Discrete Mathematics Homework 6

1. (a) Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\overline{G}$, how many edges must G have? (Such a graph is called self-complementary.) 4

(b) Find an example of a self-complementary graph on four vertices and two examples on five vertices. 5

1 2×2

(c) If G is a self-complementary graph on n vertices, where $n > 1$, prove that $n = 4k$ or $n = 4k + 1$, for some $k \in N$.

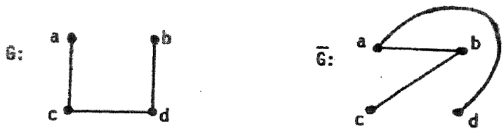
5

(d) Let G be a cycle on n vertices. Prove that G is self-complementary if and only if $n = 5$.

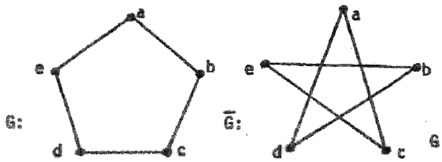
Solution 6 $\Rightarrow$: 3 $\Leftarrow$: 3

(a) Let e_1 be the number of edges in G and e_2 the number in $\overline{G}$. For any (loop-free) undirected graph G, $e_1 + e_2 = \binom{n}{2}$, the number of edges in K_n. Since G is self-complementary, $e_1 = e_2$, so $e_1 = (1/2)\binom{n}{2} = n(n-1)/4$.

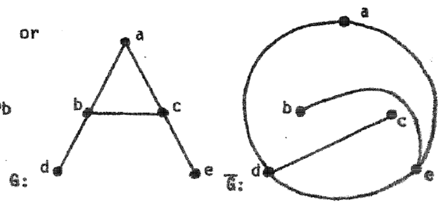
(b) Four vertices:



Five vertices:



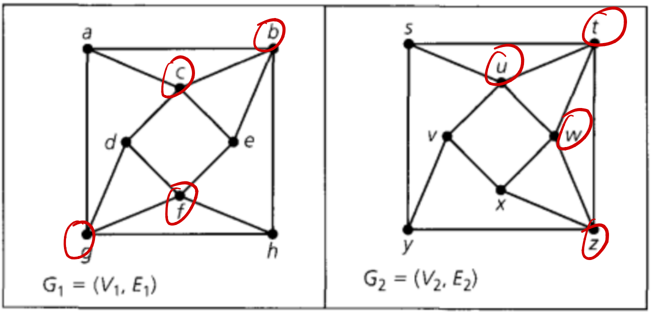
or



(c) From part (a), $4|n(n-1)$. One of n and $n-1$ is even and the other factor odd. If n is even, then $4|n$ and $n = 4k$, for some $k \in \mathbf{Z}^+$. If $n-1$ is even, then $4|(n-1)$ and $n - 1 = 4k$, or $n = 4k+1$, for some $k \in \mathbf{Z}^+$.

(d) If G is the cycle with edges $\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}$ and $\{e,a\}$, then $\overline{G}$ is the cycle with edges $\{a,c\}$, $\{c,e\}$, $\{e,b\}$, $\{b,d\}$, $\{d,a\}$. Hence, G and $\overline{G}$ are isomorphic. Conversely, if G is a cycle on n vertices and $G, \overline{G}$ are isomorphic, then $n = (1/2)\binom{n}{2}$, or $n = (1/4)(n)(n-1)$, and $n = 5$.

2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the loop-free undirected connected graphs in below figure.



(a) Determine $|V_1|, |E_1|, |V_2|,$ and $|E_2|$. 4 (4x1)

(b) Find the degree of each vertex in V_1. Do likewise for each vertex in V_2. 16 (16x1)

(c) Are the graphs G_1 and G_2 isomorphic? 10 (explain : 5)

Solution

(a) $|V_1| = 8 = |V_2|$; $|E_1| = 14 = |E_2|$.

(b) For V_1 we find that $\deg(a) = 3$, $\deg(b) = 4$, $\deg(c) = 4$, $\deg(d) = 3$, $\deg(e) = 3$, $\deg(f) = 4$, $\deg(g) = 4$, and $\deg(h) = 3$. For V_2 we have $\deg(s) = 3$, $\deg(t) = 4$, $\deg(u) = 4$, $\deg(v) = 3$, $\deg(w) = 4$, $\deg(x) = 3$, $\deg(y) = 3$, and $\deg(z) = 4$. Hence each of the two graphs has four vertices of degree 3 and four of degree 4.

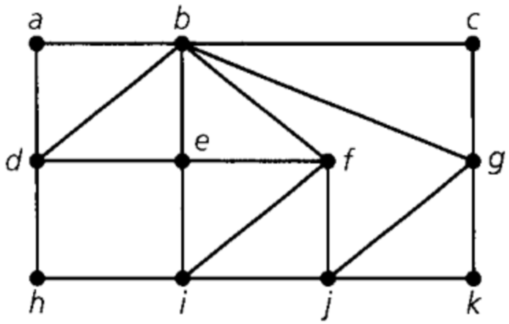
(c) Despite the results in parts (a) and (b) the graphs G_1 and G_2 are *not* isomorphic. 5

In the graph G_2 the four vertices of degree 4 — namely, $t, u, w,$ and z — are on a cycle of length 4. For the graph G_1 the vertices $b, c, f,$ and g — each of degree 4 — do not lie on a cycle of length 4. 5

A second way to observe that G_1 and G_2 are not isomorphic is to consider once again the vertices of degree 4 in each graph. In G_1 these vertices induce a disconnected subgraph consisting of the two edges $\{b, c\}$ and $\{f, g\}$. The four vertices of degree 4 in graph G_2 induce a connected subgraph that has five edges — every possible edge except $\{u, z\}$.

3 (a) Find an Euler circuit for the graph in below figure. 10

(b) If the edge $\{d, e\}$ is removed from this graph, find an Euler trail for the resulting subgraph. 10



Solution

(a) $a \to b \to c \to g \to k \to j \to g \to b \to f \to j \to i \to f \to e \to i \to h \to d \to e \to b \to d \to a$

(b) $d \to a \to b \to d \to h \to i \to e \to f \to i \to j \to f \to b \to c \to g \to k \to j \to g \to b \to e$

$\Rightarrow$: 5 $\Leftarrow$: 5

4 (a) Let $G = (V, E)$ be a directed graph or multigraph with no isolated vertices. Prove that G has a directed Euler circuit if and only if G is connected and $od(v) = id(v)$ for all $v \in V$. 10

(b) If $G = (V, E)$ is a directed graph or multigraph with no isolated vertices, prove that G has a directed Euler trail if and only if

10

(i) G is connected;

(ii) $od(v) = id(v)$ for all but two vertices x, y in V; and

(iii) $od(x) = id(x) + 1$, $id(y) = od(y) + 1$.

Solution

$\Rightarrow$: 5 $\Leftarrow$: 5

deg = od + id

(a) If $G=(V,E)$ has a directed Euler circuit, then for all $x,y \in V$ there is a directed trail from x to y (that part of the directed Euler circuit from x to y). This results in a directed path from x to y, as well as one from y to x. Hence G is connected (in fact, G is strongly connected as defined in part (b) of this exercise). Let s be the starting vertex (and terminal vertex) of the directed Euler circuit. For every $v \in V, v \neq s$, each time the circuit comes upon vertex v it must also leave the vertex, so $\text{od}(v)=\text{id}(v)$. In the case of s the last edge of the circuit is different from the first edge and $\text{od}(s)=\text{id}(s)$.

Conversely, if G satisifies the stated conditons, we shall prove by induction on $|E|$ that G has a directed Euler circuit. For $|E|=1$ the result is true (and the graph consists of a (directed) loop on one vertex). We assume the result for all such graphs with $|E|$ edges where $1 \leq |E| < n$. Now consider a directed graph $G=(V,E)$ where G satisfies the given conditions and $|E|=n$. Let $a \in V$. There exists a circuit in G that contains a. If the loop $(a,a) \notin E$, then there is an edge $(a,b) \in E$ for $b \neq a$. If not, a is isolated and this contradicts G being connected. If $(b,a) \in E$ we have the circuit $\{(a,b),(b,a)\}$ containing a. If $(b,a) \notin E$, then there is an edge of the form (b,c), $c \neq b$, $c \neq a$, because $\text{od}(b)=\text{id}(b)$. Continuing this process, since $\text{od}(a)=\text{id}(a)$ and G is finite, we obtain a directed circuit C containing a. If $C=G$ we are finished. If not, remove the edges of C from G, along with any vertex that becomes isolated. The resulting subgraph $H=(V_1,E_1)$ is such that (in H) $\text{od}(v)=\text{id}(v)$ for all $v \in V_1$. However, H is not necessarily connected. But each component of H is connected with $\text{od}(v)=\text{id}(v)$ for each vertex in a component. Consequently, by the induction hypothesis, each component of H has a directed Euler circuit, and each component has a vertex on the circuit C (from above). Hence, starting at vertex a we travel on C until we encounter a vertex v_1 on the directed Euler circuit of the component C_1 of H. Traversing C_1 we return to v_1 and continue on C to vertex v_2 on component C_2 of H. Continuing the process, with G finite we obtain a directed Euler circuit for G.

(b) Let G be a directed graph satisfying the three conditions. Add the edge (x,y). Then by part (a) the resulting graph has a directed Euler circuit C. Removing (x,y) from C yields a directed Euler trail for the given graph G. (This trail starts at y and terminates at x.) In a similar manner we find that if a directed graph G has a directed Euler trail then it satisfies the three conditions.

5. Let G be a directed graph on n vertices. If the associated undirected graph for G is K_n, prove that $\sum_{v\in V}\left(od(v)^2\right)=\sum_{v\in V}\left(id(v)^2\right)$

10

Solution

4

6

we see that $\sum_{v\in V}[\text{od}(v)-\text{id}(v)]=0$. For each $v \in V$, $\text{od}(v)+\text{id}(v)=n-1$, so $0=(n-1)\cdot 0=\sum_{v\in V}(n-1)[\text{od}(v)-\text{id}(v)]=\sum_{v\in V}[\text{od}(v)+\text{id}(v)][\text{od}(v)-\text{id}(v)]=\sum_{v\in V}[(\text{od}(v))^2-(\text{id}(v))^2]$, and the result follows.

$\sum_{V\in V}\left[\left(od(v)-id(v)\right)\left(od(V)+id(V)\right)\right]$