CS1231S TUTORIAL #3

Sets

My post on Forum

$A = \{1, \{2\}, 3\}$

Membership:

1, {2} and 3 are members/elements of A.

$1 \in A$, $\{2\} \in A$, $3 \in A$, $2 \notin A$, $\emptyset \notin A$.

Subsets:

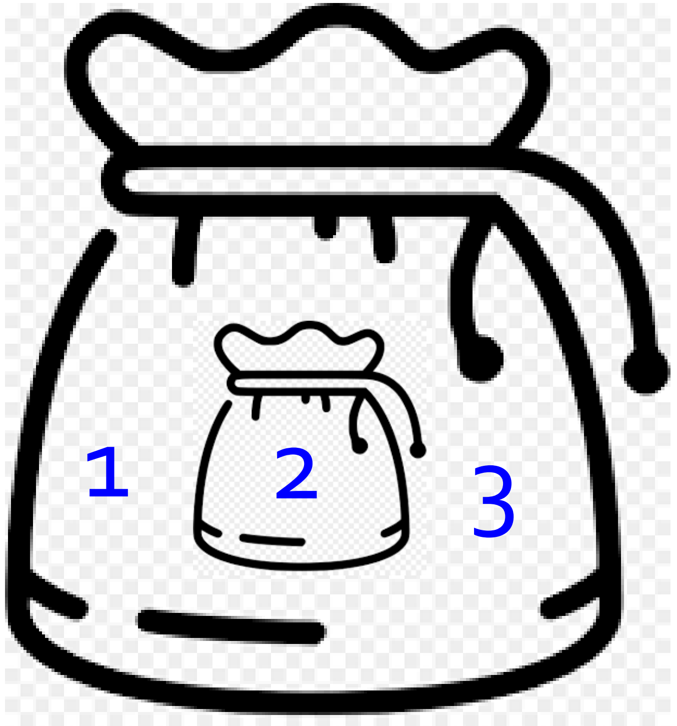
Removing no element: $\{1,\{2\},3\}$

Removing one element: $\{\{2\},3\}$, $\{1,3\}$, $\{1,\{2\}\}$

Removing two elements: $\{1\}$, $\{\{2\}\}$, $\{3\}$

Removing three elements: $\emptyset$

A



Power set:

$\wp(A)=$

$\{ \{1,\{2\},3\},$
$\{\{2\},3\}, \{1,3\}, \{1,\{2\}\},$
$\{1\}, \{\{2\}\}, \{3\},$
$\emptyset \}$

My post on Forum

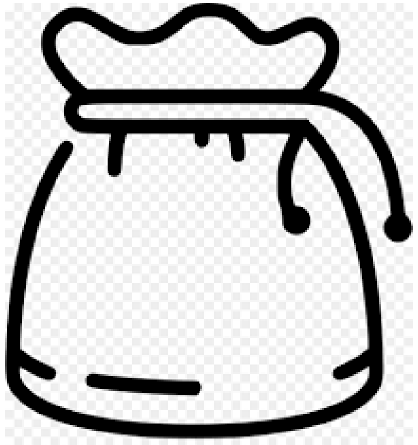
Empty set ∅

Do not call it "null set".

- In measure theory, a null set is set of measure zero (not necessarily empty)

Do not call it "null"!

An empty set is NOT equivalent to nothing. It is a set that has no members/elements.



