Some Comments on Likelihood Functions

The likelihood $\mathcal{L}$ is a function of the parameters of a statistical model. It is used to estimate the values of and uncertainties on those parameters for a given set of measurements. For an ensemble of n measurements, the likelihood is defined as

$$\mathcal{L}(x;\theta) = \prod_{i=1}^{n} \mathcal{L}_i(x;\theta) = \prod_{i=1}^{n} f(x;\theta)$$

where $f(x;\theta)$ is the probability density function for the statistical model of interest. The best value of the parameters θ can be determined by maximizing the likelihood function, or equivalently, the log of the likelihood function.

$$\begin{aligned}
\frac{\partial \ln \mathcal{L}}{\partial \theta} &= \frac{\partial}{\partial \theta} \ln \prod_{i=1}^{n} \mathcal{L}_i \\
&= \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \ln \mathcal{L}_i \\
&= 0
\end{aligned}$$

Often this procedure is described instead as minimizing $-\ln \mathcal{L}$ (minimizing the minus log likelihood).

The log likelihood can be Taylor expanded about its minimum. Since $\partial \mathcal{L}/\partial \theta|_{\theta=\theta_{min}} = 0$:

$$\begin{aligned}
\ln \mathcal{L} &= \ln \mathcal{L}_{min} + \frac{1}{2} \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\bigg|_{\theta=\theta_{min}} (\theta - \theta_{min})^2 \\
2\left(\ln \mathcal{L} - \ln \mathcal{L}_{min}\right) &= \frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\bigg|_{\theta=\theta_{min}} (\theta - \theta_{min})^2
\end{aligned}$$

In the limit of large n, the distribution $\mathcal{L}$ (due to the central limit theorem) becomes Gaussian. Since for a Gaussian distribution a change in $2\ln \mathcal{L}$ of one unit corresponds to a 1-σ variation in the parameter θ, the uncertainty on θ is given by:

$${\sigma_\theta}^2 \equiv \left\langle (\theta - \theta_{min})^2 \right\rangle = -\frac{1}{\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}}$$

Alternatively, the uncertainty on the estimated values of the parameters θ can be obtained by calculating the value of $\Delta\theta$ at which $-2\ln \mathcal{L}$ increases by 1.0. In cases where $\ln \mathcal{L}$ is not parabolic, the uncertainties can be asymmetric.

The definitions above depend on the fact that the probability density function $f(x;\theta)$ is normalized over the region of x where measurements can occur

$$\int_{xmin}^{xmax} f(x;\theta)dx = 1$$

For this reason, likelihood fits are not sensitive to the value of n. It is possible to add a Poisson term to the likelihood function to include the number of events in the likelihood fit. When a fit includes such a term, it is called an *extended likelihood fit*.

An example

Suppose a set of measurements x_i are made in an experimental setup where the number of events as a function of x follows the distribution

$$N(x) = A + Bx \quad \text{for } 0 < x < 10$$

We would like to use the likelihood method to estimate the value of $\kappa \equiv A/B$. Let's see how to setup this problem.

The total number of events N_{Tot} can be determined

$$\begin{aligned}
N_{Tot} &= \int_0^{10} A + Bx \\
&= \left(Ax + \frac{1}{2}Bx^2\right)\Big|_0^{10} \\
&= 10A + \frac{1}{2}(100B) \\
&= 10A + 50B
\end{aligned}$$

and normalized probability density function $f(x;\theta)$ is

$$\begin{aligned}
f(x;A,B) &= \frac{1}{N_{Tot}}(A + Bx) \\
&= \frac{1}{10A + 50B}(A + Bx) \\
&= \frac{A}{10A + 50B} + \frac{Bx}{10A + 50B} \\
&= 0.1\left(\frac{\kappa}{\kappa + 5} + \frac{x}{\kappa + 5}\right)
\end{aligned}$$

The overall likelihood function is therefore

$$\mathcal{L}(x;\kappa) = \prod_{i=1}^{n} 0.1\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)$$

