An Introduction to Abelian Categories

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May 2025

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Outline

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1 Background: Abelian Groups, K-Vector Spaces, and R-Modules

2 Origins and Motivation of Abelian Categories

Before we take a look at abelian categories, we need to introduce the algebraic structure they all rely on: **abelian groups**.

Definition (Abelian group)

An abelian group $(G, +)$ is a commutative monoid which allows inverses.

Abelian groups are fairly simple objects with respect to what a general non-commutative group might be. Nonetheless, they appear everywhere: $\mathbb{Z}$ and the integers modulo n form an abelian group with addition, and all vector spaces rely on abelian groups for their additive structure.

In fact, the connection between vector spaces and abelian groups is deeper than what it initially appears to be.

In order to show this, let us recall the definition of a ring and of a module.

Definition (Ring)

A ring is a set R equipped with two binary operations $+$ and $\cdot$ such that $(R, +)$ is an abelian group, $(R, \cdot)$ is a monoid and law of distributivity applies, i.e.

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$(a + b) \cdot c = a \cdot c + b \cdot c,$$

for any choice of a, b, $c \in R$.

Definition (R-module)

Given a ring R, we say that a set A equipped with a binary operation $+$ is an R-module with scalar multiplication $\cdot : R \times A \to A$ if:

1. $(A, +)$ is an abelian group,
2. $1a = a$, where 1 is the unity of the ring R and $a \in A$,
3. $\lambda(\mu a) = (\lambda\mu)a$, for each λ, $\mu \in R$, and $a \in A$,
4. $(\lambda + \mu)a = \lambda a + \mu a$ for each λ, $\mu \in R$, and $a \in A$,
5. $\lambda(a + b) = \lambda a + \lambda b$ for each $\lambda \in R$, and a, $b \in A$.

In this setting, K-vector spaces are simply K-modules and abelian groups are simply $\mathbb{Z}$-modules with the following scalar multiplication:

$$k \cdot g \triangleq \begin{cases} \underbrace{g + \ldots + g}_{k \text{ times}} & k \geq 0 \\ -(-k) \cdot g & k < 0. \end{cases}$$

Modules are indeed a great unifying concept for these two algebraic structure, and we'll see at the end of the presentation that the Freyd-Mitchell's embedding theorem will give us more reasons to believe so.

Outline

The motivation behind abelian categories

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with Vect_K the category of vector spaces over the field K, whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathsf{FinDimVect}_K$ the subcategory of Vect_K which contains only finite dimensional K-vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Zero morphisms

Zero objects also allow for the definition of zero morphisms, which will be regarded as the "trivial morphism" from A to B (e.g., sending all elements of A into a "null element", if the category is an algebraic one).

Definition

Let A and B be two objects and let 0 be a zero object of $\mathcal{C}$. Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \qquad \text{where} \quad ?_B : 0 \to B, \quad !_A : A \to 0.$$

The definition is independent of the zero object 0 and is in this sense "unique".

Preliminary steps (iv)

Moreover, the hom-set $\hom(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f + g)$, multiply a linear map by a scalar (λf), and all these operations behave "bilinearly" with the composition $(\circ)$:

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a category **enriched over a monoidal category** $(\mathcal{D}, \otimes)$ if the hom-sets of $\mathcal{C}$ are objects from $\mathcal{D}$ and if the composition of morphisms makes the composition $\circ$ bilinear over $\otimes$, namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that Vect_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if it's enriched over the category of abelian groups (Ab).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much "vector spaces over a ring": they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are $\mathbb{Z}$-modules, where:

$$n \cdot x := \underbrace{x + x + \ldots + x}_{n \text{ times}}.$$

This fact will result useful later on.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let $\mathcal{C}$ be a preadditive category. Then products and coproducts are isomorphic to one another in $\mathcal{C}$.

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

Let A and B be two objects in $\mathcal{C}$ and let $(C := A \times B, \pi_A : C \to A, \pi_B : C \to B)$ be a product of A and B. We shall determine two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that (C, ι_A, ι_B) is also a coproduct of A and B.

