An Introduction to Abelian Categories

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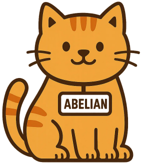
May 2025



Outline

1. Background: Abelian Groups, K-Vector Spaces, and R-Modules
2. Origins and Motivation of Abelian Categories
3. Derivation of Abelian Categories
4. Fundamental Results on Abelian Categories

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1 Background: Abelian Groups, K-Vector Spaces, and R-Modules

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Abelian Groups I

We introduce the algebraic structure all abelian categories rely on: **abelian groups**.

Definition (Abelian group)

An abelian group $(G, +)$ is a commutative monoid in which every element has an inverse.

Abelian Groups II

Though abelian groups are relatively simple compared to non-commutative groups, they appear everywhere:

- $(\mathbb{Z}, +)$,
- Integers modulo n,
- Vector spaces.

Rings and Modules I

The connection between vector spaces and abelian groups is deeper than it might first appear. Let us recall the definition of a **ring** and of a **module**.

Definition (Unital ring)

A (unital) ring is a set R equipped with two binary operations $+$ and $\cdot$ such that $(R, +)$ is an abelian group, $(R, \cdot)$ is a monoid and the distributivity laws hold, i.e., for any choice of a, b, $c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

Rings and Modules II

Definition (R-module)

Given a ring R, a set A equipped with a binary operation $+$ is said to be an R-module with scalar multiplication $\cdot$ if:

1. $(A, +)$ is an abelian group,
2. $1 \cdot a = a$, where 1 is the identity element of the ring R and $a \in A$,
3. $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$, for all λ, $\mu \in R$, and $a \in A$,
4. $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$ for all λ, $\mu \in R$, and $a \in A$,
5. $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$ for all $\lambda \in R$, and a, $b \in A$.

Abelian Groups as $\mathbb{Z}$-Modules

In this setting, K-vector spaces are K-modules and abelian groups are $\mathbb{Z}$-modules with the following scalar multiplication:

$$k \cdot g \triangleq \begin{cases} \underbrace{g + \ldots + g}_{k \text{ times}} & k \geq 0 \\ -(-k) \cdot g & k < 0. \end{cases}$$

Modules indeed unify these algebraic structures, and we further illustrate this connection through the Freyd-Mitchell embedding theorem.

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Historical Notes

In 1945 two mathematicians, Eilenberg and Mac Lane, wrote for the first time about categories in the way we look at them now.

Their goal was to apply the abstractness of category theory to homological algebra, which revolves around **chain complexes**.

Understanding chain complexes allows mathematicians to extract topological invariants from geometrical structures within a purely algebraic framework.

Chain Complexes

Chain complexes are particular concatenations of morphisms in Ab, the category of abelian groups, written as seen below

$$C_1 \xrightarrow{\quad d_1 \quad} C_2 \xrightarrow{\quad d_2 \quad} \dots \xrightarrow{\quad d_{n-1} \quad} C_n$$

where $d_{i+1} \circ d_i = 0$ for each i (i.e., $\operatorname{im} d_i \subseteq \ker d_{i+1}$).

Historical Notes II

The goal of abelian categories is to provide the most general setting to generalise and make use of chain complexes.

The first instances of the definition of abelian categories are first found in a paper from 1955 by Buchsbaum, whereas further foundations of the theory were laid in a famous paper by Grothendieck in 1957.

We derive in this presentation the modern setting of abelian categories from scratch.

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Steps to Define Abelian Categories

To derive a definition for abelian categories, we need to address the following issues:

1. We must allow for the existence of "zero morphisms", since we require $d_{i+1} \circ d_i = 0$.
2. We must allow some structure for the hom-sets, as group homomorphisms between two abelian groups A and B form an abelian group as well.
3. We need to define kernels in a pure categorical sense, since $\ker f$ is defined in a set-theoretic manner.

