Statistical Signal Processing

i.i.d: independently identically distributed

1. Math

$\pi \approx 3.14159$ $e \approx 2.71828$ $\sqrt{2} \approx 1.414$ $\sqrt{3} \approx 1.732$

Binome, Trinome

$(a \pm b)^2 = a^2 \pm 2ab + b^2$ $\quad a^2 - b^2 = (a-b)(a+b)$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

Folgen und Reihen

$\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$ Aritmetrische Summenformel $\quad \sum\limits_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$ Geometrische Summenformel $\quad \sum\limits_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ Exponentialreihe

Mittelwerte ($\sum$ von i bis N) (Median: Mitte einer geordneten Liste)

$\overline{x}_{\text{ar}} = \frac{1}{N}\sum x_i$ Arithmetisches $\geq$ $\overline{x}_{\text{geo}} = \sqrt[N]{\prod x_i}$ Geometrisches Mittel $\geq$ $\overline{x}_{\text{hm}} = \frac{N}{\sum \frac{1}{x_i}}$ Harmonisches

Ungleichungen: Bernoulli-Ungleichung: $(1+x)^n \geq 1 + nx$

$\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|$ Dreiecksungleichung $\quad \left|\underline{x}^\top \cdot \underline{y}\right| \leq \|\underline{x}\| \cdot \|\underline{y}\|$ Cauchy-Schwarz-Ungleichung

Mengen: De Morgan: $\overline{A \cap B} = \overline{A} \uplus \overline{B}$ $\quad \overline{A \uplus B} = \overline{A} \cap \overline{B}$

1.1. Exp. und Log.

$e^x := \lim\limits_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ $\quad e \approx 2{,}71828$

$a^x = e^{x \ln a}$ $\quad \log_a x = \frac{\ln x}{\ln a}$ $\quad \ln x \leq x - 1$

$\ln(x^a) = a\ln(x)$ $\quad \ln(\frac{x}{a}) = \ln x - \ln a$ $\quad \log(1) = 0$

1.2. Matrizen $\underline{A} \in \mathbb{K}^{m\times n}$

$\underline{A} = (a_{ij}) \in \mathbb{K}^{m\times n}$ hat m Zeilen (Index i) und n Spalten (Index j)

$(\underline{A} + \underline{B})^\top = \underline{A}^\top + \underline{B}^\top$ $\quad (\underline{A}\cdot\underline{B})^\top = \underline{B}^\top \cdot \underline{A}^\top$

$(\underline{A}^\top)^{-1} = (\underline{A}^{-1})^\top$ $\quad (\underline{A}\cdot\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$

$\dim \mathbb{K} = n = \operatorname{rang} \underline{A} + \dim\ker \underline{A}$ $\quad \operatorname{rang}\underline{A} = \operatorname{rang}\underline{A}^\top$

1.2.1. Quadratische Matrizen $A \in \mathbb{K}^{n\times n}$

regulär/invertierbar/nicht-singulär $\Leftrightarrow \det(\underline{A}) \neq 0 \Leftrightarrow \operatorname{rang}\underline{A} = n$

singulär/nicht-invertierbar $\Leftrightarrow \det(\underline{A}) = 0 \Leftrightarrow \operatorname{rang}\underline{A} \neq n$

orthogonal $\Leftrightarrow \underline{A}^\top = \underline{A}^{-1} \Rightarrow \det(\underline{A}) = \pm 1$

symmetrisch: $\underline{A} = \underline{A}^\top$ $\quad$ schiefsymmetrisch: $\underline{A} = -\underline{A}^\top$

1.2.2. Determinante von $\underline{A} \in \mathbb{K}^{n\times n}$: $\det(\underline{A}) = |\underline{A}|$

$\det\begin{bmatrix}\underline{A} & \underline{0}\\ \underline{C} & \underline{D}\end{bmatrix} = \det\begin{bmatrix}\underline{A} & \underline{B}\\ \underline{0} & \underline{D}\end{bmatrix} = \det(\underline{A})\det(\underline{D})$

$\det(\underline{A}) = \det(\underline{A}^T)$ $\quad \det(\underline{A}^{-1}) = \det(\underline{A})^{-1}$

$\det(\underline{A}\underline{B}) = \det(\underline{A})\det(\underline{B}) = \det(\underline{B})\det(\underline{A}) = \det(\underline{B}\underline{A})$

Hat $\underline{A}$ 2 linear abhäng. Zeilen/Spalten $\Rightarrow |\underline{A}| = 0$

1.2.3. Eigenwerte (EW) λ und Eigenvektoren (EV) $\underline{v}$

$$\boxed{\underline{A}\underline{v} = \lambda\underline{v} \qquad \det\underline{A} = \prod \lambda_i \qquad \operatorname{Sp}\underline{A} = \sum a_{ii} = \sum \lambda_i}$$

Eigenwerte: $\det(\underline{A} - \lambda\underline{1}) = 0$ Eigenvektoren: $\ker(\underline{A} - \lambda_i\underline{1}) = \underline{v}_i$

EW von Dreieck/Diagonal Matrizen sind die Elem. der Hauptdiagonale.

1.2.4. Spezialfall 2×2 Matrix A

$\det(\underline{A}) = ad - bc$ $\quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\det\underline{A}}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$\operatorname{Sp}(\underline{A}) = a + d$

$\lambda_{1/2} = \frac{\operatorname{Sp}\underline{A}}{2} \pm \sqrt{\left(\frac{\operatorname{sp}\underline{A}}{2}\right)^2 - \det\underline{A}}$

1.2.5. Differentiation

$\frac{\partial \underline{x}^\top \underline{y}}{\partial \underline{x}} = \frac{\partial \underline{y}^\top \underline{x}}{\partial \underline{x}} = \underline{y}$ $\quad \frac{\partial \underline{x}^\top \underline{A}\underline{x}}{\partial \underline{x}} = (\underline{A} + \underline{A}^\top)\underline{x}$

$\frac{\partial \underline{x}^\top \underline{A}\underline{y}}{\partial \underline{A}} = \underline{x}\underline{y}^\top$ $\quad \frac{\partial \det(\underline{B}\underline{A}\underline{C})}{\partial \underline{A}} = \det(\underline{B}\underline{A}\underline{C})\left(\underline{A}^{-1}\right)^\top$

1.2.6. Ableitungsregeln ($\forall \lambda, \mu \in \mathbb{R}$)

Linearität:	$(\lambda f + \mu g)'(x) = \lambda f'(x) + \mu g'(x_0)$
Produkt:	$(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient:	$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$ $\quad \left(\frac{\text{NAZ}-\text{ZAN}}{\text{N}^2}\right)$
Kettenregel	$(f(g(x)))' = f'(g(x))g'(x)$

1.3. Integrale $\int e^x \,\mathrm{d}x = e^x = (e^x)'$

Partielle Integration: $\int uw' = uw - \int u'w$

Substitution: $\int f(g(x))g'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t$

$F(x) - C$	$f(x)$	$f'(x)$
$\frac{1}{q+1}x^{q+1}$	x^q	qx^{q-1}
$\frac{2\sqrt{ax^3}}{3}$	$\sqrt{ax}$	$\frac{a}{2\sqrt{ax}}$
$x\ln(ax) - x$	$\ln(ax)$	$\frac{1}{x}$
$\frac{1}{a^2}e^{ax}(ax-1)$	$x\cdot e^{ax}$	$e^{ax}(ax+1)$
$\frac{a^x}{\ln(a)}$	a^x	$a^x\ln(a)$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$-\ln\lvert\cos(x)\rvert$	$\tan(x)$	$\frac{1}{\cos^2(x)}$

$$\int e^{at}\sin(bt)\,\mathrm{d}t = e^{at}\frac{a\sin(bt) + b\cos(bt)}{a^2+b^2}$$

