An Introduction to Abelian Categories

Gabriel Antonio Videtta

May 2025

Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with Vect_K the category of vector spaces over the field K, whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathrm{FinDimVect}_K$ the subcategory of Vect_K which contains only finite dimensional K-vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Preliminary steps (iv)

Moreover, the hom-set $\hom(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f+g)$, multiply a linear map by a scalar (λf), and all these operations behave "bilinearly" with the composition $(\circ)$:

$$(f+g)\circ h = f\circ h + g\circ h,$$

$$f\circ(g+h) = f\circ g + f\circ h,$$

$$(\lambda f)\circ g = \lambda(f\circ g) = f\circ(\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a category **enriched over a monoidal category** $(\mathcal{D}, \otimes)$ if the hom-sets of $\mathcal{C}$ are objects from $\mathcal{D}$ and if the composition of morphisms makes the composition $\circ$ bilinear over $\otimes$, namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that Vect_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if it's enriched over the category of abelian groups (Ab).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.