Homework 1 Solutions

1. $M(n)$ satisfies the recurrence:

$$\begin{aligned} M(1) &= 2 \\ M(n) &= 5M(n-1)+4,\ n>1 \end{aligned}$$

We guess that $M(n)$ is a linear function of 5^n, i.e.,$M(n)=\alpha 5^n+\beta$.

$$\begin{aligned} M(1) &= \alpha 5+\beta=2 \\ M(2) &= \alpha 5^2+\beta=14 \\ 20\alpha &= 12 \\ \alpha &= \frac{3}{5} \\ \beta &= -1 \end{aligned}$$

Hence $M(n)=3\cdot 5^{n-1}-1$. Verifying:

$$\begin{aligned} M(1) &= 3\cdot 5^0-1 \\ &= 2\ \checkmark \\ 5M(n-1)+4 &= 5(3\cdot 5^{n-2}-1)+4 \\ &= 3\cdot 5^{n-1}-1 \\ &= M(n)\ \checkmark \end{aligned}$$

2. The tree for $C(3^m)$ has $m+1$ levels, labeled 0, 1, ..., m. Level l has 5^l terms. On level m, each term has value $C(1)=3$. On level $l<m$, each term has value $2(3^{m-l})$. Hence

$$\begin{aligned} C(3^m) &= 3\cdot 5^m+\sum_{l=0}^{m-1}(5^l)(2)(3^{m-l}) \\ &= 3\cdot 5^m+2\cdot 3^m\sum_{l=0}^{m-1}\left(\frac{5}{3}\right)^l \\ &= 3\cdot 5^m+2\cdot 3^m\cdot\left(\frac{\left(\frac{5}{3}\right)^m-1}{\frac{5}{3}-1}\right) \\ &= 3\cdot 5^m+2\cdot 3^m\cdot\frac{3}{2}\left(\left(\frac{5}{3}\right)^m-1\right) \\ &= 3\cdot 5^m+3(5^m-3^m) \\ &= 6\cdot 5^m-3\cdot 3^m \\ &= 6n^{\log_3 5}-3n. \end{aligned}$$

Verifying:

$$\begin{aligned} C(1) &= 6(1)^{\log_3 5}-3 \\ &= 3\ \checkmark \\ 5C\left(\frac{n}{3}\right)+2n &= 5\left(6\left(\frac{n}{3}\right)^{\log_3 5}-3\left(\frac{n}{3}\right)\right)+2n \\ &= 6n^{\log_3 5}-3n \\ &= C(n)\ \checkmark \end{aligned}$$