CS1231S Discrete Structures

Tutorial 1

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Tutorial Questions (and Photo Taking)

Question D1

a)

$$(\forall d, n \in \mathbb{Z}\ d \mid a) \leftrightarrow (\exists k \in \mathbb{Z}\ n = kd)$$

(For all integers d and n; d divides n) iff (there exists an integer k; where n = kd)

b) Answer: we don't know. We are concerned about *d* and *n* being integers - $2\sqrt(2)$ is not an integer.

Question D2

Let B be the set of boys, G be the set of girls, and $Loves(x, y)$ be x loves y It can be interpreted in two ways (or more):

1. All boys love one particular girl ($\exists g \in G, \forall b \in B, Loves(b, g)$)
2. Each boy loves one girl ($\forall b \in B, \exists ginG, Loves(b, g)$)

Question D3

Statement	True when...	False when...
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Complete the table below for mixed quantifiers.

Statement	True when...	False when...
$\forall x \forall y\, P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is at least one pair x,y for which $P(x,y)$ is false.
$\forall x \exists y\, P(x,y)$	For every x, there is a y for which $P(x,y)$ is true.	There is an x for which $P(x,y)$ is false for every y.
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x, there is a y for which $P(x,y)$ is false.
$\exists x \exists y\, P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y.

Question D4

(to be discussed if there is time)

Question 1

$$
\begin{aligned}
\textit{Statement}: &\ \forall n \in \mathbb{Z}((6 \mid n) \rightarrow (2 \mid n) \wedge (3 \mid n)). && (\textit{True}) \\
\textit{Converse}: &\ \forall n \in \mathbb{Z}((2 \mid n)\ \wedge\ (3 \mid n)\ \rightarrow (6 \mid n)). && (\textit{True}) \\
\textit{Inverse}: &\ \forall n \in \mathbb{Z}(6 \nmid n \rightarrow (2 \nmid n) \vee (3 \nmid n)). && (\textit{True}) \\
\textit{Contrapositive}: &\ \forall n \in \mathbb{Z}((2 \nmid n) \vee (3 \nmid n) \rightarrow (6 \nmid n)). && (\textit{True})
\end{aligned}
$$

Question 1

$$\begin{aligned}
\textit{Statement} &: \forall r \in \mathbb{R}(r > 3 \rightarrow r^2 > 9). && (\textit{True}) \\
\textit{Converse} &: \forall r \in \mathbb{R}(r^2 > 9 \rightarrow r > 3). && (\textit{False}) \\
\textit{Inverse} &: \forall r \in \mathbb{R}(r \leq 3 \rightarrow r^2 \leq 9). && (\textit{False}) \\
\textit{Contrapositive} &: \forall r \in \mathbb{R}(r^2 \leq 9 \rightarrow r \leq 3). && (\textit{True})
\end{aligned}$$

Counterexample for converse and inverse statements: Let $r = -5$

Question 1

$$\begin{aligned} \textit{Statement} &: \forall p, q \text{ that are statements}, ((p \rightarrow q) \rightarrow \sim p). \quad (\textit{False}) \\ \textit{Converse} &: \forall p, q \text{ that are statements}, (\sim p \rightarrow (p \rightarrow q)). \quad (\textit{True}) \\ \textit{Inverse} &: \forall p, q \text{ that are statements}, (\sim (p \rightarrow q) \rightarrow p). \quad (\textit{True}) \\ \textit{Contrapositive} &: \forall p, q \text{ that are statements}, (p \rightarrow \sim (p \rightarrow q)). \quad (\textit{False}) \end{aligned}$$

Counterexample for original statement and contrapositive: Let both p and q be true.

Question 2

a) $\forall p \exists q \Big((p \neq q) \wedge Loves(p, q) \Big)$

b) $Loves(John, Mary) \wedge \forall x \Big((x \neq John) \rightarrow \sim Loves(x, Mary) \Big)$

Question 3

Proof

1.Take any integers a, b, c.

2.Suppose $a - b$ is even and $a - c$ is even.

  2.1 There is an integer s s.t. $a - b = 2s$ (defn of even numbers).

  2.2 Similarly, there is an integer t s.t. $a - c = 2t$

  2.3 Then $b - c = (a - c) - (a - b) = 2t - 2s = 2(t - s)$

3.Therefore, $b - c$ is even (by definition of even numbers).

Question 4

a) Invalid, inverse error
b) Valid, universal modus ponens
c) Invalid, converse error

Question 5

b. $\forall v \in V\ \big(G(v) \to T(v)\big)$

c. $\exists v \in V\ \big(T(v) \wedge G(v)\big)$

d. $\forall v \in V\ \big(E(v) \to {\sim}W(v)\big)$

Alternatively: $\forall v \in V\ \big({\sim}E(v) \vee {\sim}W(v)\big)$

Question 5

e. $\left(\exists v \in V\ (T(v) \wedge E(v))\right) \wedge \left(\exists u \in V\ (T(u) \wedge {\sim}E(u))\right)$

Note that the above is a conjunction (AND) of two existential clauses. Each existential variable v or u has a scope only within its own clause. Therefore, there is no ambiguity to use the same variable in both clauses, as shown below:

$$\left(\exists v \in V\ (T(v) \wedge E(v))\right) \wedge \left(\exists v \in V\ (T(v) \wedge {\sim}E(v))\right)$$

Nevertheless, you may use different variable names to avoid confusion.

