An Introduction to Abelian Categories

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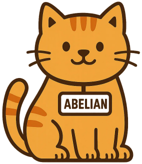
May 2025



Outline

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Before we take a look at abelian categories, we need to introduce the algebraic structure they all rely on: **abelian groups**.

Definition (Abelian group)

An abelian group $(G, +)$ is a commutative monoid which allows inverses.

Abelian groups are fairly simple objects with respect to what a general non-commutative group might be. Nonetheless, they appear everywhere: $\mathbb{Z}$ and the integers modulo n form an abelian group with addition, and all vector spaces rely on abelian groups for their additive structure.

In fact, the connection between vector spaces and abelian groups is deeper than what it initially appears to be. In order to show this, let us recall the definition of a **ring** and of a **module**.

Definition (Ring)

A ring is a set R equipped with two binary operations $+$ and $\cdot$ such that $(R, +)$ is an abelian group, $(R, \cdot)$ is a monoid and law of distributivity applies, i.e. for any choice of a, b, $c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

Definition (R-module)

Given a ring R, we say that a set A equipped with a binary operation $+$ is an R-module with scalar multiplication $\cdot$ if:

1. $(A, +)$ is an abelian group,
2. $1 \cdot a = a$, where 1 is the unity of the ring R and $a \in A$,
3. $\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$, for each λ, $\mu \in R$, and $a \in A$,
4. $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$ for each λ, $\mu \in R$, and $a \in A$,
5. $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$ for each $\lambda \in R$, and a, $b \in A$.

In this setting, K-vector spaces are simply K-modules and abelian groups are simply $\mathbb{Z}$-modules with the following scalar multiplication:

$$k \cdot g \triangleq \begin{cases} \underbrace{g + \ldots + g}_{k \text{ times}} & k \geq 0 \\ -(-k) \cdot g & k < 0. \end{cases}$$

Modules are indeed a great unifying concept for these two algebraic structure, and we'll see at the end of the presentation that the Freyd-Mitchell's embedding theorem will give us more reasons to believe so.

Outline

In 1945 two mathematicians, Eilenberg and Mac Lane, wrote for the first time about categories in the way we look at them now.

They were looking to deeply understand *natural transformations*, hence they had to deal with functors and categories first. Their goal was to apply such abstractness to homological algebra, which revolves around **chain complexes**.

Understanding chain complexes allow mathematicians to retrieve topological invariants from geometrical structures in a pure algebraic setting.

Chain complexes are particular concatenations of morphisms in Ab, the category of abelian groups, written as seen below

$$C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} C_n$$

where $d_{i+1} \circ d_i = 0$ for each i (i.e., $\operatorname{im} d_i \subseteq \ker d_{i+1}$).

We say that such a chain complex is **exact** whenever $\operatorname{im} d_i$ is exactly $\ker d_{i+1}$ for each i.

The goal of abelian categories is to provide the most general setting to generalise and make use of chain complexes.

Instances of the definition of abelian categories are first found in a paper from 1955 by Buchsbaum, whereas further foundations of the theory were laid in a famous paper from Grothendieck later in 1957.

The goal of this presentation is to derive the modern setting of abelian categories from scratch.

Outline

In order to derive a correct definition for abelian categories, we need to address the following issues:

1. We must allow for the existance of "zero morphisms", since we want $d_{i+1} \circ d_i = 0$.
2. We must allow some structure for the hom-sets, as group homomorphisms between two abelian groups A and B form an abelian group as well.
3. We need to define kernels in a pure categorical sense, since $\ker f$ is defined in a set-theoretic manner.

Zero objects I

Abelian groups, vector spaces, and modules in general have a peculiar property: *initial objects and final objects are the same!*

In both cases, such initial object is always denoted as the "trivial" or "zero" object, since it always induces a "trivial" morphism with respect to another module.

Zero objects II

Hence it is crucial to give a proper definition for **zero objects**:

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a zero object if it's both an initial and a final object.

Zero morphisms I

As anticipated, zero objects also allow for the definition of zero morphisms, which will be regarded as the "trivial morphism" from A to B (e.g., sending all elements of A into a "null element", if the category is a concrete and algebraic one).

We derive a definition for zero morphisms by combining the existance and uniqueness of initial and finals morphisms with respect to 0.

Zero morphisms II

Definition (Zero morphisms)

Let A and B be two objects and let 0 be a zero object of $\mathcal{C}$. Then we define 0_{AB} as follows

$$0_{AB} \triangleq ?_B \circ !_A, \qquad \text{where} \quad ?_B : 0 \to B, \quad !_A : A \to 0.$$

The definition is independent of the zero object 0 and is in this sense "unique".

