

An Introduction to Abelian Categories

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Outline

- 1 Background: Abelian Groups, K -Vector Spaces, and R -Modules
- 2 Origins and Motivation of Abelian Categories
- 3 Derivation of Abelian Categories



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Before we take a look at abelian categories, we need to introduce the algebraic structure they all rely on: **abelian groups**.

Definition (Abelian group)

An abelian group $(G, +)$ is a commutative monoid which allows inverses.

Abelian groups are fairly simple objects with respect to what a general non-commutative group might be. Nonetheless, they appear everywhere: \mathbb{Z} and the integers modulo n form an abelian group with addition, and all vector spaces rely on abelian groups for their additive structure.



In fact, the connection between vector spaces and abelian groups is deeper than what it initially appears to be. In order to show this, let us recall the definition of a **ring** and of a **module**.

Definition (Ring)

A ring is a set R equipped with two binary operations $+$ and \cdot such that $(R, +)$ is an abelian group, (R, \cdot) is a monoid and law of distributivity applies, i.e. for any choice of $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$



Definition (R -module)

Given a ring R , we say that a set A equipped with a binary operation $+$ is an R -module with scalar multiplication \cdot if:

- 1 $(A, +)$ is an abelian group,
- 2 $1 \cdot a = a$, where 1 is the unity of the ring R and $a \in A$,
- 3 $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$, for each $\lambda, \mu \in R$, and $a \in A$,
- 4 $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$ for each $\lambda, \mu \in R$, and $a \in A$,
- 5 $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$ for each $\lambda \in R$, and $a, b \in A$.



In this setting, K -vector spaces are simply K -modules and abelian groups are simply \mathbb{Z} -modules with the following scalar multiplication:

$$k \cdot g \triangleq \begin{cases} \underbrace{g + \dots + g}_{k \text{ times}} & k \geq 0 \\ -(-k) \cdot g & k < 0. \end{cases}$$

Modules are indeed a great unifying concept for these two algebraic structure, and we'll see at the end of the presentation that the Freyd-Mitchell's embedding theorem will give us more reasons to believe so.



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In 1945 two mathematicians, Eilenberg and Mac Lane, wrote for the first time about categories in the way we look at them now.

They were looking to deeply understand *natural transformations*, hence they had to deal with functors and categories first. Their goal was to apply such abstractness to homological algebra, which revolves around **chain complexes**.

Understanding chain complexes allow mathematicians to retrieve topological invariants from geometrical structures in a pure algebraic setting.



Chain complexes are particular concatenations of morphisms in \mathbf{Ab} , the category of abelian groups, written as seen below

$$C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} C_n$$

where $d_{i+1} \circ d_i = 0$ for each i (i.e., $\operatorname{im} d_i \subseteq \ker d_{i+1}$).

We say that such a chain complex is **exact** whenever $\operatorname{im} d_i$ is exactly $\ker d_{i+1}$ for each i .



The goal of abelian categories is to provide the most general setting to generalise and make use of chain complexes.

Instances of the definition of abelian categories are first found in a paper from 1955 by Buchsbaum, whereas further foundations of the theory were laid in a famous paper from Grothendieck later in 1957.

The goal of this presentation is to derive the modern setting of abelian categories from scratch.



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In order to derive a correct definition for abelian categories, we need to address the following issues:

- 1 We must allow the existence of “zero morphisms”, since we want $d_{i+1} \circ d_i = 0$.
- 2 We must allow some structure for the hom-sets, as group homomorphisms between two abelian groups A and B form an abelian group as well.
- 3 We need to define kernels in a pure categorical sense, since $\ker f$ is a set and not an object nor a morphism.



Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let \mathcal{C} be a category. We say 0 is a **zero object** if it's both an initial and a final object.



Zero morphisms

Zero objects also allow for the definition of zero morphisms, which will be regarded as the “trivial morphism” from A to B (e.g., sending all elements of A into a “null element”, if the category is an algebraic one).

Definition

Let A and B be two objects and let 0 be a zero object of \mathcal{C} . Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \quad \text{where } ?_B : 0 \rightarrow B, \quad !_A : A \rightarrow 0.$$

The definition is independent of the zero object 0 and is in this sense “unique”.



Preliminary steps (iv)

Moreover, the hom-set $\text{hom}(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f + g)$, multiply a linear map by a scalar (λf) , and all these operations behave “bilinearly” with the composition (\circ) :

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...



Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let \mathcal{C} be a category. We say that \mathcal{C} is a category **enriched over a monoidal category** (\mathcal{D}, \otimes) if the hom-sets of \mathcal{C} are objects from \mathcal{D} and if the composition of morphisms makes the composition \circ bilinear over \otimes , namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that Vect_K is enriched over itself!



Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let \mathcal{C} be a category. We say that \mathcal{C} is a **preadditive category** if it's enriched over the category of abelian groups (Ab).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.



Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much “vector spaces over a ring”: they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are \mathbb{Z} -modules, where:

$$n \cdot x := \underbrace{x + x + \dots + x}_{n \text{ times}}.$$

This fact will result useful later on.



Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let \mathcal{C} be a preadditive category. Then products and coproducts are isomorphic to one another in \mathcal{C} .

We only prove that products are also coproducts; the other part of the statement is proved similarly.



Proof.

Let A and B be two objects in \mathcal{C} and let $(C := A \times B, \pi_A : C \rightarrow A, \pi_B : C \rightarrow B)$ be a product of A and B . We shall determine two morphisms $\iota_A : A \rightarrow C$ and $\iota_B : B \rightarrow C$ such that (C, ι_A, ι_B) is also a coproduct of A and B .

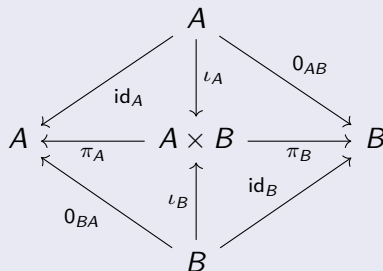
In doing so, we strive to get some “injections” of A and B into C . A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms $\iota_A, \iota_B : A, B \rightarrow C$:

- 1 $\text{id}_A, 0_{AB} \rightsquigarrow \iota_A,$
- 2 $0_{BA}, \text{id}_B \rightsquigarrow \iota_B.$



Proof.

ι_A and ι_B yield the following commutative diagram:



Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from \mathcal{C} and let $f, g : A, B \rightarrow D$ be morphisms.

Let's define $h_{f,g} : C \rightarrow D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the “connecting morphism” from C to D .



Proof.

We then expect that h_{ι_A, ι_B} – the connecting morphism generated by the “injections” – will behave as the identity on C .

Since the following identities hold:

$$\pi_A \circ h_{\iota_A, \iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$

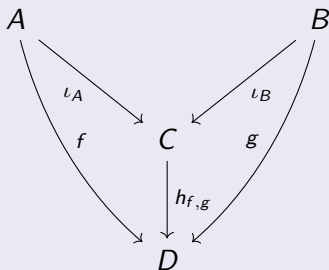
$$\pi_B \circ h_{\iota_A, \iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A, ι_B} is indeed id_C by the universal property of the product.



Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:



The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\text{hom}(A, B)$ and $\text{hom}(B, A)$, respectively.



Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned}
 h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\
 &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\
 &= \dots \\
 &= 0_{C,D}.
 \end{aligned}$$



We're missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let's derive kernels from an example, and cokernels will be defined as the dual of kernels.

In linear algebra, given a linear map $f : V \rightarrow W$, we define $\ker f$ as follows:

$$\ker f \triangleq \{v \in V \mid f(v) = 0\}.$$

Of course we're not allowed to define kernels as sets, but only as morphisms. An elementary property of $\ker f$ is that $\ker f$ is a subspace of V , hence there exists a natural injection map $\iota : \ker f \rightarrow V$ such that:

$$\ker f \xrightarrow{\iota} V \xrightarrow{f} W, \quad f \circ \iota = 0_{VW}.$$



It is then natural to define the kernel of $f : A \rightarrow B$ as the “biggest morphism k ” that annihilates f .

Definition

Let $f : A \rightarrow B$ be a morphism. Then a kernel k of f is a morphism $k : K \rightarrow A$ such that:

- 1 $f \circ k = 0_{KB}$,
- 2 If $k' : K' \rightarrow A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \rightarrow K$ such that $k' = k \circ \iota_{K'}$.



Dually, a cokernel of $f : A \rightarrow B$ is the “smallest morphism j ” that f annihilates.

Definition

Let $f : A \rightarrow B$ be a morphism. Then a cokernel j of f is a morphism $j : B \rightarrow J$ such that:

- 1 $j \circ f = 0_{AJ}$,
- 2 If $j' : B \rightarrow J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \rightarrow J'$ such that $j' = \pi_{J'} \circ j$.



Definition

A pre-additive category is pre-abelian if it allows kernels and cokernels for every morphism f and permits products for a finite family of objects.

Since a pre-abelian category allows products, it also implicitly allows coproduct, as we have shown earlier.



We're missing just one piece to get to the definition of an abelian category. When we looked at the morphism $\iota : \ker f \rightarrow A$ in the context of linear algebra, we generalised ι thinking of it as an “immersion”, namely an injective function.

The opposite is also true in linear algebra: an injective linear map gives always rise to a kernel. In our categorical framework, this will translate to the following definition:

Definition

A monomorphism is said **normal** if it's a kernel of a morphism. An epimorphism is said **conormal** if it's a cokernel of a morphism.



Definition

An abelian category is a pre-abelian category in which all monomorphisms are normal and all epimorphisms are conormal.

Summing up, an abelian category is a category \mathcal{C} such that:

- 1 The hom-sets $\text{hom}(A, B)$ of \mathcal{C} are abelian groups in a way such that composition is respected;
- 2 \mathcal{C} allows for products and coproducts of finite families of objects;
- 3 Each morphism $f : A \rightarrow B$ has a kernel and a cokernel;
- 4 Each monomorphism is a kernel of a morphism;
- 5 Each epimorphism is a cokernel of another morphism.



Abelian categories are the most general setting for developing homological algebra and exact sequences.

