TOWARDS GLOBAL HOMOLOGICAL MIRROR SYMMETRY FOR THETA DIVISORS IN PRINCIPALLY POLARIZED ABELIAN VARIETIES

Abstract. This is an informal description of [ACLLc, ACLLd, ACLLb].

A *principally polarized abelian variety* (ppav) V_τ of complex dimension g can be defined by $V_\tau := (\mathbb{C}^*)^g/\tau\mathbb{Z}^g$, where $\tau\mathbb{Z}^g$ acts multiplicatively on $(\mathbb{C}^*)^g$. The parameter τ is an element of the genus-g Siegel upper half-space

$$\mathcal{H}_g := \{\tau = B + i\Omega \text{ a } g\times g \text{ complex matrix} \mid \tau \text{ symmetric, } \Omega := \mathrm{Im}(\tau) \text{ positive definite}\}, \tag{0.1}$$

which is the moduli space of complex g-dimensional ppav's with Torelli structure (to be explained more in Section 1.3). A *theta divisor* $\Theta_\tau \subset V_\tau$ can be described as the image of the infinite-type hypersurface

$$\widetilde{\Theta}_\tau = \left\{ \vartheta(\tau, x) = \sum_{m\in\mathbb{Z}^g} x_1^{m_1}\cdots x_g^{m_g} e^{\pi i n^T \tau n} = 0 \right\} \subset (\mathbb{C}^*)^g \tag{0.2}$$

under the covering map $(\mathbb{C}^*)^g \to V_\tau = (\mathbb{C}^*)^g/\tau\mathbb{Z}^g$, where $\vartheta : \mathcal{H}_2 \times (\mathbb{C}^*)^g \to \mathbb{C}$ is the Riemann theta function. When $g = 2$, for a generic τ, the theta divisor Θ_τ is a smooth genus two curve with V_τ being its Jacobian variety.

1. Homological mirror symmetry (HMS)

For hypersurfaces in toric varieties, [AAK16] gave a mirror construction in the SYZ [SYZ96] mirror symmetry framework, based on proposals and earlier work by [Giv95, HV00, Gro01], etc. As a generalization, since a ppav is a quotient of the toric variety $(\mathbb{C}^*)^g$, they also outlined [AAK16, Section 10.2] the construction of SYZ mirrors to hypersurfaces in ppavs. The SYZ mirror for a theta divisor Θ_τ in the complex g-dimensional ppav V_τ is a *Landau-Ginzburg (LG) model* of the form (Y_τ, W), where Y_τ is a noncompact locally toric Calabi-Yau manifold of complex dimension $g+1$, and the superpotential $W : Y_\tau \to \mathbb{C}$ is a holomorphic function with a singular fiber above $0 \in \mathbb{C}$.

Cannizzo [Can20] proved a HMS result with a genus two curve $\Theta_\tau \subset V_\tau$ (i.e. dimension $g = 2$) on the complex side and with the candidate mirror (Y_τ, W) proposed in [AAK16] on the symplectic side, provided that τ belongs to the following 1-parameter family

$$\tau = \frac{i}{2\pi}\log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad t > 0 \text{ large.} \tag{1.1}$$

Here, Y_τ is of real dimension 6 and W is a symplectic fibration away from the singular fiber $W^{-1}(0)$, in which the critical locus is a "banana" configuration of three 2-spheres (see Fig. 1). When τ satisfies Equation (1.1), these three 2-spheres all have the same symplectic area; this is the one symplectic parameter mirror to the one parameter of complex structures determined by τ on the genus-2 curve.

