# CS1231S TUTORIAL #4

Functions

Q1. $f : \mathbb{Q} \to \mathbb{Q}$ ?

YES

NO

Why?

Why?

(a) $f(n) = \pm n$

(b) $f(n) = 2\sqrt{n}$

(c) $f(n) = \frac{1}{n^2+1}$

(d) $f(n) = \lfloor \sin n \rfloor$

What is the range of sine $n$?

**Q2.** Let $U$ be a set and $A \subseteq U$ such that $\emptyset \neq A \neq U$.

Define $\chi: U \to \mathbb{Z}$ by setting, for all $x \in U$,

$$\chi(x) = \begin{cases} \boxed{\begin{matrix} 0, \\ 1, \end{matrix}} & \begin{matrix} if\ x \notin A; \\ if\ x \in A. \end{matrix} \end{cases}$$

Domain: $U$

Codomain: $\mathbb{Z}$

Image: $\{0, 1\}$

Find the domain, the codomain, and the image of $\chi$.

## Definition of a function (recap)

**Definition 6.1.1 (lecture slide):**
A function from set $A$ to set $B$ is an assignment to each element $x \in A$ <u>exactly one element</u> $y \in B$, i.e. $y = f(x)$.

From above, we can infer two properties of a function:

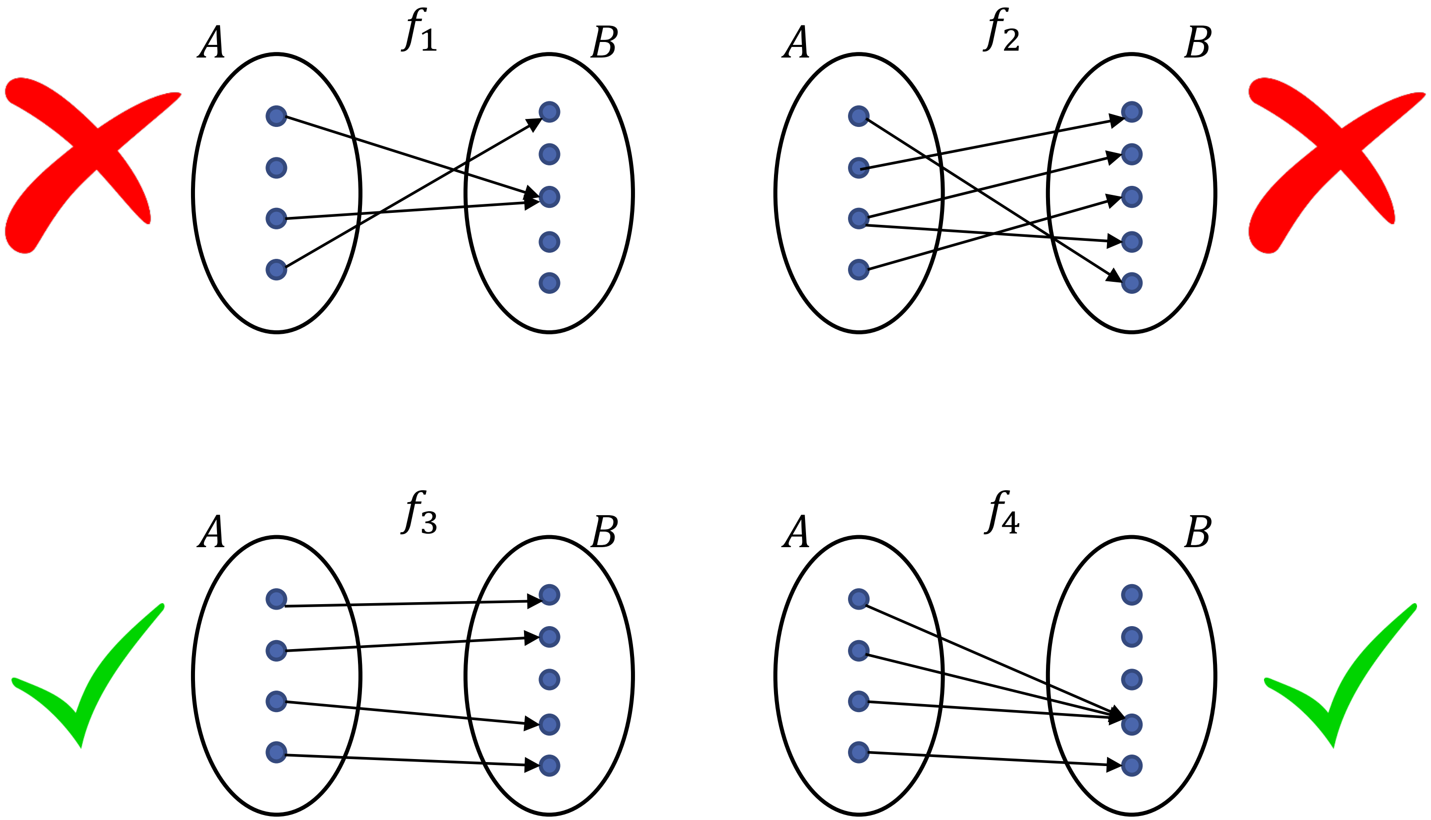
(F1) $\forall x \in A, \exists y \in B \left(y = f(x)\right)$.

