Global HMS for Genus Two Curves

(with Haniya Azam, Catherine Cannizzo, Chiu-Chu Melissa Liu)

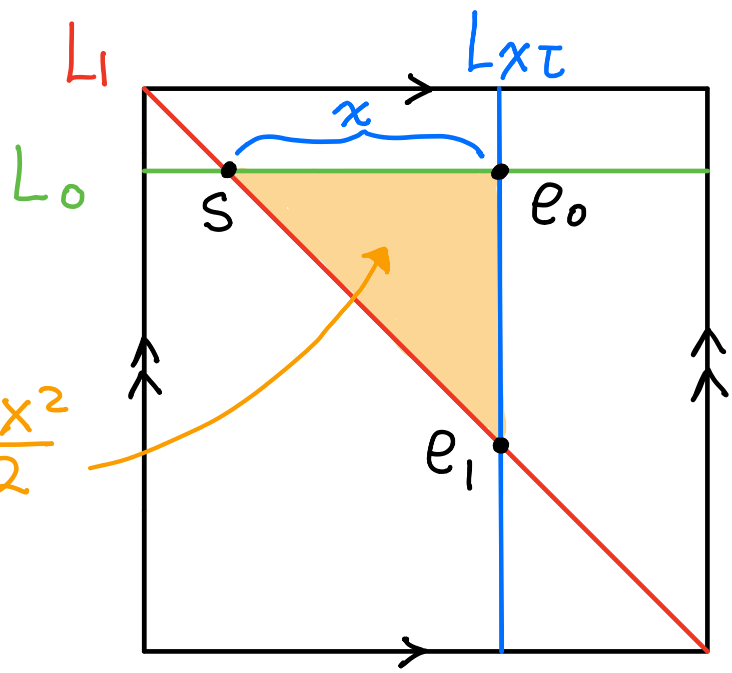
Torus (Polishchuk - Zaslow 1998)

$$E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \quad \longleftrightarrow \quad \left(T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2, \quad \omega_\tau = \tau\, dr\wedge d\theta\right)$$

$$\tau = B + i\Omega \in \text{upper half plane}, \ \Omega > 0$$

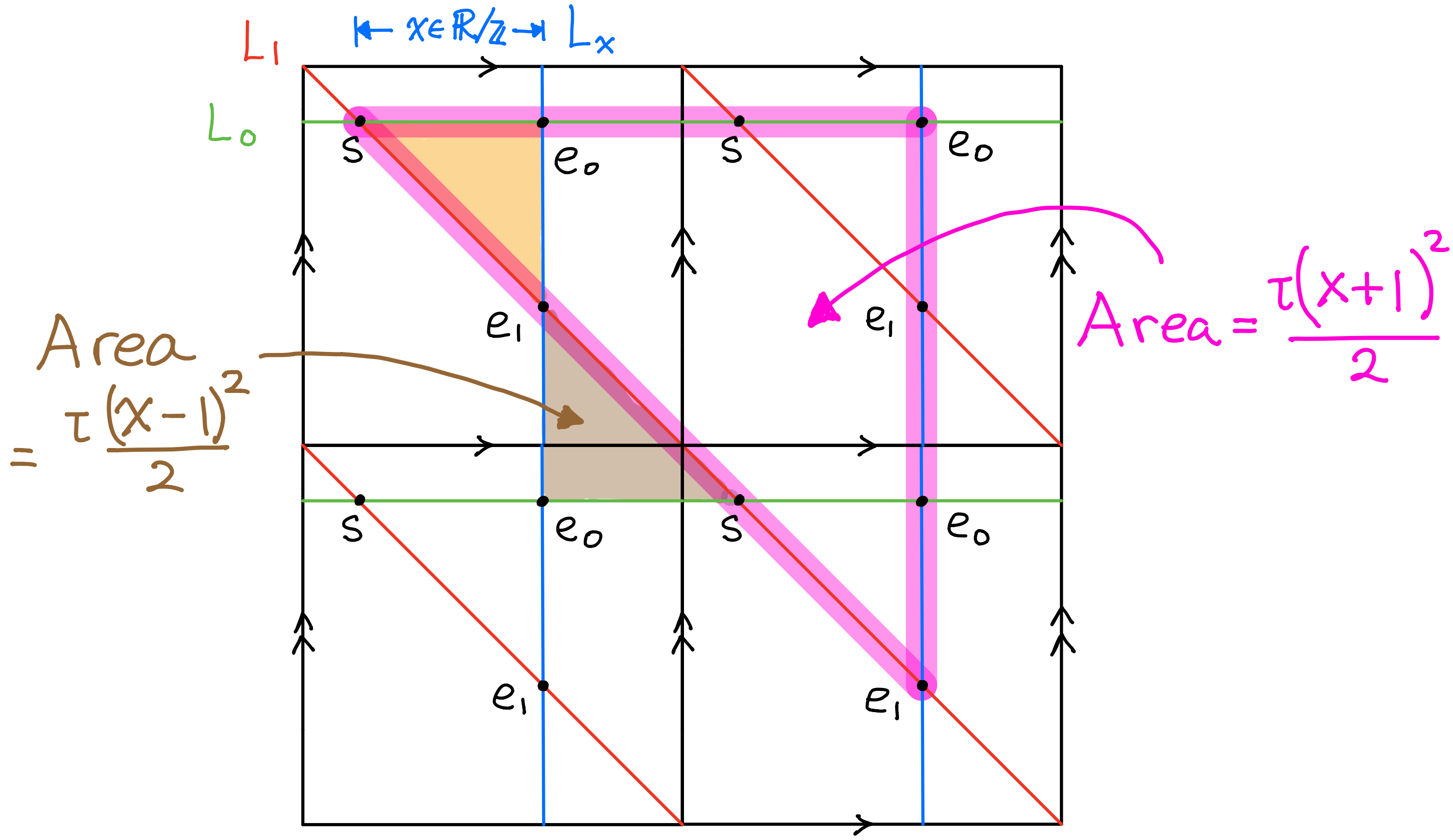
$$D^b(\mathrm{Coh}(E_\tau)) = \mathrm{Fuk}(T^2, \omega_\tau)$$

