

An Introduction to Abelian Categories

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Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with \mathbf{Vect}_K the category of vector spaces over the field K , whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathbf{FinDimVect}_K$ the subcategory of \mathbf{Vect}_K which contains only finite dimensional K -vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let \mathcal{C} be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Zero morphisms

Zero objects also allow for the definition of zero morphisms, which will be regarded as the “trivial morphism” from A to B (e.g., sending all elements of A into a “null element”, if the category is an algebraic one).

Definition

Let A and B be two objects and let 0 be a zero object of \mathcal{C} . Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \quad \text{where } ?_B : 0 \rightarrow B, \quad !_A : A \rightarrow 0.$$

The definition is independent of the zero object 0 and is in this sense “unique”.

Preliminary steps (iv)

Moreover, the hom-set $\text{hom}(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f + g)$, multiply a linear map by a scalar (λf) , and all these operations behave “bilinearly” with the composition (\circ) :

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let \mathcal{C} be a category. We say that \mathcal{C} is a category **enriched over a monoidal category** (\mathcal{D}, \otimes) if the hom-sets of \mathcal{C} are objects from \mathcal{D} and if the composition of morphisms makes the composition \circ bilinear over \otimes , namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that \mathbf{Vect}_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let \mathcal{C} be a category. We say that \mathcal{C} is a **preadditive category** if it's enriched over the category of abelian groups (\mathbf{Ab}).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much “vector spaces over a ring”: they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are \mathbb{Z} -modules, where:

$$n \cdot x := \underbrace{x + x + \dots + x}_{n \text{ times}}.$$

This fact will result useful later on.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let \mathcal{C} be a preadditive category. Then products and coproducts are isomorphic to one another in \mathcal{C} .

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

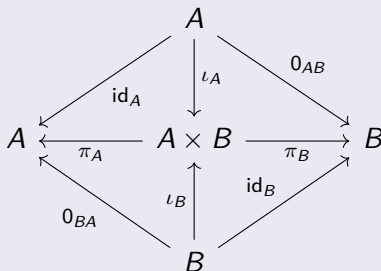
Let A and B be two objects in \mathcal{C} and let $(C := A \times B, \pi_A : C \rightarrow A, \pi_B : C \rightarrow B)$ be a product of A and B . We shall determine two morphisms $\iota_A : A \rightarrow C$ and $\iota_B : B \rightarrow C$ such that (C, ι_A, ι_B) is also a coproduct of A and B .

In doing so, we strive to get some “injections” of A and B into C . A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms $\iota_A, \iota_B : A, B \rightarrow C$:

- 1 $\text{id}_A, 0_{AB} \rightsquigarrow \iota_A,$
- 2 $0_{BA}, \text{id}_B \rightsquigarrow \iota_B.$

Proof.

ι_A and ι_B yield the following commutative diagram:



Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from \mathcal{C} and let $f, g : A, B \rightarrow D$ be morphisms.

Let's define $h_{f,g} : C \rightarrow D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the “connecting morphism” from C to D .

Proof.

We then expect that h_{ι_A, ι_B} – the connecting morphism generated by the “injections” – will behave as the identity on C .

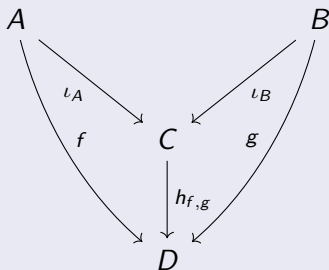
Since the following identities hold:

$$\begin{aligned}\pi_A \circ h_{\iota_A, \iota_B} &= \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A, \\ \pi_B \circ h_{\iota_A, \iota_B} &= \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.\end{aligned}$$

then h_{ι_A, ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:



The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\text{hom}(A, B)$ and $\text{hom}(B, A)$, respectively.

Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned}h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\&= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\&= \dots \\&= 0_{C,D}.\end{aligned}$$



We're missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let's derive kernels from an example, and cokernels will be defined as the dual of kernels.

In linear algebra, given a linear map $f : V \rightarrow W$, we define $\ker f$ as follows:

$$\ker f \triangleq \{v \in V \mid f(v) = 0\}.$$

Of course we're not allowed to define kernels as sets, but only as morphisms. An elementary property of $\ker f$ is that $\ker f$ is a subspace of V , hence there exists a natural injection map $\iota : \ker f \rightarrow V$ such that:

$$\ker f \xrightarrow{\iota} V \xrightarrow{f} W, \quad f \circ \iota = 0_{VW}.$$

It is then natural to define the kernel of $f : A \rightarrow B$ as the “biggest morphism k ” that annihilates f .

Definition

Let $f : A \rightarrow B$ be a morphism. Then a kernel k of f is a morphism $k : K \rightarrow A$ such that:

- 1 $f \circ k = 0_{KB}$,
- 2 If $k' : K' \rightarrow A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \rightarrow K$ such that $k' = k \circ \iota_{K'}$.

Dually, a cokernel of $f : A \rightarrow B$ is the “smallest morphism j ” that f annihilates.

Definition

Let $f : A \rightarrow B$ be a morphism. Then a cokernel j of f is a morphism $j : B \rightarrow J$ such that:

- 1 $j \circ f = 0_{AJ}$,
- 2 If $j' : B \rightarrow J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \rightarrow J'$ such that $j' = \pi_{J'} \circ j$.

Definition

A pre-additive category is pre-abelian if it allows kernels and cokernels for every morphism f and permits products for a finite family of objects.