1.1. Kähler structure on (Y_τ, W).

A sophisticated construction of a symplectic form on Y_τ was given in [Can20], though its method is specific to τ satisfying Equation (1.1). In [ACLLc, ACLLd], for any g and τ, we equip Y_τ with a Kähler structure in the following way. We first construct a mirror LG model $(\widetilde{Y}_\tau, \widetilde{W})$ to $\widetilde{\Theta}_\tau$, where $\widetilde{Y}_\tau$ is a toric variety of infinite type whose complex structure is invariant under the complex torus $(\mathbb{C}^*)^{g+1}$ action; moreover, $\widetilde{W}$ and the complex structure on $\widetilde{Y}_\tau$ are preserved by the $\tau\mathbb{Z}^g$-action. We then consider a neighborhood $\widetilde{Y}_\tau^\epsilon = \widetilde{W}^{-1}(\{|z| < \epsilon\}) \subset \widetilde{Y}_\tau$ of the singular fiber, for a sufficiently small $\epsilon > 0$. On $\widetilde{Y}_\tau^\epsilon$, the $\tau\mathbb{Z}^g$-action is free, so the complex structure descends to the quotient $Y_\tau := \widetilde{Y}_\tau^\epsilon/\tau\mathbb{Z}^g$ and $\widetilde{W}|_{\widetilde{Y}_\tau^\epsilon}$ descends to a holomorphic function that is the superpotential $W : Y_\tau \to \mathbb{C}$. Furthermore, using the construction in [KL19] (based on the finite type construction of [Gui94]), $\widetilde{Y}_\tau^\epsilon$ can be equipped with a Kähler form $\widetilde{\omega}_\tau$ that is invariant under action of the compact torus $U(1)^{g+1}$ and of $\tau\mathbb{Z}^g$, so it descends to a Kähler form ω_τ on the quotient Y_τ.

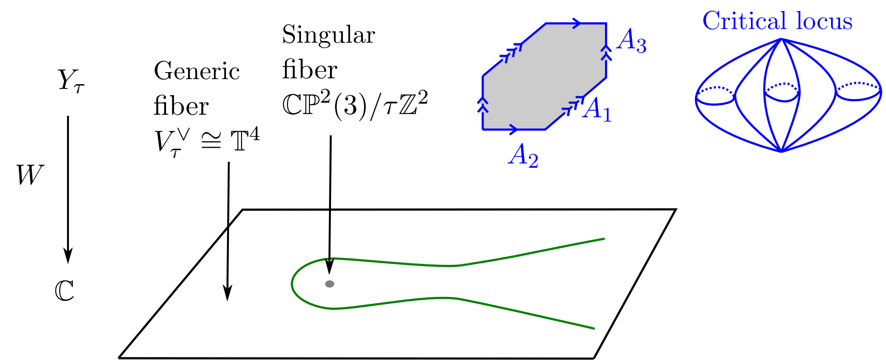

FIGURE 1. Mirror (Y_τ, W) to a genus-2 curve $\Theta_\tau \subset V_\tau = \mathbb{C}^2/\tau\mathbb{Z}^2 \cong \mathbb{T}^4$. Each generic fiber is a 4-torus isomorphic to the mirror $V_\tau^\vee$ of V_τ. The singular fiber above $0 \in \mathbb{C}$ is a blow-up of $\mathbb{CP}^2$ at 3 points, quotiented by $\tau\mathbb{Z}^2$. The moment polytope for $\mathbb{CP}^2(3)$ is a hexagon and the quotient by $\tau\mathbb{Z}^2$ is depicted as having the edges of the hexagon identified as shown in the figure. The critical locus corresponds to the boundary of the hexagon (drawn in blue), comprising three line segments with ends identified. This is the moment polytope for a "banana" configuration of three spheres as depicted in the picture of the critical locus, where each edge of the hexagon is the moment polytope for one sphere. The symplectic areas, A_1, A_2, and A_3, of these spheres are labelled next to their respective moment polytopes. Objects of the Fukaya-Seidel category are Lagrangian submanifolds in Y_τ obtained by parallel transport of a Lagrangian in a fiber $\mathbb{T}^4$, along the U-shaped curved in the base drawn in green.

Only the imaginary part Ω of τ contributes to the construction of $\widetilde{Y}_\tau$ and the Kähler form $\widetilde{\omega}_\tau$ on $\widetilde{Y}_\tau^\epsilon$. The real part B of τ contributes to the B-field, which is a closed real 2-form on Y_τ, that together with ω_τ form the complexified Kähler form on Y_τ. B-field is not present in [Can20] as Equation (1.1) is purely imaginary. The space of all Kähler classes on Y mirror to the theta divisors, called the *Kähler space* $K(Y)$, is identified with the space of Ω's known as the pure tropical Siegel space

$$\mathcal{H}_g^{\mathrm{trop,p}} := \{\Omega \in S_g(\mathbb{R}) = \text{symmetric real } g \times g \text{ matrices} \mid \Omega \text{ positive definite}\}\,, \tag{1.2}$$