$\int \frac{\mathrm{d}t}{\sqrt{at+b}} = \frac{2\sqrt{at+b}}{a}$ $\quad \int t^2 e^{at}\,\mathrm{d}t = \frac{(ax-1)^2+1}{a^3}e^{at}$

$\int te^{at}\,\mathrm{d}t = \frac{at-1}{a^2}e^{at}$ $\quad \int xe^{ax^2}\,\mathrm{d}x = \frac{1}{2a}e^{ax^2}$

1.3.1. Volumen und Oberfläche von Rotationskörpern um x-Achse

$V = \pi\int_a^b f(x)^2\,\mathrm{d}x$ $\quad O = 2\pi\int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x$

2. Probability Theory Basics

2.1. Kombinatorik

Mögliche Variationen/Kombinationen um k Elemente von maximal n Elementen zu wählen bzw. k Elemente auf n Felder zu verteilen:

	Mit Reihenfolge	Reihenfolge egal
Mit Wiederholung	n^k	$\binom{n+k-1}{k}$
Ohne Wiederholung	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Permutation von n mit jeweils k gleichen Elementen: $\frac{n!}{k_1!\cdot k_2!\cdot\ldots}$

Binomialkoeffizient $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!\cdot(n-k)!}$

$\binom{n}{0} = 1$ $\quad \binom{n}{1} = n$ $\quad \binom{4}{2} = 6$ $\quad \binom{5}{2} = 10$ $\quad \binom{6}{2} = 15$

2.2. Der Wahrscheinlichkeitsraum $(\Omega, \mathbb{F}, \mathsf{P})$

Ergebnismenge	$\Omega = \{\omega_1, \omega_2, \ldots\}$	Ergebnis $\omega_j \in \Omega$
Ereignisalgebra	$\mathbb{F} = \{A_1, A_2, \ldots\}$	Ereignis $A_i \subseteq \Omega$
Wahrscheinlichkeitsmaß	$\mathsf{P}: \mathbb{F} \to [0,1]$	$\mathsf{P}(A) = \frac{\lvert A\rvert}{\lvert\Omega\rvert}$

2.3. Wahrscheinlichkeitsmaß P

$\mathsf{P}(A) = \frac{|A|}{|\Omega|}$ $\quad \mathsf{P}(A\cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A\cap B)$

2.3.1. Axiome von Kolmogorow

Nichtnegativität: $\mathsf{P}(A) \geq 0 \Rightarrow \mathsf{P}: \mathbb{F} \mapsto [0,1]$

Normiertheit: $\mathsf{P}(\Omega) = 1$

Additivität: $\mathsf{P}\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty}\mathsf{P}(A_i)$,

wenn $A_i \cap A_j = \emptyset, \forall i \neq j$

2.4. Bedingte Wahrscheinlichkeit

Bedingte Wahrscheinlichkeit für A falls B bereits eingetreten ist:

$\mathsf{P}_B(A) = \mathsf{P}(A|B) = \frac{\mathsf{P}(A\cap B)}{\mathsf{P}(B)}$

2.4.1. Totale Wahrscheinlichkeit und Satz von Bayes

Es muss gelten: $\bigcup\limits_{i\in I} B_i = \Omega$ für $B_i \cap B_j = \emptyset, \forall i \neq j$

Totale Wahrscheinlichkeit: $\mathsf{P}(A) = \sum\limits_{i\in I}\mathsf{P}(A|B_i)\,\mathsf{P}(B_i)$

Satz von Bayes: $\mathsf{P}(B_k|A) = \frac{\mathsf{P}(A|B_k)\,\mathsf{P}(B_k)}{\sum\limits_{i\in I}\mathsf{P}(A|B_i)\,\mathsf{P}(B_i)}$

Multiplikationssatz: $\mathsf{P}(A\cap B) = \mathsf{P}(A|B)\,\mathsf{P}(B) = \mathsf{P}(B|A)\,\mathsf{P}(A)$

2.5. Zufallsvariable

$X: \Omega \mapsto \Omega'$ ist Zufallsvariable, wenn für jedes Ereignis $A' \in \mathbb{F}'$ im Bildraum ein Ereignis A im Urbildraum $\mathbb{F}$ existiert, sodass $\{\omega \in \Omega | \, X(\omega) \in A'\} \in \mathbb{F}$

2.6. Distribution

Bezeichnung	Abk.	Zusammenhang
Wahrscheinlichkeitsdichte	pdf	$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$
Kumulative Verteilungsfkt.	cdf	$F_X(x) = \int\limits_{-\infty}^{x} f_X(\xi)\,\mathrm{d}\xi$

Joint CDF: $F_{X,Y}(x,y) = \mathsf{P}(\{X \leq x, Y \leq y\})$

2.7. Relations between $f_{\mathsf{X}}(x)$, $f_{\mathsf{X},\mathsf{Y}}(x,y)$, $f_{\mathsf{X}\,|\,\mathsf{Y}}(x|y)$

$$\boxed{\begin{gathered}\underbrace{f_{X,Y}(x,y)}_{\text{Joint PDF}} = f_{X\,|\,Y}(x,y)f_Y(y) = f_{Y\,|\,X}(y,x)f_X(x)\\ \underbrace{\int\limits_{-\infty}^{\infty} f_{X,Y}(x,\xi)\,\mathrm{d}\xi}_{\text{Marginalization}} = \underbrace{\int\limits_{-\infty}^{\infty} f_{X\,|\,Y}(x,\xi)f_Y(\xi)\,\mathrm{d}\xi}_{\text{Total Probability}} = f_X(x)\end{gathered}}$$

2.8. Bedingte Zufallsvariablen

Ereignis A gegeben: $F_{X\,|\,A}(x|A) = \mathsf{P}(\{X \leq x\}|A)$

ZV Y gegeben: $F_{X\,|\,Y}(x|y) = \mathsf{P}(\{X \leq x\}|\{Y = y\})$

$p_{X\,|\,Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

$f_{X\,|\,Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathrm{d}F_{X|Y}(x|y)}{\mathrm{d}x}$

2.9. Unabhängigkeit von Zufallsvariablen

$X_1, \cdots, X_n$ sind stochastisch unabhängig, wenn für jedes $\underline{x} \in \mathbb{R}^n$ gilt:

$F_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \prod\limits_{i=1}^{n} F_{X_i}(x_i)$

$p_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \prod\limits_{i=1}^{n} p_{X_i}(x_i)$

$f_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \prod\limits_{i=1}^{n} f_{X_i}(x_i)$

3. Common Distributions

3.1. Binomialverteilung $\mathcal{B}(n,p)$ mit $p \in [0,1], n \in \mathbb{N}$

Folge von n Bernoulli-Experimenten

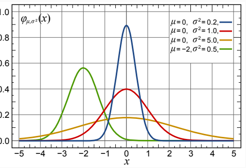
p: Wahrscheinlichkeit für Erfolg $\quad k$: Anzahl der Erfolge

$$p_X(k) = B_{n,p}(k) = \begin{cases}\binom{n}{k}p^k(1-p)^{n-k} & k \in \{0,\ldots,n\}\\ 0 & \text{sonst}\end{cases}$$

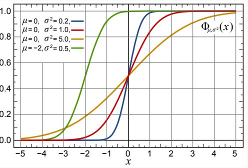
$\mathsf{E}[X] = np$ Erwartungswert	$\mathsf{Var}[X] = np(1-p)$ Varianz	$G_X(z) = (pz + 1 - p)^n$ Wahrscheinlichkeitserz. Funktion

3.2. Normalverteilung

WDF/PDF:



KVF/CDF:



WDF: $\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad x \in \mathbb{R}}$ $\quad \mu \in \mathbb{R}$, $\sigma > 0$

