Chapter 6 Pricing Stock Options

- Motivating example
- The arbitrage theorem
- The Black-Scholes option pricing formula (reading)
- Use the arbitrage theorem to price arbitrary payoff

Future value with constant interest rate

- Suppose the **annual interest rate** (constant) is r. Principal = 1. After T years,
  - Interest compounded once a year, the **future value** (FV) $= 1 \times (1 + r)^T$
  - Interest compounded n times a year, after T years,
    $FV = (1 + r/n)^{nT}$
  - Continuous compounding:
    - After one year, FV = $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nT} = e^{rT}$
    - Recall

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$r = 6\%$, interest compounded twice a year

$T=0$ $T=0.5$

1 $1+\frac{r}{2}$ $(1+\frac{r}{2})^2$

The interest rate for each half year is $\frac{r}{2} = 3\%$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}} = e$$

$\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{\frac{n}{r}\cdot rt} = e^{rt}$

Price as Present Value (PV)

- If one can surely having \$$X(T)$ at time T by investing \$$X(0)$ today, then today's price of $X(T)$ is $X(0)$
- For deterministic $X(T)$, the **present value** is $\mathrm{X}(0) = PV(X(T))$
  - e.g., when the interest rate r is compounded n times a year,

$$X(T) = X(0)\boxed{\left(1+\frac{r}{n}\right)^{nT}}$$

The present value of $X(T)$ is $X(0) = \frac{X(T)}{(1+r/n)^{nT}}$ ✓

The discount factor: $1/(1+r/n)^{nT}$, e^{-rT}

Recall Geometric Brownian motion

$$Z(t) = \sigma B(t) + \left(\mu - \frac{\sigma^2}{2}\right)t$$

- $X(t) = X(0)e^{Z(t)}, E\big(X(t)\big) = X(0)\, e^{\mu t}$
- The GBM model assumes that both μ and σ are constant over time.
- $\mu = \frac{1}{t} lnE[\frac{X(t)}{X(0)}]$
- This equation can be used as a basis for estimating μ from historical data.

$\frac{X(t)}{X(0)} = e^{Z(t)}$

$\ln\left(\frac{X(t)}{X(0)}\right) = Z(t)$

$E\left(\frac{X(t)}{X(0)}\right) = e^{\mu t}$

$\frac{1}{t} \ln\left(E\left(\frac{X(t)}{X(0)}\right)\right) = \mu$

Recall Geometric Brownian motion

- When σ is close to 0, $X(t) = X(0)e^{\mu t}$
- The stock price will behave like an almost risk-free bank account with interest rate μ.
- σ measures the riskiness of a stock, called volatility.
- Recall $ln\frac{X(t+s)}{X(t)} \sim N(\left(\mu - \frac{\sigma^2}{2}\right)s, \sigma^2 s)$ (P157), let $t = 0, s = \Delta t$,

$$\sigma^2 = \frac{1}{\Delta t} Var[ln\frac{X(\Delta t)}{X(0)}]$$

- σ^2 is the variance of log-returns per unit time.
- This equation can be used as a basis for estimating σ^2 from historical data.

Estimating Volatility from Historical Data

- Stock price during 21 consecutive trading days.
- Usually, assume 1 year=252 trading days.
- $\Delta t = \frac{1}{252}$ years
- Sample s.d. of daily return u_i is 0.0121593.
- The estimated volatility is

$$\frac{0.0121593}{\sqrt{\frac{1}{252}}} = 0.1930 = 19.30\% \text{ p.a.}$$

- p.a. = per annum

Day i	Closing stock price (dollars), S_i	Price relative S_i/S_{i-1}	Daily return $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

Call Options

- Suppose that the price of a stock is given by $X(t)$ for $t \geq 0$. A *call option* on the stock with maturity T (years) and strike price K operates as follows:
  - At time 0, the investor pays $\$c$ per share to buy an option on one share of the stock.
  - At time T, the payoff to the investor is

$$\text{payoff} = \begin{cases} X(T) - K, & \text{if } X(T) > K \\ 0, & \text{otherwise} \end{cases} = (X(T) - K)^{+}.$$

- For example, consider a 6-month call option with strike price \$80. The payoff will be \$5 if the stock price is \$85 after 6 months. The payoff will be \$0 if the stock price drops below \$80 after 6 months.

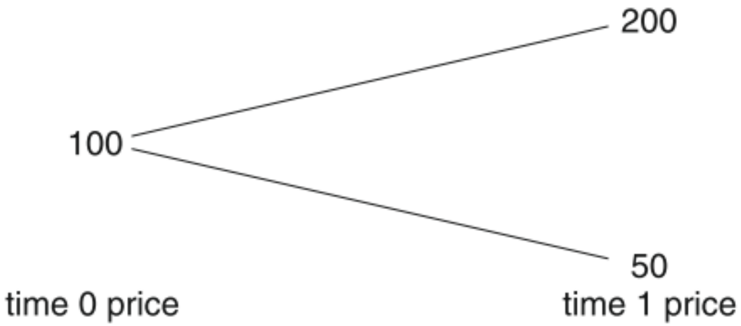
- **In this course, simply treat** pay off of **an option as a function of** $X(T)$. Different options mean different functions of $X(T)$.