which is the moduli space of tropical ppav's with Torelli structure studied in [CMV13, Cha12]. Together with the B-fields (which contribute modulo $\mathbb{Z}$ in each component), we get the identification between the complexified Kähler space $K_{\mathbb{C}}(Y)$ with $\mathcal{H}_g/S_g(\mathbb{Z}) \cong (S^1)^{g(g+1)/2} \times \mathcal{H}_g^{\mathrm{trop,p}}$, as $\mathcal{H}_g = S_g(\mathbb{R}) \times \mathcal{H}_g^{\mathrm{trop,p}} \cong \mathbb{R}^{g(g+1)/2} \times \mathcal{H}_g^{\mathrm{trop,p}}$.

1.2. **Floer theory for symplectic LG models and HMS.** In [ACLLd], we consider any dimension $g \geq 2$ and any generic $\tau \in \mathcal{H}_g$ (a generic τ is one for which Θ_τ is smooth, excluding the the degenerated cases such as when a genus-2 curve becomes two elliptic curves joint at a node when τ is diagonal). We prove a HMS result, at the level of cohomology, with the complex side being the theta divisor Θ_τ, described by its derived category of coherent sheaves, and the symplectic side being the LG model (Y_τ, W), described by its derived Fukaya-Seidel category; more precisely stated as

$$D^b\,\mathrm{Coh}(\Theta_\tau) \cong D^b\,\mathrm{FS}_{\mathrm{aff}}(Y_\tau, W). \tag{1.3}$$

The objects of the Fukaya-Seidel category $\mathrm{FS}(Y_\tau, W)$ are Lagrangian submanifolds in Y_τ obtained by parallel transport of a Lagrangian in the fiber $W^{-1}(-\varepsilon) \cong \mathbb{T}^g$ (for some $0 < \varepsilon < \epsilon$), along a U-shaped curve in the base as illustrated in Fig. 1. We only consider the full subcategory $\mathrm{FS}_{\mathrm{aff}}(Y_\tau, W)$ generated by those obtained from the affine Lagrangians in the fiber $\mathbb{T}^g = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ (i.e. those defined by affine functions on $\mathbb{R}^{2g}$). It is expected, although at this point unclear, that $\mathrm{FS}_{\mathrm{aff}}(Y_\tau, W)$ generates $\mathrm{FS}(Y, W)$. This is the same HMS result proved in [Can20] for $g = 2$ and τ satisfying Equation (1.1).

The LG model (Y_τ, W) has a rich geometry. The symplectic manifold Y_τ is non-exact and non-monotone, and as mentioned be before, the critical locus of the singular fiber of W is a configuration of spheres (which is more complicated than the more familiar Lefschetz fibrations where the critical locus consists of points). The work by [Can20] was the first construction and computation of a Fukaya-Seidel category for a LG model with these features (using some of the ideas from [AA]).

In [ACLLd], we follow the general strategy of [Can20] and develop many of its methods for studying the Floer theory of (Y_τ, W) to be more systematic so they can be generalized to all g and τ. Most notably, the Kähler structure we define on Y_τ is more general as explained in Section 1.1. A key ingredient is the computation of the monodromy, i.e. the symplectic parallel transport of a fiber around a circle in the base enclosing the critical point, carried out in the general case with respect to our Kähler structure. The monodromy computation is needed in describing the differentials on the Floer complexes. Other elements we contributed to include studying the changes to the A_∞ structure maps (which includes contributions from the B-field) under Lagrangian isotopies, describing the grading structures on fibered Lagrangians and their morphisms, etc. Some of these tools can be applied more generally and we portioned them out into [ACLLa, ACLLe].

