Chapter 3: Vector Space

3.1 Def: let V be a set of elements, on which the operations if addition $+$ and scalar multiplication $\circ$ are defined that satisfy the two closure properties and eight axioms below

C1 For each, $x, y \in V$, $x+y \in V$

C2 For each $\alpha \in \mathbb{R}$, and $y \in V$, $\alpha y \in V$

A1 $x+y = y+x$, for any $x, y \in V$

A2 $x+(y+z) = (x+y)+z$, for all $x, y, z \in V$

A3 There exists a zero element $0 \in V$ such that $x+0=x$ for all $x \in V$

A4 for each $x \in V$, there exists an additive $y \in V$, such that $x+y=0$

A5 $\alpha(x+y) = \alpha x + \alpha y$, for $x, y \in V$, $\alpha \in \mathbb{R}$

A6 $(\alpha+\beta)x = \alpha x + \beta x$, for $x \in V$, $\alpha, \beta \in \mathbb{R}$

A7 $(\alpha\beta)x = \alpha(\beta x)$, for all $x \in V$, $\alpha, \beta \in \mathbb{R}$

A8 $1x = x$, for all $x \in V$

Then the set V with two operations forms a vector space.

② Example:

$\mathbb{R}^n$, $\mathbb{R}^{m\times n}$

$P_n = \{a_0 + a_1 x + \cdots a_n x^n \mid a_i \in \mathbb{R}\}$ $\{set +, \cdot\}$

Ex: $V = \{\binom{x}{y} \mid x, y \in \mathbb{R}\}$

Define $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x_1+y_1 \\ x_2+y_2 \end{Bmatrix}$

$\alpha \in \mathbb{R}$, $\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ 0 \end{bmatrix}$

Q: is $\{V, +, \cdot\}$ a vector space?

A8: if $\alpha = 1$, $1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ doesn't satisfy

V isn't an vector space

③ Theorem

Theorem 3.1.1 *If V is a vector space and* $\mathbf{x}$ *is any element of V, then*

(i) $0\mathbf{x} = \mathbf{0}$.
(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ *implies that* $\mathbf{y} = -\mathbf{x}$ *(i.e., the additive inverse of* $\mathbf{x}$ *is unique).*
(iii) $(-1)\mathbf{x} = -\mathbf{x}$.

(i) Proof: $0 = -x + x = -x + (1+0)x = -x + x + 0x = 0 + 0x = 0x$

So. $\underset{\text{sclar}}{0}x = \underset{\text{element}}{0}$

3.2 subspace

① Def.

If S is a nonempty subset of a vector space V, and S satisfies the conditions

(i) $\alpha\mathbf{x} \in S$ whenever $\mathbf{x} \in S$ for any scalar α

(ii) $\mathbf{x}+\mathbf{y} \in S$ whenever $\mathbf{x} \in S$ and $\mathbf{y} \in S$

then S is said to be a **subspace** of V.

Theorem:

A subspace of a vector space is a vector space

Ex: Is $S=\left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_1+x_2+x_3=1\right\}$ a subspace

let $x_1+x_2+x_3=1$, $y_1+y_2+y_3=1$

$z_i = x_i + y_i$

$z_1+z_2+z_3=2 \notin S$

So S isn't a subspace

Two trivial subspace

① Zero subspace $\{0\}$

② V is a subspace of itself

② the Null space

Let A be a $m \times n$ matrix

The null space of A: $N(A) = \{x \in \mathbb{R}^n \mid Ax=0\} \subset \mathbb{R}^n$

Theroem: The null space of $A_{m\times n}$ is a subspace of $\mathbb{R}^n$

Proof: (i) $0 \in N(A)$ (ii) for any $u, v \in N(A)$, $Au=0$, $Av=0$

$Au+Av = A(u+v)=0$, which shows $u+v$ is a null space

(iii) for every $u \in Null(A)$ $Au=0$, the $A(\alpha u)=0$, so $\alpha u \in Null(A)$

Thus we can prove $Null(A)$ is a subspace

③ Spanning space (subspace)

Def.

> Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space V. A sum of the form $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$, where $\alpha_1, \ldots, \alpha_n$ are scalars, is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ will be denoted by $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Ex: $A_{m\times n} = \{a_1, a_2, a_3, \cdots a_n\}$

$\mathrm{Span}\{a_1\ a_2 \cdots a_n\} \subset \mathbb{R}^m$

For any vector $V \in \mathrm{Span}\{a_1, a_2 \cdots a_n\}$

$$V = \alpha_1 a_1 + \alpha_2 a_2 + \cdots \alpha_n a_n$$

$$= A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$Ax = b$ has a sol. if and only if $b \in \mathrm{Span}\{a_1, a_2 \cdots a_n\}$

Ex: let $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$ let $Ax = 0$

$U = \begin{pmatrix} 1 & 0 & 3 & 0 & 7 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$x = \begin{pmatrix} -3\alpha - 7\beta \\ -\alpha - 3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + \beta \underbrace{\begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{V_2}$$

$\in \mathrm{span}\{V_1, V_2\}$

④ Spanning set

> The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a **spanning set** for V if and only if every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Ex: $\mathbb{R}^2 = \mathrm{span}\left\{ \overset{e_1}{\binom{1}{0}}, \overset{e_2}{\binom{0}{1}} \right\}$

$\quad \parallel$

$\left\{ \binom{x_1}{x_2} \mid x_i \in \mathbb{R} \right\}$

for any $x \in \mathbb{R}^2$, $x = x_1 e_1 + x_2 e_2$

(e_1, e_2) is a spanning set of $\mathbb{R}^2$

Ex: let $P_3(x)$ be a polynomial set of degree no less than 3

Is $\{x^2, x+x^2, -x-x^2\}$ a spanning set of P_3

let $P_3(x) = \{a_0 + a_1 x + a_2 x^2\}$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 x^2 + \alpha_2 (x + x^2) + \alpha_3 (-x - x^2)$$

$$= (\alpha_1 + \alpha_2 - \alpha_3) x^2 + (\alpha_2 - \alpha_3) x$$

$$1 \notin (a_1 + a_2 - a_3) x^2 + (a_2 - a_3) x$$

⑤ linear system

$Ax = b$

Theorem

Theorem 3.2.2 *If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{x}_0$ is a particular solution, then a vector $\mathbf{y}$ will also be a solution if and only if $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ where $\mathbf{z} \in N(A)$.*

3.3 Linear independence

① Def,

Definition The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are said to be **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

implies that all the scalars $c_1, \ldots, c_n$ must equal 0.

② In $\mathbb{R}^m$

let $v_1 = \{1, 2, \ldots\}$ $v_2 = \{2, 1, \ldots\}$ $v_3 = \{4, -1, \ldots\}$

Is $\{v_1, v_2, v_3\}$ linear independence

Given $v_1, v_2 \cdots v_k \in \mathbb{R}^m$

The system $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$

is equivalent to

$$[v_1, v_2 \cdots v_k] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = 0$$

$$\Updownarrow$$

$$Ac = 0$$

③ Theorem

Theorem 3.3.1 *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{R}^n$ and let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular.*

Lemma: Given $v_1, v_2 \cdots v_n$, one is linear dependent if and only if one is linear Combanation of others.

Proof: If $v_1 = \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n$,

then $(-1)v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$

$$\begin{bmatrix} -1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \neq 0 \quad \text{is a solution}$$

So $\{v_1, v_2 \cdots v_n\}$ is linear dependent.

$\Rightarrow$ If $(v_1, v_2 - v_n)$ are linear dependent

the System $c_1v_1 + c_2v_2 + \cdots c_nv_n = 0$

has a nonzero sol.

* let $\{v_1, v_2, \cdots v_k\}$ be linearly independent

$\{v_1, v_2 \cdots v_{k-1}\}$ is still linearly independent proof: the system $\alpha_1 v_1 + \cdots \alpha_k v_k = 0$ has only zero sol.

If $\{v_1, v_2 \cdots v_{k-1}\}$ is linearly dependent

there exist $\{\beta_1, \beta_2 \cdots \beta_{k-1}\}$, not all zero

Such that $\beta_1 v_1 + \beta_2 v_2 + \cdots \beta_{k-1} v_{k-1} + 0 \cdot v_k = 0$

which shows that $\{v_1, v_2 \cdots v_k\}$ is linearly dependent

and contradicts with the assumption

Theorem 3.3.2 *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. A vector $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ can be written uniquely as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.*

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$, uniquely $\alpha_1, \alpha_2 \cdots \alpha_k$

proof: Asume that $\{v_1, v_2 \cdots v_k\}$ is linear independent

If for $u \in \mathrm{span}\{v_1, v_2 \cdots v_k\}$

and $\begin{cases} u = \alpha_1 v_1 + \cdots \alpha_k u_k \\ u = \beta_1 v_1 + \cdots \beta_k v_k \end{cases}$