$\mathsf{E}(X) = \mu$ Erwartungswert	$\mathsf{Var}(X) = \sigma^2$ Varianz	$\varphi_X(\omega) = e^{j\omega\mu - \frac{\omega^2\sigma^2}{2}}$ Charakt. Funktion

3.3. Sonstiges

Gammadistribution $\Gamma(\alpha,\beta)$: $\mathsf{E}[X] = \frac{\alpha}{\beta}$

Exponential: $f(x,\lambda) = \lambda e^{-\lambda x}$ $\quad \mathsf{E}[X] = \lambda^{-1}$ $\quad \mathsf{Var}[X] = \lambda^{-2}$

4. Wichtige Parameter

4.1. Erwartungswert (1. zentrales Moment)

gibt den mittleren Wert einer Zufallsvariablen an

$$\mu_X = \mathrm{E}[X] = \underbrace{\sum_{x\in\Omega'} x \cdot \mathrm{P}_X(x)}_{\text{diskrete } X:\Omega\to\Omega'} \;\hat{=}\; \underbrace{\int_{\mathbb{R}} x \cdot f_X(x)\,\mathrm{d}x}_{\text{stetige } X:\Omega\to\mathbb{R}}$$

$\mathrm{E}[\alpha X + \beta Y] = \alpha\,\mathrm{E}[X] + \beta\,\mathrm{E}[Y]$ $\qquad X \le Y \Rightarrow \mathrm{E}[X] \le \mathrm{E}[Y]$

$\mathrm{E}[X^2] = \mathrm{Var}[X] + \mathrm{E}[X]^2$

$\mathrm{E}[X\,Y] = \mathrm{E}[X]\,\mathrm{E}[Y]$, falls X und Y stochastisch unabhängig

Umkehrung nicht möglichich: Unkorreliertheit $\not\Rightarrow$ Stoch. Unabhängig!

4.1.1. Für Funktionen von Zufallsvariablen $g(x)$

$\mathrm{E}[g(X)] = \sum_{x\in\Omega'} g(x)\,\mathrm{P}_X(x) \;\hat{=}\; \int_{\mathbb{R}} g(x) f_X(x)\,\mathrm{d}x$

4.2. Varianz (2. zentrales Moment)

ist ein Maß für die Stärke der Abweichung vom Erwartungswert

$$\sigma_X^2 = \mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

$\mathrm{Var}[\alpha X + \beta] = \alpha^2\,\mathrm{Var}[X]$ $\qquad \mathrm{Var}[X] = \mathrm{Cov}[X, X]$

$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + \sum_{j\neq i} \mathrm{Cov}[X_i, X_j]$

Standard Abweichung: $\sigma = \sqrt{\mathrm{Var}[X]}$

4.3. Kovarianz

Maß für den linearen Zusammenhang zweier Variablen

$$\mathrm{Cov}[X, Y] = \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])^\top] = \\ = \mathrm{E}[X\,Y^\top] - \mathrm{E}[X]\,\mathrm{E}[Y]^\top = \mathrm{Cov}[Y, X]$$

$\mathrm{Cov}[\alpha X + \beta, \gamma Y + \delta] = \alpha\gamma\,\mathrm{Cov}[X, Y]$

$\mathrm{Cov}[X + U, Y + V] = \mathrm{Cov}[X, Y] + \mathrm{Cov}[X, V] + \mathrm{Cov}[U, Y] + \mathrm{Cov}[U, V]$

4.3.1. Korrelation = standardisierte Kovarianz

$\rho(X, Y) = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\cdot\mathrm{Var}[Y]}} = \frac{C_{x,y}}{\sigma_x\cdot\sigma_y} \qquad \rho(X, Y) \in [-1; 1]$

4.3.2. Kovarianzmatrix für $\underline{z} = (\underline{x}, \underline{y})^\top$

$\mathrm{Cov}[\underline{z}] = \underline{C}_{\underline{z}} = \begin{bmatrix} C_X & C_{XY} \\ C_{XY} & C_Y \end{bmatrix} = \begin{bmatrix} \mathrm{Cov}[X, X] & \mathrm{Cov}[X, Y] \\ \mathrm{Cov}[Y, X] & \mathrm{Cov}[Y, Y] \end{bmatrix}$

Immer symmetrisch: $C_{xy} = C_{yx}$! Für Matrizen: $\underline{C}_{\underline{x}\underline{y}} = \underline{C}_{\underline{y}\underline{x}}^\top$

5. Estimation

5.1. Estimation

Statistic Estimation treats the problem of inferring underlying characteristics of unknown random variables on the basis of observations of outputs of those random variables.

Sample Space Ω	nonempty set of outputs of experiment
Sigma Algebra $\mathbb{F} \subseteq 2^\Omega$	set of subsets of outputs (events)
Probability $\mathrm{P} : \mathbb{F} \mapsto [0, 1]$	
Random Variable $X : \Omega \mapsto \mathbb{X}$	mapped subsets of Ω
Observations: $x_1, \dots, x_N$	single values of X
Observation Space $\mathbb{X}$	possible observations of X
Unknown parameter $\theta \in \Theta$	parameter of propability function
Estimator $T : \mathbb{X} \mapsto \Theta$	$T(X) = \hat{\theta}$, finds $\hat{\theta}$ from X

unknown parm. θ	estimation of param. $\hat{\theta}$
R.V. of param. Θ	estim. of R.V. of parm $T(X) = \hat{\Theta}$

5.2. Quality Properties of Estimators

Consistent: If $\lim_{N\to\infty} T(x_1, \dots, x_N) = \theta$

Bias $\mathrm{Bias}(T) := \mathrm{E}[T(X_1, \dots, X_N)] - \theta$

unbiased if $\mathrm{Bias}(T) = 0$ (biased estimators can provide better estimates than unbiased estimators.)

Variance $\mathrm{Var}[T] := \mathrm{E}\left[(T - \mathrm{E}[T])^2\right]$

5.3. Mean Square Error (MSE)

The MSE is an extension of the Variance $\mathrm{Var}[T] := \mathrm{E}\left[(T - \mathrm{E}[T])^2\right]$:

$$\text{MSE: } \varepsilon[T] = \underbrace{\mathrm{E}\left[(T - \theta)^2\right]}_{=\mathrm{E}[(\hat{\theta}-\theta)^2]} = \mathrm{Var}(T) + (\mathrm{Bias}[T])^2$$

If Θ is also r.v. $\Rightarrow$ mean over both (e.g. Bayes est.):

Mean MSE: $\mathrm{E}[(T(X) - \Theta)^2] = \mathrm{E}\left[\mathrm{E}\left[(T(X) - \Theta)^2 | \Theta = \theta\right]\right]$

5.3.1. Minimum Mean Square Error (MMSE)

Minimizes mean square error: $\arg\min_{\hat{\theta}} \mathrm{E}\left[(\hat{\theta} - \theta)^2\right]$

$\mathrm{E}\left[(\hat{\theta} - \theta)^2\right] = \mathrm{E}[\theta^2] - 2\hat{\theta}\,\mathrm{E}[\theta] + \hat{\theta}^2$

Solution: $\frac{\mathrm{d}}{\mathrm{d}\hat{\theta}}\,\mathrm{E}\left[(\hat{\theta} - \theta)^2\right] \overset{!}{=} 0 = -2\,\mathrm{E}[\theta] + 2\hat{\theta} \quad \Rightarrow \hat{\theta}_{\mathrm{MMSE}} = \mathrm{E}[\theta]$

5.4. Maximum Likelihood

Given model $\{\mathbb{X}, \mathbb{F}, \mathrm{P}_\theta; \theta \in \Theta\}$, assume $\mathrm{P}_\theta(\underline{x})$ or $f_X(\underline{x}, \theta)$ for observed data $\underline{x}$. Estimate parameter θ so that the likelihood $L(\underline{x}, \theta)$ or $L(\theta | X = \underline{x})$ to obtain $\underline{x}$ is maximized.