Picture from Auroux's talks:



$$\text{Area} = \frac{\tau x^2}{2}$$

$$\begin{tikzcd} L_0 \arrow[r, "s"] \arrow[rr, bend right, "\left(\sum_{\triangle} e^{2\pi i\int \omega_\tau^* \triangle}\right) e_0"'] & L_1 \arrow[r, "e_1"] & L_{x\tau} \end{tikzcd}$$



$$\begin{tikzcd} L_0 \arrow[r, "s"] \arrow[rr, bend right, "\left(\sum_{n\in\mathbb{Z}} e^{2\pi i \tau \frac{(n+x)^2}{2}}\right) e_0"'] & L_1 \arrow[r, "e_1"] & L_{x\tau} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"] \arrow[rr, bend right] & \mathcal{L}_1 \arrow[r, "ev"] & \mathcal{O}_{\tau x} \end{tikzcd}$$

$$\vartheta(\tau x;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n x \tau} = e^{-\pi i \tau x^2} \sum_{n\in\mathbb{Z}} e^{2\pi i \tau \frac{(n+x)^2}{2}}$$

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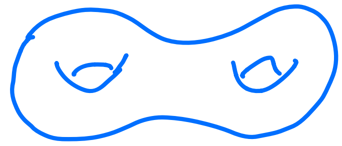
deg j line bundle $\longleftrightarrow$ Lagrangian slope $-j$

skyscraper sheaf $\longleftrightarrow$ Vertical Lagrangian

Genus two complex curve



$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \ , \quad \text{symmetric} \ , \quad \operatorname{Im}\tau > 0$$

$$\Theta_\tau = \left\{ \vartheta(x_1, x_2; \tau) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} x_1^{-n_1} x_2^{-n_2} e^{-\pi i n^T \tau n} = 0 \right\}$$

$$\cap$$

$$V_\tau = (\mathbb{C}^*)^2 / \tau\mathbb{Z}^2 = \{(x_1, x_2) \in (\mathbb{C}^*)^2\} \big/ (x_1,x_2) \sim (\tau n)\cdot(x_1,x_2)$$

$$\uparrow \ \wr \begin{pmatrix} x_1 = e^{2\pi i v_1} \\ x_2 = e^{2\pi i v_2} \end{pmatrix}$$

$$\tau\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\cdot(x_1, x_2) = \begin{pmatrix} \tau_{11} n_1 + \tau_{12} n_2 \\ \tau_{21} n_1 + \tau_{22} n_2 \end{pmatrix}\cdot(x_1, x_2)$$

