1. The following system describes a weakly-nonlinear oscillator, where a is a parameter of the system which we can choose to vary, and $x(t)$ and $y(t)$ are real functions of t:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x + a\left(y - \frac{y^3}{3}\right). \end{aligned}$$

(a) The only critical point is the origin. Use `SymPy` to find the eigenvalues/vectors of the system linearised about the origin. Describe how the behaviour of the critical point changes (e.g. node, centre etc.) as the value of a varies.

(b) Draw the phase portrait first for $a = 0.1$, then for $a = -0.1$. In each case overlay the trajectory for the initial conditions $(x(0), y(0)) = (0.1, 0.1)$ and $(x(0), y(0)) = (2, 2)$ (you should experiment with the length of time to run the trajectories; you may need a mix of large and small durations). What can you observe? Does the behaviour of the critical point at the origin look consistent with the eigenvalues you found for the linearised system?

2. Certain types of chemical reaction can be modelled by the system of equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= a - x - bx + x^2 y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= bx - x^2 y, \end{aligned}$$

$$\frac{b \pm \sqrt{b^2 - 2b + 1}}{2}$$

$b = \pm\sqrt{b^2 - 2b + 1}$

where a and b are parameters of the system which we can choose to vary, and x, y denote different chemical concentrations.

$b = 1/2$

$b^2 = -b^2 \pm 2b \mp 1$

$-2b + 1 = 0$

$-2b^2 + 2b - 1 = 0$

$2b^2 - 2b + 1 = 0$

(a) Use `SymPy` to find the one critical point of the system, linearise the system about its critical point and find an expression for the eigenvalues and eigenvectors of the linearised system in terms of the parameters a and b.

$\times 2 \; + 2$

(b) Now fix the parameter $a = 1$. Use `NumPy` to plot the phase portrait of the system on the domain $0 \leq x \leq 5$ and $0 \leq y \leq 5$, first for the case $b = 0.5$, then for $b = 3$. In each case overlay the trajectories having initial condition $(x(0), y(0)) = (0, 0)$ and $(x(0), y(0)) = (2, 3)$. Describe the difference in behaviour shown in the two plots.

(c) Clearly when $a = 1$ the behaviour for small b and large b is quite different. In fact a bifurcation occurs as the value of b is increased. Experiment with different values of b and produce two plots illustrating the bifurcation (one with b below the critical value, and one with b above it). Can you identify anything important that happens to the eigenvalues obtained in part (a) for these values of a and b?

(d) In (b) you should have found some behaviour which appears to be periodic. For this trajectory, plot $x(t)$ and $y(t)$ as functions of t, to more clearly demonstrate this periodic behaviour. (Show these as two subplots on the same figure.)

3. This problem concerns the *modified Euler formula*, as described in problems 22-26 in section 8.2 of Boyce & DiPrima.

(a) Write a function `ModifiedEuler` which implements the modified Euler formula. This should take the same arguments as the `Euler` function from Lab 2, which you may wish to use as a starting point.

(b) Use your `ModifiedEuler` function to find approximate values of the solution to the initial value problem $y' = 5t - 2\sqrt{y}, y(0) = 2$, with $h = 0.05$, as in problem 23 from Boyce & DiPrima.

(c) Use `SymPy` to find the exact solution of the IVP.

(d) Using these results, produce a table like the following (showing the values of the solutions when $t = 0.1, 0.2, 0.3, 0.4, 0.5, 1.0$), and also a plot of the three solutions over the range $0 \leq t \leq 1$.

	Euler, h=0.05	Modified Euler, h=0.05	Exact
0.0	2.000000	2.000000	2.000000
0.1	1.734749	1.751741	1.751547
0.2	1.537944	1.569657	1.569274
⋮	⋮	⋮	⋮

4. Consider the following system of three equations, which is a particular instance of the Rössler system:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -y - z, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x + \tfrac{1}{5}y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \tfrac{1}{5} + (x - \tfrac{5}{2})z. \end{aligned}$$

(a) Produce a plot of the trajectory in 3d phase space (x, y, z) with initial condition $(x(0), y(0), z(0)) = (0, 0, 0)$ (some cursory research on the Rössler system will suggest what this should look like!).

(b) Show that the solution appears to become periodic (in a similar way to Q2(d)). Produce a 2d plot of the limit cycle, by plotting the coordinates $(x(t), y(t))$ as t varies (to get a clear plot, you should ignore small values of t, when the periodic behaviour has not yet established).

(c) What happens when you replace the coefficient $\frac{5}{2}$ appearing in the third equation with 3?