Q1

TRUE | FALSE

(a) $\emptyset \in \emptyset$

(b) $\emptyset \subseteq \emptyset$

(c) $\emptyset \in \{\emptyset\}$

(d) $\emptyset \subseteq \{\emptyset\}$

(e) $\{\emptyset, 1\} = \{1\}$

(f) $1 \in \{\{1,2\}, \{2,3\}, 4\}$

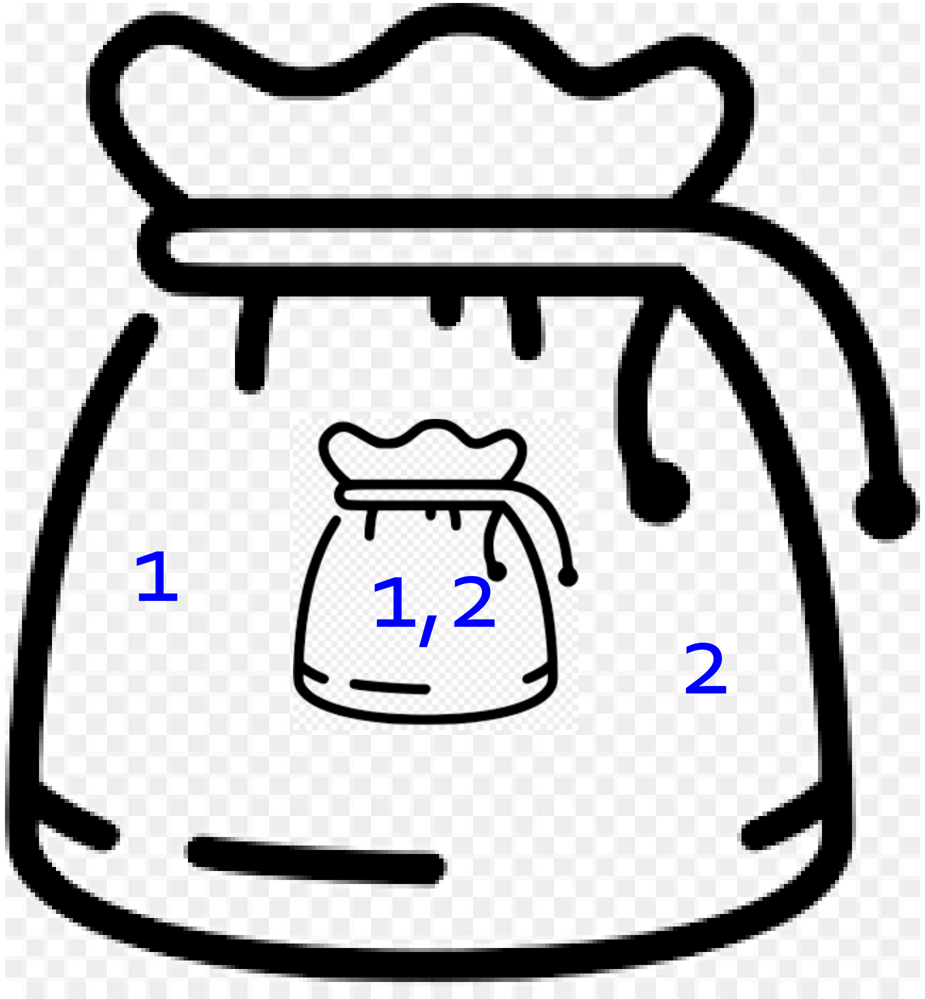
(g) $\{1,2\} \subseteq \{3,2,1\}$

(h) $\{3,3,2\} \subsetneq \{3,2,1\}$

Q2. Let A = {1, {1, 2}, 2, $\cancel{\{1, 2\}}$}. Find |A|.

$A = \{1, \{1,2\}, 2\}$

$|A| = 3$



A

3. $A = \{0,1,4,5,6,9\}$, $B = \{0,2,4,6,8\}$

Find |A|, |B|, |A ∩ B| , |A ∪ B|

|S|: cardinality of set S (#elements in S)

|A| = 6

|B| = 5

$A \cap B = \{0,4,6\}$

$|A \cap B| = 3$

$A \cup B = \{0,1,2,4,5,6,8,9\}$

$|A \cup B| = 8$

Q4. Let $A = \{2n+1 : n \in \mathbb{Z}\}$ and $B = \{2n-1 : n \in \mathbb{Z}\}$.

Is A = B? Yes or No? To show A=B, we show A⊆B and B ⊆A.

1. (⊆) From context, this means A⊆B
   - 1.1 Let $a \in \mathrm{A}$
   - 1.2 Use the definition of A to find $n \in \mathbb{Z}$ such that a = 2n+1
   - 1.3 Then a = 2(n+1) – 1 (basic algebra)
   - 1.4 As $n \in \mathbb{Z}$ we know that $n+1 \in \mathbb{Z}$ (by closure of integers)
   - 1.5 So a ∈ B (by definition of B)

2. (⊇) From context, this means A ⊇B
   - 2.1 Let $\mathrm{b} \in \mathrm{B}$
   - 2.2 Use the definition of B to find $n \in \mathbb{Z}$ such that b = 2n-1
   - 2.3 Then b = 2(n-1) + 1 (basic algebra)
   - 2.4 As $n \in \mathbb{Z}$ we know that $n-1 \in \mathbb{Z}$ (by closure of integers)
   - 2.5 So b ∈ A (by definition of A)

3. Hence A = B by definition of set equality

Q5. A = $\{x \in \mathbb{Z}: 2 \leq x \leq 5\}$, B = $\{x \in \mathbb{R}: 2 \leq x \leq 5\}$.

Is A=B? Yes or No?