$0 = (\alpha_1 - \beta_1) v_1 + \cdots (\alpha_k - \beta_k) v_k$

Since $\{v_1, \cdots v_k\}$ is linear independent

$\alpha_1 - \beta_1 = 0 \cdots \alpha_k - \beta_k = 0$

So we prove that's uniquely combined

3.4 Basis and dimension

① Def

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a **basis** for a vector space V if and only if

(i) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

(ii) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V.

what about $N(A)$?

Ex: let $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$ $\quad N(A) = \{x \mid Ax = 0\}$

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow x = \begin{Bmatrix} -3\alpha - 7\beta \\ -\alpha - 3\beta \\ \alpha \\ \beta \\ 0 \end{Bmatrix} = \alpha \underbrace{\begin{Bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}}_{v_1} + \beta \underbrace{\begin{Bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{Bmatrix}}_{v_2} = \text{span}\{v_1, v_2\} = N(A)$$

and v_1, v_2 are linear independent, so they are basis of $N(A)$

② Theorem

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.

proof: let $u_1, u_2 \cdots u_m \in V$

Consider $c_1u_1 + c_2u_2 + \cdots c_mu_m = 0$, $u_i \in V$, so $u_i = \alpha_{i1}v_1 + \alpha_{i2}v_2 + \cdots \alpha_{in}v_n$

$$u_i = \sum_{j=1}^{n} \alpha_{ij} v_j$$

$$\sum_{i=1}^{m} c_i u_i = \sum_{i=1}^{m} c_i \sum_{j=1}^{n} \alpha_{ij} v_j = 0$$

$$\sum_{j=1}^{n} v_j \boxed{\sum_{i=1}^{m} \alpha_{ij} c_i} = 0$$ enforced to be 0, possible?

Let $\sum_{i=1}^{m} \alpha_{ij} c_i = 0$ $\quad A = (a_{ij}) \quad Ac = \sum a_{ij} c_i$

$A^T c = 0$, $A^T = n \times m$, $m > n$, $\boxed{\quad}[\,] = 0$, so this system has non zero c

since $m > n$, there exists $c \neq 0$, so $\{u_1, u_2 \cdots u_m\}$ is linear dependent

Corollary 3.4.2 *If both $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ are bases for a vector space V, then $n = m$.*

③ Dimenson

If $(v_1, v_2 - v_n)$ is a basis of V, then the dimension of V is $\dim(V) = n$

Theorem 3.4.3 *If V is a vector space of dimension $n > 0$, then*

(I) *any set of n linearly independent vectors spans V.*
(II) *any n vectors that span V are linearly independent.*

proof: (i) let $v_1, v_2 \cdots v_n$ be linearly independent $\Rightarrow V = \text{span}\{v_1, v_2 - v_n\}$

For any $x \in V$. $\{v_1, v_2 \cdots v_n\ x\}$ must be linearly dependent

(other wish, $\dim(V) = n+1$)

$c_1v_1 + c_2v_2 + \cdots c_{n+1} \cdot x = 0$, c_j are not alway equal to 0

which shows $c_{n+1} \neq 0$, since $\{v_1, v_2 - v_n\}$ are linear independent

So $x = -\frac{c_1}{c_{n+1}}v_1 - \frac{c_2}{c_{n+1}}v_2 - \cdots \frac{c_n}{c_{n+1}}v_n \in \text{span}(v_1, v_2 \cdots v_n)$

Therefore $V = \text{span}\{v_1, v_2 \cdots v_n\}$ (a basis)

(ii) let $V = \text{span}(u_1, u_2 - u_n)$

if $u_1, u_2 \cdots u_n$ are not linear independent

$c_1u_1 + c_2u_2 + - c_nu_n = 0$ for c_j not all 0

Assume that $c_n \neq 0$, then u_n is the linear combination of others

$u_n = -\frac{c_1}{c_n}u_1 - \frac{c_2}{c_n}u_2 - \cdots - \frac{c_{n-1}}{c_n}u_{n-1}$