Zero objects I

Abelian groups, vector spaces, and modules in general have a peculiar property: *initial objects and final objects are the same!*

In both cases, this initial object is always denoted as the "trivial" or "zero" object, since it always induces a "trivial" morphism with respect to another module.

Zero objects II

It is therefore crucial to provide a proper definition of **zero objects**:

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a zero object if it's both an initial and a final object.

Zero morphisms I

As anticipated, zero objects also allow for the definition of zero morphisms, which will be regarded as the “trivial morphism” from A to B.

We derive a definition for zero morphisms by combining the existence and uniqueness of initial and final morphisms with respect to 0.

Zero morphisms II

Definition (Zero morphisms)

Let A and B be two objects and let 0 be a zero object of $\mathcal{C}$. Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \qquad \text{where} \quad ?_B : 0 \to B, \quad !_A : A \to 0.$$

The definition is independent of the zero object 0 and is in this sense "unique".

Enrichment of RMod over Ab

Hom-sets in Ab are not just sets, they have a natural structure of abelian group as well:

- We can sum morphisms $(f + g)$;
- The morphism 0_{VW} acts as the identity element;
- The sum behaves "bilinearly" with respect to the composition $(\circ)$:

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h.$$

Preadditive Categories

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if each hom-set in $\mathcal{C}$ is an abelian group onto which the composition of morphisms acts bilinearly.

In short, a preadditive category is one in which morphisms can be added and subtracted in a way compatible with composition.

(Co)products in Preadditive Categories

Products and coproducts behave in the same way in a preadditive category, as shown below.

Proposition

Let $\mathcal{C}$ be a preadditive category. Then products and coproducts are isomorphic in $\mathcal{C}$.

We only prove that products are also coproducts; the other part of the statement is proved similarly.

$\sqcup$ and $\times$ coincide in Preadditive Categories I

Proof.

Let A and B be two objects in $\mathcal{C}$ and let $C := A \times B$ equipped with projections π_A, $\pi_B : C \to A, B$ be a product of A and B.

We shall determine two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that (C, ι_A, ι_B) is also a coproduct of A and B.

In doing so, we aim to construct two "injections" of A and B into C.

$\sqcup$ and $\times$ coincide in Preadditive Categories II

Proof.

A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms ι_A, $\iota_B : A, B \to C$:

1. id_A, $0_{AB} \rightsquigarrow \iota_A$,
2. 0_{BA}, $\mathrm{id}_B \rightsquigarrow \iota_B$.

$\sqcup$ and $\times$ coincide in Preadditive Categories III

Proof.

ι_A and ι_B yield the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\iota_A"] \arrow[dr, "0_{AB}"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \\
 & B \arrow[ul, "0_{BA}"] \arrow[u, "\iota_B"] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

$\sqcup$ and $\times$ coincide in Preadditive Categories IV

Proof.

Let us now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from $\mathcal{C}$ and let $f, g : A, B \to D$ be morphisms.

Let us define $h_{f,g} : C \to D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the "connecting morphism" from C to D.

$\sqcup$ and $\times$ coincide in Preadditive Categories V

Proof.

We expect that h_{ι_A,ι_B} – the connecting morphism generated by the "injections" – will behave as the identity on C.

Since the following identities hold:

$$\pi_A \circ h_{\iota_A,\iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$
$$\pi_B \circ h_{\iota_A,\iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A,ι_B} is indeed id_C by the universal property of the product.

$\sqcup$ and $\times$ coincide in Preadditive Categories VI

Proof.

Let us now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"] \\
& C \arrow[d, "h_{f,g}"] & \\
& D &
\end{tikzcd}$$

$\sqcup$ and $\times$ coincide in Preadditive Categories VII

Proof.

Commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\hom(A, B)$ and $\hom(B, A)$, respectively.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned} h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\ &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\ &= \dots \\ &= 0_{C,D}. \end{aligned}$$

$\square$

(Co)kernels in Category Theory I

We are missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let us derive kernels from an example, and cokernels will be defined as the dual of kernels.

