MATH4205/MATH7620 Probability Theory and Stochastic Processes 2025-2026, Semester 1

Assignment 4

Name Student ID Marks

Instructions:

- Due date: 11:59 PM 23 Nov 2025 (Sun)
- 20 marks will be deducted for every 24 hours (rounded up) of late submission
- Submit a soft copy (in PDF) to Moodle
- Layout the intermediate steps systematically

1. (20 marks) The present price of a stock is \$400. The price at time 1 is assumed to (in present value dollars) be either \$300, \$400, or \$500. An option to purchase one share of the stock at time 1 for the (present value) price K = \$450 costs c dollars. Determine the range of c so that there is no arbitrage opportunity.
2. (20 marks) Let $X(t)$ be the price of a stock at time t. The current price of the stock is $X(0) = 80$, the stock's volatility is $\sigma = 20\%$ per annum, and the interest rate is $r = 4\%$ per annum. Suppose that the payoff of an investment at time $T = 0.25$ is 5 if $70 < X(0.25) < 90$ and 0 otherwise. Determine the current price f of the investment, assuming a geometric Brownian motion model for $X(t)$.
3. (20 marks) The goal of this exercise is to show that $\int_0^t B_s ds \sim N(0, \frac{t^3}{3})$.

   (a) Show that

   $$\int_0^t B_s ds = tB_t - \int_0^t s dB_s$$

   (b) Use part (a) to explain why $\int_0^t B_s ds$ is normally distributed.

   (c) Use part (a) and Proposition 8.7 to show that

   $$E\left[\int_0^t B_s ds\right] = 0$$

   (d) Use part (a) and Proposition 8.7 to show that

   $$E\left[\left(\int_0^t B_s ds\right)^2\right] = \frac{t^3}{3}$$

4. (10 marks) Suppose that X_t is an $It\hat{o}$ process such that $dX_t = B_t dt + t dB_t$. Determine the differential $d(te^{X_t})$.
5. (10 marks) Determine the integral $\int_0^t e^{s/2} \sin B_s dB_s$.
6. (10 marks) Determine if the following stochastic differential equation is exact. If yes, solve it for $t \geq 0$ and for an arbitrary initial value X_0.

   $$dX_t = (3t^2 B_t^2 + t^3)dt + (2t^3 B_t + 1)dB_t$$

7. (10 marks) Find the solution of the stochastic differential equation

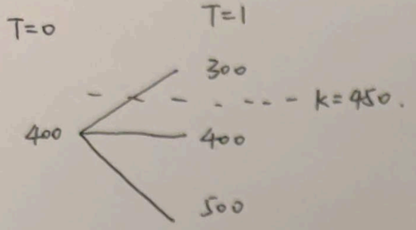
   $$dX_t = (2X_t + 1)dt + e^{2t} dB_t$$

   For $t \geq 0$ and for an arbitrary initial value X_0.

~End~

Assignment 4.

1.



Suppose $p(S_1=300)=p$

$p(S_1=400)=q \Rightarrow p+q+r=1.$

$p(S_1=500)=r$

$E[S_1]=300p+400q+500r=400$

$$\begin{cases} 300p+400q+500r=400 \\ p+q+r=1 \end{cases} \Rightarrow. \begin{cases} r=p \\ q=1-2p \end{cases}$$

since $p,q,r>0$,

$1-2p>0$. $p<\frac{1}{2}$.

so $0<p<\frac{1}{2}$. $c=(500-450)\cdot r=50r\in(0,25)$.

so $c\in(0,25)$.

2. The stock price follows a geometric Brownian motion under risk-neutral measure.

$dX(t)=r\,X(t)dt+\sigma X(t)dW(t)$. $r=0.04$, $\sigma=0.2$, $W(t)$ is standard Brownian emotion.

$X(t)=X(0)\exp\{(r-\frac{\sigma^2}{2})T+\sigma W(T)\}$.

the current price $f=e^{-rT}\cdot E^Q[\text{payoff}]=e^{-rT}\cdot 5\cdot p^Q(70<X(T)<90)$.

$$\begin{cases} \text{mean of } \ln X(T):\ \ln X(0)+(r-\frac{\sigma^2}{2})T=\ln(80)+0.02\times0.25=\ln 80+0.005\ \cancel{\approx 4.387} \\ \text{Variance of } \ln X(T):\ \sigma^2T=(0.2)^2\times0.25=0.01. \end{cases}$$

So $\ln X(T)\sim N(\overset{\ln 80+0.05}{\cancel{4.387}},0.01)$.

So $p^Q(70<X(T)<90)=p^Q(\ln 70<\ln X(T)<\ln 90)$.

$=p\left(\frac{\ln 70-(\ln 80+0.05)}{\cancel{0.07}\ 0.1}<Z<\frac{\ln 90-\cancel{4.387}(\ln 80+0.05)}{\cancel{0.07}\ 0.1}\right)$.