(F2) $\forall x \in A, \forall y_1, y_2 \in B, (y_1 = f(x) \land y_2 = f(x) \rightarrow y_1 = y_2)$
(in other words, the $y$ in (F1) is unique)

Or, combining (F1) and (F2):

$$\forall x \in A, \exists!\, y \in B \left(y = f(x)\right).$$

# Quick check: Which of the following are functions and which are not?




Informally, for a function, every element in the <u>domain</u> must have **exactly one arrow coming out** of it.

Note that nothing is said about elements in the codomain.

# Recap: Definition 6.2.5 (lecture slide)

A function $f: A \to B$ is surjective iff

$$\forall y \in B, \exists x \in A \left(y = f(x)\right).$$

A function $f: A \to B$ is injective iff

$$\forall x_1, x_2 \in A \left(f(x_1) = f(x_2) \to x_1 = x_2\right).$$

A function is bijective iff it is surjective and injective.

(Surjective) Informally, every element in the codomain must have **at least one arrow** going into it.

$\wedge$

(Injective) Informally, every element in the codomain must have **at most one arrow** going into it.

$\equiv$

(Bijective) Informally, every element in the codomain must have **exactly one arrow** going into it.

# Q3. Is the function injective? Surjective? Prove. If it is bijective, find the inverse function. Here denote by **Bool** the set **{true; false}**.

## Quick check before we go into the details:

Tutors: Ask students what their answers are.

| Function | Injective? | Surjective? |
|---|---|---|
| $f: \mathbb{Q} \to \mathbb{Q};$ $x \mapsto 12x + 31$ | ✓ | ✓ |
| $g: \mathrm{Bool}^2 \to \mathrm{Bool};$ $(p, q) \mapsto p \wedge {\sim} q$ | ✗ | ✓ |
| $h: \mathrm{Bool}^2 \to \mathrm{Bool}^2;$ $(p, q) \mapsto (p \wedge q, p \vee q)$ | ✗ | ✗ |
| $k: \mathbb{Z} \to \mathbb{Z};$ $x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$ | ✓ | ✗ |

## Q3.

What does the previous table show?

A function can be

(a) injective but not surjective;

(b) surjective but not injective;

(c) injective and surjective (i.e. bijective); or

(d) neither injective nor surjective.

All four cases are possible!

# Q3. $f: \mathbb{Q} \to \mathbb{Q};$ $x \mapsto 12x + 31$

Part 1

## Both Injective and Surjective

1. Note that for all $x, y \in \mathbb{Q}$,
$$y = 12x + 31 \iff x = \frac{y - 31}{12}$$

2. Define $f^*: \mathbb{Q} \to \mathbb{Q}$ by setting, for all $y \in \mathbb{Q}$,
$$f^*(y) = \frac{y - 31}{12}.$$

3. Then whenever $x, y \in \mathbb{Q}$,
$$y = f(x) \iff x = f^*(y).$$

4. Thus $f^*$ is the inverse of $f$.

5. Hence $f$ is both injective and surjective by Theorem 6.2.18.

Students to note:

You see that for this solution, we didn't show that $f$ is injective and surjective separately, and then conclude that $f$ is bijective (we could have done so).

Instead, we find the inverse of $f$, and since its inverse exists, it must be bijective (by Theorem 6.2.18.)

# Q3. Part 2

$g: \text{Bool}^2 \to \text{Bool};$
$(p, q) \mapsto p \wedge {\sim}q$

Not Injective But Surjective

1. $g(false, true) \;=\; false \;=\; g(false, false)$,
   where $(false, true) \neq (false, false)$.
2. So $g$ is not injective.
3. $g(true, false) \;=\; true$
4. So every element in the codomain Bool is in the image of $g$ by lines 1 and 3.
5. This says $g$ is surjective.

# Q3.
Part 3

$h\colon \mathrm{Bool}^2 \to \mathrm{Bool}^2;$
$(p, q) \mapsto (p \wedge q, p \vee q)$

Not Injective, Not Surjective

1. $h(true, false) = (false, true) = h(false, true),$
   $where\ (true, false) \neq (false, true).$
2. So $h$ is not injective.
3. If $p, q, r \in \mathrm{Bool}$ such that $h(p, q) = (true, r)$, then
   3.1. $p \wedge q = true$ by the definition of $h$;
   3.2. $\therefore p = true$
   3.3. $\therefore r = p \vee q = true$ by the definition of $h$.
4. So $(\boldsymbol{true}, \boldsymbol{false})$ in the codomain is not in the image of $h$.
5. This says $h$ is not surjective.