Option pricing: a single time period model

FV: $200(1+r)$
$50(1+r)$

- The present price of a stock is \$100 per unit share
- After one time period, it will be, **in present value dollars**, either \$200 or \$50.

- At time 0, we purchase x units of the stock with a cost of $100x$
- At time 0, we purchase y call options, each with a strike price of $K = \$150$. The cost per option is c, resulting in a total expenditure of cy for the call options. This purchase grants us the right to buy y shares of the underlying stock at a strike price of \$150 per share at Time 1.



Option pricing: a single time period model

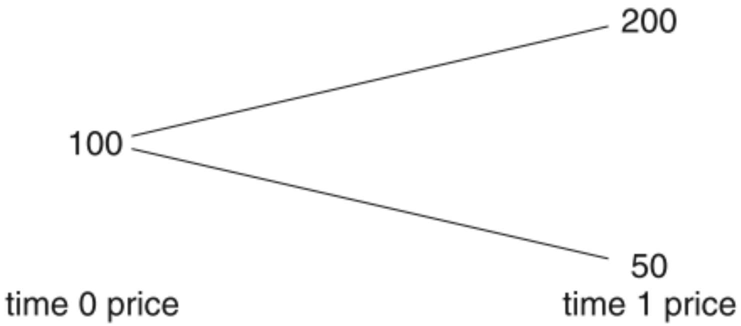
At time 1,

$X(t)-k$

- If the stock price rises to \$200, we exercise option, and realize a gain of \$(200-150) for each of the y option we purchased

  50

- If the stock price is \$50, stock is worth $50x$, and option at Time 1 is worthless



Option pricing: a single time period model

- Suppose both x and y can either be positive or negative or zero, i.e., buy or sell stock or option.
- To find the option price c, suppose at time 0 we "buy x units of stock and buy y options"
- The **value of holding** at **time 1**

express in the PV units

$$value = \begin{cases} 200x + 50y & \text{if stock price is \$200} \\ 50x & \text{if stock price is \$50} \end{cases}$$

- We choose y so that the preceding value **is the same** no matter what the price of stock at time 1 is $y = -3x$.

$200x + 50y = 50x$.

- In this way, we construct a riskless portfolio.

1. Stock + option
2. Saving in the bank

no arbitrage requires that two portfolios with identical payoffs at maturity must have the same price today; otherwise, we could earn a risk-free profit from the price difference.

Option pricing by **no arbitrage argument**

- The original cost of purchasing x units of stock and $-3x$ option is

Original cost = $100x - 3xc$

- The gain on the transaction is

Gain = $50x - (100x - 3xc) = x(3c - 50)$ $=0$

- If $3c \neq 50$, we can **guarantee a positive gain** by letting x be positive when $3c > 50$ and letting it be negative when $3c < 50$, no matter the price of stock at time 1.
- A sure win betting scheme is called an arbitrage
- The only cost c that **does not result in an arbitrage** is $c = 50/3$.

- Consider an experiment with possible outcomes $\Omega = \{1, 2, \ldots, m\}$.
- Suppose that n wagers are available.
- The (net) return of the ith wager is $r_i(j)$ per unit bet if the outcome of the experiment is j.

Theorem 7.1 (The Arbitrage Theorem) Exactly one of the following is true: Either

(i) there exists a probability vector $\mathbf{q} = (q_1, \ldots, q_m)$ for which

$$E^{\mathbf{q}}[r_i(X)] = \sum_{j=1}^{m} q_j \cdot r_i(j) = 0 \qquad \text{for all } i = 1, 2, \ldots, n$$

or

(ii) there exists a betting scheme $\mathbf{x} = (x_1, \ldots, x_n)$ for which

$$\sum_{i=1}^{n} x_i \cdot r_i(j) > 0 \qquad \text{for all } j = 1, 2, \ldots, m.$$

Revisit: option pricing by no arbitrage argument

- The outcome of experiment is the value of the stock at time 1.
- Two different wagers: to buy (or sell) the stock and to buy (or sell) the option.
- By the arbitrage theorem, there will be no sure win if there is a probability vector $(p, 1-p)$ that makes the **expected return** under both wagers equal to 0.
- The return of purchasing 1 unit of **stock** is

$$return = \begin{cases} 200 - 100 = 100 & \text{if stock price is \$200} \\ 50 - 100 = -50 & \text{if stock price is \$50} \end{cases}$$

- Let p denote the probability that price is 200 at time 1,

$$E(return) \; = \; 100p - 50(1-p)$$

Revisit: option pricing by no arbitrage argument

- Setting the expected return to 0 yields
$$p = 1/3$$
- The probability vector $(\frac{1}{3}, \frac{2}{3})$ for wager 1 (stock) yields an expected return 0.
- The return of purchasing 1 **option** is
$$return = \begin{cases} 50 - c & \text{if stock price is \$200} \\ -c & \text{if stock price is \$50} \end{cases}$$
- Hence, the expected return of wager 2 (option) when $p = 1/3$ is
$$E(return) = (50 - c) \times \frac{1}{3} - c \times \frac{2}{3} = 0$$

So $c = 50/3$.