≈ 0.7873.

So $f=5\cdot e^{-0.04\times0.25}\times0.7873\approx3.8975$.

3. (a) for Itô's formula, $df(t,B_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial B^2} dt$.

let $f(t,B_t) = tB_t$, then $\frac{\partial f}{\partial t} = B_t$, $\frac{\partial f}{\partial B} = t$, $\frac{\partial^2 f}{\partial B^2} = 0$.

$$d(tB_t) = B_t\,dt + t\,dB_t.$$

$$\int_0^t d(sB_s) = \int_0^t B_s\,ds + \int_0^t s\,dB_s$$

Since $B_0 = 0$, left integral $= tB_t - 0\cdot B_0 = tB_t$

so $$tB_t = \int_0^t B_s\,ds + \int_0^t s\,dB_s$$

$$\int_0^t B_s\,ds = tB_t - \int_0^t s\,dB_s$$

(b). Since $B_t \sim N(0,t)$, so $tB_t \sim N(0,t^3)$

$\int_0^t s\,dB_s$ is an Itô integral with deterministic integrand, so it's normally distributed with mean 0, variance $\int_0^t s^2\,ds = \frac{t^3}{3}$.

So both terms are normally distributed, the linear combination is also distributed.

(c) $E[\int_0^t B_s\,ds] = E[tB_t - \int_0^t s\,dB_s] = tE[B_t] - E[\int_0^t s\,dB_s] = 0 - 0 = 0$.

(d) $E[(tB_t - \int_0^t sB_s)^2] = E[t^2B_t^2 - 2tB_t\int_0^t s\,dB_s + (\int_0^t sB_s)^2]$

$E[t^2B_t^2] = t^2E[B_t^2] = t^2\cdot t = t^3$. $E[(\int_0^t s\,dB_s)^2] = \int_0^t s^2\,ds = \frac{t^2}{3}$

$E[B_t\int_0^t s\,dB_s] = \int_0^t s\,ds = \frac{t^2}{2}$

so $E[(tB_t - \int_0^t sB_s)^2] = t^3 - 2t\cdot\frac{t^2}{3} + \frac{t^3}{3} = \frac{t^3}{3}$

so $E[(\int_0^t B_s\,ds)^2] = \frac{t^2}{3}$

4. let $f(t, X_t) = te^{X_t}$, for Itô's formula, $df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x} dt$

$\frac{\partial f}{\partial t} = e^x$, $\frac{\partial f}{\partial x} = te^x$, $\frac{\partial^2 f}{\partial x^2} = te^x$.

$$d(te^{X_t}) = e^{X_t} dt + te^{X_t} dt + \frac{1}{2}(te^{X_t})(t^2 dt).$$

Since $dX_t = B_t dt + t dB_t$

$$d(te^{X_t}) = e^{X_t} dt + te^{X_t} (B_t dt + t dB_t) + \frac{1}{2} te^{X_t} \cdot t^2 dt.$$

$$= e^{X_t} (1 + tB_t + \frac{1}{2}t^3) + t^2 e^{X_t} dB_t.$$

5. let $G(x,t)$, $\frac{\partial G}{\partial x} = e^{\frac{t}{2}} \sin x$.

so $G(x,t) = \int e^{\frac{t}{2}} \sin x \, dx = e^{\frac{t}{2}} \int \sin x \, dx = -e^{\frac{t}{2}} \cos x + h(t)$

$\cancel{\frac{\partial^2 G}{\partial t}}$

Since for Itô's formula, we have

$$dG(x,t) = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial x} dB_t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} dt.$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}(e^{\frac{t}{2}} \sin x) = e^{\frac{t}{2}} \cos x.$$

$$\frac{\partial G}{\partial t} = (-e^{\frac{t}{2}} \cos x + h'(t)) = -\frac{1}{2} e^{\frac{t}{2}} \cos x + h'(t).$$

so $dG(x,t) = h'(t) dt + e^{\frac{t}{2}} \cancel{\sin} \sin(B_t) dB_t$, set $h'(t) = 0$, the $dG = e^{\frac{t}{2}} \cancel{\sin} \sin(B_t) dB_t$

$G(t,x) = -e^{\frac{t}{2}} \cos x + C$, by the defination of Itô integral

$$\int_0^t e^{\frac{s}{2}} \cancel{\sin(B_s)} \sin(B_s) ds = G(t, B_t) - G(0, B_0)$$

$$= -e^{\frac{t}{2}} \cos B_t + C - (-1 + C) = 1 - e^{\frac{t}{2}} \cos B_t.$$

at $t = 0$, the integral $1 - e^0 \cdot \cos 0 = 1 - 1 \cdot 1 = 0$.