# Q3.

Part 4

$k: \mathbb{Z} \to \mathbb{Z};$

$$x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$$

## Injective, Not Surjective

1. We first show that $x$ is even $\to k(x)$ is even.
   1.1. Let $x$ be an even integer.
   1.2. Then $k(x) = x$ by the definition of $k$.
   1.3. So $k(x)$ is even.
2. Next we show that $x$ is an odd $\to k(x)$ is odd.
   2.1. Let $x$ be an odd integer.
   2.2. Then $k(x) = 2x - 1 = 2(x-1) + 1$, where $x - 1$ is an integer.
   2.3. So $k(x)$ is odd.
3. Since every integer is either even or odd but not both, lines 1 and 2 tell us that, for every $x \in \mathbb{Z}$,
   3.1. $x$ is even iff $k(x)$ is even; and
   3.2. $x$ is odd iff $k(x)$ is odd.
4. Now we show that $k$ is injective.
   4.1. Let $x, x' \in \mathbb{Z}$ such that $k(x) = k(x')$.
   4.2. Case 1: $k(x)$ is even.
      4.2.1. Then both $x$ and $x'$ are even by line 3.1.
      4.2.2. So $x = k(x) = k(x') = x'$ by the definition of $k$.
   4.3. Case 2: $k(x)$ is odd.
      4.3.1. Then both $x$ and $x'$ are odd by line 3.2.
      4.3.2. So $2x - 1 = k(x) = k(x') = 2x' - 1$ by the definition of $k$.
      4.3.3. Thus $x = x'$.
   4.4. Since $k(x)$ is either even or odd, we conclude that $x = x'$ in any case.

Continued …

# Q3. Part 4

$k: \mathbb{Z} \to \mathbb{Z}$;

$$x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$$

## Injective, Not Surjective

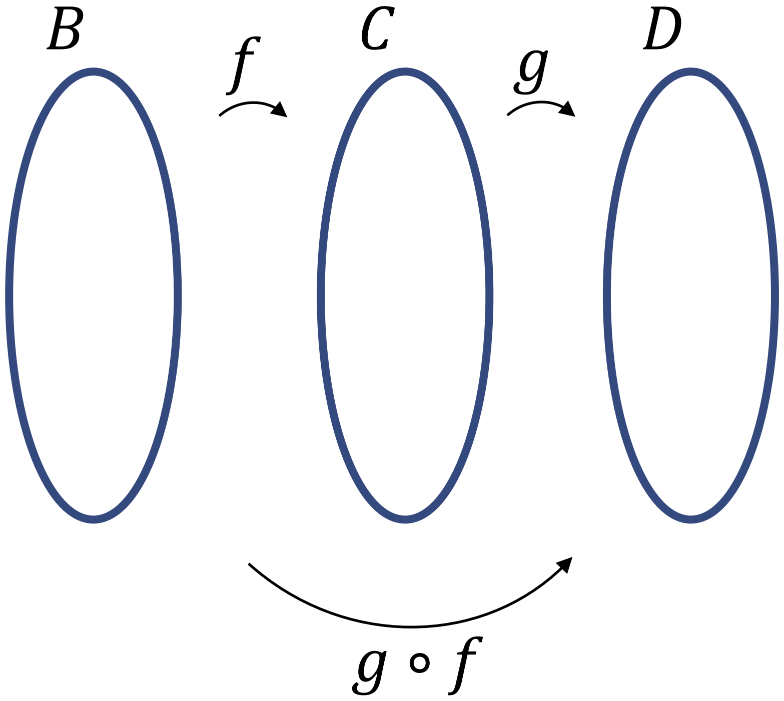
5. Finally, we show that $k$ is not surjective.
   - 5.1. We prove this by contradiction.
     - 5.1.1. Suppose $k$ is surjective.
     - 5.1.2. Note 3 is in the codomain $\mathbb{Z}$.
     - 5.1.3. Use the surjectivity of $k$ to find $x \in \mathbb{Z}$ such that $k(x) = 3$.
     - 5.1.4. Note $k(x) = 3 = 2 \times 1 + 1$ is odd.
     - 5.1.5. So $x$ is odd by line 3.2.
     - 5.1.6. Thus $3 = k(x) = 2x - 1$ by the choice of $x$ and the definition of $k$.
     - 5.1.7. Solving gives $x = \frac{3+1}{2} = 2 = 2 \times 1$, which is even.
     - 5.1.8. This contradicts line 5.1.5.
   - 5.2. Hence $k$ is not surjective.

4. Now we show that $k$ is injective.
   - 4.1. Let $x, x' \in \mathbb{Z}$ such that $k(x) = k(x')$.
   - 4.2. Case 1: $k(x)$ is even.
     - 4.2.1. Then both $x$ and $x'$ are even by line 3.1.
     - 4.2.2. So $x = k(x) = k(x') = x'$ by the definition of $k$.
   - 4.3. Case 2: $k(x)$ is odd.
     - 4.3.1. Then both $x$ and $x'$ are odd by line 3.2.
     - 4.3.2. So $2x - 1 = k(x) = k(x') = 2x' - 1$ by the definition of $k$.
     - 4.3.3. Thus $x = x'$.
   - 4.4. Since $k(x)$ is either even or odd, we conclude that $x = x'$ in any case.

