Asg. 3 Solutions

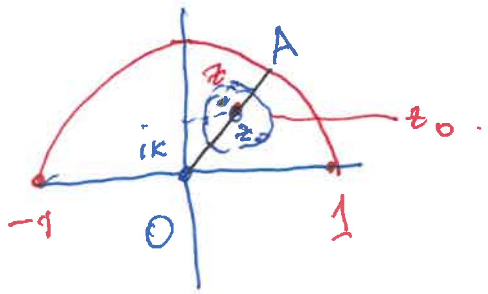
1. $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \ldots = \frac{e^{i2z}+e^{-i2z}}{2}$$

$$= \cos(2z).$$

2. Let $z_0 = h+ik \in S$. Draw the line OA and choose

$$r < \min\{k, |z_0 - A|\}.$$



Consider the ball

$$B(z_0, r) := \{z : |z_0 - z| < r\}.$$

and take any $z \in B(z_0, r)$.

We have that

$$\operatorname{Im} z > \operatorname{Im} z_0 - r = k - r > 0 \qquad (1).$$

In addition, by triangle inequality

$$|z| \le |z - z_0| + |z_0| < r + 1 - |z_0 - A| < \cancel{r} + 1 - \cancel{r} < 1 \quad (2).$$

It follows from (1) and (2) that $z \in S$ for any $z \in B(z_0, r)$. Hence, any point $z_0 \in S$ is inner point. $\Rightarrow$ S is open.

2

3. (b). $z = x+iy$

$$f(z) = \frac{z}{1+z} = \frac{x+iy}{(1+x)+iy} = \frac{(x+iy)((1+x)-iy)}{((1+x)+iy)((1+x)-iy)}$$

$$= \frac{(x(1+x)+y^2) + i(y(1+x)-yx)}{(1+x)^2+y^2} = u(x,y)+i\,v(x,y)$$

$$u(x,y) = \frac{x(1+x)+y^2}{(1+x)^2+y^2}, \quad v(x,y) = \frac{y(1+x)-yx}{(1+x)^2+y^2}$$

4 use the inequality

$$|\operatorname{Re} f(z) - \operatorname{Re} L| \leq \sqrt{|\operatorname{Re} f(z) - \operatorname{Re} L|^2 + |\operatorname{Im} f(z) - \operatorname{Im} L|^2}$$

$$= |f(z) - L| < \varepsilon.$$

5. (a). Let $z_1(t) = t$, $t \in R$. Then

$$\lim_{t\to 0} \frac{\operatorname{Re} z_1(t)}{z_1(t)} = \lim_{t\to 0} \frac{t}{t} = 1. \qquad (1)$$

Choose $z_2(t) = t+it$, $t \in \mathbb{R}$. Then

$$\lim_{t\to 0} \frac{\operatorname{Re} z_2(t)}{z_2(t)} = \lim_{t\to 0} \frac{t}{t+it} = \frac{1}{1+i}. \qquad (2)$$

Both $z_1(t)$ and $z_2(t)$ tend to 0 as $t \to 0$, but the limit in (1) and (2) are different therefore $\lim_{z\to 0} \frac{\operatorname{Re} z}{z}$ does not exist.