Maximum Likelihood Estimation 1.4

Heather Ann Dye

11/15/2022

Maximum Likelihood Estimation 1.4

We have a population and know that population has a particular distribution: normal, uniform, exponential. However, we do not know the specific parameters.

Example 1 The height of adult men in the US is normally distributed. We do not know the mean μ or the variance σ^2 and can not identify the specific function modeling the population.

Example 2 Consider data from an exponential distribution. The parameter θ determines which exponential distribution. The parameter θ is greater than zero, so the parameter space consists of all reals greater then zero.

In general, we say that the distribution depends on θ, an element of the parameter space Ω.

We repeat our random experiment n times to draw a sample:

$$X_1, X_2, \ldots X_n.$$

We calculate the statistic $\mu(x_1, x_2, \ldots x_n)$ to estimate the parameter θ. This is the **point estimate**.

Method 1: Maximum Likelihood Estimators

Maximum likelihood estimators maximize the value of the probability density function for the given sample. That is, the likelihood of the sample is maximized by the calculated parameter.

Example 3 A population is described by a Bernoulli trial $(Bin(1, p))$ with the distribution

$$P(X = x) = f(x) = p^x(1-p)^{1-x}, \ x = 0, 1.$$

If we repeat the experiment n times, then

$$P(X_1 = a_1, X_2 = a_2, \ldots X_n = a_n) = p^{\sum a_i}(1-p)^{n - \sum a_i}.$$

Since p is unknown, the likelihood function is

$$L(p) = p^{\sum a_i}(1-p)^{n - \sum a_i}.$$

From the sample, we know that $X_i = a_i$ (constants) and the only variable in $L(p)$ is p:

$$L(p) = p^{(\sum a_i)}(1-p)^{n - \sum a_i}.$$

We choose p so that $L(p)$ is maximized; our sample has the largest possible probability. We follow the maximization technique from Calculus 1 (take the derivative).

$$L'(p) = \left(\sum a_i\right)p^{(\sum a_i)-1}(1-p)^{n - \sum a_i} - [n - \sum a_i]p^{(\sum a_i)-1}(1-p)^{(n-1) - \sum a_i}.$$

Now,
$$L'(p) = 0$$
when
$$\frac{\sum a_i}{p} - \frac{(n - \sum a_i)}{1 - p} = 0$$
or
$$p = \frac{\sum a_i}{n}.$$
Note that there is a special case if $\sum a_i = 0$ or $\sum a_i = n$.

Definition Maximum likelihood estimates have the form:
$$L(\theta) = \Pi f(x_i, \theta).$$

We maximize $L(\theta)$ with respect to θ. Then we substitute in values for the X_i from the sample.

- Note that $Log(L(\theta))$ and $L(\theta)$ have maximums at the same values of θ.
- It is easier to calculate the maximum of $Log(L(\theta))$.
- The value $Log(L(\theta))$ is usually written as $LL(\theta)$.

Example 4 Let $X_i \sim Exp(\theta)$. The exponential pdf has the form
$$f(x) = \frac{1}{\theta} Exp\left(\frac{-x}{\theta}\right), \ 0 \le x < \infty.$$

We draw a sample of size n then the maximum likelihood function is:
$$L(\theta) = \frac{1}{\theta^n} Exp\left(\frac{-\sum x_i}{\theta}\right).$$

The Log likelihood function is:
$$LL(\theta) = Log(L(\theta)) = Log\left(\frac{1}{\theta^n}\right) - \frac{\sum x_i}{\theta}.$$

The derivative of $LL(\theta)$:
$$LL'(\theta) = \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2}.$$

If $LL'(\theta) = 0$ then
$$\theta = \frac{\sum x_i}{n}.$$

Note that
$$LL''(\theta) = \frac{n}{\theta^2} + \frac{-2\sum x_i}{\theta^3}$$

so that
$$LL'\left(\frac{\sum x_i}{n}\right) = \frac{n^3}{(\sum x_i)^2} - 2\frac{n^3}{(\sum x_i)^2} < 0.$$

Problem Given the following five item sample, estimate θ. We use the formula just derived to estimate θ.
$$3, 4, 4, 1, 2$$

```
mydata<-c(3,4,4,1,2)
mean(mydata)
```

```
## [1] 2.8
```

2

Estimator Method 2: Method of Moments

Suppose that the pdf of the distribution is known and $E(x) = \mu$ and $Var(X) = \sigma^2$. Then, we can set the distribution estimate (from the sample) equal to the theoretical moment and solve for the parameter.

Example 5   Let X_i have the distribution

$$f(x, \theta) = \theta x^{\theta - 1}, \text{for } 0 \le x < 1, 0 \le \theta < \infty.$$

We calculate that

$$E(x) = \int_0^1 x \theta x^{\theta - 1} dx = \int_0^1 \theta x^{\theta} dx.$$

Then

$$E(x) = \frac{\theta}{\theta + 1}.$$

But $E(x)$ is the mean and is estimated by $\bar{x}$. Equating the two estimates:

$$\bar{x} = \frac{\theta}{\theta + 1}.$$

We can solve to determine our estimated value of θ:

$$\tilde{\theta} = \frac{\bar{x}}{1 - \bar{x}}.$$

Problem   Using the data and the formula that we derived, estimate $\tilde{\theta}$.

X	Frequency
0	5
1/4	3
1/2	10
3/4	9

```
mydata2<-c( rep(0,5), rep(1/4,3), rep(1/2, 10), rep(3/4, 9))
mean(mydata2)/(1 -mean(mydata2))
```

```
## [1] 0.862069
```