# Q4. $f: B \to C$

(a) Suppose $f$ is injective. Show that $g \circ f$ is injective whenever $g$ is an injective function with domain $C$.

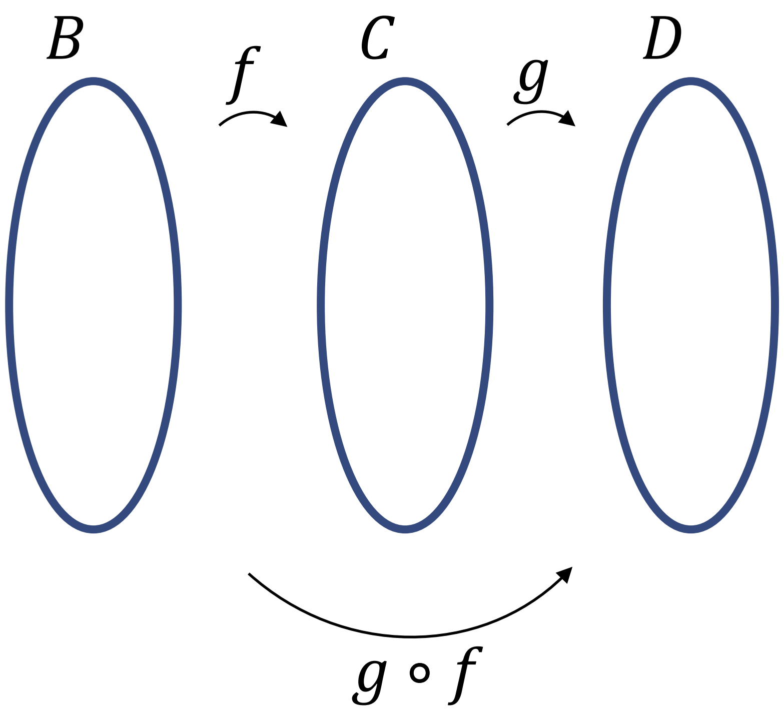
1. Suppose $f$ is injective.
2. Let $g$ be an injective function with domain $C$.
3. Take $x, x' \in B$ s.t. $(g \circ f)(x) = (g \circ f)(x')$.
4. Then $g\big(f(x)\big) = g\big(f(x')\big)$ by the definition of $g \circ f$.
5. $\therefore f(x) = f(x')$ as $g$ is injective.
6. $\therefore x = x'$ as $f$ is injective.



## Q4. $f: B \to C$

(b) Suppose we have a function $g$ with domain $C$ such that $g \circ f$ is injective. Show that $f$ is injective.

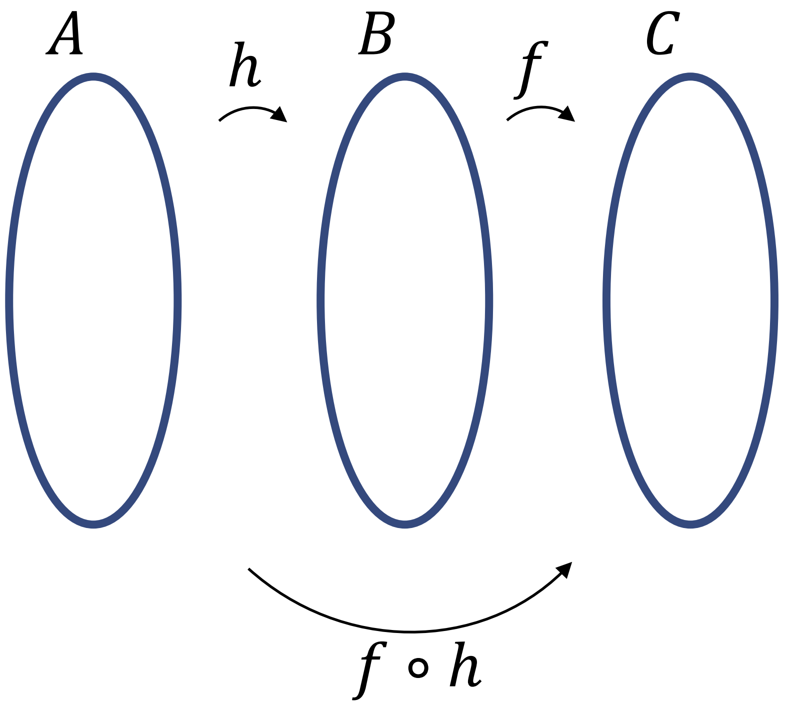
1. Suppose $g$ is a function with domain $C$ such that $g \circ f$ is injective.
2. Let $x, x' \in B$ such that $f(x) = f(x')$.
3. Then $(g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x')$ by definition of $g \circ f$.
4. So $x = x'$ as $g \circ f$ is injective by the choice of $g$.



# Q5. $f: B \to C$

(a) Suppose $f$ is surjective. Show that $f \circ h$ is surjective whenever $h$ is a surjective function with codomain $B$.