Likelihood Function: (Prob. for θ given $\underline{x}$)

Discrete: $\quad L(x_1, \dots, x_N; \theta) = \mathrm{P}_\theta(x_1, \dots, x_N)$

Continuous: $\quad L(x_1, \dots, x_N; \theta) = f_{X_1,\dots,X_N}(x_1, \dots, x_N, \theta)$

If N observations are Identically Independently Distributed (i.i.d.):

$L(\underline{x}, \theta) = \prod_{i=1}^{N} \mathrm{P}_\theta(x_i) = \prod_{i=1}^{N} f_{X_i}(x_i)$

ML Estimator (Picks θ): $T_{\mathrm{ML}} : X \mapsto \underset{\theta\in\Theta}{\operatorname{argmax}}\{L(X, \theta)\} =$

$= \underset{\theta\in\Theta}{\operatorname{argmax}}\{\log L(\underline{X}, \theta)\} \overset{\text{i.i.d.}}{=} \underset{\theta\in\Theta}{\operatorname{argmax}}\{\sum \log L(x_i, \theta)\}$

Find Maximum: $\frac{\partial L(\underline{x}, \theta)}{\partial\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta} \log L(x; \theta)\Big|_{\theta=\hat{\theta}} \overset{!}{=} 0$

Solve for θ to obtain ML estimator function $\hat{\theta}_{\mathrm{ML}}$

Check quality of estimator with MSE

Maximum-Likelihood Estimator is Asymptotically Efficient. However, there might be not enough samples and the likelihood function is often not known.

5.5. Uniformly Minimum Variance Unbiased (UMVU) Estimators (Best unbiased estimators)

Best unbiased estimator: Lowest Variance of all estimators.

Fisher's Information Inequality: Estimate lower bound of variance if

- $L(x, \theta) > 0, \forall x, \theta$
- $L(x, \theta)$ is diffable for θ
- $\int_{\mathbb{X}} \frac{\partial}{\partial\theta} L(x, \theta)\,\mathrm{d}x = \frac{\partial}{\partial\theta} \int_{\mathbb{X}} L(x, \theta)\,\mathrm{d}x$

Score Function:

$g(x, \theta) = \frac{\partial}{\partial\theta} \log L(x, \theta) = \frac{\frac{\partial}{\partial\theta} L(x,\theta)}{L(x,\theta)} \qquad \mathrm{E}[g(x, \theta)] = 0$

Fischer Information:

$I_{\mathrm{F}}(\theta) := \mathrm{Var}[g(X, \theta)] = \mathrm{E}[g(x, \theta)^2] = -\,\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2} \log L(X, \theta)\right]$

Cramér-Rao Lower Bound (CRB): (if T is unbiased)

$$\mathrm{Var}[T(X)] \ge \left(\frac{\partial\,\mathrm{E}[T(X)]}{\partial\theta}\right)^2 \frac{1}{I_{\mathrm{F}}(\theta)} \qquad \mathrm{Var}[T(X)] \ge \frac{1}{I_{\mathrm{F}}(\theta)}$$

For N i.i.d. observations: $I_{\mathrm{F}}^{(N)}(x, \theta) = N \cdot I_{\mathrm{F}}^{(1)}(x, \theta)$

5.5.1. Exponential Models

If $f_X(x) = \frac{h(x)\exp\left(a(\theta)t(x)\right)}{\exp(b(\theta))}$ then $I_F(\theta) = \frac{\partial a(\theta)}{\partial\theta}\,\frac{\partial E[t(X)]}{\partial\theta}$

Some Derivations: (check in exam)

Uniformly: Not diffable $\Rightarrow$ no $I_F(\theta)$

Normal $\mathcal{N}(\theta, \sigma^2)$: $g(x, \theta) = \frac{(x-\theta)}{\sigma^2} \qquad I_{\mathrm{F}}(\theta) = \frac{1}{\sigma^2}$

Binomial $\mathcal{B}(\theta, K)$: $g(x, \theta) = \frac{x}{\theta} - \frac{K-x}{1-\theta} \qquad I_{\mathrm{F}}(\theta) = \frac{K}{\theta(1-\theta)}$

5.6. Bayes Estimation (Conditional Mean)

A Priori information about θ is known as probability $f_\Theta(\theta; \sigma)$ with random variable Θ and parameter σ. Now the conditional pdf $f_{X|\Theta}(x, \theta)$ is used to find θ by minimizing the mean MSE instead of uniformly MSE.

Mean MSE for Θ: $\mathrm{E}\left[\mathrm{E}[(T(X) - \Theta)^2 | \Theta = \theta]\right]$

Conditional Mean Estimator:

$T_{\mathrm{CM}} : x \mapsto \mathrm{E}[\Theta | X = x] = \int_\Theta \theta \cdot f_{\Theta|X}(\theta|x)\,\mathrm{d}\theta$

Posterior $f_{\Theta|\underline{X}}(\theta|\underline{x}) = \frac{f_{\underline{X}|\Theta}(\underline{x}) f_\theta(\theta)}{\int_\Theta f_{\underline{X},\xi}(\underline{x}, \xi)\,\mathrm{d}\xi} = \frac{f_{\underline{X}|\theta}(\underline{x}) f_\theta(\theta)}{f_X(x)}$

Hint: to calculate $f_{\Theta|\underline{X}}(\theta|\underline{x})$: Replace every factor not containing θ, such as $\frac{1}{f_X(x)}$ with a factor γ and determine γ at the end such that $\int_\Theta f_{\Theta|\underline{X}}(\theta|\underline{x})\,\mathrm{d}\theta = 1$

MMSE: $\mathrm{E}[\mathrm{Var}[X | \Theta = \theta]]$

Multivariate Gaussian: $X, \Theta \sim \mathcal{N} \quad \Rightarrow \sigma_X^2 = \sigma_{X|\Theta=\theta}^2 + \sigma_\Theta$

$T_{\mathrm{CM}} : x \mapsto \mathrm{E}[\Theta | X = x] = \underline{\mu}_\Theta + \underline{C}_{\Theta,X}\underline{C}_X^{-1}(\underline{x} - \underline{\mu}_X)$

MMSE:

$\mathrm{E}\left[\| T_{\mathrm{CM}} - \Theta\|_2^2\right] = \mathrm{tr}(\underline{C}_{\theta|X}) = \mathrm{tr}(\underline{C}_\Theta - \underline{C}_{\Theta,X}\underline{C}_X^{-1}\underline{C}_{X,\Theta})$

Orthogonality Principle:

$T_{\mathrm{CM}}(\underline{X}) - \Theta \perp h(\underline{X}) \quad \Rightarrow \quad \mathrm{E}[(T_{\mathrm{CM}}(\underline{X}) - \Theta)h(\underline{X})] = 0$

MMSE Estimator: $\hat{\theta}_{\mathrm{MMSE}} = \arg\min_{\theta\in\Theta} \mathrm{MSE}$

minimizes the MSE for all estimators

5.7. Example:

Estimate mean θ of X with prior knowledge $\theta \in \Theta \sim \mathcal{N}$:

$X \sim \mathcal{N}(\theta, \sigma_{X|\Theta=\theta}^2)$ and $\Theta \sim \mathcal{N}(m, \sigma_\Theta^2)$

$\hat{\theta}_{\mathrm{CM}} = \mathrm{E}[\Theta | \underline{X} = \underline{x}] = \frac{N\sigma_\Theta^2}{\sigma_{X|\Theta=\theta}^2 + N\sigma_\Theta^2}\hat{\theta}_{\mathrm{ML}} + \frac{\sigma_{X|\Theta=\theta}^2}{\sigma_{X|\Theta=\theta}^2 + N\sigma_\Theta^2} m$

For N independent observations x_i: $\hat{\theta}_{\mathrm{ML}} = \frac{1}{N}\sum x_i$

Large $N \Rightarrow$ ML better, small $N \Rightarrow$ CM better

6. Linear Estimation

t is now the unknown parameter θ, we want to estimate y and $\underline{x}$ is the input vector... review regression problem $\underline{y} = \underline{A}\underline{x}$ (we solve for $\underline{x}$), here we solve for $\underline{t}$, because $\underline{x}$ is known (measured)! Confusing...

