1. Let $G = (V, E)$ be a loop-free connected graph with $|V| = v$. If $|E| > (v/2)^2$, prove that G cannot be bipartite.

15

Solution 5

Partition V as $V_1 \cup V_2$ with $|V_1| = m$, $|V_2| = v - m$. If G is bipartite, then the maximum number of edges that G can have is $m(v-m) = -[m-(v/2)]^2 + (v/2)^2$, a function of m. For a given value of v, when v is even, $m = v/2$ maximizes $m(v-m) = (v/2)[v-(v/2)] = (v/2)^2$. For v odd, $m = (v-1)/2$ or $m = (v+1)/2$ maximizes $m(v-m) = [(v-1)/2][v-((v-1)/2)] = [(v-1)/2][(v+1)/2] = [(v+1)/2][v-((v+1)/2)] = (v^2-1)/4 = \lfloor (v/2)^2 \rfloor < (v/2)^2$. Hence if $|E| > (v/2)^2$, G cannot be bipartite. 5

$-[m-(\frac{v}{2})]^2 + (\frac{v}{2})^2 \le \frac{v^2}{4}$ 5

2. a) Let $G = (V, E)$ be a loop-free connected graph with $|V| \ge 11$. Prove that either G or its complement $\overline{G}$ must be nonplanar. 10

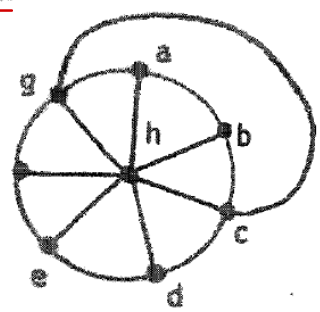
b) Find a counterexample to part (a) for $|V| = 8$. 10

Solution

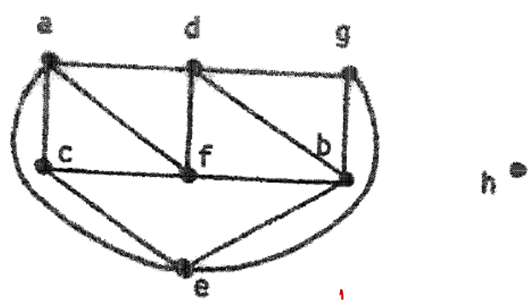
(a) Suppose that $G = (V,E)$ with $|V| = 11$. Then $\overline{G} = (V, E_1)$ where $\{a,b\} \in E_1$ iff $\{a,b\} \notin E$. Let $e = |E|$, $e_1 = |E_1|$. If both G and $\overline{G}$ are planar, then by Corollary 11.3 (and part (b) of Exercise 20, if necessary), $e \le 3|V| - 6 = 33 - 6 = 27$ and $e_1 \le 3|V| - 6 = 27$. But with $|V| = 11$, there are $\binom{11}{2} = 55$ edges in K_{11}, so $|E| + |E_1| = 55$ and either $e \ge 28$ or $e_1 \ge 28$. Hence, one of G, $\overline{G}$ must be planar. 4 4

If $G = (V,E)$ and $|V| > 11$, consider an induced subgraph of G on $V' \subset V$ where $|V'| = 11$. 2

(b) G:



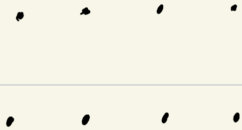
$\overline{G}$:



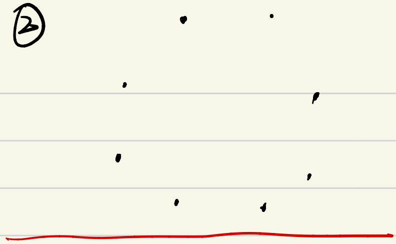
→ example

(Give G and $\overline{G}$. Only give a planar G, I don't know whether $\overline{G}$ is planar, so 3 points.)