Thus $V = \text{span}\{u_1, u_2 - u_{n-1}\}$

if $u_1, u_2 - u_{n-1}$ are linearly independent

then $u_1, u_2 \cdots u_n$ is a basis, which contradicts with $\dim(V) = n$

Theorem 3.4.4 *If V is a vector space of dimension $n > 0$, then*

(i) *no set of fewer than n vectors can span V.*
(ii) *any subset of fewer than n linearly independent vectors can be extended to form a basis for V.*
(iii) *any spanning set containing more than n vectors can be pared down to form a basis for V.*

3.5 change of Basis

① Def Let V be a vector space, $\dim(V)=n$

$\{v_1, v_2, \cdots v_n\}$ is a basis of V

for any $x \in V = \text{span}\{v_1, v_2 \cdots v_n\}$

there exists unique $\alpha_1, \alpha_2 \cdots \alpha_n$ such that

$X = \alpha_1 V_1 + \alpha_2 V_2 + \cdots \alpha_n V_n$

Suppose that $\beta = \{v_1, v_2 \cdots v_n\}$ is a ordered basis for V

and $v \in V$

then $[v]_\beta = \begin{Bmatrix} c_1 \\ \vdots \\ c_n \end{Bmatrix}$ is called coordinate vector of V with respect to β

Ex: for a polynomial $P(x) = x(x-1)(x-2) = x^3 - 3x + 2x$

$P_4(x) = \text{Span}\{1, x, x^2, x^3\}$

$[P]_B = \begin{Bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{Bmatrix}$

② Transition matrix

let $\beta = \{u_1, u_2 \cdots u_n\}$ and $\gamma = \{w_1, w_2 \cdots w_n\}$

let two Bases form a vector space V

The $n \times n$ matrix

$U = \{[w_1]_\gamma, [w_2]_\gamma \cdots [w_n]_\gamma\}$ is called the transition matrix

from β to γ

for $u_1 \in V$

$u_1 = \alpha_1 w_1 + \alpha_2 w_2 + \cdots \alpha_n w_n$

$[u_1]_\gamma = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$v \in V$, $v = u_1\beta_1 + \cdots u_n\beta_n$

$[v]_\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

$v = w_1\gamma_1 + \cdots w_n\gamma_n$

$[v]_\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$

Theorem, $\beta = [u_1, u_2 \cdots u_n]$, $\gamma = [w_1, \cdots w_n]$: bases of V

U: transition matrix, For any $v \in V$

(i) $[v]_\gamma = U[v]_\beta$ (ii) U: nonsingular

Since $u_j \in V$

$u_j = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{nj}w_n$

for any $v \in V$

$v = c_1u_1 + c_2u_2 + \cdots + c_nu_n = \Sigma c_ju_j = \sum_{j=1}^{n} c_j \sum_{i=1}^{n} a_{ij}w_i$

$= \sum_{i=1}^{n} w_i \underbrace{\sum_{j=1}^{n} a_{ij}c_j}_{b_i}$

$= \sum_{i=1}^{n} w_i b_i$

$= w_1b_1 + w_2b_2 + \cdots + w_nb_n \Rightarrow [v]_\gamma = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$[v]_\gamma = A[v]_\beta = U[v]_\beta$

(ii) If U is singular

there exists $x \neq 0$, $Ux = 0$

Choose $[v_\beta] = x$ to get $v = x_1v_1 + x_2v_2 + \cdots x_nv_n$

$\Rightarrow [v_\beta] = 0$

* $v = c_1u_1 + \cdots c_nu_n$ uniquely

$\Rightarrow v = 0$ $\qquad 0 = x_1 u_1 + x_2 u_2 + \cdots x_n u_n$

$0 = 0 \cdot u_1 + \cdots 0 \cdot u_n$

$\Rightarrow x = 0$

Ex: $V = \mathbb{R}^n$

$(u_1 \cdots u_n) = \tilde{U}$ $\quad n \times n$ matrix

$(w_1 \cdots w_n) = \tilde{W}$ $\quad n \times n$ matrix

For any $v \in \mathbb{R}^n$, $v = \tilde{U} x = x_1 u_1 + x_2 u_2 + \cdots x_n u_n$

$v = \tilde{W} y = y_1 w_1 + y_2 w_2 + \cdots y_n w_n$

$\tilde{U} x = \tilde{W} y$

$y = \underbrace{\tilde{W}^{-1} \tilde{U}}\, x$ $\quad$ transition matrix

$y = \underbrace{U}\, x$

3.6 Row space and column space ☆

① Def

> If A is an $m \times n$ matrix, the subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the **row space** of A. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A.

$A_{m\times n} = (a_1 \cdots a_n)$, span$(a_1, \cdots a_n)$ = Column space of A

$A_{m\times n} = \begin{pmatrix} \tilde{a_1} \\ \vdots \\ \tilde{a_m} \end{pmatrix}$, span$(\tilde{a_1} \cdots \tilde{a_m})$ = row space of A

② Theorem

Theorem 3.6.1 *Two row equivalent matrices have the same row space.*

let $A = \begin{pmatrix} \tilde{a_1} \\ \vdots \\ \tilde{a_m} \end{pmatrix}$ and $B = \begin{pmatrix} \tilde{b_1} \\ \vdots \\ \tilde{b_m} \end{pmatrix}$ be two row equivalent matrix

$A = UB$, $A^T = B^T U^T$

$(\tilde{a_1}^T \cdots \tilde{a_m}^T) = (\tilde{b_1}^T \cdots \tilde{b_m}^T) U$

$\tilde{a_i}^T = u_{1i}\tilde{b_1}^T + u_{2i}\tilde{b_2}^T + \cdots u_{mi}\tilde{b_m}^T$

$\tilde{a_i} \in$ span$(\tilde{b_1} \cdots \tilde{b_m})$ = row space of B

row space of A $\subset$ row space of B

$U^{-1}A = B \Rightarrow$ row space of B $\subset$ row space of A

③ Dimension

$\begin{cases} \text{dim}(N(A)) = \text{nullity of } A \\ \text{dim (row space)} = \text{rank}(A) \\ \text{dim (Column space)} \end{cases}$

$\dim(N(A^T))$

④ Base

Given a matrix A

$E_k \cdots E_2 E_1 A = U$ echelon form, A and U have same row/null space

$$\begin{matrix} v_1 \leftarrow \\ v_2 \leftarrow \\ v_3 \leftarrow \end{matrix} \begin{bmatrix} 1 & \text{---} & & \\ & 1 & \text{---} & \\ & & 1 & \text{---} \\ 0 & \cdots & & 0 \end{bmatrix}$$

base $= \{v_1, v_2, v_3\}$ of row space

Ex: Find a basis of row space of $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -4 \end{pmatrix}$

sol.

$$A \xrightarrow{\text{row operation}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

So $\{(1,-2,3), (0,1,5)\}$ is a basis of row space

and Rank is 2

Question ?

For Given $v_1, v_2 - v_k \in \mathbb{R}^n$, find a basis of $\operatorname{span}\{v_1, v_2 - v_k\}$

$\operatorname{span}\{v_1 -- v_k\}$ is the column space of a matrix $(v_1 -- v_k)$

Find $\dim\{\operatorname{span}(v_1 -- v_k)\} = \operatorname{rank}(B)$

let $v \in \operatorname{span}(v_1 -- v_k)$, $v = x_1 v_1 + \cdots x_k v_k$, $v = Ax$

were $A = (v_1, v_2 -- v_k)$

Ex: $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$ Find a basis for

(i) row space (ii) Column space

(iii) $N(A)$ (iv) $\operatorname{Rank}(A)$

$$A \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 3.6.5 The Rank–Nullity Theorem

If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

Proof Let U be the reduced row echelon form of A. The system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $U\mathbf{x} = \mathbf{0}$. If A has rank r, then U will have r nonzero rows, and consequently the system $U\mathbf{x} = \mathbf{0}$ will involve r lead variables and $n - r$ free variables. The dimension of $N(A)$ will equal the number of free variables. ■

Theorem 3.6.6 *If A is an $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.*

③ linear system

Theorem: (consistency theorem for linear system)

A linear system $Ax=b$ is consist if and only if b is in the column space of A

$A = (a_1, a_2 \cdots a_n)$ $\quad Ax=b \Leftrightarrow x_1a_1 + x_2a_2 + \cdots x_na_n = b$

Theorem 3.6.2 (i) The system $Ax=b$ is constant for any $b \in R^m$ if and only if the column vectors of A spans R^m (Column space of A is R^m)

proof: ? (ii) $Ax=b$ has at most one solution for every $b \in R^m$ if and only if the column vectors of A are linearly independent.

proof: let x_1, x_2 be two solution $\begin{cases} Ax_1 = b \\ Ax_2 = b \end{cases} \Rightarrow A(x_1 - x_2) = 0$

if the column vectors of A is linearly independent

then $Ax=0$ has only zero sol so $x_1 = x_2$. $Ax=b$ has only one solution.