1. Training $\rightarrow$ 2. Estimation

Training: We observe y and $\underline{x}$ (knowing both) and then based on that we try to estimate y given $\underline{x}$ (only observe $\underline{x}$) with a linear model $\hat{y} = \underline{x}^\top\underline{t}$

$$\text{Estimation: } \hat{y} = \underline{x}^\top\underline{t} + m \quad \text{or} \quad \hat{y} = \underline{x}^\top\underline{t}$$

Given: N observations $(y_i, \underline{x}_i)$, unknown parameters $\underline{t}$, noise m

$\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} \underline{x}_1^\top \\ \vdots \\ \underline{x}_n^\top \end{bmatrix} \qquad$ Note: $\hat{y} \neq y$!!

Problem: Estimate y based on given (known) observations $\underline{x}$ and unknown parameter $\underline{t}$ with assumed linear Model: $\hat{y} = \underline{x}^\top\underline{t}$

Note $y = \underline{x}^\top\underline{t} + m \rightarrow y = \underline{x}'^\top\underline{t}'$ with $\underline{x}' = \begin{pmatrix} \underline{x} \\ 1 \end{pmatrix}$, $\quad t' = \begin{pmatrix} \underline{t} \\ m \end{pmatrix}$

Sometimes in Exams: $\hat{y} = \underline{x}^\top\underline{t} \Leftrightarrow \hat{\underline{x}} = \underline{T}^\top\underline{y}$

estimate $\underline{x}$ given $\underline{y}$ and unknown $\underline{T}$

6.1. Least Square Estimation (LSE)

Tries to minimize the square error for linear Model: $\hat{y}_{\mathrm{LS}} = \underline{x}^\top\underline{t}_{\mathrm{LS}}$

Least Square Error: $\min\left[\sum_{i=1}^{N}(y_i - \underline{x}_i^\top\underline{t})^2\right] = \min_{\underline{t}}\left\|\underline{y} - \underline{X}\underline{t}\right\|$

$$\underline{t}_{\mathrm{LS}} = (\underline{X}^\top\underline{X})^{-1}\underline{X}^\top\underline{y}$$

$\hat{\underline{y}}_{\mathrm{LS}} = \underline{X}\underline{t}_{\mathrm{LS}} \in span(X)$

Orthogonality Principle: N observations $\underline{x}_i \in \mathbb{R}^d$

$\underline{Y} - \underline{X}\underline{T}_{\mathrm{LS}} \perp \mathrm{span}[\underline{X}] \Leftrightarrow \underline{Y} - \underline{X}\underline{T}_{\mathrm{LS}} \in \mathrm{null}[\underline{X}^\top]$, thus

$\underline{X}^\top(\underline{Y} - \underline{X}\underline{T}_{\mathrm{LS}}) = 0$ and if $N > d \wedge \mathrm{rang}[\underline{X}] = d$:

$\underline{T}_{\mathrm{LS}} = (\underline{X}^\top\underline{X})^{-1}\underline{X}^\top\underline{Y}$

6.2. Linear Minimum Mean Square Estimator (LMMSE)

Estimate y with linear estimator $\underline{t}$, such that $\hat{y} = \underline{t}^\top\underline{x} + m$

Note: the Model does not need to be linear! The estimator is linear!

$$\hat{y}_{\mathrm{LMMSE}} = \arg\min_{\underline{t}, m} \mathrm{E}\left[\left\|\underline{y} - (\underline{t}^\top\underline{x} + m)\right\|_2^2\right]$$

If Random joint variable $\underline{z} = \begin{pmatrix} \underline{x} \\ y \end{pmatrix}$ with

$\underline{\mu}_{\underline{z}} = \begin{pmatrix} \underline{\mu}_{\underline{x}} \\ \mu_y \end{pmatrix}$ and $\underline{C}_{\underline{z}} = \begin{bmatrix} \underline{C}_{\underline{x}} & \underline{c}_{\underline{x}y} \\ c_{y\underline{x}} & c_y \end{bmatrix}$ then

LMMSE Estimation of y given x is

$\hat{y} = \mu_y + \underline{c}_{y\underline{x}}\underline{C}_{\underline{x}}^{-1}(\underline{x} - \underline{\mu}_{\underline{x}}) = \underbrace{\underline{c}_{y\underline{x}}\underline{C}_{\underline{x}}^{-1}}_{=\underline{t}^\top}\underline{x} - \underbrace{\mu_y + \underline{c}_{y\underline{x}}\underline{C}_{\underline{x}}^{-1}\underline{\mu}_{\underline{x}}}_{=m}$

Minimum MSE: $\mathrm{E}\left[\left\|\underline{y} - (\underline{x}^\top\underline{t} + m)\right\|_2^2\right] = c_y - c_{y\underline{x}}C_{\underline{x}}^{-1}\underline{c}_{\underline{x}y}$

Hint: First calculate $\hat{y}$ in general and then set variables according to system equation.

Multivariate: $\hat{\underline{y}} = \underline{T}_{\mathrm{LMMSE}}^\top\underline{x} \qquad \underline{T}_{\mathrm{LMMSE}}^\top = \underline{C}_{\underline{y}\underline{x}}\underline{C}_{\underline{x}}^{-1}$

If $\underline{\mu}_{\underline{z}} = \underline{0}$ then

Estimator $\hat{y} = \underline{c}_{y,\underline{x}}\underline{C}_{\underline{x}}^{-1}\underline{x}$

Minimum MSE: $\mathrm{E}[c_{y,\underline{x}}] = c_y - \underline{t}^\top\underline{c}_{\underline{x},y}$

6.3. Matched Filter Estimator (MF)

For channel $\underline{y} = \underline{h}x + \underline{v}$, Filtered: $\underline{t}^\top \underline{y} = \underline{t}^\top \underline{h}x + \underline{t}^\top \underline{v}$

Find Filter $\underline{t}^\top$ that maximizes SNR $= \frac{\|\underline{h}x\|}{\|\underline{v}\|}$

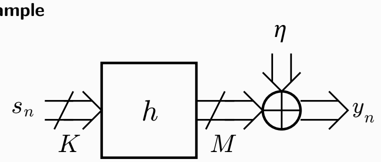
$\underline{t}_{\mathrm{MF}} = \max_{t} \left\{ \frac{\mathrm{E}\left[(\underline{t}^\top \underline{h}x)^2\right]}{\mathrm{E}\left[(\underline{t}^\top \underline{v})^2\right]} \right\}$

In the lecture (estimate $\underline{h}$):

$\boldsymbol{T}_{\mathrm{MF}} = \max_{T} \left\{ \frac{\left|\mathrm{E}\left[\hat{\underline{h}}^H \underline{h}\right]\right|^2}{\mathrm{tr}\left[\mathrm{Var}[\boldsymbol{T}\underline{n}]\right]} \right\}$

$\hat{\underline{h}}_{\mathrm{MF}} = \boldsymbol{T}_{\mathrm{MF}}\underline{y}$ $\quad \boldsymbol{T}_{\mathrm{MF}} \propto \boldsymbol{C}_{\underline{h}} \boldsymbol{S}^H \boldsymbol{C}_{\underline{n}}^{-1}$

6.4. Example



System Model: $\underline{y}_n = \boldsymbol{H}\underline{s}_n + \eta_n$

with $\boldsymbol{H} = (h_{m,k}) \in \mathbb{C}^{M\times K}$ $\quad (m \in [1, M], k \in [1, K])$

