TP2: Polynomial Interpolation

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1. Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that
$$|f(x) - P(x)| < \epsilon,$$
for all $x \in [a, b]$.

2. If $x_0, x_1, \cdots x_{n+1}$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with
$$f(x_k) = P(x_k)$$
, for each $k = 0, 1, \cdots, n$. Then this polynomial is given by
$$P(x) = f(x_0)L_0(x) + \cdots + f(x_n)L_n(x),$$
where $L_k(x) = \prod_{i=0}^{n} \frac{(x - x_i)}{(x_k - x_i)}, \ i \neq k$.

3. Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0), (0.5, y), (1, 3)$ and $(2, 2)$. The coefficient of x_3 in $P_3(x)$ is 6. Find y.

4. Suppose $x_0, x_1, \cdots x_n$ are distinct numbers in the interval $[a, b]$ and $f \in \mathbf{C}^{n+1}[a, b]$. Then, for each $x \in [a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots x_n$ and hence in (a, b) exists with
$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \tag{1}$$
where $P(x)$ is the interpolating polynomial.

5. Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.

   a. $f(x) = e^{2x} \cos 3x, \ x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$.

   b. $f(x) = \sin(\ln x), \ x_0 = 2, x_1 = 2.4, x_2 = 2.6, n = 2$.

   c. $f(x) = \ln x, \ x_0 = 1, x_1 = 1.1, x_2 = 1.3, x_3 = 1.4, n = 3$.

   d. $f(x) = \cos x + \sin x, \ x_0 = 1, x_1 = 0.25, x_2 = 0.5, x_3 = 1, n = 3$.

6. Let e^x, for $0 \leq x \leq 2$.

   a. Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$

   b. Approximate $f(0.75)$ using linear interpolation with $x_0 = 0.5$ and $x_1 = 1$

   c. Approximate $f(0.25)$ and $f(0.75)$ using the second interpolating polynomial with $x_0 = 0, x_1 = 1$ and $x_2 = 2$.

   d. Which approximations are better and why?

7. The error function is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

   a. Integrate the Maclaurin series for e^{-x^2} to show that

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$$

   b. Use the Maclaurin series to construct a table for $erf(x)$ that is accurate to within 10^{-4} for $erf(x)$, where $x_i = 0.2i$, for $i = 0, 1, \cdots, 5$.

   c. Use both linear interpolation and quadratic interpolation to obtain an approximation to $erf(13)$. Which approach seems most feasible?

8. Let f be defined at $x_0, x_1, \cdots x_k$ and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\cdots,j-1,j+1,\cdots,k}(x) - (x - x_i)P_{0,1,\cdots,i-1,i+1,\cdots,k}(x)}{(x_i - x_j)} \tag{2}$$

   is the kth Langrage polynomial that interpolates f at the $k+1$ points $x_0, x_1, \cdots x_k$.

9. Use Neville's method to obtain the approximations for Lagrange interpolating poly nomials of degrees one, two, and three to approximate each of the following:

   a. $f(0.43)$ if $f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.48169$.

   b. $f(0)$ if $f(0) = 1, f(-0.5) = 0.93750, f(-0.25) = 1.33203, f(0.25) = 0.800781, f(0.5) = 0.687500$.

10. Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0), (0.5, y), (1, 3)$ and $(2, 2)$. Use Neville's method to find y if $P_3(1.5) = 0$.

11. Naville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$			
$x_1 = 0.25$	$P_1 = 2$	$P_{0,1} = 2.6$		
$x_2 = 0.5$	P_2	$P_{1,2}$	$P_{0,1,2}$	
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$

12. Suppose $f \in \mathbf{C^1}[a, b]$, $f'(x) \neq 0$ on $[a, b]$ and f has a zero p in $[a, b]$. Let $x_0, \cdots, x_n$ be $n+1$ distinct numbers in $[a, b]$ with $f(x_k) = y_k$, for each $k = 0, 1, \cdots, n$. To approximate p construct interpolating polynomial of degree n on the nodes $y_0, \cdots, y_n$ for f^{-1}. Since $y_k = f(x_k)$ and $f(p) = 0$, it follows that $f^{-1}(y_k) = x_k$ and $p = f^{-1}(0)$. Using iterated interpolation to approximate $f^{-1}(0)$ is called iterated inverse interpolation. Use iterated inverse interpolation to find an approximation to the solution of $x - e^{-x} = 0$, using the data

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

13. Use the Newton forward-difference formula to construct interpolating polynomials of degree one,two, and three for the following data. Approximate the specified value using each of the polynomials.

    a. $f(0.43)$ if $f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.48169$

    b. $f(0.18)$ if $f(0.1) = -0.29004986, f(0.2) = -0.56079734, f(0.3) = -0.81401972, f(0.4) = -1.0526302$

14. (a.) Use the New to Divided-Difference method to construct the interpolating polynomial of degree three for the unequally spaced points given in the following table:

x	$f(x)$
-0.1	5.30000
0.0	2.000000
0.2	3.19000
0.3	1.0000

    (b.) Add $f(0.35) = 0.97260$ to the table, and construct the interpolating polynomial of degree 4.

15. Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

16. a. Show that the following two cubic polynomials

$$P(x) = 3 - 2(x+1) + O(x+1)x + x(x+1)(x-1)$$
$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + x(x+1)(x+2)$$

    both interpolate the data

x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

    b. Why does part (a) not violate the uniqueness property of interpolating polynomials?

17. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3
$P(x)$	4	9	15	18

17. For a function f, the Newton divided-difference formula gives the interpolating polynomial

$$P_3(x) = 1 + 4x + 4x(x-0.25) + \frac{16}{3}x(x-0.25)(x-0.25), \tag{3}$$

on the nodes $x_0 = 0, x_1 = 0.25, x_2 = 0.5$ and $x_3 = 0.75$. Find $f(0.75)$.

18. For a function f, the forward-divided differences are given by

$$\begin{array}{llll} x_0 = 0 & f[x_0] & & \\ x_1 = 0.4 & f(x_1) & f[x_0, x_1] & \\ x_2 = 0.7 & f[x_2] = 6 & f[x_0, x_1] = 10 & f[x_0, x_1, x_2] = \dfrac{50}{7} \end{array}$$

19. A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population (thousands)	151326	179323	203302	226542	249633	281422

   a. Use appropriately divided differences to approximate the population in the years $1940, 1975$, and 2020.

   b. The population in 1940 was approximately 132165000. How accurate do you think your 1975 and 2020 figures are ?

20. Determine the natural cubic spline S that interpolates the data $f(0) = 0, f(1) = 1$, and $f(2) = 2$.

21. Determine the clamped cubic spline s that interpolates the data $f(0) = 0, f(1) = 1, f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$.

22. Construct the natural cubic spline for the following data.

a.

x	$f(x)$
0.8	0.22363362
1.0	0.65809197

b.

x	$f(x)$
0.1	-0.62049958
0.2	-0.28398668
0.3	0.00660095
0.4	0.24842440

23. A natural cubic spline on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if} \quad 0 \leq x \leq 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$