A SHORT DESCRIPTION OF [ACLLa]

As a complex manifold, a genus-2 curve Θ_τ is a hypersurface in its Jacobian variety V_τ, which is a principally polarized abelian surface. The parameter τ is an element of the genus-2 Siegel upper half space

$$\mathcal{H}_2 := \{\tau = B + i\Omega \text{ a } 2\times 2 \text{ complex matrix} : \tau \text{ symmetric}, \ \Omega \text{ positive definite}\}, \tag{0.1}$$

which is the moduli space of 2-dimensional principally polarized abelian varieties with Torelli structure (i.e. a choice of integral symplectic basis). In other words, we can describe $\Theta_\tau \subset V_\tau = (\mathbb{C}^*)^2/\tau\mathbb{Z}^2$ as the image of the hypersurface

$$\widetilde{\Theta}_\tau = \left\{ \vartheta(\tau, x) = \sum_{m=(m_1,m_2)\in\mathbb{Z}^2} x_1^{m_1} x_2^{m_2} e^{\pi i n^T \tau n} = 0 \right\} \subset (\mathbb{C}^*)^2 \tag{0.2}$$

under the covering map $(\mathbb{C}^*)^2 \to V_\tau = (\mathbb{C}^*)^2/\tau\mathbb{Z}^2$, where $\tau\mathbb{Z}^2$ acts multiplicatively on $(\mathbb{C}^*)^2$. In the above formula, $\vartheta : \mathcal{H}_2 \times (\mathbb{C}^*)^2 \to \mathbb{C}$ is the Riemann theta function.

In her thesis, Cannizzo [Can20] proved a HMS result for genus two curves and their candidate mirrors constructed by [AAK16] in a generalized SYZ [SYZ96] mirror symmetry framework. The mirror to a genus two complex curve Θ_τ is a Landau-Ginzburg model (Y_τ, W), where Y_τ is a 6-dimensional symplectic manifold that is a locally toric Calabi-Yau 3-fold, and $W : Y_\tau \to \mathbb{C}$ is a symplectic fibration with a singular fiber above $0 \in \mathbb{C}$. See Figure 1 for a description of this fibration. In the singular fiber, the critical locus is a "banana" configuration of three 2-spheres. Note that in a Lefschetz fibration for which Fukaya-Seidel category was first constructed, the critical locus consists of points, so the critical locus in this example is more complicated. This work by [Can20] is the first construction and computation of a Fukaya-Seidel category for a non-exact, non-monotone, and non-Lefschetz symplectic fibration.

The Kähler (so symplectic) structure on Y_τ constructed in [Can20] requires that the three spheres in the critical locus to all have the same symplectic area. This one area parameter gives rise to a 1-parameter family of symplectic structures which is mirror to a 1-parameter family of complex structures on the genus-2 curve, i.e. the family where $\tau = \frac{i}{2\pi} \log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ with $t > 0$ large. In [ACLLa], we extend this construction to prove a global HMS result that covers the 3 complex dimensional moduli space $\mathcal{H}_2$ of complex structures on the genus-2 curve. Below are the two main ingredients of [ACLLa] that are not present in [Can20].

(1) We need to use more general symplectic structures on Y_τ where the areas of the three 2-spheres in the critical locus may vary independently instead of all being the same, as well as doing Floer theory computations with these symplectic structures.

