An Introduction to Abelian Categories

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Preliminary steps (i)

The intuition for abelian categories comes from the behaviour of a kind of mathematical object which is found everywhere in each science: **vector spaces**.

Vector spaces are considered to be well understood and have interesting categorical properties.

Preliminary steps (ii)

First of all, we will denote with Vect_K the category of vector spaces over the field K, whose objects are vector spaces and morphisms are linear maps.

We will denote with $\mathrm{FinDimVect}_K$ the subcategory of Vect_K which contains only finite dimensional K-vector spaces.

Preliminary steps (iii): zero objects

Vector spaces have a peculiar property: initial objects are isomorphic to final objects! This tells use that 0 (the zero dimensional vector space) is a special object.

Definition (Zero object)

Let $\mathcal{C}$ be a category. We say 0 is a **zero object** if it's both an initial and a final object.

Preliminary steps (iv)

Moreover, the hom-set $\hom(V, W)$ has an additional structure: it's not just a set, it has a natural structure of a vector space as well!

We can sum linear maps $(f + g)$, multiply a linear map by a scalar (λf), and all these operations behave "bilinearly" with the composition $(\circ)$:

$$(f + g) \circ h = f \circ h + g \circ h,$$

$$f \circ (g + h) = f \circ g + f \circ h,$$

$$(\lambda f) \circ g = \lambda(f \circ g) = f \circ (\lambda g).$$

This gives rise to an important definition...

Preliminary steps (v): enriched categories

Definition (Enriched categories)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a category **enriched over a monoidal category** $(\mathcal{D}, \otimes)$ if the hom-sets of $\mathcal{C}$ are objects from $\mathcal{D}$ and if the composition of morphisms makes the composition $\circ$ bilinear over $\otimes$, namely:

$$(F \otimes G) \circ H = (F \circ H) \otimes (G \circ H),$$

$$F \circ (G \otimes H) = (F \circ G) \otimes (F \circ H).$$

Therefore, we can say that Vect_K is enriched over itself!

Preliminary steps (vi): preadditive categories

Recall that an abelian group is a monoid which allows inverses and satisfies the law of commutativity. For example, a vector space V is itself an abelian group.

Definition (Preadditive category)

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **preadditive category** if it's enriched over the category of abelian groups (Ab).

In short, a preadditive category is such that its morphisms can be added and subtracted in a way that respects composition.

Preliminary steps (vii): modules and relationship with Ab

Before we properly discuss abelian categories, let's introduce the last fundamental algebraic structure we're going to talk about in this seminary: **modules**.

Modules are pretty much "vector spaces over a ring": they have the same axioms as a vector space, except they are built over a ring, which does not have to allow inverses.

Notice that abelian groups are $\mathbb{Z}$-modules, where:

$$n \cdot x := \underbrace{x + x + \ldots + x}_{n \text{ times}}.$$

This fact will result useful later on.

Products and coproducts behave in the same way in a pre-additive category, as shown below.

Proposition

Let $\mathcal{C}$ be a preadditive category. Then products and coproducts are isomorphic to one another in $\mathcal{C}$.

We only prove that products are also coproducts; the other part of the statement is proved similarly.

Proof.

Let A and B be two objects in $\mathcal{C}$ and let $(C := A \times B, \pi_A : C \to A, \pi_B : C \to B)$ be a product of A and B. We shall determine two morphisms $\iota_A : A \to C$ and $\iota_B : B \to C$ such that (C, ι_A, ι_B) is also a coproduct of A and B.

In doing so, we strive to get some "injections" of A and B into $A \times B$. A way of doing that is to use the universal property of $A \times B$ and extend the following morphisms to two morphisms ι_A, $\iota_B : A, B \to C$:

1. id_A, $0_{AB} \rightsquigarrow \iota_A$,
2. 0_{BA}, $\mathrm{id}_B \rightsquigarrow \iota_B$.

Proof.

ι_A and ι_B yield the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\iota_A"] \arrow[dr, "0_{AB}"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \\
 & B \arrow[ul, "0_{BA}"] \arrow[u, "\iota_B"'] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Proof.

Let's now prove that (C, ι_A, ι_B) is a coproduct. Let D be an object from $\mathcal{C}$ and let $f,\ g : A,\ B \to D$ be morphisms.

Let's define $h_{f,g} : C \to D$ such that:

$$h_{f,g} = f \circ \pi_A + g \circ \pi_B.$$

$h_{f,g}$ will play the role of the "connecting morphism" from C to D.

Proof.

We then expect that h_{ι_A,ι_B} – the connecting morphism generated by the "injections" – will behave as the identity on C.

Since the following identities hold:

$$\pi_A \circ h_{\iota_A,\iota_B} = \pi_A \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_A,$$
$$\pi_B \circ h_{\iota_A,\iota_B} = \pi_B \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) = \pi_B.$$

then h_{ι_A,ι_B} is indeed id_C by the universal property of the product.

Proof.

Let's now prove that $h_{f,g}$ is the unique morphism which makes the following diagram commute:

$$
\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "f"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "g"] \\
& C \arrow[d, "h_{f,g}"] & \\
& D &
\end{tikzcd}
$$

The commutativity can easily be proved by hand, since 0_{AB} and 0_{BA} are the zeroes of $\hom(A, B)$ and $\hom(B, A)$, respectively.

Proof.

On the other hand, uniqueness is proved as follows:

$$\begin{aligned} h_{f,g} - h' &= (h_{f,g} - h') \circ h_{\iota_A, \iota_B} \\ &= (h_{f,g} - h') \circ (\iota_A \circ \pi_A + \iota_B \circ \pi_B) \\ &= \dots \\ &= 0_{C,D}. \end{aligned}$$

□