Pricing the option : continuous-time scenario

- We regard the price of the stock **over time** as our experiment, so the outcome of the experiment is the **value of the function** $X(t), 0 \leq t \leq T$.
- Wagers: (buy or sell) stock and (buy or sell) option
- Assume we can generalize the arbitrage theorem:

no arbitrage $\leftrightarrow$ there exists a probability measure over the set of outcomes under which all of the wagers have expected return 0.

- Let **P** denote the probability measure **on the set of outcomes**.

Pricing the option: continuous-time scenario

- Consider the wager of observing the stock for a time s and then purchasing one share with the intention of selling it at time t, $0 \le s < t \le T$.
- The present value of the amount paid for stock is $e^{-\alpha s}X(s)$
- The present value of the amount received is $e^{-\alpha t}X(t)$
- To make the expected return of this wager to be 0, we must have

$$E_P[e^{-\alpha t}X(t)|X(u), 0 \le u \le s] = e^{-\alpha s}X(s)$$

- Martingale probability measure

Pricing the option : continuous-time scenario

- Recall Example 6.10 (b), $Z(t) = \sigma B(t) + \left(\mu - \frac{\sigma^2}{2}\right)t, \quad X(t) = X(0)e^{Z(t)}$

$$E[X(t)|\mathcal{F}_s] = X(s)e^{\mu(t-s)}$$

- Let $\mu = r$, i.e., the drift coefficient μ in the GBM model is set as the risk-free rate. The risk-free rate is the theoretical return on an investment with zero risk.

$$E[e^{-rt}X(t)|\mathcal{F}_s] = X(s)e^{-rs}$$

- The **discount rate** used for the expected payoff of stock is the risk-free rate.

Pricing the option : continuous-time scenario

- **The probability measure is the one that governors the process $\{\boldsymbol{Z(t)}, \mathbf{0} \leq \boldsymbol{t} \leq \boldsymbol{T}\}$, which is the Brownian motion with drift $\boldsymbol{r}$- $\frac{\boldsymbol{\sigma}^2}{\mathbf{2}}$, and variance $\boldsymbol{\sigma}^2$ (P177).**
- Consider the wager of purchasing an European call option whose strike price is K.

$$value\ of\ option\ at\ time\ t = \begin{cases} X(t) - K & \text{if } X(t) \geq K \\ 0 & \text{if } X(t) < K \end{cases}$$

- The present value of the option is $e^{-\alpha t}(X(t) - K)^+$
- Let c denote the time 0 cost of the option, to have expected return 0,

$$E_P[e^{-rt}(X(t) - K)^+] = c \qquad \text{P178}$$

Summary: Risk-neutral valuation

- Prices are **expected** values, but not under the “real-world” or “true” probability; rather, they are expected values under an “artificial” probability, called risk-neutral probability or equivalent martingale measure (EMM).

- A risk-neutral world has two features that simplify the pricing of derivatives:

1. The expected rate of return (μ) on a stock is the risk-free rate.

2. The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.

Example

- Let X(t) be the price of a stock at time t. The current price of the stock is X(0) = 20, the stock's volatility is σ = 10% per annum, and the interest rate is r = 4% per annum. Suppose that the payoff of an investment at time 1 is 3 if X(1) > 24 and 0 otherwise. Determine the current price f of the investment, assuming a geometric Brownian motion model for X(t).

Time 0 — Time 1

3 $X(1) > 24$

0 otherwise

$$X(t) = X(0).\exp\{\sigma B(t) + (r - \frac{\sigma^2}{2})t\}$$

$$X(t) = 20.\exp\{0.1\, B(t) + (0.04 - \frac{0.01}{2})t\}$$

$$f = e^{-rT}\underbrace{(p.3 + (1-p).0)}_{\text{expected pay off of the investment}}$$

discounted with r

$$P = P\{X(1) > 24\}$$

$$= P\{20 \cdot \exp\{0.1 B(1) + 0.035\} > 24\}$$

$$= P\{\exp\{0.1 B(1) + 0.035\} > 1.2\}$$

$$= P\{0.1 B(1) + 0.035 > \log 1.2\}$$

$$= P\{\underbrace{B(1)}_{Z} > \frac{\log 1.2 - 0.1}{0.035}\}$$

$$= P\{Z > 1.4732\} = \Phi(-1.4732)$$

$$=$$

$$f = e^{-0.04}$$