and the log likelihood is

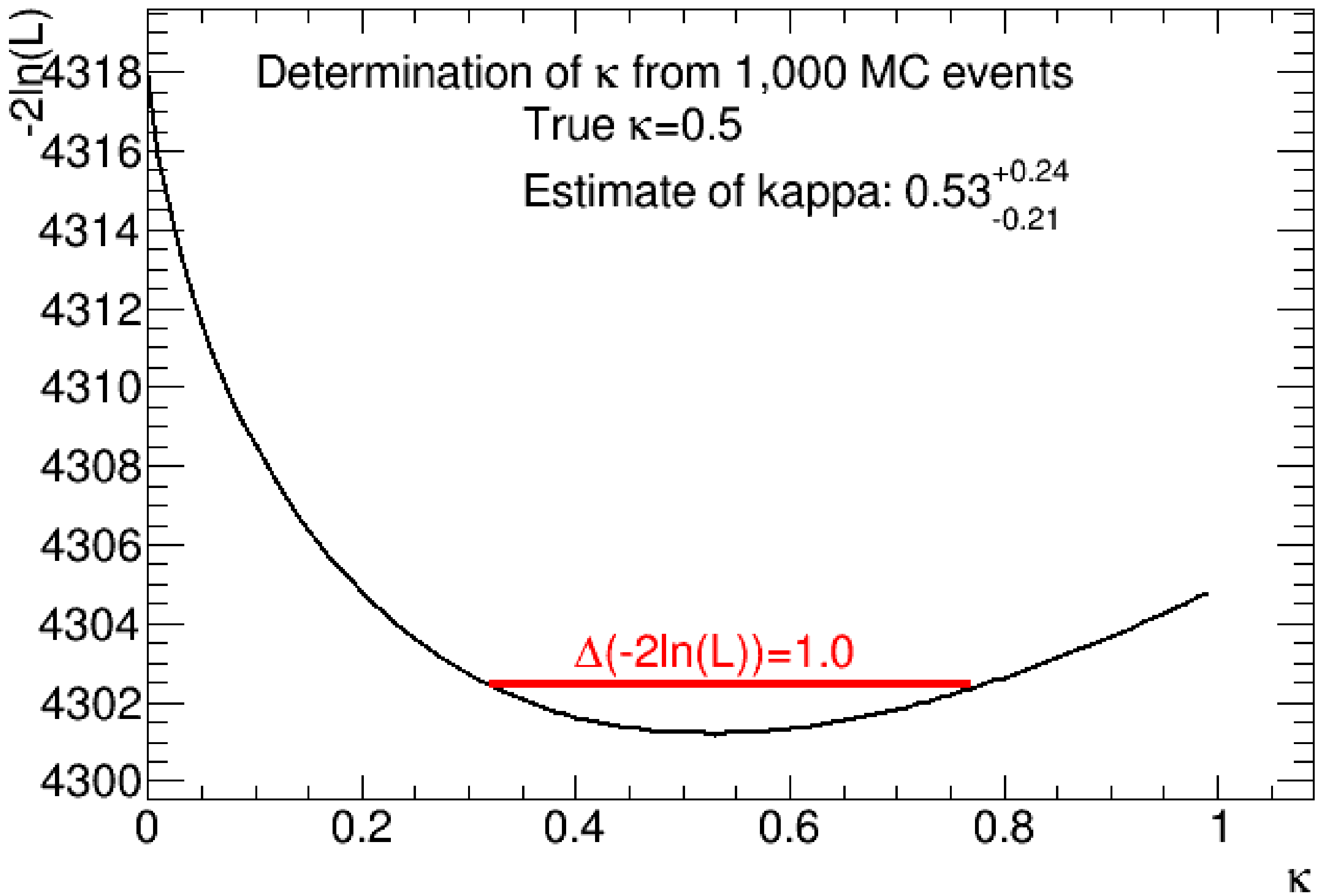
$$\begin{aligned}
\ln\left(\mathcal{L}(x;\kappa)\right) &= \sum_{i=1}^{n} \ln\left(0.1\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)\right) \\
&= \sum_{i=1}^{n} \ln\left(\left(\frac{\kappa}{\kappa+5} + \frac{x_i}{\kappa+5}\right)\right) + n\ln(0.1)
\end{aligned}$$

If the values x_i are known, then $-\ln\left(\mathcal{L}(x;\kappa)\right)$ can be minimized with respect to κ. Because the last term in independent of x_i, it merely adds a constant term to the log likelihood and is irrelevant for the minimization.

There are many programs available to do such minimization (Root for example, has a good interface to the Minuit minimization package). However, you can also find the minimum by by seeing how $\ln\left(\mathcal{L}(x;\kappa)\right)$ changes when κ is varied. An example root macro to generate fake data for the case $A = 1$, $B = 2$ and to use these data to determine κ can be found here:

http://physics.lbl.gov/shapiro/Physics226/myLikelihoodFit.C

The output of this macro is provided on the next page:


-2ln(L)
4300
4302
4304
4306
4308
4310
4312
4314
4316
4318
0
0.2
0.4
0.6
0.8
1
κ
Determination of κ from 1,000 MC events
True κ=0.5
Estimate of kappa: 0.53+0.24 -0.21
Δ(-2ln(L))=1.0