Counterexample:

1. Let x = 3.14.
2. x ∈ B as x ∈ $\mathbb{R}$ and $2 \leq x \leq 5$
3. But x ∉ $\mathbb{Z}$
4. So x ∉ A by definition of A
5. Lines 2 and 4 imply A≠B by the definition of set equality.

Q6. Let $U = \{5,6,7 \ldots,12\}$ and

$$M_k = \{n \in \mathbb{Z} : n = km \text{ for some } m \in \mathbb{Z}\}, \forall k \in \mathbb{Z}$$

(a) $\{n \in U : n \; is \; even\}$

(b) $\{n \in U : n = m^2 \text{ for some } m \in \mathbb{Z}\}$

(c) $\{-5, -4, -3, \ldots, 5\} \setminus \{1, 2, 3, \ldots, 10\}$

Q6. Let $\mathrm{U} = \{5,6,7\ldots,12\}$ and

$$M_k = \{n \in \mathbb{Z} : n = km \text{ for some } m \in \mathbb{Z}\}, \forall k \in \mathbb{Z}$$

(d) $\overline{\{5,7,9\} \cup \{9,11\}}$ (U is the universal set)

What is this called?

(e) $\{x, y \ \in \{1,3,5\} \times \{2,4\} : \ x + y \geq 6\}$

(f) $\wp(\{2,4\})$

$\{1,3,5\} \times \{2,4\} =$
$\{ (1,2),$
$(1,4),$
$(3,2),$
$(3,4),$
$(5,2),$
$(5,4) \}$

Ordered pairs

Is $(3,2) = (2,3)$?

Q7. Show that for all sets A,B, C
$A \cap (B \setminus C) = (A \cap B) \setminus C$

1. $A \cap (B \backslash C) = \{x : x \in A \ \wedge \ x \in B \backslash C\}$ by the definition of ∩
2. $= \{x : \ x \in A \wedge (x \in B \wedge x \notin C)\}$ by the definition of \
3. $= \{x : (x \in A \wedge x \in B) \wedge x \notin C\}$ by associativity of ∧
4. $= \{x : x \in A \cap B \wedge x \notin C\}$ by the definition of ∩
5. $= (A \cap B) \backslash C$ by the definition of \

Q8. Prove that for all sets A and B, $(A \cup \bar{B}) \cap (\bar{A} \cup B) = (A \cap B) \cup (\bar{A} \cap \bar{B})$

$(A \cup \bar{B}) \cap (\bar{A} \cup B)$

$= ((\mathrm{A} \cup \bar{B}) \cap \bar{A}) \cup ((\mathrm{A} \cup \bar{B}) \cap B)$ Distributive law

$= \big((\mathrm{A} \cap \bar{A}) \cup (\bar{B} \cap \bar{A})\big) \cup \big((\mathrm{A} \cap \mathrm{B}) \cup (\bar{B} \cap B)\big)$ Distributive law

$= (\emptyset \cup (\bar{B} \cap \bar{A})) \cup ((\mathrm{A} \cap \mathrm{B}) \cup \emptyset)$ Complement law

$= (\bar{B} \cap \bar{A}) \cup (\mathrm{A} \cap \mathrm{B})$ Identity Law

$= (A \cap B) \cup (\bar{A} \cap \bar{B})$ Commutative Law

Q9. Let A, B be sets. Show that A ⊆ B iff A ∪ B = B

This is the "only if" part

1. (⇒) Suppose $A \subseteq B$
   1.1 (To show $A \cup B \subseteq B$)
      1.1.1 Let $z \in A \cup B$.
      1.1.2 Then $z \in A$ or $z \in B$ (by defn of ∪)
      1.1.3 Case 1: Suppose $z \in A$
         1.1.3.1 Then $z \in B$ as $A \subseteq B$ (from 1)
      1.1.4 Case 2: Suppose $z \in B$
         1.1.4.1 Then $z \in B$
      1.1.5 In all cases, we have $z \in B$
   1.2 (To show $B \subseteq A \cup B$)
      1.2.1 Let $z \in B$.
      1.2.2 Then $z \in A \vee z \in B$ (by defn of ∨)
      1.2.3 So $z \in A \cup B$ (by definition of ∪)
   1.3 Lines 1.1 and 1.2 imply $A \cup B = B$ (by definition of set equality)

Q9. Let A, B be sets.
Show that A ⊆ B iff A ∪ B = B

This is the "if" part

2. (⇐) Suppose $A \cup B = B$

    2.1 Let $z \in A$

    2.2 Then $z \in A \vee z \in B$ (by definition of ∨)

    2.3 So $z \in A \cup B$ (by definition of ∪)

    2.4 This implies $z \in B$ as $A \cup B = B$ (by line 2)

    2.5 Therefore, $A \subseteq B$ (by definition of ⊆)

Q10. For sets A and B, define $A \oplus B = (A \backslash B) \cup (B \backslash A)$

(a) Let A = {1, 4, 9, 16} and B = {2, 4, 6, 8, 10, 12, 14, 16}.
Find $A \oplus B$.