①

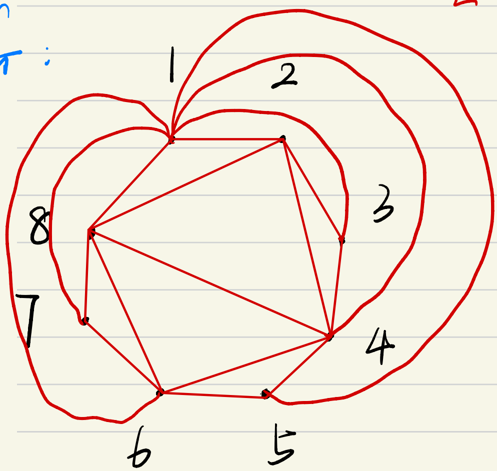


②

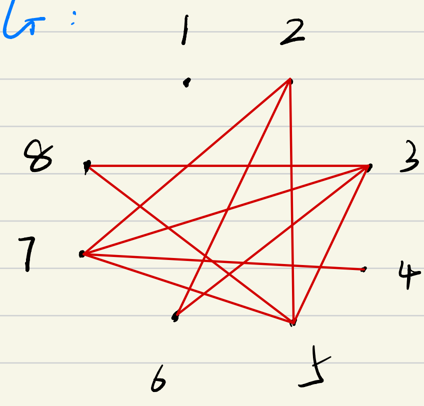


③

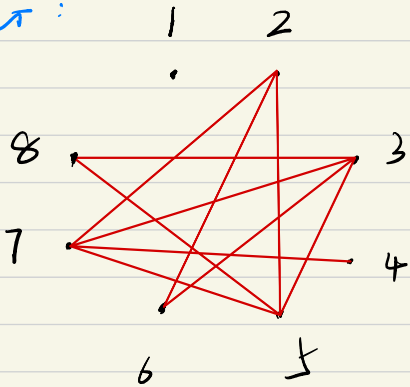
G:

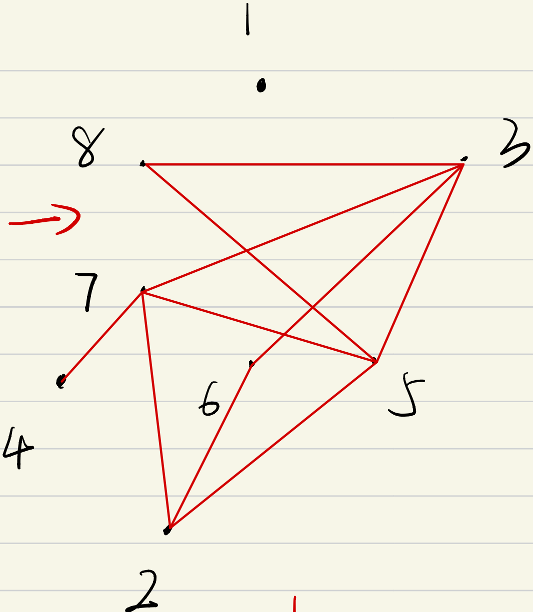


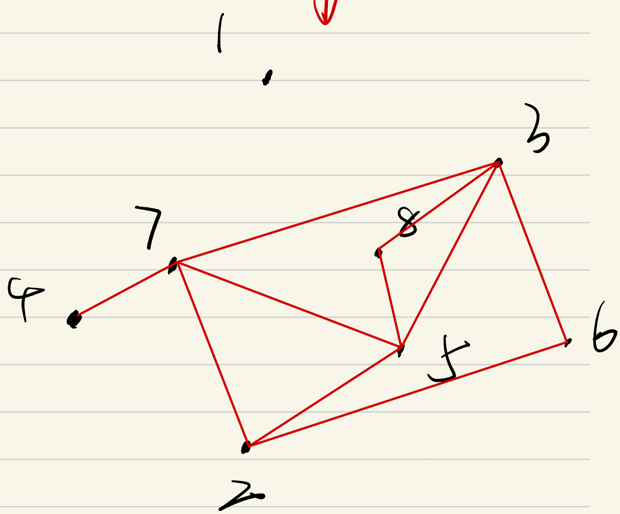
$\bar{G}$:



$\overline{G}$:





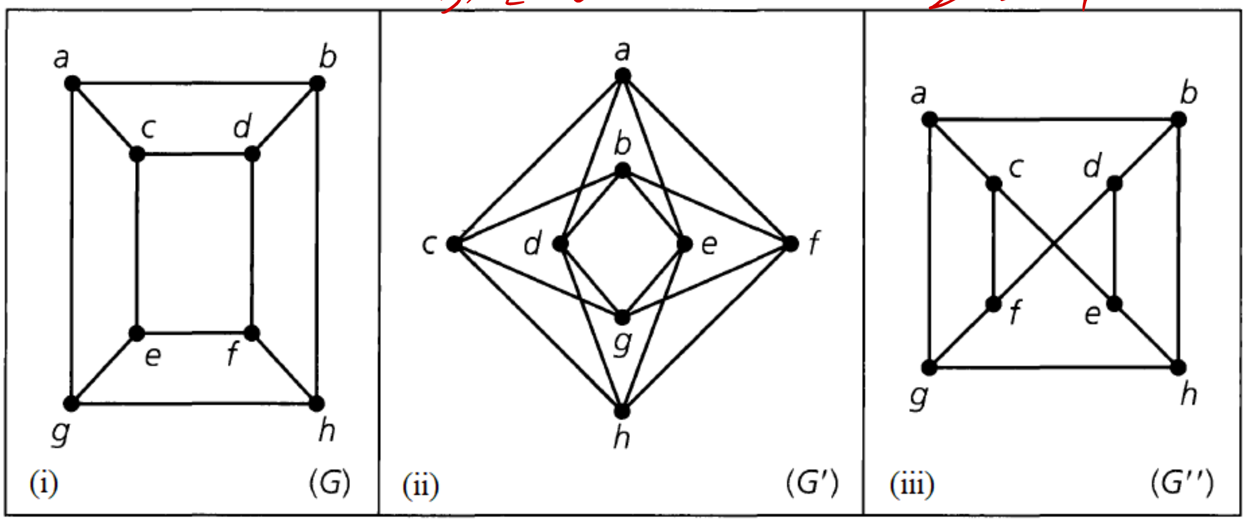


$\Rightarrow$: 10 $\Leftarrow$: 10

3 (a) Prove that a graph is bipartite if and only if it contains no odd cycles.

(b) Which is of the following graph is bipartite? If bipartite, give 2 subsets V_1 and V_2 such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. 10

3×2=6 2×2=4



Solution

(a) Suppose G is bipartite. Let $V = A \cup B$ such that $A \cap B = \emptyset$ and that all edges $e \in E$ are such that $e = \{a, b\}$ where $a \in A$ and $b \in B$. 5

Suppose G has (at least) one odd cycle C. Let the length of C be $2n + 1$.

$$C : \underset{\in A}{v_1} \to \underset{\in B}{v_2} \to \cdots \to \underset{\in A}{v_{2n-1}} \to \underset{\in B}{v_{2n}} \to \underset{\in A}{v_{2n+1}} \to \underset{\in A}{v_1}$$ 5

$\underset{\in A}{v_{2n+1}} \to \underset{\in A}{v_1}$ contradicts to the assumption that G is bipartite. Therefore G contains no odd cycles.

Suppose G has no odd cycles. Choose any vertex $v \in G$. Divide $V(G)$ into two sets of vertices like this:

$$A = \{w \in V : d(v,w) \text{ is even}\} \text{ and } B = \{w \in V : d(v,w) \text{ is odd}\}$$ 5

where $d(v, w)$ is the length of shortest path between v and w.

Clearly, $V(G) = A \cup B$ and $A \cap B = \emptyset$.

Suppose $a_1, a_2 \in A$ are adjacent. Then there would be a closed walk of odd length

$$\underbrace{v \to \cdots \to a_1}_{\text{even length}} \to \underbrace{a_2 \to \cdots \to v}_{\text{even length}}$$

5

It follows that G would then contain an odd cycle. This contradicts our assumption that G contains no odd cycles.

By the same argument, neither can any two vertices in B be adjacent.

Thus A and B satisfy the conditions for G to be bipartite.

(b)(i) Let $V_1 = \{a, d, e, h\}$ and $V_2 = \{b, c, f, g\}$. Then every vertex of G is in $V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Also every edge in G may be written as $\{x, y\}$ where $x \in V_1$ and $y \in V_2$. Consequently, the graph G in part (i) of the figure is bipartite.

(ii) Let $V_1' = \{a, b, g, h\}$ and $V_2' = \{c, d, e, f\}$. Then every vertex of G' is in $V_1' \cup V_2'$ and $V_1' \cap V_2' = \emptyset$. Since every edge of G' may be written as $\{x, y\}$, with $x \in V_1'$ and $y \in V_2'$, it follows that this graph is bipartite. In fact G' is (isomorphic to) the complete bipartite graph $K_{4,4}$.