Linear Channel Model $\underline{y} = \boldsymbol{S}\underline{h} + \underline{n}$ with

$\underline{h} \sim \mathcal{N}(0, \boldsymbol{C}_{\underline{h}})$ and $\underline{n} \sim \mathcal{N}(0, \boldsymbol{C}_{\underline{n}})$

Linear Estimator $\boldsymbol{T}$ estimates $\hat{\underline{h}} = \boldsymbol{T}\underline{y} \in \mathbb{C}^{MK}$

$\boldsymbol{T}_{\mathrm{MMSE}} = \boldsymbol{C}_{\underline{h}\underline{y}} \boldsymbol{C}_{\underline{y}}^{-1} = \boldsymbol{C}_{\underline{h}} \boldsymbol{S}^{\mathrm{H}} (\boldsymbol{S}\boldsymbol{C}_{\underline{h}}\boldsymbol{S}^{\mathrm{H}} + \boldsymbol{C}_{\underline{n}})^{-1}$

$\boldsymbol{T}_{\mathrm{ML}} = \boldsymbol{T}_{\mathrm{Cor}} = (\boldsymbol{S}^{\mathrm{H}} \boldsymbol{C}_{\underline{n}}^{-1} \boldsymbol{S})^{-1} \boldsymbol{S}^{\mathrm{H}} \boldsymbol{C}_{\underline{n}}^{-1}$

$\boldsymbol{T}_{\mathrm{MF}} \propto \boldsymbol{C}_{\underline{h}} \boldsymbol{S}^{\mathrm{H}} \boldsymbol{C}_{\underline{n}}^{-1}$

For Assumption $\boldsymbol{S}^{\mathrm{H}}\boldsymbol{S} = N\sigma_s^2 \mathbf{1}_{K\times M}$ and $\boldsymbol{C}_{\underline{n}} = \sigma_\eta^2 \mathbf{1}_{N\times M}$

Estimator	Averaged Squared Bias	Variance
ML/Correlator	0	$KM\frac{\sigma_\eta^2}{N\sigma_s^2}$
Matched Filter	$\sum_{i=1}^{KM} \lambda_i \left(\frac{\lambda_i}{\lambda_1} - 1\right)^2$	$\sum_{i=1}^{KM} \left(\frac{\lambda_i}{\lambda_1}\right)^2 \frac{\sigma_\eta^2}{N\sigma_s^2}$
MMSE	$\sum_{i=1}^{KM} \lambda_i \left(\frac{1}{1 + \frac{\sigma_\eta^2}{\lambda_i N\sigma_s^2}} - 1\right)^2$	$\sum_{i=1}^{KM} \frac{1}{\left(1 + \frac{\sigma_\eta^2}{\lambda_i N\sigma_s^2}\right)^2} \frac{\sigma_\eta^2}{N\sigma_s^2}$

6.5. Estimators

Upper Bound: Uniform in $[0;\theta]$: $\hat{\theta}_{\mathrm{ML}} = \frac{2}{N}\sum x_i$

Probability p for $\mathcal{B}(p, N)$: $\hat{p}_{\mathrm{ML}} = \frac{x}{N}$ $\quad \hat{p}_{\mathrm{CM}} = \frac{x+1}{N+2}$

Mean μ for $\mathcal{N}(\mu, \sigma^2)$: $\hat{\mu}^2_{\mathrm{ML}} = \frac{1}{N}\sum_{i=1}^{N} x_i$

Variance σ^2 for $\mathcal{N}(\mu, \sigma^2)$: $\hat{\sigma}^2_{\mathrm{ML}} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

7. Gaussian Stuff

7.1. Gaussian Channel

Channel: $Y = hs_i + N$ with $h \sim \mathcal{N}, N \sim \mathcal{N}$

$L(y_1, \dots, y_N) = \prod_{i=1}^{n} f_{Y_i}(y_i, h)$

$f_{Y_i}(y_i, h) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(y_i - hs_i)^2\right)$

$\hat{h}_{ML} = \underset{h}{\operatorname{argmin}}\{\left\|\underline{y} - h\underline{s}\right\|^2\} = \frac{\underline{s}^\top \underline{y}}{\underline{s}^\top \underline{s}}$

If multidimensional channel: $\underline{y} = \boldsymbol{S}\underline{h} + \underline{n}$:

$L(\underline{y}, \underline{h}) = \frac{1}{\sqrt{\det(2\pi\boldsymbol{C})}}\exp\left(-\frac{1}{2}(\underline{y} - \boldsymbol{S}\underline{h})^\top \boldsymbol{C}^{-1}(\underline{y} - \boldsymbol{S}\underline{h})\right)$

$l(\underline{y}, \underline{h}) = \frac{1}{2}\left(\log(\det(2\pi\boldsymbol{C})) - (\underline{y} - \boldsymbol{S}\underline{h})^\top \boldsymbol{C}^{-1}(\underline{y} - \boldsymbol{S}\underline{h})\right)$

$\frac{\mathrm{d}}{\mathrm{d}h}(\underline{y} - \boldsymbol{S}\underline{h})^\top \boldsymbol{C}^{-1}(\underline{y} - \boldsymbol{S}\underline{h}) = -2\boldsymbol{S}^\top \boldsymbol{C}^{-1}(\underline{y} - \boldsymbol{S}\underline{h})$

Gaussian Covariance: if $Y \sim \mathcal{N}(0, \sigma^2), N \sim \mathcal{N}(0, \sigma^2)$:

$\boldsymbol{C}_Y = \mathrm{Cov}[Y, Y] = \mathrm{E}[(Y - \mu)(Y - \mu)^\top] = \mathrm{E}[Y\,Y^\top]$

For Channel $Y = Sh + N$: $\mathrm{E}[Y\,Y^\top] = S\,\mathrm{E}[hh^\top]S^\top + \mathrm{E}[NN^\top]$

7.2. Multivariate Gaussian Distributions

A vector $\mathbf{x}$ of n independent Gaussian random variables x_i is jointly Gaussian. If $\mathbf{x} \sim \mathcal{N}(\underline{\mu}_{\mathbf{x}}, \boldsymbol{C}_{\mathbf{x}})$:

$$\boxed{f_{\mathbf{x}}(\underline{x}) = f_{x_1,\dots,x_n}(x_1, \dots, x_n) = \frac{1}{\sqrt{\det(2\pi\boldsymbol{C}_{\mathbf{x}})}}\exp\left(-\frac{1}{2}\left(\underline{x} - \underline{\mu}_{\mathbf{x}}\right)^\top \boldsymbol{C}_{\mathbf{x}}^{-1}\left(\underline{x} - \underline{\mu}_{\mathbf{x}}\right)\right)}$$

Affine transformations $\mathbf{y} = \boldsymbol{A}\mathbf{x} + \underline{b}$ are jointly Gaussian with

$\mathbf{y} \sim \mathcal{N}(\boldsymbol{A}\underline{\mu}_{\mathbf{x}} + \underline{b}, \boldsymbol{A}\boldsymbol{C}_{\mathbf{x}}\boldsymbol{A}^\top)$

All marginal PDFs are Gaussian as well

Contour Lines

Ellipsoid with central point $\mathrm{E}[\underline{y}]$ and main axis are the eigenvectors of $\boldsymbol{C}_{\underline{y}}^{-1}$

7.3. Conditional Gaussian

$\underline{A} \sim \mathcal{N}(\underline{\mu}_A, \boldsymbol{C}_A), \underline{B} \sim \mathcal{N}(\underline{\mu}_B, \boldsymbol{C}_B)$

$\Rightarrow (\underline{A}|\underline{B} = b) \sim \mathcal{N}(\underline{\mu}_{A|B}, \boldsymbol{C}_{A|B})$

Conditional Mean:

$\mathrm{E}[\underline{A}|\underline{B} = \underline{b}] = \underline{\mu}_{A|B=\underline{b}} = \underline{\mu}_A + \boldsymbol{C}_{AB}\boldsymbol{C}_{BB}^{-1}\left(\underline{b} - \underline{\mu}_B\right)$

Conditional Variance:

$\boldsymbol{C}_{A|B} = \boldsymbol{C}_{AA} - \boldsymbol{C}_{AB}\boldsymbol{C}_{BB}^{-1}\boldsymbol{C}_{BA}$

7.4. Misc

If CDF of gaussian distribution given $\Phi(z) \sim \mathcal{N}(0, 1)$ then for $X \sim \mathcal{N}(1, 1)$ the CDF is given as $\Phi(x - \mu_x)$

8. Sequences

8.1. Random Sequences

Sequence of a random variable. Example: result of a dice is RV, roll a dice several times is a random sequence.

