_2) = \int_0^\infty P(X_2 > x \mid X_1 = x) f_{X_1}(x)\, dx$$

Since X_1, X_2 are independent, so we have

$$P(X_2 > x \mid X_1 = x) = P(X_2 > x) = e^{-\lambda_2 x}$$

So $$P(X_1 < X_2) = \int_0^\infty e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \lambda_1 \cdot \int_0^\infty e^{-(\lambda_1+\lambda_2)x}\, dx$$

$$= \lambda_1 \left[-\frac{1}{\lambda_1+\lambda_2} \cdot e^{-(\lambda_1+\lambda_2)x} \right]_0^\infty$$

$$= \lambda_1 \left[0 - \left(-\frac{1}{\lambda_1+\lambda_2}\right) \right]$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}$$

2. Let $\{N(t)|t \geq 0\}$ be a Poisson process with rate λ. Let S_n be the arrival time of the n^{th} event. Find $P\{S_2 < s|N(t) = 2\}$ for any $0 \leq s \leq t$.

$$P\{S_2 < s \mid N(t)=2\} = \frac{P\{S_2 < s\} \cap \{N(t)=2\}}{P\{N(t)=2\}}$$

if $N(s)=2$, then $S_1, S_2 < S$

$$P\{N(t)=2\} = e^{-\lambda t} \frac{(\lambda t)^2}{2!}$$

$$P\{N(s)=2, N(t)-N(s)=0\} = P\{N(s)=2\} \cdot P\{N(t)-N(s)=0\}$$

$$= \left(e^{-\lambda s} \frac{(\lambda t)^2}{2!}\right)\left(e^{-\lambda(t-s)} \cdot \frac{(\lambda(t-s))^0}{0!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda s)^2}{2}$$

$$P\{S<2 \mid N(t)=2\} = \frac{e^{-\lambda t} \frac{(\lambda s)^2}{2}}{e^{-\lambda t} \frac{(\lambda t)^2}{2}} = \left(\frac{s}{t}\right)^2$$

Marks: