An Introduction to Abelian Categories

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Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with Vect_K the category of vector spaces over the field K, whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathrm{FinDimVect}_K$ the subcategory of Vect_K which contains only finite dimensional K-vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Zero morphisms

Zero objects also allow for the definition of zero morphisms, which will be regarded as the "trivial morphism" from A to B (e.g., sending all elements of A into a "null element", if the category is an algebraic one).

Definition

Let A and B be two objects and let 0 be a zero object of $\mathcal{C}$. Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \qquad \text{where} \quad ?_B : 0 \to B, \qquad !_A : A \to 0.$$

The definition is independent of the zero object 0 and is in this sense "unique".

Preliminary steps (iv)

Moreover, the hom-set $\hom(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f+g)$, multiply a linear map by a scalar (λf), and all these operations behave "bilinearly" with the composition $(\circ)$:

$$(f+g)\circ h = f\circ h + g\circ h,$$

$$f\circ (g+h) = f\circ g + f\circ h,$$

$$(\lambda f)\circ g = \lambda(f\circ g) = f\circ(\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a category **enriched over a monoidal category** $(\mathcal{D}, \otimes)$ if the hom-sets of $\mathcal{C}$ are objects from $\mathcal{D}$ and if the composition of morphisms makes the composition $\circ$ bilinear over $\otimes$, namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that Vect_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if it's enriched over the category of abelian groups (Ab).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much "vector spaces over a ring": they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are $\mathbb{Z}$-modules, where:

$$n \cdot x := \underbrace{x + x + \ldots + x}_{n \text{ times}}.$$

This fact will result useful later on.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let $\mathcal{C}$ be a preadditive category. Then products and coproducts are isomorphic to one another in $\mathcal{C}$.

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

Let A and B be two objects in $\mathcal{C}$ and let $(C := A \times B, \pi_A : C \to A, \pi_B : C \to B)$ be a product of A and B. We shall determine two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that (C, ι_A, ι_B) is also a coproduct of A and B.

In doing so, we strive to get some "injections" of A and B into C. A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms ι_A, ι_B : A, $B \to C$:

1. id_A, $0_{AB} \rightsquigarrow \iota_A$,
2. 0_{BA}, $\mathrm{id}_B \rightsquigarrow \iota_B$.

Proof.

ι_A and ι_B yield the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\iota_A"] \arrow[dr, "0_{AB}"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \\
 & B \arrow[ul, "0_{BA}"] \arrow[u, "\iota_B"'] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from $\mathcal{C}$ and let $f, g : A, B \to D$ be morphisms.

Let's define $h_{f,g} : C \to D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the "connecting morphism" from C to D.

Proof.

We then expect that h_{ι_A,ι_B} – the connecting morphism generated by the "injections" – will behave as the identity on C.

Since the following identities hold:

$$\pi_A \circ h_{\iota_A,\iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$
$$\pi_B \circ h_{\iota_A,\iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A,ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"] \\
& C \arrow[d, "h_{f,g}"] & \\
& D &
\end{tikzcd}$$

The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\hom(A, B)$ and $\hom(B, A)$, respectively.

Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned} h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A,\iota_B} \\ &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\ &= \ldots \\ &= 0_{C,D}. \end{aligned}$$

$\square$

We're missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let's derive kernels from an example, and cokernels will be defined as the dual of kernels.

In linear algebra, given a linear map $f : V \to W$, we define $\ker f$ as follows:

$$\ker f \triangleq \{v \in V \mid f(v) = w\}.$$

Of course we're not allowed to defined kernels as sets, but only as morphisms. An elementary property of $\ker f$ is that $\ker f$ is a subspace of V, hence there exists a natural injection map $\iota : \ker f \to V$ such that:

$$\ker f \xhookrightarrow{\iota} V \xrightarrow{f} W \ , \qquad f \circ \iota = 0_{VW}.$$

It is then natural to define the kernel of $f : A \to B$ as the "biggest morphism k" that annihilates f.

Definition

Let $f : A \to B$ be a morphism. Then a kernel k of f is a morphism $k : K \to A$ such that:

1. $f \circ k = 0_{KB}$,
2. If $k' : K' \to A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \to K$ such that $k' = k \circ \iota_{K'}$.

Dually, a cokernel of $f : A \to B$ is the "smallest morphism j" that f annihilates.

Definition

Let $f : A \to B$ be a morphism. Then a cokernel j of f is a morphism $j : B \to J$ such that:

1. $j \circ f = 0_{AJ}$,
2. If $j' : B \to J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \to J'$ such that $j' = \pi_{J'} \circ j$.

Definition

A pre-additive category is pre-abelian if it allows kernels and cokernels for every morphism f and permits products for a finite family of objects.