Hom-sets of the form $\hom(V, W)$ in Ab (and RMod, the category of R-modules) have an additional structure: they are not just a set, they have a natural structure of abelian group as well!

We can sum morphisms $(f+g)$, 0_{VW} is indeed the unity, and the sum behave "bilinearly" with respect to the composition $(\circ)$:

$$(f+g)\circ h = f\circ h + g\circ h,$$

$$f\circ(g+h) = f\circ g + f\circ h.$$

This gives rise to an important definition...

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if each hom-set in $\mathcal{C}$ is an abelian group onto which the composition of morphisms acts bilinearly.

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let $\mathcal{C}$ be a preadditive category. Then products and coproducts are isomorphic to one another in $\mathcal{C}$.

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

Let A and B be two objects in $\mathcal{C}$ and let $C := A \times B$ equipped with projections π_A, $\pi_B : C \to A, B$ be a product of A and B.

We shall determine two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that (C, ι_A, ι_B) is also a coproduct of A and B.

In doing so, we strive to get some “injections” of A and B into C.

Proof.

A way of doing that is to use the universal property of C and extend the following morphisms to two morphisms ι_A, $\iota_B : A, B \to C$:

1. $\mathrm{id}_A, 0_{AB} \rightsquigarrow \iota_A$,
2. $0_{BA}, \mathrm{id}_B \rightsquigarrow \iota_B$.

Proof.

ι_A and ι_B yield the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\iota_A"] \arrow[dr, "0_{AB}"] & \\
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
 & B \arrow[ul, "0_{BA}"] \arrow[u, "\iota_B"'] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from $\mathcal{C}$ and let $f, g : A, B \to D$ be morphisms.

Let's define $h_{f,g} : C \to D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the "connecting morphism" from C to D.

Proof.

We then expect that h_{ι_A,ι_B} – the connecting morphism generated by the "injections" – will behave as the identity on C.

Since the following identities hold:

$$\pi_A \circ h_{\iota_A,\iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$
$$\pi_B \circ h_{\iota_A,\iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A,ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:

$$
\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"'] \\
& C \arrow[d, "h_{f,g}"] & \\
& D &
\end{tikzcd}
$$

Proof.

Commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\hom(A, B)$ and $\hom(B, A)$, respectively.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned} h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\ &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\ &= \dots \\ &= 0_{C,D}. \end{aligned}$$

$\square$

We're missing just two ingredients for the definition of pre-abelian categories: **kernels** and **cokernels**. Let's derive kernels from an example, and cokernels will be defined as the dual of kernels.

In abstract and linear algebra, given a morphism $f : A \to B$, we define $\ker f$ as follows:

$$\ker f \triangleq \{a \in B \mid f(a) = 0\}.$$

Of course we're not allowed to defined kernels as sets, but only as objects or morphisms. An elementary property of $\ker f$ is that $\ker f$ is subordinate to A, hence there exists a natural injection map $\iota : \ker f \to A$ such that:

$$\ker f \xhookrightarrow{\iota} A \xrightarrow{f} B \ , \qquad f \circ \iota = 0_{AB}.$$

It is then natural to define the kernel of $f : A \to B$ as the "biggest morphism k" that annihilates f.

Definition

Let $f : A \to B$ be a morphism. Then a kernel k of f is a morphism $k : K \to A$ such that:

1. $f \circ k = 0_{KB}$,
2. If $k' : K' \to A$ is a morphism such that $f \circ k' = 0_{K'B}$, then there exists a unique morphism $\iota_{K'} : K' \to K$ such that $k' = k \circ \iota_{K'}$.

Dually, a cokernel of $f : A \to B$ is the "smallest morphism j" that f annihilates.

Definition

Let $f : A \to B$ be a morphism. Then a cokernel j of f is a morphism $j : B \to J$ such that:

1. $j \circ f = 0_{AJ}$,
2. If $j' : B \to J'$ is a morphism such that $j' \circ f = 0_{AJ'}$, then there exists a unique morphism $\pi_{J'} : J \to J'$ such that $j' = \pi_{J'} \circ j$.

(Co)kernels of the same morphism f have the desidered property of being "unique" and can always be thought of being the same object.

Proposition

Kernels are cokernels of a morphism f are unique up to isomorphism.

We prove the statement only for kernels, then the statement is obtained for cokernels dually.

Proof.

Let k, $k' : K, K' \to A$ be two kernels of $f : A \to B$. The situation is represented in the following commutative diagram:

$$
\begin{tikzcd}
K \arrow[dr, "k"'] \arrow[drr, bend left, "0_{KB}"] & & \\
& A \arrow[r, "f"] & B \\
K' \arrow[ur, "k'"] \arrow[urr, bend right, "0_{K'B}"'] & &
\end{tikzcd}
$$

Proof.