(iii) G'' contains a cycle of length 5, namely, $a \to c \to e \to d \to b \to a$. Therefore, G'' is not bipartite.

4. Let G be a connected planar graph such that degree of every vertex is 3. The length of boundary of every region in G is 4, 6 or 8. Every vertex lies on one region of length 4, one region of length 6, and one region of length 8. 20

Determine the number of regions of each length and the number of regions of G (including infinite region).

Solution

Let v be the number of vertices in the graph. Since each vertex is incident to one region of length 4, one region of length 6, and one region of length 8, there are v incidences of vertices with regions of each length. Since every region of length l is incident with l vertices, there are thus $v/4$, $v/6$, and $v/8$ faces of lengths 4, 6, 8, respectively. Hence there are $r = \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right) v = \frac{13}{24} v$ regions.

1 1 1 2

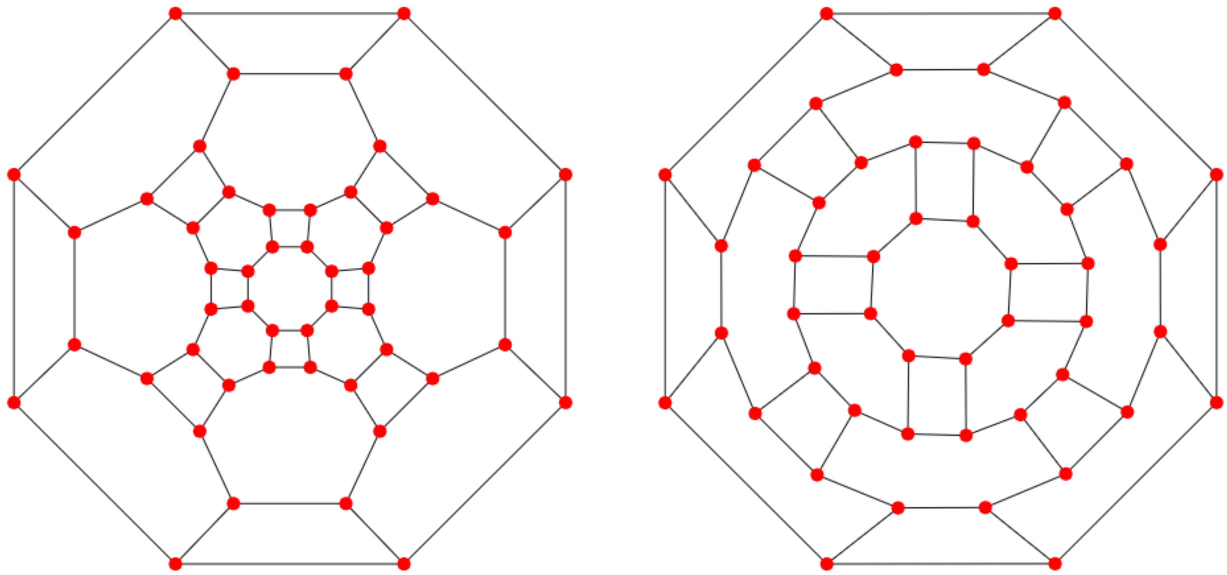
$2e = 3v$ by handshaking theorem 2

$2 = v - e + r = v - \frac{3}{2} v + \frac{13}{24} v = \frac{1}{24} v$ by Euler's theorem 2

$v = 48$ 2

$r = \frac{13}{24} v = 26$ faces, 12 of them are of length 4, 8 of them are of length 6 and 6 of them are of length 8

3 2 2 2



Truncated Cuboctahedron

5. Let $G = (V, E)$ be an undirected connected loop-free graph. Suppose further that G is planar and determines 53 regions. If, for some planar embedding of G, each region has at least five edges in its boundary, prove that $|V| \geq 82$.

7 15

Solution

Proof: Since each region has at least five edges in its boundary, $2|E| > 5(53)$, or $|E| \geq (1/2)(5)(53)$. And from Theorem 11.6 we have $|V| = |E| - 53 + 2 = |E| - 51 \geq (1/2)(5)(53) - 51 = (265/2) - 51 = 81\frac{1}{2}$. Hence $|V| \geq 82$.

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