

An Introduction to Abelian Categories

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May 2025

Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with \mathbf{Vect}_K the category of vector spaces over the field K , whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathbf{FinDimVect}_K$ the subcategory of \mathbf{Vect}_K which contains only finite dimensional K -vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let \mathcal{C} be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Preliminary steps (iv)

Moreover, the hom-set $\text{hom}(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f + g)$, multiply a linear map by a scalar (λf) , and all these operations behave “bilinearly” with the composition (\circ) :

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let \mathcal{C} be a category. We say that \mathcal{C} is a category **enriched over a monoidal category** (\mathcal{D}, \otimes) if the hom-sets of \mathcal{C} are objects from \mathcal{D} and if the composition of morphisms makes the composition \circ bilinear over \otimes , namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that \mathbf{Vect}_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let \mathcal{C} be a category. We say that \mathcal{C} is a **preadditive category** if it's enriched over the category of abelian groups (\mathbf{Ab}).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much “vector spaces over a ring”: they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are \mathbb{Z} -modules, where:

$$n \cdot x := \underbrace{x + x + \dots + x}_{n \text{ times}}.$$

This fact will result useful later on.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let \mathcal{C} be a preadditive category. Then products and coproducts are isomorphic to one another in \mathcal{C} .

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

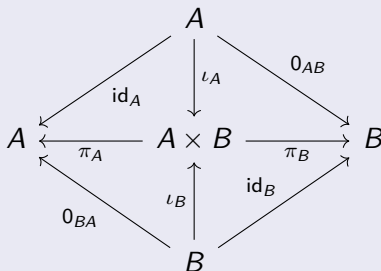
Let A and B be two objects in \mathcal{C} and let $(C := A \times B, \pi_A : C \rightarrow A, \pi_B : C \rightarrow B)$ be a product of A and B . We shall determine two morphisms $\iota_A : A \rightarrow C$ and $\iota_B : B \rightarrow C$ such that (C, ι_A, ι_B) is also a coproduct of A and B .

In doing so, we strive to get some “injections” of A and B into $A \times B$. A way of doing that is to use the universal property of $A \times B$ and extend the following morphisms to two morphisms $\iota_A, \iota_B : A, B \rightarrow C$:

- 1 $\text{id}_A, 0_{AB} \rightsquigarrow \iota_A,$
- 2 $0_{BA}, \text{id}_B \rightsquigarrow \iota_B.$

Proof.

ι_A and ι_B yield the following commutative diagram:



Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from \mathcal{C} and let $f, g : A, B \rightarrow D$ be morphisms.

Let's define $h_{f,g} : C \rightarrow D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the “connecting morphism” from C to D .

Proof.

We then expect that h_{ι_A, ι_B} – the connecting morphism generated by the “injections” – will behave as the identity on C .

Since the following identities hold:

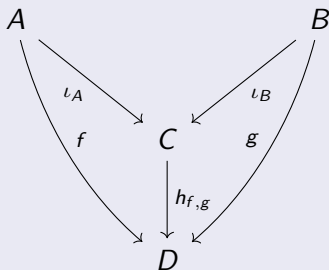
$$\pi_A \circ h_{\iota_A, \iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$

$$\pi_B \circ h_{\iota_A, \iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A, ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:



The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\text{hom}(A, B)$ and $\text{hom}(B, A)$, respectively.

Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned}h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\&= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\&= \dots \\&= 0_{C,D}.\end{aligned}$$