Since k is a kernel of f, there exists a unique morphism $p : K' \to K$ such that $k' = k \circ p$.

$$\begin{tikzcd}
K \arrow[dr, "k"'] \arrow[drr, bend left, "0_{KB}"] & & \\
& A \arrow[r, "f"] & B \\
K' \arrow[uu, dashed, "p"] \arrow[ur, "k'"] \arrow[urr, bend right, "0_{K'B}"'] & &
\end{tikzcd}$$

Proof.

The same goes in the other way: there exists a unique morphism $p' : K \to K'$ such that $k = k' \circ p'$.

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[rrd, bend left, "0_{KB}"] \arrow[dd, dashed, "p'"'] & & \\
& A \arrow[r, "f"] & B \\
K' \arrow[ur, "k'"] \arrow[rru, bend right, "0_{K'B}"'] & &
\end{tikzcd}$$

Proof.

We note that

$$k = k' \circ p' = \underbrace{k \circ p}_{k'} \circ p'.$$

Since id_K is such that $k = k \circ \mathrm{id}_K$ as well, by the universal property of kernels $p \circ p' = \mathrm{id}_K$. Similarly, $p' \circ p = \mathrm{id}_{K'}$, hence p and p' are isomorphisms.

□

Definition

A pre-additive category is pre-abelian if it allows kernels and cokernels for every morphism f and permits products for a finite family of objects.

Since a pre-abelian category allows products, it also implicitly allows coproduct, as we have shown earlier.

We're missing just one piece to get to the definition of an abelian category. When we looked at the morphism $\iota : \ker f \to A$ in the context of linear algebra, we generalised ι thinking of it an "immersion", namely an injective function.

The opposite is also true for abelian groups, vector spaces and modules: an injective linear map gives always rise to a kernel.

In our categorical framework, this will translate to the following definition:

Definition

A monomorphism is said **normal** if it's a kernel of a morphism. An epimorphism is said **conormal** if it's a cokernel of a morphism.

Definition

An abelian category is a pre-abelian category in which all monomorphisms are normal and all epimorphisms are conormal.

Summing up, an abelian category is a category $\mathcal{C}$ such that:

1. The hom-sets of $\mathcal{C}$ are abelian groups in a way such that composition is respected;
2. $\mathcal{C}$ allows for products and coproducts of finite families of objects, and they are the same;
3. Each morphism $f : A \to B$ has a kernel and a cokernel;
4. Each monomorphism is a kernel of a morphism;
5. Each epimorphism is a cokernel of another morphism.

Outline

In this final section of the presentation, we illustrate two fundamental results about abelian categories, which motivate even more their existance.

Before starting out, let's define what a **bimorphism** is:

Definition (Bimorphism)

Given a category $\mathcal{C}$, a morphism $f : A \to B$ which is both monic and epic is called a bimorphism.

In a general category, bimorphisms and isomorphisms do not coincide. For example, in Mon, $\iota : \mathbb{N} \to \mathbb{Z}$ is a bimorphism, even though $\mathbb{N} \ncong \mathbb{Z}$.

This is not the case for abelian categories:

Theorem

Given an abelian category $\mathcal{C}$, bimorphisms are always isomorphisms.

Proof.

Let $f : A \to B$ be a bimorphism. Then, since in $\mathcal{C}$ all monomorphisms are normal, f is the kernel of a morphism $g : B \to C$.

Observe that

$$g \circ f = 0_{AC} = 0_{BC} \circ f.$$

Since f is an epimorphism, this implies that $g = 0_{BC}$.

Proof.

Then $\mathrm{id}_B : B \to B$ is a kernel of g as well:

- $g \circ \mathrm{id}_B = 0_{BC} \circ \mathrm{id}_B = 0_{BC}$,
- If $k' : K' \to B$ is a kernel of g, then the only morphism k satisfying $k' = \mathrm{id}_B \circ k$ is k' itself, hence it is unique.

Proof.

Since $f : A \to B$ and $\mathrm{id}_B : B \to B$ are both kernels of the same morphism, the morphism $p : A \to B$ connecting f to id_B is an isomorphism (see Proposition 2). Since $p = \mathrm{id}_B \circ f = f$, this concludes the proof. $\square$

References

[1] Peter J Freyd. *Abelian categories*. Vol. 1964. Harper & Row New York, 1964.

[2] Sandro M. Roch. *A brief introduction to abelian categories*. 2020.