8.2. Markov Sequence $X_n : \Omega \to X_n$

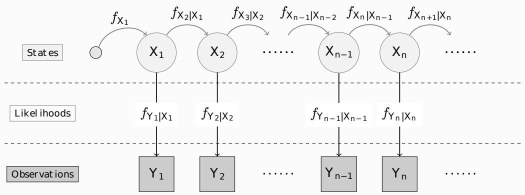
Sequence of memoryless state transitions with certain probabilities.

1. state: $f_{X_1}(x_1)$
2. state: $f_{X_2\,|\,X_1}(x_2|x_1)$

n. state: $f_{X_n\,|\,X_{n-1}}(x_n|x_{n-1})$

8.3. Hidden Markov Chains

Problem: states X_i are not visible and can only be guessed indirectly as a random variable Y_i.



Conditional pdf $f_{\mathbf{X}_n|\mathbf{Y}_n}$ $\quad$ Likelihood pdf $f_{Y_n\,|\,X_n}$

State-transition pdf $f_{X_n\,|\,X_{n-1}}$

Estimation:

$f_{\mathbf{X}_n|\mathbf{Y}_n} \propto f_{\mathbf{Y}_n|\mathbf{X}_n} \cdot \int_{\mathbb{X}} f_{\mathbf{X}_n|\mathbf{X}_{n-1}} \cdot f_{\mathbf{X}_{n-1}|\mathbf{Y}_{n-1}}\,\mathrm{d}\underline{x}_{n-1}$

9. Recursive Estimation

9.1. Kalman-Filter

recursively calculates the most likely state from previous state estimates and current observation. Shows optimum performance for Gauss-Markov Sequences.

State space:

$\underline{x}_n = \boldsymbol{G}_n\underline{x}_{n-1} + \boldsymbol{B}\underline{u}_n + \underline{v}_n$

$\underline{y}_n = \boldsymbol{H}_n\underline{x}_n + \underline{w}_n$

With gaussian process/measurement noise $\underline{v}_n/\underline{w}_n$

Short notation: $\mathrm{E}[\underline{x}_n|\underline{y}_{n-1}] = \hat{\underline{x}}_{n|n-1}$ $\quad \mathrm{E}[\underline{x}_n|\underline{y}_n] = \hat{\underline{x}}_{n|n}$

$\mathrm{E}[\underline{y}_n|\underline{y}_{n-1}] = \hat{\underline{y}}_{n|n-1}$ $\quad \mathrm{E}[\underline{y}_n|\underline{y}_n] = \hat{\underline{y}}_{n|n}$

1. step: Prediction

Mean: $\hat{\underline{x}}_{n|n-1} = \boldsymbol{G}_n\hat{\underline{x}}_{n-1|n-1}$

Covariance: $\boldsymbol{C}_{\underline{x}_{n|n-1}} = \boldsymbol{G}_n\boldsymbol{C}_{\underline{x}_{n-1|n-1}}\boldsymbol{G}_n^\top + \boldsymbol{C}_{\underline{v}}$

2. step: Update

Mean: $\hat{\underline{x}}_{n|n} = \hat{\underline{x}}_{n|n-1} + \boldsymbol{K}_n\left(\underline{y}_n - \boldsymbol{H}_n\hat{\underline{x}}_{n|n-1}\right)$

Covariance: $\boldsymbol{C}_{\underline{x}_{n|n}} = \boldsymbol{C}_{\underline{x}_{n|n-1}} + \boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{C}_{\underline{x}_{n|n-1}}$

$$\hat{\underline{x}}_{n|n} = \underbrace{\hat{\underline{x}}_{n|n-1}}_{\text{estimation } \mathrm{E}[X_n\,|\,Y_{n-1}=y_{n-1}]} + \overbrace{\boldsymbol{K}_n\underbrace{\left(\underline{y}_n - \boldsymbol{H}_n\hat{\underline{x}}_{n|n-1}\right)}_{\text{innovation}:\Delta y_n}}^{\text{correction: } \mathrm{E}[X_n\,|\Delta Y_n=\Delta y_n]}$$

With optimal **Kalman-gain** (prediction for $\underline{x}_n$ based on Δy_n):

$\boldsymbol{K}_n = \boldsymbol{C}_{\underline{x}_{n|n-1}}\boldsymbol{H}_n^\top(\underbrace{\boldsymbol{H}_n\boldsymbol{C}_{\underline{x}_{n|n-1}}\boldsymbol{H}_n^\top + \boldsymbol{C}_{\underline{w}_n}}_{\boldsymbol{C}_{\delta y_n}})^{-1}$

Innovation: closeness of the estimated mean value to the real value

$\Delta\underline{y}_n = \underline{y}_n - \hat{\underline{y}}_{n|n-1} = \underline{y}_n - \boldsymbol{H}_n\hat{\underline{x}}_{n|n-1}$

Init: $\hat{\underline{x}}_{0|-1} = \mathrm{E}[X_0]$ $\quad \sigma^2_{0|-1} = \mathrm{Var}[X_0]$

MMSE Estimator: $\hat{\underline{x}} = \int \underline{x}_n f_{X_n\,|\,Y_{(n)}}(\underline{x}_n|\underline{y}_{(n)})\,\mathrm{d}\underline{x}_n$

For non linear problems: Suboptimum nonlinear Filters: Extended KF, Unscented KF, ParticleFilter

9.2. Extended Kalman (EKF)

Linear approximation of non-linear g, h

$\underline{x}_n = g_n(\underline{x}_{n-1}, \underline{v}_n)$ $\quad \underline{v}_n \sim \mathcal{N}$

$\underline{y}_n = h_n(\underline{x}_{n-1}, \underline{w}_n)$ $\quad \underline{w}_n \sim \mathcal{N}$

9.3. Unscented Kalman (UKF)

Approximation of desired PDF $f_{X_n\,|\,Y_n}(x_n|y_n)$ by Gaussian PDF.