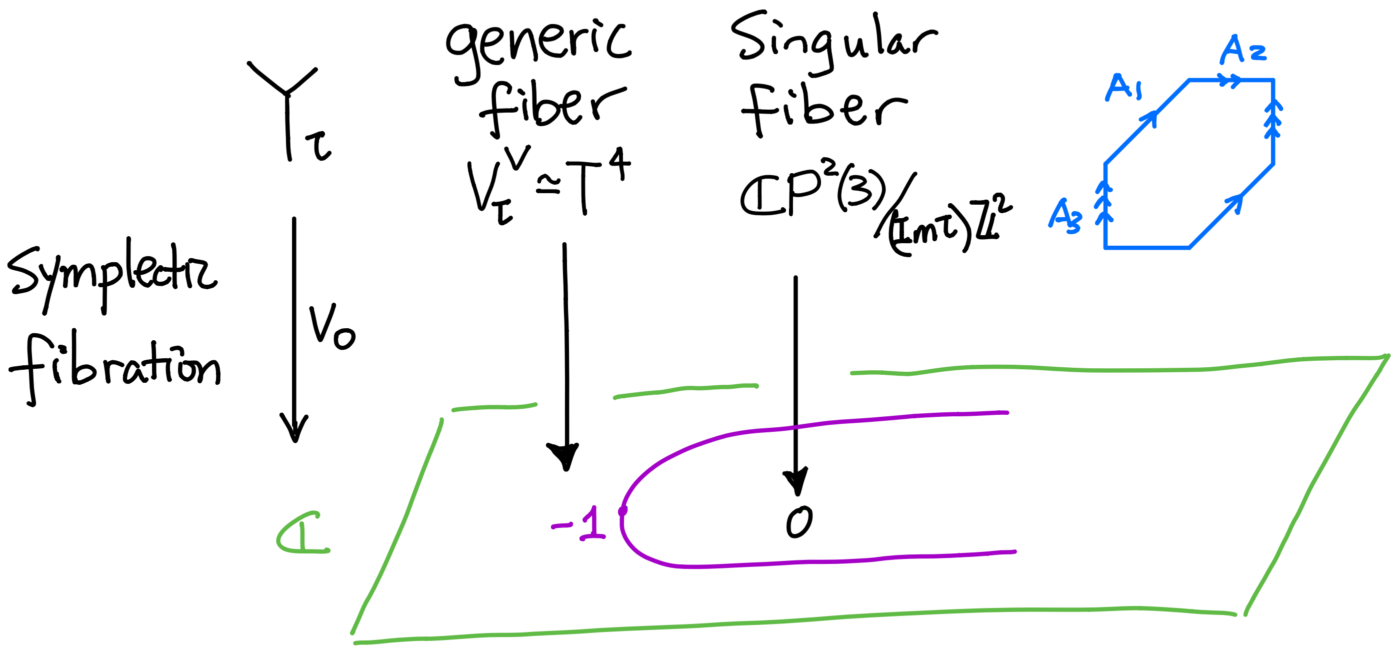
$$= \left(e^{2\pi i(\tau_{11} n_1 + \tau_{12} n_2)} x_1 \ , \ e^{2\pi i(\tau_{21} n_1 + \tau_{22} n_2)} x_2 \right)$$

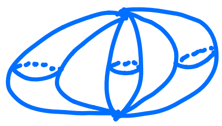
$$V_\tau^+ = \mathbb{C}^2 / \mathbb{Z}^2 + \tau\mathbb{Z}^2 = \{(v_1, v_2) \in \mathbb{C}^2\} / \mathbb{Z}^2 + \tau\mathbb{Z}^2$$

$$\cong T^4 \ (4 \text{ torus})$$

SYZ Mirror symmetry: Abouzaid - Auroux - Katzarkov 2012

Replace Θ_τ by $X_\tau = Bl_{\Theta_\tau \times \{0\}}(V_\tau \times \mathbb{C})$





$\mathrm{Crit}(v_0) = C_1 \cup C_2 \cup C_3 \subseteq Y$

$C_j \cong \mathbb{CP}^1$ with symplectic area A_j

HMS:

$$\begin{tikzcd}
\mathrm{Coh}(V_\tau) \arrow[rr, "\iota^* : \mathcal{L}^j \mapsto \mathcal{L}^j|_{\Theta_\tau}"] \arrow[d, "\wr"', "\substack{\text{Fukaya}\\ 2002}"'] \arrow[drr, phantom, "\circlearrowright"] & & \mathrm{Coh}(\Theta_\tau) \arrow[d, "\wr", "\substack{\text{Cannizzo}\\ 2019}"'] \\
\mathrm{Fuk}(V_\tau^\vee) \arrow[rr, "\cup"] & & \mathrm{FS}(Y_\tau, v_0)
\end{tikzcd}$$