So the final answer is $1 - e^{\frac{t}{2}} \cos(B_t)$

6. The equation is said to be exact if

$$dX(t) = M(t, B_t)dt + N(t, B_t)\, dB_t.$$

let $M(t,x) = 3t^2x^2 + t^3$, $N(t,x) = 2t^3x + 1$, wher $x = B_t$

$$\frac{\partial F}{\partial x} = N(t,x) = 2t^3x+1. \quad F = \int (2t^3x+1)dx = t^3x^2 + x + g(t).$$

then $\frac{\partial F}{\partial t} = 3t^2x^2 + g'(t)$. $\frac{\partial^2 F}{\partial x^2} = 2t^3$.

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 3t^2x^2 + g'(t) + \frac{1}{2}\cdot 2t^3 = 3t^2x^2 + g'(t) + t^3$$

$= M(t,x)$. So $g'(t) = 0$, $g(t) = C$.

$F(t,x) = t^3x^2 + x + C$ the SDE is exact.

so $dX_t = dF(t, B_t)$, so $X(t) = F(t, B_t) + C_1 = t^3B_t^2 + B_t + C_1 + C$

$X_0 = C_1 + C + B_0 = 0$. so $X_t = X_0 + B_t + t^3 B_t^2$

7. the SDE is of the form $dX_t = (a(t)X_t + b(t))dt + \sigma(t)dB_t$.

$a(t) = 2$, $b(t) = 1$, $\sigma(t) = e^{2t}$, the solution is

$$X_t = \Phi(t)\left[X_0 + \int_0^t \Phi(s)^{-1} b(s)\, ds + \int_0^t \Phi(s)^{-1}\sigma(s)\, dB_s\right].$$

$$\Phi(t) = e^{\int_0^t a(s)\, ds} = e^{\int_0^t 2\, ds} = e^{2t}, \quad \Phi^{-1}(t) = e^{-2t}$$

$$X(t) = e^{2t}\left[X_0 + \int_0^t e^{-2s}\cdot 1\, ds + \int_0^t e^{-2s}\cdot e^{2s}\, dB_s\right].$$

$$\int_0^t e^{-2s}\, ds = \left[-\frac{1}{2}e^{-2s}\right]_0^t = \frac{1}{2}(1 - e^{-2t}).$$

$$\int_0^t dB_s = B_t - B_0 = B_t.$$

$$\text{So } X_t = e^{2t}\left[X_0 + \frac{1}{2}(1 - e^{-2t}) + B_t\right].$$

Q2: payoff of investment

$$\text{payoff} = \begin{cases} 5 & (P_1) \quad 70 < X(0.25) < 90 \\ 0 & (1-P_1) \quad \text{otherwise} \end{cases}$$

$$f = [P_1 \cdot 5 + (1-P_1)\cdot 0]\cdot e^{-rT} \qquad r = 0.04,\ T = 0.25$$

$$P_1 = P\{70 < X(0.25) < 90\}$$

$$X(0.25) = X(0)\cdot \exp\{Z(0.25)\} \qquad Z(t) = \sigma B(t) + (r - \frac{\sigma^2}{2})t$$

$$Z(0.25) = 0.2B(0.25) + (0.04 - \frac{0.04}{2})\cdot 0.25$$

$$= 0.2B(0.25) + 0.005$$

$$P_1 = P\{70 < 80\cdot \exp\{0.2B(0.25) + 0.005\} < 90\}.$$

$$= 0.7873$$

$$f = e^{-0.04\times 0.25}\cdot 0.7873 \cdot 5 = 3.8975.$$

Q3 (b) $\int_0^t B_s\,ds = tB_t - \int_0^t s\,dB_s$

$=$

$(tB_t,\ \int_0^t s\,dB_s)$ $\qquad \alpha tB_t + \beta\int_0^t s\,dB_s = \underline{\int_0^t(\alpha t + \beta s)\,dB_s}$

Q5. $d(f(t, \overset{x}{B_t})) = f_t\,dt + f_x\,dB_t + \frac{1}{2}f_{xx}\,dt$

$$f_x = e^{\frac{t}{2}}\cdot \sin x$$

$$f(t,x) = e^{\frac{t}{2}}(-\cos x) \qquad f_t = -\frac{1}{2}e^{\frac{t}{2}}\cos x \qquad f_{xx} = e^{\frac{t}{2}}\cos x$$

$$d(-e^{\frac{t}{2}}\cos B_t) = (-\frac{1}{2}e^{\frac{t}{2}}\cdot\cos B_t + \frac{1}{2}\cdot e^{\frac{t}{2}}\cdot\cos B_t)\,dt + e^{\frac{t}{2}}\sin B_t\,dB_t$$

$$-e^{\frac{t}{2}}\cdot\cos B_t - (-e^0\cdot\cos 0) = \int_0^t e^{\frac{s}{2}}\sin B_s\,dB_s$$

$$\int_0^t e^{\frac{s}{2}}\sin B_s\,dB_s = 1 - e^{\frac{t}{2}}\cos B_t$$