In abstract and linear algebra, given a morphism $f : A \to B$, we define $\ker f$ as follows:

$$\ker f \triangleq \{a \in A \mid f(a) = 0\}.$$

(Co)kernels in Category Theory II

We are not allowed to define kernels as sets, but only as objects or morphisms. An elementary property of $\ker f$ is that $\ker f$ is subordinate to A, hence there exists a natural injection map $\iota : \ker f \to A$ such that:

$$\ker f \xhookrightarrow{\iota} A \xrightarrow{f} B \ , \qquad f \circ \iota = 0_{AB}.$$

(Co)kernels in Category Theory III

It is then natural to define the kernel of $f : A \to B$ as the "biggest morphism k" that is annihilated by f.

Definition

Let $f : A \to B$ be a morphism. Then a kernel k of f is a morphism $k : K \to A$ such that:

1. $f \circ k = 0_{KB}$,
2. If $k' : K' \to A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \to K$ such that $k' = k \circ \iota_{K'}$.

(Co)kernels in Category Theory IV

Dually, a cokernel of $f : A \to B$ is the "smallest morphism j" that annihilates f.

Definition

Let $f : A \to B$ be a morphism. Then a cokernel j of f is a morphism $j : B \to J$ such that:

1. $j \circ f = 0_{AJ}$,
2. If $j' : B \to J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \to J'$ such that $j' = \pi_{J'} \circ j$.

(Co)kernels Are Unique up to Isomorphism I

(Co)kernels of the same morphism f have the desired property of being "unique" and can always be thought of being the same object.

Proposition

Kernels and cokernels of a morphism f are unique up to isomorphism.

We prove the statement only for kernels, then the other part is obtained for cokernels dually.

(Co)kernels Are Unique up to Isomorphism II

Proof.

Let k, $k' : K$, $K' \to A$ be two kernels of $f : A \to B$. The situation is represented in the following commutative diagram:

$$
\begin{tikzcd}
K \arrow[dr, "k"'] \arrow[drr, bend left, "0_{KB}"] & & \\
& A \arrow[r, "f"] & B \\
K' \arrow[ur, "k'"] \arrow[urr, bend right, "0_{K'B}"'] & &
\end{tikzcd}
$$

(Co)kernels Are Unique up to Isomorphism III

Proof.

Since k is a kernel of f, there exists a unique morphism $p : K' \to K$ such that $k' = k \circ p$.

$$
\begin{tikzcd}
K \arrow[dr, "k"] \arrow[drr, bend left, "0_{KB}"] & & \\
 & A \arrow[r, "f"] & B \\
K' \arrow[uu, dashed, bend left, "p"] \arrow[ur, "k'"] \arrow[urr, bend right, "0_{K'B}"'] & &
\end{tikzcd}
$$

(Co)kernels Are Unique up to Isomorphism IV

Proof.

The same holds in the other direction: there exists a unique morphism $p' : K \to K'$ such that $k = k' \circ p'$.

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[drr, bend left, "0_{KB}"] \arrow[dd, dashed, "p'"'] & & \\
& A \arrow[r, "f"] & B \\
K' \arrow[ur, "k'"] \arrow[urr, bend right, "0_{K'B}"'] & &
\end{tikzcd}$$

(Co)kernels Are Unique up to Isomorphism V

Proof.

We note that

$$k = k' \circ p' = \underbrace{k \circ p}_{k'} \circ p'.$$

Since id_K is such that $k = k \circ \mathrm{id}_K$ as well, by the universal property of kernels $p \circ p' = \mathrm{id}_K$. Similarly, $p' \circ p = \mathrm{id}_{K'}$, hence p and p' are isomorphisms. $\square$

Pre-abelian Categories

Definition

A preadditive category is called pre-abelian if it admits kernels and cokernels for every morphism and has finite products.