$A \backslash B$ =

$B \backslash A$ =

$A \oplus B$=

Q10. For sets A and B, define $A \oplus B = (A \backslash \mathrm{B}) \cup (B \backslash \mathrm{A})$

(b) Show that for all sets A, B: $A \oplus B = (A \cup B) \backslash (A \cap B)$.

$A \oplus B$	
$= (A \backslash B) \cup (B \backslash \mathrm{A})$	definition of $\oplus$
$= (A \cap \bar{B}) \cup (B \cap \bar{A})$	Set Difference law
$= ((A \cap \bar{B}) \cup B) \cap ((A \cap \bar{B}) \cup \bar{A})$	Distributive law
$= (A \cup B) \cap (\bar{B} \cup B) \cap (A \cup \bar{A}) \cap (\bar{B} \cup \bar{A})$	Distributive law
$= (A \cup B) \cap U \cap U \cap (\bar{B} \cup \bar{A})$	Complement law
$= (A \cup B) \cap U \cap U \cap (\overline{B \cap A})$	De Morgan's law
$= (A \cup B) \cap (\overline{B \cap A})$	Identity law
$= (A \cup B) \cap (\overline{A \cap B})$	Commutative law
$= (A \cup B) \backslash (A \cap B)$	Set Difference law

11. (2015/16 Semester 1 exam question 16(a)) Denote by $|x|$ the absolute value of the integer x, i.e.,

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

(a) $\exists z \in S\ \forall x, y \in S\ \ z > |x - y|$.

False

1. It suffices to show its negation is true
$$\forall z \in S\ \exists x, y \in S\ z \leq |x - y|.$$
2. Take any $z \in S$.
3. Let $x = 8$ and $y = -9$.
4. Then $x, y \in S$ and $|x - y|$ $= |8 - (-9)| = 17 > 8 = \max S \geq z$.
5. Since this is true, the original statement is false.

(b) $\exists z \in S\ \forall x, y \in S\ \ z < |x - y|$.

True

1. $-1 \in S$
2. $|x - y| \geq 0 > -1$ for all $x, y \in S$

Q12. For sets $A_{m,} A_{m+1}, \dots . A_n$ define

$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \cdots \cup A_n$ and

$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \cdots \cap A_n$

(a) Let $A_i = \{x \in \mathbb{Z}: x \geq i\}$ for each $i \in \mathbb{Z}$.

Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in roster notation.

$$\bigcup_{i=2}^{5} A_i =$$

$$\bigcap_{i=2}^{5} A_i =$$

For sets $A_m, A_{m+1}, \dots . A_n$ define
$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \dots \cup A_n$ and
$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \dots \cap A_n$

(b) Let $B_1, B_2, \dots, B_k, C_1, C_2, \dots, C_l$ be sets such that $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$

Show that $B_i \subseteq C_j$ for all $i \in \{1,2, \dots, k\}$ and all $j \in \{1,2, \dots, l\}$

1. Let $B_1, B_2, \dots, B_k, C_1, C_2, \dots, C_l$ be sets such that $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$
2. 2.1 Let $r \in \{1,2, \dots, k\}$ and $s \in \{1,2, \dots, l\}$

   2.2 Take any $z \in B_r$

   2.3 Then $z \in B_1$ or $z \in B_2$ or … or $z \in B_k$ by definition of "or", as $r \in \{1,2, \dots, k\}$

   2.4 So $z \in B_1 \cup B_2 \cup \dots \cup B_k = \bigcup_{i=1}^{k} B_i$ by definition of ∪ and ⋃

   2.5 Hence z $\in \bigcap_{j=1}^{l} C_j = C_1 \cap C_2 \cap \dots \cap C_l$ *as* $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$ by line 1

   2.6 Thus $z \in C_1$ and $z \in C_2$ and … and $z \in C_l$ by the definition of ∩

   2.7 In particular, we know that $z \in C_s$ as $s \in \{1,2, \dots, l\}$
3. So $B_i \subseteq C_j$ for any $i \in \{1,2, \dots, k\}$ and any $j \in \{1,2, \dots, l\}$

THE END