# Homework 5 Solutions

1. We construct the solution iteratively; in each step we add $[x, x+1]$ where $x$ is the leftmost uncovered point.

```
p1(X[1..n])
{
     ans = {};
     last = -infinty;
     mergesort(X[1..n]);
     for (i = 1; i <= n; ++i)
         if (X[i] > last + 1)   // outside the last added interval [last, last+1]
         {
             last = X[i];
             ans += [last, last+1];
         }
     return ans;
}
```

The algorithm runs in time $\Theta(n \lg n + n) = \Theta(n \lg n)$.

We now prove that there is an optimal solution containing the first interval selected by this algorithm. By repeating the argument, this shows that the algorithm returns an optimal solution.

Consider an optimal solution $O$, and let $i = [l, l+1]$ be the first interval selected by the algorithm; $l$ must be the leftmost point, since $X$ is sorted in nondecreasing order, and `last` is initialized to $-\infty$.

If $O$ contains the interval $i$, then we are done. If not, let $i' = [l', l'+1]$ be the leftmost interval in $O$. $i'$ must contain the leftmost point, i.e., $l' \leqslant l \leqslant l'+1 \leqslant l+1$. Since there are no points to the left of $l$, $i = [l, l+1]$ covers at least as many points in $X$ as $i' = [l', l'+1]$. Hence $O - i' + i$ is another optimal solution.

2. Starting at vertex 0, Prim's algorithm picks the following edges:

```
(0, 7): 16
(7, 1): 19
(0, 2): 26
(2, 3): 17
(7, 5): 28
(5, 4): 35
(2, 6): 50
MST has total weight: 181
```

Recall that the cut theorem says that the lightest cross edge for any cut $(A, B)$ of a weighted graph $G = (V, E, W)$ must be in some MST. The correctness of Prim's algorithm follows immediately from the cut theorem applied to the cut $(T, V - T)$.

3. Consider the following graph:

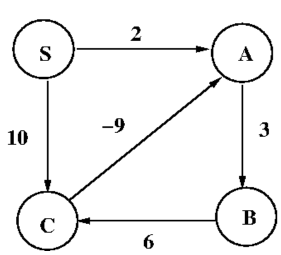


**Figure 1.**

Dijkstra's algorithm computes shortest distances from $S$ as follows:

| S | A | B | C |
|---|---|---|---|
| 0 | inf | inf | inf |
| 0 | 2 | inf | 10 |
| 0 | 2 | 5 | 10 |
| 0 | 2 | 5 | 10 |
| 0 | 1 | 5 | 10 |

However, the true shortest distances from $S$ to $B$ is 4.