1.3. **Identifications of complex and Kähler moduli spaces under HMS.** Given $\tau \in \mathcal{H}_g$, it determines an abelian variety $V_\tau = (\mathbb{C}^*)^g/\tau\mathbb{Z}^g$, a principal polarization $[\omega_{V_\tau}] \in H^2(V_\tau;\mathbb{Z})$ represented by a positive, integral, and antisymmetric real $(1,1)$-form ω_{V_τ}, and an integral symplectic basis $\{\alpha_j, \beta_j\}_{j=1}^g$ of $(H_1(V_\tau;\mathbb{Z}), [\omega_{V_\tau}])$, or equivalently, an isometry $(H_1(V_\tau;\mathbb{Z}); [\omega_{V_\tau}]) \cong (\mathbb{Z}^{2g}, J_g)$, where $J_g = \left(\begin{smallmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{smallmatrix}\right)$ and $\mathbb{I}_g$ is the identity matrix. This choice of the integral symplectic basis is the Torelli structure. Forgetting the Torelli structure gives us the moduli space $\mathcal{A}_g$ of ppav's. More explicitly, $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathcal{H}_g$ on the left by $\left(\begin{smallmatrix} A & C \\ D & E \end{smallmatrix}\right) \circ \tau = (A\tau + C)(D\tau + E)^{-1}$, and the quotient is $\mathcal{A}_g := [\mathcal{H}_g/\mathrm{Sp}(2g,\mathbb{Z})]$, which is also the moduli space of complex structures on the theta divisor.

In between $\mathcal{H}_g$ and $\mathcal{A}_g$, there are also have several moduli spaces of interest that fit together into the following commutative diagram, with the vertical arrows being covering maps

$$\begin{tikzcd}
\mathcal{H}_g = S_g(\mathbb{R}) \times \mathcal{H}_g^{\mathrm{trop,p}} \cong \mathbb{R}^{g(g+1)/2} \times \mathcal{H}_g^{\mathrm{trop,p}} \arrow[d] \arrow[dr] & \\
\mathcal{H}_g/S_g(\mathbb{Z}) \cong (S^1)^{g(g+1)/2} \times \mathcal{H}_g^{\mathrm{trop,p}} \arrow[r] \arrow[d] & \mathcal{H}_g^{\mathrm{trop,p}} \arrow[d] \\
\mathcal{A}_g^F = [\mathcal{H}_g/P_g(\mathbb{Z})] \arrow[r] \arrow[d] & \mathcal{A}_g^{\mathrm{trop,p}} = [\mathcal{H}_g^{\mathrm{trop,p}}/\mathrm{GL}(g,\mathbb{Z})] \\
\mathcal{A}_g = [\mathcal{H}_g/\mathrm{Sp}(2g,\mathbb{Z})]. &
\end{tikzcd} \tag{1.4}$$

The Siegel parabolic subgroup $P_g(\mathbb{Z})$ of $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathcal{H}_g$ by $\tau \mapsto (A\tau + C)A^T$, with $A \in \mathrm{GL}(g,\mathbb{Z})$ and $C \in M_g(\mathbb{Z})$. The group $\mathrm{Sp}(2g;\mathbb{Z})$ is generated by $P_g(\mathbb{Z})$ and another element that acts on $\mathcal{H}_g$ by $\tau \mapsto -\tau^{-1}$.

Under mirror symmetry, as mentioned in Section 1.1, the Kähler space $K(Y)$ is identified with $\mathcal{H}_g^{\mathrm{trop,p}}$, and the complexified Kähler space $K_{\mathbb{C}}(Y)$ is identified with $\mathcal{H}_g/S_g(\mathbb{Z})$. The Kähler moduli space (defined to be $K(Y)/\mathrm{Aut}(Y)$) is identified with the moduli space of pure tropical ppav's $\mathcal{A}_g^{\mathrm{trop,p}} = \mathcal{H}_g^{\mathrm{trop,p}}/\mathrm{GL}(g,\mathbb{Z})$, where the $\mathrm{GL}(g,\mathbb{Z})$ acts by $A \cdot \Omega = A\Omega A^T$.

The complexified Kähler moduli of Y is identified with the space $\mathcal{A}_g^F := [\mathcal{H}_g/P_g(\mathbb{Z})]$. We introduced $\mathcal{A}_g^F$ in [ACLLb] as the moduli of ppav's together with a SYZ fibration $\pi^{\mathrm{SYZ}} : V_\tau = (\mathbb{C}^*)^g/\tau\mathbb{Z}^g \to \mathbb{R}^g/\Omega\mathbb{Z}^g$, coming from a logarithmic map on $(\mathbb{C}^*)^g$ that is equivariant with respect to the $\tau\mathbb{Z}^g$ and $\Omega\mathbb{Z}^g$ actions. The group $P_g(\mathbb{Z})$ is precisely what preserves the sublattice $\Gamma_F := H_1(T_F;\mathbb{Z}) \cong \mathbb{Z}^g$ of $H_1(V_\tau;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, where T_F is the fiber of the SYZ fibration π^{SYZ}. The map $\mathcal{A}_g^F \to \mathcal{A}_g^{\mathrm{trop,p}}$ sends a SYZ fibred ppav to its base.