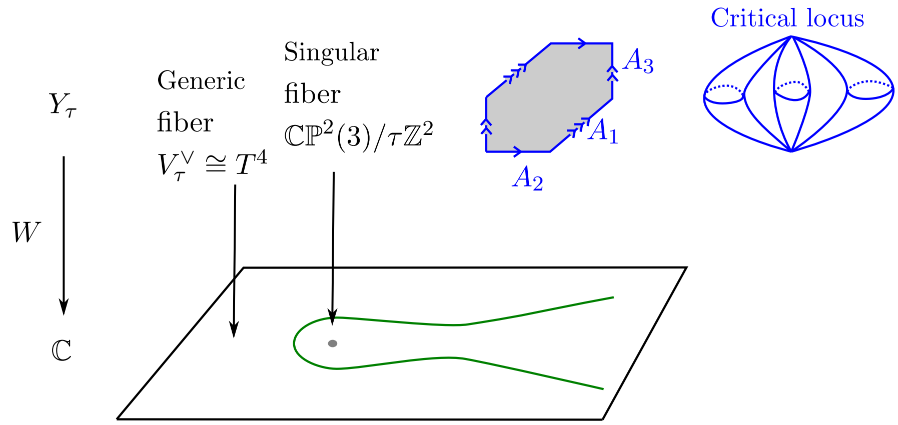

FIGURE 1. The symplectic fibration $W : Y_\tau \to \mathbb{C}$. Each generic fiber is a 4-torus isomorphic to the mirror $V_\tau^\vee$ of V_τ. The singular fiber above $0 \in \mathbb{C}$ is a blow-up of $\mathbb{CP}^2$ at 3 points, quotiented by $\tau\mathbb{Z}^2$. The moment polytope for $\mathbb{CP}^2(3)$ is a hexagon and the quotient by $\tau\mathbb{Z}^2$ is depicted as having the edges of the hexagon identified as shown in the figure. The critical locus is the boundary of the hexagon, drawn in blue, which is a union of three line segments with ends identified. This is the moment polytope for a "banana" configuration of three spheres as depicted in the picture of the critical locus, where each edge of the hexagon is the moment polytope for one sphere. The symplectic areas, A_1, A_2, and A_3, of these spheres are labelled next to their respective moment polytopes. The Lagrangian objects in Y_τ that we consider are of the form $\cup\ell$, which is the Lagrangian obtained by parallel transport of a Lagrangian ℓ in the fiber T^4 above $-\epsilon \in \mathbb{C}$ along the U-shaped curved in the base drawn in green.

(2) The complex moduli space $\mathcal{H}_2$ and the global Kähler moduli space of the mirror should match, however the global stringy Kähler moduli space cannot be covered by a single Kähler cone in this case, so we need to describe how it is glued from the individual Kähler cones and how the structure maps of the Fukaya-Seidel category changes through the transitions.

Below we discuss these two points in a bit more detail.

When the symplectic areas A_j, $j = 1, 2, 3$, of the three spheres can vary independently, they give rise to a 3-parameter space of symplectic structures on Y_τ, which via mirror symmetry corresponds to a 3-parameter space of $\Omega = \operatorname{Im}\tau$'s in $\mathcal{H}_2$. (The other 3 parameters of $B = \operatorname{Re}\tau$'s in $\mathcal{H}_2$ correspond to what is known as the B-fields, which complexify the symplectic structure and play a role in the Floer theory for (Y_τ, W). We also incorporate these in [ACLLa].) The hexagon drawn in Figure 1 is a representative example of what the moment polytope of the singular fiber looks like when $\Omega_{12} = \Omega_{21} > 0$, $\Omega_{11} - \Omega_{12} > 0$, and $\Omega_{22} - \Omega_{12} > 0$. In this case, we have the identification

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1 + A_2 & A_1 \\ A_1 & A_1 + A_3 \end{pmatrix} \iff \begin{array}{c} A_1 = \Omega_{12} = \Omega_{21} \\ A_2 = \Omega_{11} - \Omega_{12} \\ A_3 = \Omega_{21} - \Omega_{12} \end{array}. \tag{0.3}$$

From this identification, one can see that simply varying the symplectic areas will not completely cover all the possible Ω's to give us a global correspondence with the complex moduli space. It'll only cover the part where the areas $A_j > 0$ for $j = 1, 2, 3$. As $A_j \to 0$ for example, we go through transitions known as Atiyah flops into different chambers of the moduli space where and hexagons change shapes by actions of $\mathrm{GL}(2, \mathbb{Z})$. Formally, $\mathrm{GL}(2, \mathbb{Z})$ acts on the tropical Siegel space (i.e. the space of Ω's), which is the moduli space of principally polarized tropical abelian varieties with Torelli structure; this was defined and studied in [CMV13, Cha12].

In fact, by writing

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \begin{bmatrix} z+y & x \\ x & z-y \end{bmatrix}, \tag{0.4}$$

we can identify the space of positive definite real symmetric 2×2 matrices with

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid z > \sqrt{x^2 + y^2}\}, \tag{0.5}$$

which is a cone over $C' \times \{1\} \subset \mathbb{R}^3$, where $C' \subset \mathbb{R}^2$ is the open unit disk centered at the origin:

$$C' = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}. \tag{0.6}$$

Indeed, this space cannot be covered by a single top-dimensional Kähler cone, which we call a chamber. Each chamber is in fact a cone over a triangular region in C'. There are infinitely many of them and their corresponding triangles form a triangulation of the disk C'. All of them are in the same $\mathrm{GL}(2, \mathbb{Z})$ orbit. Figure 2 shows 6 different chambers and how they fit together.

