Chapter 2: Determinat of matrices

2.1 Def The determinet of matrix is scalar
$|A|$

For $n=1$ $\quad A=(a_{11})$ $\quad \det(A)=a_{11}$

For $n=2$ $\quad A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad \det(A) \overset{def}{=} a_{11}a_{22} - a_{21}a_{22}$

For $n=3$ $\quad A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\quad \det(A) \underset{\sim}{\overset{def}{=}} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

⋮

Expansion based on the first row

Def:

The **determinant** of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j}\det(M_{1j}) \qquad j = 1, \ldots, n$$

are the cofactors associated with the entries in the first row of A.

2.1 Determinat of a matrix

① Def: $\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$ — expansion on the first row

$A_{ij} = (-1)^{i+j}\det(M_{ij})$: cofactor of A

M_{ij}: determinat of an $(n-1)\times(n-1)$ matrix obtained from A by deleting ith row and jth column

② theorem

2.1.1 *If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A.*

Expansion on ith row

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
$$= \underline{a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}}$$

Expansion on jth column

for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

proof:

Theorem 2.1.2 *If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.*

proof: let A be an $n\times n$ matrix, we prove the theorem by math induction

(i) for $n=1$ $A^T = A$, $\det(A^T) = \det(A)$

(ii) Assume the theorem hold for any $k\times k$ matrix, we study the case $k+1$

By Def $\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{k+2}\det(M_{k+1})$

$= a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + \cdots (-1)^{k+2}\det(M_{k+1}^T)$

$= \det(A^T)$ (expansion on the first row)

Theorem 2.1.3 *If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A.*

$$\det\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & a_{33} & \\ * & & & \ddots\ a_{nn} \end{bmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}$$

Theorem 2.1.4 *Let A be an $n \times n$ matrix.*

(i) *If A has a row or column consisting entirely of zeros, then* $\det(A) = 0$.
(ii) *If A has two identical rows or two identical columns, then* $\det(A) = 0$.

③ Calculation

proof: $\det(EA) = \det(E) \times \det(A)$, for any elementary matrix

Lemma 2.2.1 *Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then*

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$

Elementary matrix

Type I : $\det(E_{I}) = -1$ (interchange two rows)

Type II : $\det(E_{II}) = k$ ($r_n = kr_n$)

Type III : $\det(E_{III}) = 1$ ($r_i = r_i + kr_j$)

2.2 properties of Determinant

① operations

$\det(EA) = \det(E)\det(A)$

② Calculation

$E_k \cdots E_2 E_1 A = U$

$\det(E_k) \cdots \det(E_1)\det(A) = \det(U)$

2.2 properties of Determinat

① operation $\Rightarrow \det(EA) = \det(E)\det(A)$ $\qquad \det(E) = \begin{cases} -1 \\ k \\ 1 \end{cases}$

② calculation

$E_k \cdots E_1 A = U$

$$\det(A) = \frac{\det(U)}{\det(E_k)\det(E_{k-1})\cdots\det(E_1)}$$

③ Theorm

Theorem 2.2.2 *An $n \times n$ matrix A is singular if and only if*

$$\det(A) = 0$$

proof: $E_k \cdots E_1 A = U$, if A is singular, the last row of U must be 0

so $\det(U) = 0$, and $\det(E_1) \cdots \det(E_k)\det(A) = \det(U) = 0$

So $\det(A) = 0$

if $\det(A) = 0$, and A is nonsingular, $E_k \cdots E_1 A = I$

$\det(E_1)\det(E_2)\cdots\det(E_k)\det(A) = 1$

abide contradict with the assumption $\det(A) = 0$

The following are equivalent

Theorem 1.5.2 Equivalent Conditions for Nonsingularity

Let A be an $n \times n$ matrix. The following are equivalent:

(a) *A is nonsingular.*
(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.*
(c) *A is row equivalent to I.*
(d) $Ax = b$ has a unique solution for any b
(e) $\det(A) \neq 0$

Theorem 2.2.3 *If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B)$$

proof: if A is nonsingular

$E_k \cdots E_1 A = I$

$A = E_1^{-1} \cdots E_k^{-1}$

$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1} B) = \det(E_1^{-1} \cdots E_k^{-1}) \det(B) = \det(A) \det(B)$

if A is singular

$E_k \cdots E_1 A = U$ $\quad A = E_1^{-1} \cdots E_k^{-1} U$, and the last row of U must be 0

$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1} UB) = \det(E_1^{-1}) \cdots \det(E_k^{-1}) \det(UB)$

since the last row of UB is also zero

then $\det(AB) = 0 = \det(A)\det(B)$

④ Properties:

(i) $\det(A^n) = [\det(A)]^n$

(ii) if A invertible $\quad \det(A^{-1}A) = \det(A^{-1})\det(A) = 1$

$\det(A^{-1}) = \frac{1}{\det(A)}$

(iii) $\det(kA) = k^n \det(A)$

(iv) $\det(A^T A) = \det(A^T)\det(A) = [\det(A)]^2$ only if A is a square matrix

if A is a $m \times n$ matrix

$$\begin{vmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{vmatrix} = \det(A_{11}) \det(A_{22})$$

2.3 Cramer's rule $\quad A_{n\times n}\, x = b$

① Adjoint matrix

cofactor: $A_{ij} = (-1)^{i+j} \det(M_{ij}) \quad i,j = 1,2,\cdots n$

Adjont of A: $\operatorname{adj}(A) = (A_{ij})^T$

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{n1} & - & - & A_{nn} \end{pmatrix}^T$$

② Theorm let A be an $n\times n$ nonsingular matrix

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A \quad \text{when } \det(A) \neq 0$$

$A \cdot \operatorname{adj}(A) = (C_{ij})$

$$C_{ij} = \sum a_{il} A_{jl} = \begin{cases} 0 & i\neq j \\ \det(A) & i=j \end{cases} \Rightarrow \text{so } C = \det(A)\cdot I = \begin{pmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{pmatrix}$$

so $A \cdot \operatorname{adj}(A) = \det(A) \cdot I$, Therefore. $A \cdot \frac{\operatorname{adj}(A)}{\det(A)} = I$ $\Rightarrow A^{-1}$

③ Cramer's rule

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$$A_i = \begin{pmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & b_{2i} & \cdots & \vdots \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{pmatrix}$$: replace nth column by b

proof: $Ax = b \Rightarrow x = A^{-1}b \Rightarrow x = \frac{1}{\det(A)} \operatorname{adj}(A) \cdot b$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ A_{12} & \cdots & A_{n2} \\ - & - & - \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_i = \frac{1}{\det A} (A_{1i} b_1 + A_{2i} b_2 + \cdots A_{ni} b_n)$$