In doing so, we strive to get some "injections" of A and B into C. A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms ι_A, $\iota_B : A, B \to C$:

1. $\mathrm{id}_A, 0_{AB} \rightsquigarrow \iota_A$,
2. $0_{BA}, \mathrm{id}_B \rightsquigarrow \iota_B$.

Proof.

ι_A and ι_B yield the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\iota_A"] \arrow[dr, "0_{AB}"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \\
 & B \arrow[ul, "0_{BA}"] \arrow[u, "\iota_B"] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from $\mathcal{C}$ and let f, $g : A$, $B \to D$ be morphisms.

Let's define $h_{f,g} : C \to D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the "connecting morphism" from C to D.

Proof.

We then expect that h_{ι_A,ι_B} – the connecting morphism generated by the "injections" – will behave as the identity on C.

Since the following identities hold:

$$\pi_A \circ h_{\iota_A,\iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$
$$\pi_B \circ h_{\iota_A,\iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A,ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"] \\
& C \arrow[d, "h_{f,g}"] & \\
& D &
\end{tikzcd}$$

The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\hom(A, B)$ and $\hom(B, A)$, respectively.

Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned} h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A,\iota_B} \\ &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\ &= \ldots \\ &= 0_{C,D}. \end{aligned}$$

$\square$

We're missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let's derive kernels from an example, and cokernels will be defined as the dual of kernels.

In linear algebra, given a linear map $f : V \to W$, we define $\ker f$ as follows:

$$\ker f \triangleq \{v \in V \mid f(v) = w\}.$$

Of course we're not allowed to defined kernels as sets, but only as morphisms. An elementary property of $\ker f$ is that $\ker f$ is a subspace of V, hence there exists a natural injection map $\iota : \ker f \to V$ such that:

$$\ker f \xhookrightarrow{\iota} V \xrightarrow{f} W \,, \qquad f \circ \iota = 0_{VW}.$$

It is then natural to define the kernel of $f : A \to B$ as the "biggest morphism k" that annihilates f.

Definition

Let $f : A \to B$ be a morphism. Then a kernel k of f is a morphism $k : K \to A$ such that:

1. $f \circ k = 0_{KB}$,
2. If $k' : K' \to A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \to K$ such that $k' = k \circ \iota_{K'}$.

Dually, a cokernel of $f : A \to B$ is the "smallest morphism j" that f annihilates.

Definition

Let $f : A \to B$ be a morphism. Then a cokernel j of f is a morphism $j : B \to J$ such that:

1. $j \circ f = 0_{AJ}$,
2. If $j' : B \to J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \to J'$ such that $j' = \pi_{J'} \circ j$.

Definition

A pre-additive category is pre-abelian if it allows kernels and cokernels for every morphism f and permits products for a finite family of objects.

Since a pre-abelian category allows products, it also implicitly allows coproduct, as we have shown earlier.

We're missing just one piece to get to the definition of an abelian category. When we looked at the morphism $\iota : \ker f \to A$ in the context of linear algebra, we generalised ι thinking of it an "immersion", namely an injective function.

The opposite is also true in linear algebra: an injective linear map gives always rise to a kernel. In our categorical framework, this will translate to the following definition:

Definition

A monomorphism is said **normal** if it's a kernel of a morphism. An epimorphism is said **conormal** if it's a cokernel of a morphism.

Definition

An abelian category is a pre-abelian category in which all monomorphisms are normal and all epimorphisms are conormal.

Summing up, an abelian category is a category $\mathcal{C}$ such that:

1. The hom-sets $\hom(A, B)$ of $\mathcal{C}$ are abelian groups in a way such that composition is respected;
2. $\mathcal{C}$ allows for products and coproducts of finite families of objects;
3. Each morphism $f : A \to B$ has a kernel and a cokernel;
4. Each monomorphism is a kernel of a morphism;
5. Each epimorphism is a cokernel of another morphism.

Abelian categories are the most general setting for developing homological algebra and exact sequences.