Since a pre-abelian category allows products, it also implicitly allows coproduct, as we have shown earlier.

Monomorphisms as Kernels in Abelian Categories

When we looked at the morphism $\iota : \ker f \to A$ in the context of linear algebra, we generalised ι thinking of it as an "immersion", namely an injective function.

The opposite is also true for abelian groups, vector spaces and modules: an injective linear map can always be realised as the kernel of some other morphism.

Normal Monomorphisms and Conormal Epimorphisms

In our categorical framework, this will translate to the following definition:

Definition

A monomorphism is said to be **normal** if it is a kernel of a morphism. An epimorphism is said to be **conormal** if it is a cokernel of a morphism.

Definition of an Abelian Category

Definition

An abelian category is a pre-abelian category in which all monomorphisms are normal and all epimorphisms are conormal.

Summing up, an abelian category is a category $\mathcal{C}$ such that:

1. The hom-sets of $\mathcal{C}$ are abelian groups compatible with the operation of composition;
2. $\mathcal{C}$ has finite products and coproducts, which coincide;
3. Each morphism $f : A \to B$ has a kernel and a cokernel;
4. Each monomorphism is a kernel of a morphism;
5. Each epimorphism is a cokernel of another morphism.

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Bimorphisms

In this final section of the presentation, we present two fundamental results concerning abelian categories, which motivate even more their introduction.

Let us define what a **bimorphism** is:

Definition (Bimorphism)

Given a category $\mathcal{C}$, a morphism $f : A \to B$ which is both monic and epic is called a bimorphism.

Bimorphisms are Isomorphisms in Abelian Categories I

In a general category, bimorphisms and isomorphisms do not coincide. For example, in Mon, $\iota : \mathbb{N} \to \mathbb{Z}$ is a bimorphism, even though $\mathbb{N}$ is not isomorphic to $\mathbb{Z}$.

This is not the case for abelian categories:

Theorem

Given an abelian category $\mathcal{C}$, bimorphisms are always isomorphisms.

Bimorphisms are Isomorphisms in Abelian Categories II

Proof.

Let $f : A \to B$ be a bimorphism. Then, since in $\mathcal{C}$ all monomorphisms are normal, f is the kernel of a morphism $g : B \to C$.

Observe that

$$g \circ f = 0_{AC} = 0_{BC} \circ f.$$

Since f is an epimorphism, this implies that $g = 0_{BC}$.

Bimorphisms are Isomorphisms in Abelian Categories III

Proof.

Then $\mathrm{id}_B : B \to B$ is a kernel of g as well:

- $g \circ \mathrm{id}_B = 0_{BC} \circ \mathrm{id}_B = 0_{BC}$,
- If $k' : K' \to B$ is a kernel of g, then the only morphism k satisfying $k' = \mathrm{id}_B \circ k$ is k' itself, hence it is unique.

Bimorphisms are Isomorphisms in Abelian Categories IV

Proof.

Since $f : A \to B$ and $\mathrm{id}_B : B \to B$ are both kernels of the same morphism, the morphism $p : A \to B$ connecting f to id_B is an isomorphism (see Proposition 2). Since $p = \mathrm{id}_B \circ f = f$, this concludes the proof.

$\square$

The Freyd-Mitchell Embedding Theorem I

One of the most important results about abelian categories states that each small abelian category is a full subcategory of RMod, for some suitable ring R.

Theorem (Freyd-Mitchell embedding theorem)

Each small abelian category admits a full, faithful and exact functor to RMod, *for some suitable ring* R.

The Freyd-Mitchell Embedding Theorem II

The Freyd-Mitchell embedding theorem allows one to study all small abelian categories in a concrete and well-studied setting, that of R-modules, with the most advanced algebraic tools.

This also heavily connects abelian groups and R-modules in general, giving an insight into how deep the link between these two algebraic structures is.

References

[1] Peter J Freyd. *Abelian categories*. Vol. 1964. Harper & Row New York, 1964.

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