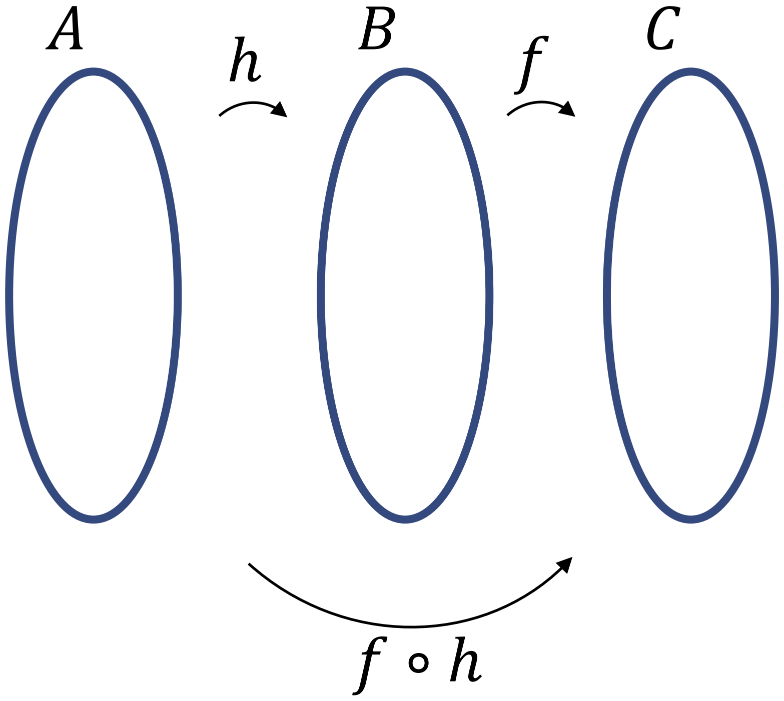
1. Suppose $f$ is surjective.
2. Let $h$ be a surjective function with codomain $B$.
3. Take any $y \in C$.
4. Apply the surjectivity of $f$ to find $x \in B$ such that $y = f(x)$.
5. Apply the surjectivity of $h$ to find $w$ in the domain of $h$ such that $x = h(w)$.
6. Then $y = f(x) = f\big(h(w)\big) = (f \circ h)(w)$ by the definition of $f \circ h$.



# Q5. $f: B \to C$

(b) Suppose we have a function $h$ with codomain $B$ such that $f \circ h$ is surjective. Show that $f$ is surjective.

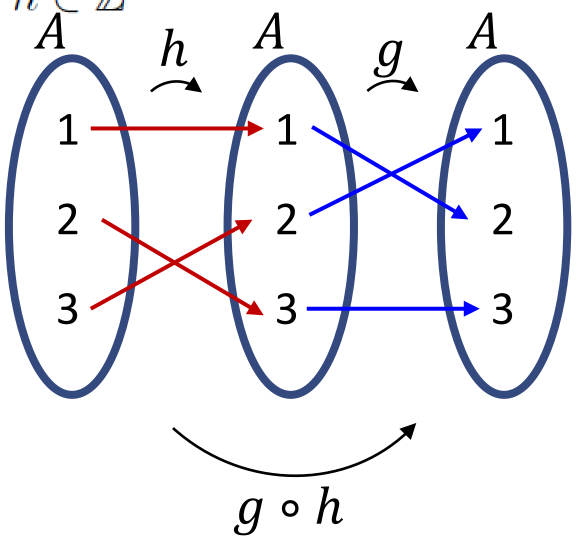
1. Suppose $h$ is a function with codomain $B$ such that $f \circ h$ is surjective.
2. Take any $y \in C$.
3. Apply the surjectivity of $f \circ h$ to find $w$ in the domain of $h$ such that $y = (f \circ h)(w)$.
4. Let $x = h(w)$.
5. Then $x \in B$ and $y = (f \circ h)(w) = f(h(w)) = f(x)$ by the definition of $f \circ h$.



# Q6.

Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that

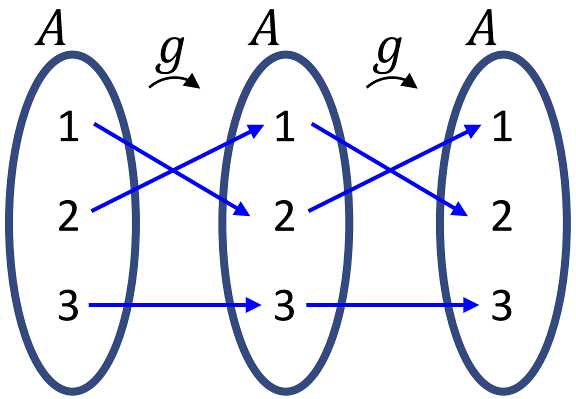
$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$




Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

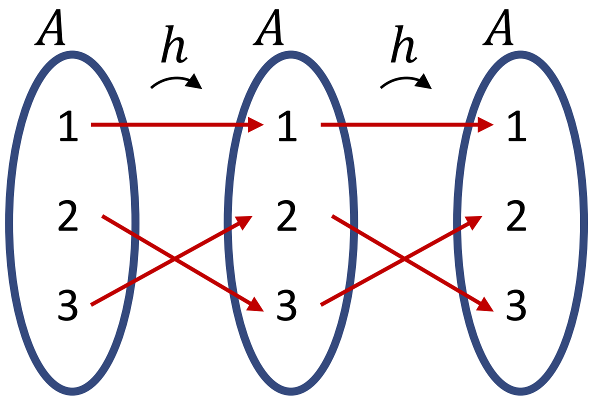
$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise}, \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise}. \end{cases}$$

Find the orders of $g$, $h$, $g \circ h$, and $h \circ g$.