Theorem (Equivalent conditions)

(i) A is nonsingular (ii) $Ax=0$ has only zero sol.

(iii) A is row equivalent to I (iv) $Ax=b$ has exact only one sol.

(v) $\det(A) \neq 0$ (vi) Column vectors of A forms a basis for $\mathbb{R}^n$

(vii) row vector of A forms a basis for $\mathbb{R}^{1\times n}$

(viii) the nullity of A is 0 $\Leftrightarrow \dim(N(A))=0$

④ Remarks

Given $(v_1, v_2 \sim v_n) \in \mathbb{R}^n$, span$(v_1 \sim v_n)$, Find the basis

(ii) $A_{m\times n}$

Column space of $A = \text{Span}(a_1 -- a_n) = \{y' = Ax \mid x \in \mathbb{R}^n\}$

For any $z \in$ Column space of A

the z is a linear combination of A, or there exist a $\alpha \in \mathbb{R}^n$

such that $z = Ax$

$N(A^T) = \{y \mid A^T y = 0\}$, then $A^T y = 0$

Question: what's the relation between z and y?

$y^T z = y^T A x = \underbrace{(A^T y)^T}_{0} x = 0$

⑤ Review

Concept:
- linear combination, span space
- linear independent/dependent
- basis
- demension
- Rank

- $v_1 \sim v_k$ are linearly dependent. then one can be written as the linear combination of others
- $v_1 \cdots v_k$ are linearly independent. the subset is linearly independent
- $v_1 v_2 \cdot v_k \in V$ are linearly independent, for any $v \in V$
  $v_1 \cdots v_k, v$ are linearly dependent
- $\{v_1 \cdots v_k\}$ is a basis for V, for any $y \in V$
  y can be written as the linear combination of $v_1 \sim v_k$ uniquely
- $\dim(V) = n$, then any n linearly independent vectors forms a basis
- $W \subset V$, $\dim(W) \leq \dim(V)$
- let $v_1 \cdots v_k \in R^n$ be linearly independent

and A is a $n\times n$ nonsingular matrix

Then $\{Av_1 \cdots Av_k\}$ are also linearly independent

Ex: let $v_1 \cdots v_m$ be linearly dependent

with $v_1 \neq 0$, Then there exist v_k for index k such that

v_k is a linearly combanation of $\{v_1 - v_{k-1}\}$

⑥ Rank

Theorem: let A be a $m\times n$ matrix

(i) $\mathrm{Rank}(A) + \dim(N(A)) = n$

(ii) $\mathrm{Rank}(A) = \mathrm{Rank}(A^T)$

(iii) $0 \leq \mathrm{Rank}(A) \leq \min(n,m)$

(iv) $\mathrm{Rank} = 0$ if and only if $A=0$

(v) $\mathrm{Rank}(A) = \mathrm{Rank}(A^TA) = \mathrm{Rank}(AA^T)$

(vi) if P is $m\times n$ and Q is $n\times m$, and both are nonsingular

the $\mathrm{Rank}(PAQ) = \mathrm{Rank}(A)$

(vii) $\mathrm{Rank}(A+B) \leq \mathrm{Rank}(A) + \mathrm{Rank}(B)$

(viii) $\mathrm{Rank}(AB) \leq \min\{\mathrm{Rank}(A), \mathrm{Rank}(B)\}$

(ix) $\mathrm{Rank}(A, B) \leq \mathrm{Rank}(A) + \mathrm{Rank}(B)$

proof of (v): $N(A) = N(A^TA)$

since $\mathrm{Rank}(A) + \dim(N(A)) = n$

$\mathrm{Rank}(A^TA) + \dim(N(A^TA)) = n$

so $\mathrm{Rank}(A) = \mathrm{Rank}(A^TA)$

Proof of (vi): let $B = AQ$

so B and A have the same column space

dim (column space of B) = dim (column space of A)

so $\text{Rank}(B) = \text{Rank}(A)$

$\text{Rank}(AQ) = \text{Rank}(A)$

$\text{Rank}(PAQ) = \text{Rank}(PAQ)^T = \text{Rank}(Q^T A^T P^T)$

$= \text{Rank}(Q^T A^T) = \text{Rank}(AQ) = \text{Rank}(A)$