

# An Introduction to Abelian Categories

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# Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

## Preliminary steps (ii)

First of all, we will denote with  $\mathbf{Vect}_K$  the category of vector spaces over the field  $K$ , whose objects are vector spaces and morphisms are linear maps.

We will denote with  $\mathbf{FinDimVect}_K$  the subcategory of  $\mathbf{Vect}_K$  which contains only finite dimensional  $K$ -vector spaces.

## Preliminary steps (iii)

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that  $0$  (the zero dimensional vector space) is a special object.

### Definition (Zero object)

Let  $\mathcal{C}$  be a category. We say  $0$  is a **zero object** if it's both an initial and a final object.

## Preliminary steps (iv)

Moreover, the hom-set  $\text{hom}(V, W)$  has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps ( $f + g$ ), multiply a linear map by a scalar ( $\lambda f$ ), and all these operations behave “bilinearly” with the composition ( $\circ$ ):

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...

# Preliminary steps (v)

## Definition (Enriched categories)

Let  $\mathcal{C}$  be a category. We say that  $\mathcal{C}$  is a category **enriched over a category**  $\mathcal{D}$  if the hom-sets of  $\mathcal{C}$  are objects from  $\mathcal{D}$ .

We restrict  $\mathcal{D}$  to be a monoidal category (e.g., cartesian category,  $\mathbf{Ab}$ ,  $\mathbf{Vect}_K$ ,  $\mathbf{Mod}_R$ ).

Therefore, we can say that  $\mathbf{Vect}_K$  is enriched over itself!

# Preliminary steps (vi)

## Definition (Preadditive category)

Let  $\mathcal{C}$  be a category. We say that  $\mathcal{C}$  is a **preadditive category** if it's enriched over the category of abelian groups ( $\mathbf{Ab}$ ).