2. Kähler cones and Voronoi decompositions

Besides establishing HMS results all τ and identifying the complex and Kähler moduli spaces as discussed in Section 1, another aspect of understanding *global* mirror symmetry involves understanding the structure of the Kähler space $K(Y)$, which as mentioned in Section 1.1, is identified with the pure tropical Siegel space $\mathcal{H}_g^{\mathrm{trop,p}}$. On a complex manifold, the set of Kähler forms is a Kähler cone, which is a convex cone in a linear space; however, $K(Y)$ and $\mathcal{H}_g^{\mathrm{trop,p}}$ are not linear in nature. We would like to describe the Kähler cones in $K(Y)$ as well as how the Floer theoretic structure maps change as we cross from one cone to another. When dimension $g = 2$, we answer these two questions in [ACLLc] and [ACLLd], respectively. In particular, we show that when $g = 2$, the Kähler cones in $K(Y)$ are in one-to-one correspondence with the 3-dimensional cones in the Voronoi decomposition of $\mathcal{H}_2^{\mathrm{trop}}$ [Vor08, Vor09]. We explain this more below.

When τ belongs to the 1-parameter family in Equation (1.1), the symplectic areas A_j, $j = 1, 2, 3$, of the three spheres in the critical locus of the singular fiber are all the same. When these three areas vary independently, they give rise to a 3-parameter space of symplectic structures on Y_τ, which via mirror symmetry corresponds to a 3-parameter space of complex structures on the genus-2 curve determined by $\Omega = \operatorname{Im}\tau$'s in $\mathcal{H}_2$. The other 3 parameters of $B = \operatorname{Re}\tau$'s correspond to the B-fields.

The hexagon drawn in Figure 1 is a representative example of what the moment polytope of the singular fiber looks like when $\Omega_{12} = \Omega_{21} = A_1 > 0$, $\Omega_{11} - \Omega_{12} = A_2 > 0$, and $\Omega_{22} - \Omega_{12} = A_3 > 0$. From this identification of the areas with Ω, one can see that simply varying the symplectic areas will not completely cover all the possible Ω's to give us a global correspondence with the complex moduli space. It'll only cover the part where each area $A_j > 0$. As an $A_j \to 0$ for example, we go through transitions known as Atiyah flops into different cones in $K(Y)$ and hexagons change shapes by actions of $\mathrm{GL}(2, \mathbb{Z})$.

In fact, by writing

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \begin{bmatrix} z + y & x \\ x & z - y \end{bmatrix}, \tag{2.1}$$

we can identify the space of positive definite real symmetric 2×2 matrices with a circular cone $(z > \sqrt{x^2 + y^2})$ over $C' \times \{1\} \subset \mathbb{R}^3$, where $C' \subset \mathbb{R}^2$ is the open unit disk centered at the origin:

$$C' = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}. \tag{2.2}$$

Indeed, this space cannot be covered by a single Kähler cone. Each Kähler cone is in fact a cone over a triangular region in C'. There are infinitely many of them and their corresponding triangles form a triangulation of the disk C'. All of them are in the same $\mathrm{GL}(2, \mathbb{Z})$ orbit. Figure 2 shows 6 different cones and how they fit together.

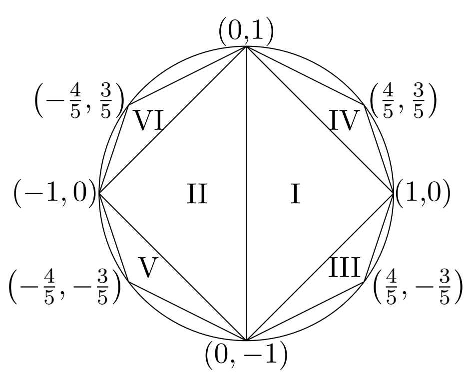

Figure 2. The open disk C' with 6 triangles, I-VI, corresponding to the different Kähler cones. There are actually infinitely many cones that correspond to a triangulation of the disk.

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