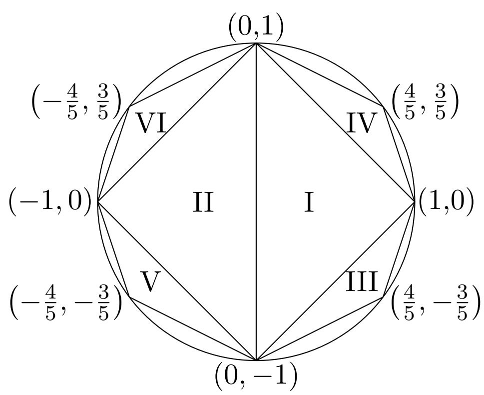

FIGURE 2. The open disk C' and six different chambers, I-VI. There are actually infinitely many chambers that form a triangulation of the disk.

The manifold Y_τ can be described as $\widetilde{Y}'_\tau / \tau\mathbb{Z}^2$, where $\widetilde{Y}_\tau$ is a toric variety of infinite type (i.e. its moment polyhedron has infinitely many facets) and $\widetilde{Y}'_\tau \subset \widetilde{Y}_\tau$ is a small neighborhood of the singular fiber. We use the construction by [KL19] (based on the finite type construction of [Gui94]) for the Kähler structure on $\widetilde{Y}_\tau$, which descends to a Kähler structure on the quotient Y_τ. Then we write the superpotential $W : Y_\tau \to \mathbb{C}$ in terms of the action-angle coordinates on Y_τ to do computations such as studying the monodromy of a fiber around a circle in the base enclosing the critical point. The monodromy computation is needed for computing the differentials in the Floer theory for (Y_τ, W).

The precise concluding HMS statement is the completion of the following commutative diagram, with [Can20] proving it for a 1-parameter family of τ's and [ACLLa] proving it for all $\tau \in \mathcal{H}_2$,

$$
\begin{tikzcd}
D^b_{\mathcal{L}}\operatorname{Coh}(V_\tau) \arrow[r, "\iota^*"] \arrow[d, hook, "{[\text{Fuk02}]}"'] & D^b_{\mathcal{L}}\operatorname{Coh}(\Theta_\tau) \arrow[d, hook, "{[\text{Can20, ACLLa}]}"] \\
H^0(\operatorname{Fuk}(V_\tau^\vee)) \arrow[r, "\cup"] & H^0(FS(Y_\tau, W))
\end{tikzcd}
\quad . \tag{0.7}
$$

The vertical maps are fully-faithful embeddings. The B-model $D^b_{\mathcal{L}}\operatorname{Coh}(\Theta_\tau)$ is a full subcategory of the bounded derived category of coherent sheaves generated by powers of an ample line bundle $\mathcal{L}$ on the genus-2 two curve Θ_τ. The A-model is the cohomological Fukaya-Seidel category $FS(Y_\tau, W)$. The left vertical arrow in this diagram is HMS for abelian varieties due to Fukaya [Fuk02]. The top arrow is the restriction functor given by the pullback of the inclusion $\iota : \Theta_\tau \to V_\tau$. The bottom $\cup$ operator is given by parallel-transporting Lagrangian objects in the fiber $V_\tau^\vee$ along a U-shaped curve in the base of W (the green curve in Figure 1) to obtain Lagrangian objects in Y_τ.

For a hypersurface that is a theta divisor

$$
\Theta_\tau = \left\{ \vartheta(\tau, x) = \sum_{m\in\mathbb{Z}^g} \prod_{j=1}^{g} x_j^{m_j} e^{\pi i n^T \tau n} = 0 \right\} \subset V_\tau = (\mathbb{C}^*)^g/\tau\mathbb{Z}^g \tag{0.8}
$$

in a principally polarized abelian variety V_τ of any dimension $g \geq 2$, where τ is an element of the genus-g Siegel upper half space $\mathcal{H}_g$, we will establish the right vertical map in (0.7) and show that the diagram commutes in [ACLLb] for this higher dimensional generalization.

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