Also, note that it is incorrect to write

$$\exists v \in V\ \left((T(v) \wedge E(v)) \wedge (T(v) \wedge {\sim}E(v))\right)$$

because the scope of v here covers both $(T(v) \wedge E(v))$ as well as $(T(v) \wedge {\sim}E(v))$. This means that if there exists a visitor who took the Transformers Ride and visited Ancient Egypt, the same visitor took the Transformers Ride but did not visit Ancient Egypt! The statement becomes false (by applying commutativity and associativity of conjunction).

Question 6

a) False, none of the titles is read by all three female readers
b) False, Mr. Dueet doesn't read any Fantasy title
c) True, Mr. Dueet reads all the Mystery titles
d) True, none of the Fantasy titles is read by Dueet (and Fandi)

Question 7

a.

3. If an object is black, then it is a square.
2. (Contrapositive form) If an object is a square, then it is above all the gray objects.
4. If an object is above all the gray objects, then it is above all the triangles.
1. If an object is above all the triangles, then it is above all the blue objects.

$\therefore$ If an object is black, then it is above all the blue objects.

b.

Let O, the domain, be the set of objects.

3. $\forall x \in O, \{Black(x) \to Square(x)\}$.
2. (Contrapositive form) $\forall x \in O, \{Square(x) \to \{\forall y \in O[Gray(y) \to Above(x,y)]\}\}$.
4. $\forall x \in O, \{\{\forall y \in O, [Gray(y) \to Above(x,y)]\} \to \{\forall z \in O, [Triangle(z) \to Above(x,z)]\}\}$.
1. $\forall x \in O, \{\{\forall z \in O, [Triangle(z) \to Above(x,z)]\} \to \{\forall w \in O, [Blue(w) \to Above(x,w)]\}\}$.

$\therefore \forall x \in O, \{Black(x) \to \{\forall w \in O, [Blue(w) \to Above(x,w)]\}\}$.

Question 8

a. To prove an "if and only if" statement, you need to prove both directions.

($\Rightarrow$) 1. Suppose $\left(\forall x \in D\ P(x)\right) \wedge \left(\forall x \in D\ Q(x)\right)$ is true.

2. Consider any $a \in D$.

   2.1 Since $\forall x \in D\ P(x)$ is true, we have $P(a)$ is true. (universal instantiation)

   2.2 Similarly, $Q(a)$ is true.

   2.3 Therefore, $P(a) \wedge Q(a)$ is true for any $a \in D$.

3. Therefore, $\forall x \in D\ \left(P(x) \wedge Q(x)\right)$ is true.

($\Leftarrow$) 1. Suppose $\forall x \in D\ \left(P(x) \wedge Q(x)\right)$ is true.

2. Consider any $a \in D$.

   2.1 Then $P(a) \wedge Q(a)$ is true.

   2.2 So, $P(a)$ is true and $Q(a)$ is true.

   2.3 Since $P(a)$ is true for any $a \in D$, we have $\forall x \in D\ P(x)$ is true.

   2.4 Similarly, since $Q(a)$ is true for any $a \in D$, we have $\forall x \in D\ Q(x)$ is true.

3. Therefore, $\left(\forall x \in D\ P(x)\right) \wedge \left(\forall x \in D\ Q(x)\right)$ is true.

Question 8

b. To claim that $(\exists x \in D\ P(x)) \wedge (\exists x \in D\ Q(x))$ and $\exists x \in D\ (P(x) \wedge Q(x))$ are equivalent is to claim that they have the same truth values for any, D, P and Q, i.e. there is an implicit universal quantification over D, P and Q. To prove inequivalence, it therefore suffices to give a counterexample. There are many possible counterexamples. Here's one:

Let $D = \mathbb{N}$, $P(x)$ be "$x^2 = 0$" and $Q(x)$ be "$x^2 = 1$".

Then $(\exists x \in \mathbb{N}\ x^2 = 0) \wedge (\exists x \in \mathbb{N}\ x^2 = 1)$ is true, but $\exists x \in \mathbb{N}\ (x^2 = 0 \wedge x^2 = 1)$ is false.

Question 9

a) $\sim \left(\forall x, y, \in \mathbb{R}(x > y \to x^2 > y^2)\right) \equiv \exists x, y \in \mathbb{R}(x > y \wedge x^2 \leq y^2)$ which is true: one example is $(x = 1, y = -2)$, so the original statement is false

b) Irrelevant counterexample; it does not satisfy the hypothesis $(x > y)$