$g \circ g(1) = 1$
$g \circ g(2) = 2$
$g \circ g(3) = 3$

∴ order of $g$ = 2




$h \circ h(1) = 1$
$h \circ h(2) = 2$
$h \circ h(3) = 3$

∴ order of $h$ = 2

# Q6.

Let $A = \{1, 2, 3\}$. The *order* of a bijection $f : A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

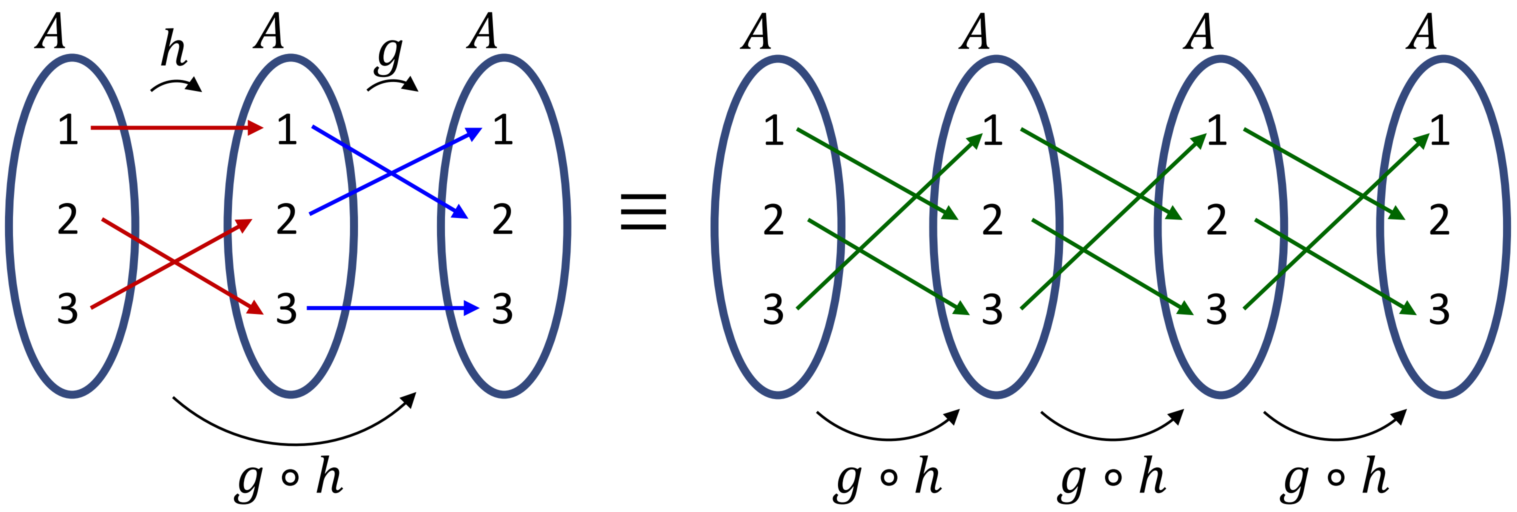
Define functions $g, h : A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Is $(g \circ h) \circ (g \circ h) = \mathrm{id}_A$?

Find the orders of $g$, $h$, $g \circ h$, and $h \circ g$.

Is $(g \circ h) \circ (g \circ h) \circ (g \circ h) = \mathrm{id}_A$?



# Q6.

Let $A = \{1, 2, 3\}$. The *order* of a bijection $f: A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