9.4. Particle-Filter

For non linear state space *and* non-gaussian noise

Non-linear State space:

$\underline{x}_n = g_n(\underline{x}_{n-1}, \underline{v}_n)$

$\underline{y}_n = h_n(\underline{x}_{n-1}, \underline{w}_n)$

Posterior Conditional PDF: $f_{X_n\,|\,Y_n}(x_n|y_n) \propto \overbrace{f_{Y_n|X_n}(y_n|x_n)}^{\text{likelihood}}\cdot$

$\cdot\int_{\mathbb{X}} \underbrace{f_{X_n|X_{n-1}}(x_n|x_{n-1})}_{\text{state transition}}\underbrace{f_{X_{n-1}|Y_{n-1}}(x_{n-1}|y_{n-1})}_{\text{last conditional PDF}}\,\mathrm{d}x_{n-1}$

N random Particles with particle weight w_n^i at time n

Monte-Carlo-Integration: $I = \mathrm{E}[g(X)] \approx I_N = \frac{1}{N}\sum_{i=1}^{N}\tilde{g}(x^i)$

Importance Sampling: Instead of $f_X(x)$ use **Importance Density** $q_X(x)$

$I_N = \frac{1}{N}\sum_{i=1}^{N}\tilde{w}^i g(x^i)$ with weights $\tilde{w}^i = \frac{f_X(x^i)}{q_X(x^i)}$

If $\int f_{X_n}(x)\,\mathrm{d}x \neq 1$ then $I_N = \sum_{i=1}^{N} w^i g(x^i)$ with $w^i = \frac{\tilde{w}^i}{\sum_{i=1}^{N}\tilde{w}^i}$

9.5. Conditional Stochastic Independence

$\mathrm{P}(A \cap B|E) = \mathrm{P}(A|E)\cdot\mathrm{P}(B|E)$

Given Y, X and Z are independent if

$f_{Z\,|\,Y,X}(z|y,x) = f_{Z\,|\,Y}(z|y)$ or

$f_{X,Z\,|\,Y}(x,z|y) = f_{Z\,|\,Y}(z|y)\cdot f_{X\,|\,Y}(x|y)$

$f_{Z\,|\,X,Y}(z|x,y) = f_{Z\,|\,Y}(z|y)$ or $f_{X\,|\,Z,Y}(x|z,y) = f_{X\,|\,Y}(x|y)$

10. Hypothesis Testing

making a decision based on the observations

10.1. Definition

Null hypothesis $H_0 : \theta \in \Theta_0$ (Assumed first to be true)
Alternate hypothesis $H_1 : \theta \in \Theta_1$ (The one to proof)
Descision rule $\varphi : \mathbb{X} \to [0,1]$ with
$\varphi(x) = 1$: decide for H_1, $\varphi(x) = 0$: decide for H_0 Error level α with $\mathsf{E}[d(X)|\theta] \leq \alpha, \forall \theta \in \Theta_0$

Error Type	Decision \ Reality	H_1 false (H_0 true)	H_1 true (H_0 false)
1 (FA) False Alarm	H_1 rejected (H_0 accepted)	**T**rue **N**egative $\mathsf{P} = 1 - \alpha$	**F**alse **N**egative (Type 2) $\mathsf{P} = \beta$
2 (DE) Detection Error	H_1 accepted (H_0 rejected)	**F**alse **P**ositive (Type 1) $\mathsf{P} = \alpha$	**T**rue **P**ositive $\mathsf{P} = 1 - \beta$

Power: Sensitivity/Recall/Hit Rate: $\frac{\mathsf{TP}}{\mathsf{TP+FN}} = 1 - \beta$
Specificity/True negative rate: $\frac{\mathsf{TN}}{\mathsf{FP+TN}} = 1 - \alpha$
Precision/Positive Prediciton rate: $\frac{\mathsf{TP}}{\mathsf{TP+FP}}$
Accuracy: $\frac{\mathsf{TP+TN}}{\mathsf{P+N}} = \frac{2-\alpha-\beta}{2}$

10.1.1. Design of a test

Cost criterion $G_\varphi : \Theta \to [0,1], \theta \mapsto \mathsf{E}[d(X)|\theta]$
False Positive lower than α: $G_d(\theta)|_{\theta \in \Theta_0} \leq \alpha, \forall \theta \in \Theta_0$
False Negative small as possible: $\max\{G_d(\theta)|_{\theta \in \Theta_1}\}, \forall \theta \in \Theta_1$

10.2. Sufficient Statistics

Sufficiency for a test $T(X)$ means that no other test statistic, i.e., function of the observations $\underline{x}$, contains additional information about the parameter θ to be estimated:
$f_{X|T}(x|T(x) = t, \theta) = f_{X|T}(x|T(x) = t)$

11. Tests

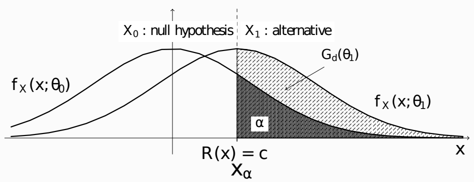
11.1. Neyman-Pearson-Test

The best test of P_0 against P_1 is

$$d_{\mathsf{NP}}(x) = \begin{cases} 1 & R(x) > c \\ \gamma & R(x) = c \\ 0 & R(x) < c \end{cases}$$

Likelihood-Ratio: $R(x) = \frac{f_X(x;\theta_1)}{f_X(x;\theta_0)}$

$\gamma = \frac{\alpha - \mathsf{P}_0(\{R > c\})}{\mathsf{P}_0(\{R = c\})}$ Errorlevel α
Steps: For α calculate x_α, then $c = R(x_\alpha)$



Maximum Likelihood Detector: $d_{\mathsf{ML}}(x) = \begin{cases} 1 & R(x) > 1 \\ 0 & \text{otherwise} \end{cases}$

ROC Graphs: plot $G_d(\theta_1)$ as a function of $G_d(\theta_0)$

11.2. Bayes Test (MAP Detector)

Prior knowledge on possible hypotheses: $\mathsf{P}(\{\theta \in \Theta_0\}) + \mathsf{P}(\{\theta \in \Theta_1\}) = 1$, minimizes the probability of a wrong decision.

$$d_{\mathsf{Bayes}} = \begin{cases} 1 & \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} > \frac{c_0\,\mathsf{P}(\theta_0|x)}{c_1\,\mathsf{P}(\theta_1|x)} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \mathsf{P}(\theta_1|x) > \mathsf{P}(\theta_0|x) \\ 0 & \text{otherwise} \end{cases}$$

Risk weights c_0, c_1 are 1 by default.
If $\mathsf{P}(\theta_0) = \mathsf{P}(\theta_1)$, the Bayes test is equivalent to the ML test

Loss Function $L(d(x), \theta) = \begin{cases} c_0 & \text{type 1 } d(x) = 1, \text{ but } \theta = \theta_0 \\ c_1 & \text{type 2 } d(x) = 0, \text{ but } \theta = \theta_1 \end{cases}$

$\mathrm{risk}(d) = \mathsf{E}[L(d(X), \theta)] = \mathsf{E}[\mathsf{E}[L(d(x), \theta)|x = X]]$

Multiple Hypothesis $d_{\mathsf{Bayes}} = \begin{cases} 0 & x \in \mathbb{X}_0 \\ 1 & x \in \mathbb{X}_1 \\ 2 & x \in \mathbb{X}_2 \end{cases}$

11.3. Linear Alternative Tests

$d : \mathbb{X} \to \mathbb{R}, \underline{x} \mapsto \begin{cases} 1 & \underline{w}^\top \underline{x} - w_0 > 0 \\ 0 & \text{otherwise} \end{cases}$

Estimate normal vector $\underline{w}^\top$, which separates $\mathbb{X}$ into $\mathbb{X}_0$ and $\mathbb{X}_1$

$\log R(\underline{x}) = \frac{\ln(\det(\underset{\sim}{C}_0))}{\ln(\det(\underset{\sim}{C}_1))} + \frac{1}{2}(\underline{x} - \underline{\mu}_0)^\top \underset{\sim}{C}_0^{-1}(\underline{x} - \underline{\mu}_0) -$
$-\frac{1}{2}(\underline{x} - \underline{\mu}_1)^\top \underset{\sim}{C}_1^{-1}(\underline{x} - \underline{\mu}_1) = 0$

For 2 Gaussians, with $\underset{\sim}{C}_0 = \underset{\sim}{C}_1 = \underset{\sim}{C}$: $\underline{w}^\top = (\underline{\mu}_1 - \underline{\mu}_0)^\top \underset{\sim}{C}$

and constant translation $w_0 = \frac{(\underline{\mu}_1 - \underline{\mu}_0)^\top \underset{\sim}{C}(\underline{\mu}_1 - \underline{\mu}_0)}{2}$