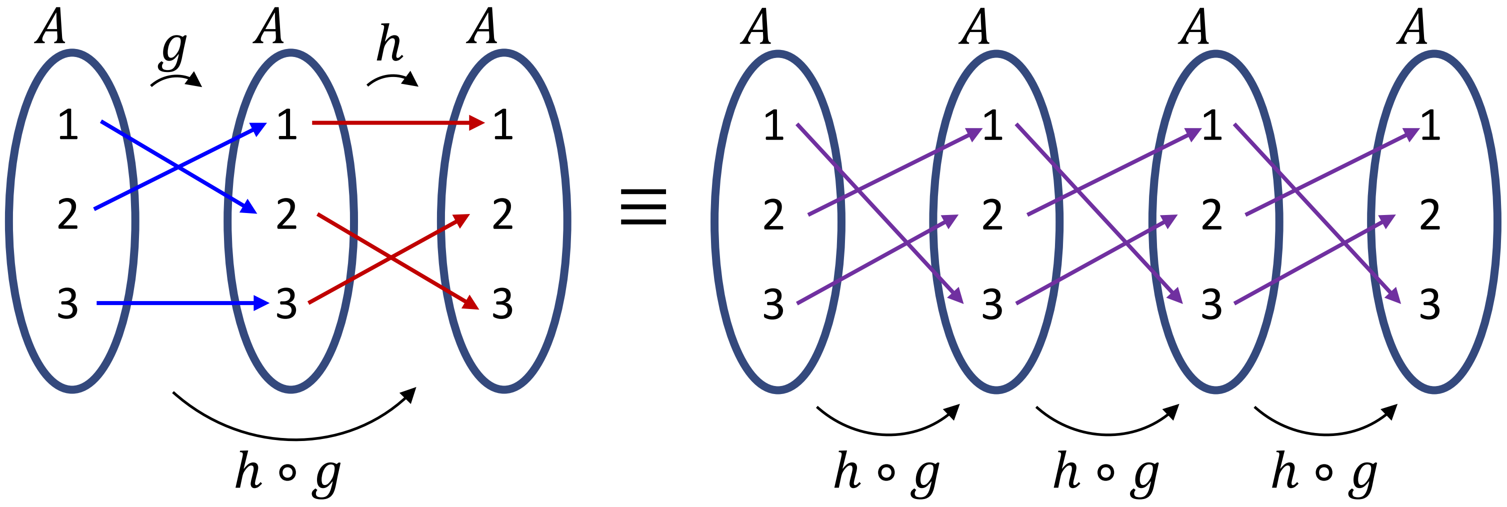
Define functions $g, h: A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

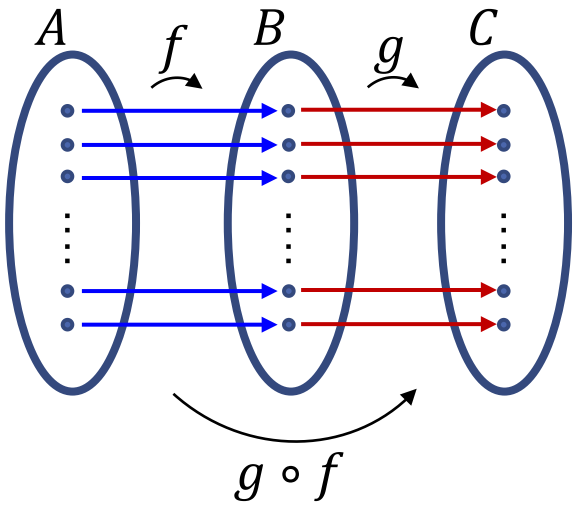
Is $(h \circ g) \circ (h \circ g) = \mathrm{id}_A$?

Find the orders of $g$, $h$, $g \circ h$, and $h \circ g$.

Is $(h \circ g) \circ (h \circ g) \circ (h \circ g) = \mathrm{id}_A$?



## Q7. Let $A, B, C$ be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f: A \to B$ and all bijections $g: B \to C$.




1. For all $x \in B$ and all $z \in C$,
   1.1. $z = (g \circ f)(x)$
   1.2. $\Leftrightarrow$ $z = g\big(f(x)\big)$ by the definition of $g \circ f$
   1.3. $\Leftrightarrow$ $g^{-1}(z) = f(x)$ by the definition of $g^{-1}$
   1.4. $\Leftrightarrow$ $f^{-1}(g^{-1}(z)) = x$ by the definition of $f^{-1}$
   1.5. $\Leftrightarrow$ $(f^{-1} \circ g^{-1})(z) = x$ by the definition of $f^{-1} \circ g^{-1}$
2. So $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ by the definition of $(g \circ f)^{-1}$

## Q8. $A, B$ are sets. The *graph* of a function $f: A \to B$ is

$$\{(x, y) \in A \times B : y = f(x)\}$$

(a) Assuming $A \neq \emptyset$, find a subset $S \subseteq A \times B$ that cannot be the *graph* of any function $A \to B$.

We claim that $S = \emptyset$ works.

1. We prove this by contradiction.
   1.1 Suppose $f: A \to B$ whose graph is $S$.
   1.2 Since $A \neq \emptyset$, it has an element, say $x$.
   1.3 Then $\big(x, f(x)\big) \in S$ by the definition of graphs.
   1.4 This contradicts the fact that $S = \emptyset$.
2. So $S$ cannot be the graph of any function $A \to B$.

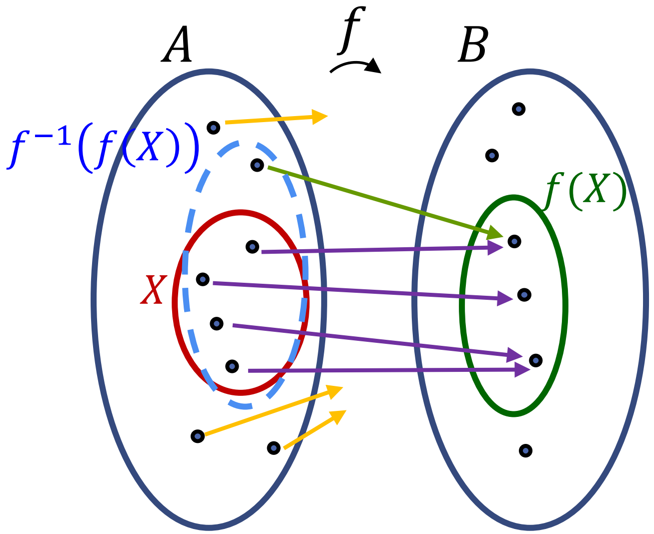
Q8. (b) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if $\forall x \in \mathrm{A}\ \exists!\, y \in B\ (x, y) \in S$

1. ("Only if")
   1.1. Suppose $S$ is the graph of a function $f: A \to B$.
   1.2. Pick any $x \in A$.
   1.3. ("Existence part")
      1.3.1. $f(x) \in B$ as $B$ is the codomain of $f$.
      1.3.2. As $S$ is the graph of $f$, we know $\big(x, f(x)\big) \in \mathrm{S}$.
      1.3.3. So $(x, y) \in \mathrm{S}$ for some $y \in B$.
   1.4. ("Uniqueness part")
      1.4.1. Let $y \in B$ such that $(x, y) \in S$.
      1.4.2. As $S$ is the graph of $f$, we know $y = f(x)$.
   1.5. So there is a unique $y \in B$ s.t. $(x, y) \in S$.

2. ("If")
   2.1. Suppose $\forall x \in \mathrm{A}\ \exists!\, y \in B\ (x, y) \in S$.
   2.2. Define $f: A \to B$ by setting $f(x)$ to be the unique $y \in B$ s.t. $(x, y) \in S$, for every $x \in A$.
   2.3. This function is well-defined by line 2.1.
   2.4. By this definition of $f$, for all $(x, y) \in A \times B$, $(x, y) \in S \Leftrightarrow y = f(x)$.
   2.5. So $S$ is indeed the graph of $f$.

## Q9. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(a) Compare the sets $X$ and $f^{-1}(f(X))$.
Is one always a subset of the other?




$X \subseteq f^{-1}(f(X))$ is always true.

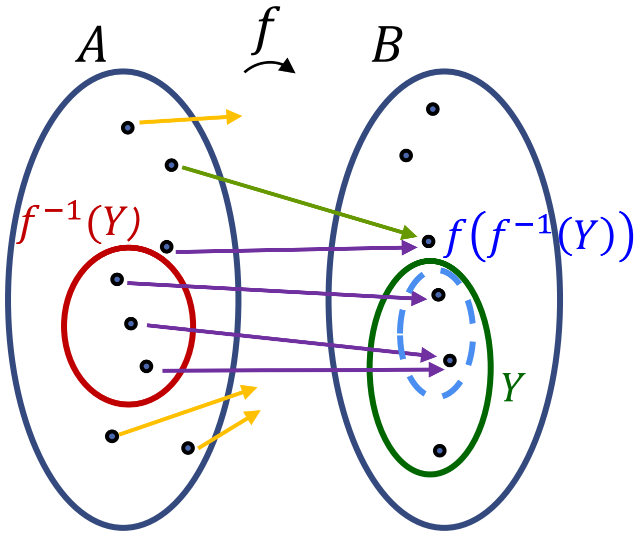
1. Let $x \in X$.
2. Then $f(x) \in f(X)$ by the definition of $f(X)$.
3. So $x \in f^{-1}(f(X))$ by the definition of $f^{-1}(f(X))$.

Possible that $f^{-1}(f(X)) \nsubseteq X$.

1. Consider $f: \{-1, 1\} \to \{0\}$ where $f(-1) = 0 = f(1)$, and $X = \{1\}$.
2. Note $f(X) = \{f(1)\} = \{0\}$.
3. Since $f(-1) = 0$, we know $-1 \in f^{-1}(\{0\}) = f^{-1}\big(f(X)\big)$.
4. As $-1 \notin \{1\} = X$, we deduce that $f^{-1}\big(f(X)\big) \nsubseteq X$.

# Q9. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(b) Compare the sets $Y$ and $f(f^{-1}(Y))$.
Is one always a subset of the other?




## $f(f^{-1}(Y)) \subseteq Y$ is always true.

1. Take any $y \in f^{-1}(f(Y))$.
2. There is some $x \in f^{-1}(Y)$ s.t. $y = f(x)$
   by the definition of $f^{-1}\left(f(Y)\right)$.
3. As $x \in f^{-1}(Y)$, we get $y' \in Y$ which makes $y' = f(x)$.
4. Since $f$ is a function, $y = f(x) = y' \in Y$.

## Possible that $Y \not\subseteq f(f^{-1}(Y))$.

1. Consider $f: \{0\} \to \{-1, 1\}$ where $f(0) = 1$, and $Y = \{-1\}$.
2. Note that no $x \in \{0\}$ makes $f(X) = -1$.
3. So $f^{-1}(Y) = \emptyset$ by the definition of $f^{-1}(Y)$.
4. This entails $f\left(f^{-1}(Y)\right) = \emptyset \not\supseteq \{-1\} = Y$.

THE END