On the singular values of matrices with high displacement rank

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Nov. 6, 2017

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Matrices with low (A,B)-displacement rank

real Vandermonde
Cauchy
Pick
Löwner
real pos. def. Hankel

Large scale dynamical systems
(reduced order models)

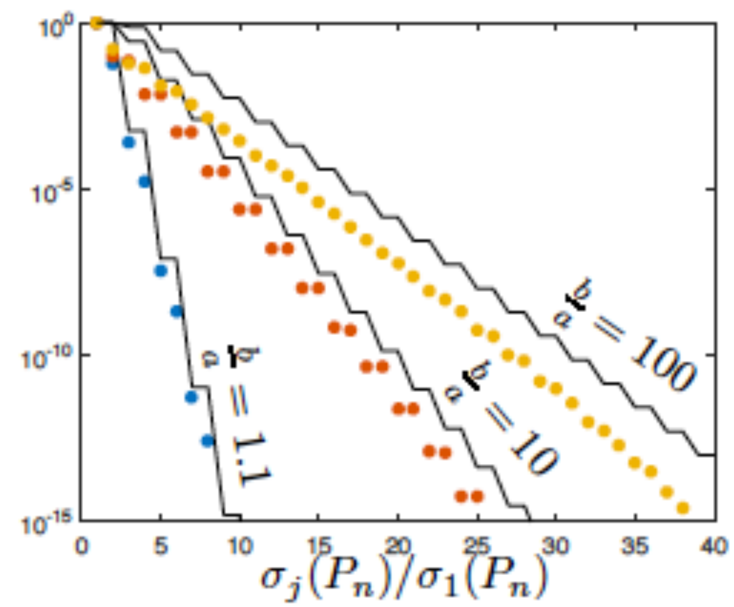
$$AX + XA^T = BB^T$$

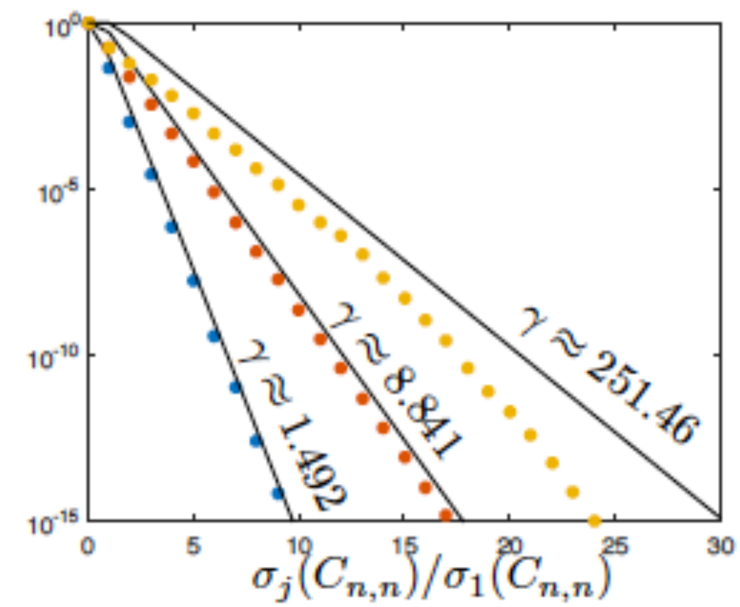
Poisson's equation

$$u_{xx} + u_{yy} = f,\ X_{i,j} = u(i/n, j/n)$$

Sylvester matrix equation

$$AX - XB = F$$
$$AX - XB = UV^*$$
$$X \in \mathbb{C}^{m\times n}, U \in \mathbb{C}^{m\times \rho}, V \in \mathbb{C}^{n\times \rho}$$
$$(A, B)\text{-displacement rank} = \rho$$





$$\boxed{\sigma_{j\rho+1}(X) \le C\mu^{-j}\|X\|_2}$$

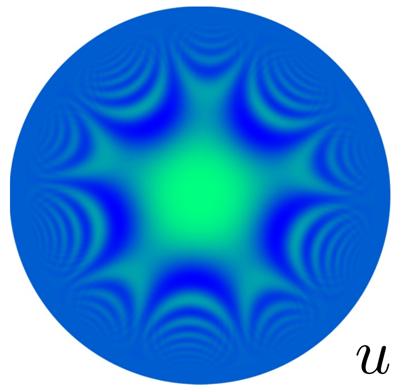
(Beckermann & Townsend, 2017), (Antoulas, Sorenson & Zhou, 2002), (Sabino, 2008), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002),
(Druskin, Knizhnerman & Simonici, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Kressner & Sirkovic, 2015),
(W., Townsend, & Wright, 2017), (Townsend & Fortunato, 2017)

Matrices with high (A,B)-displacement rank

Poisson's equation with smooth right-hand sides

$$u_{xx} + u_{yy} = f$$

$$D_{xx}X - XD_{yy} = F$$



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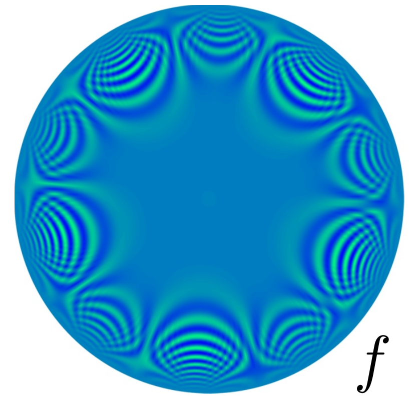

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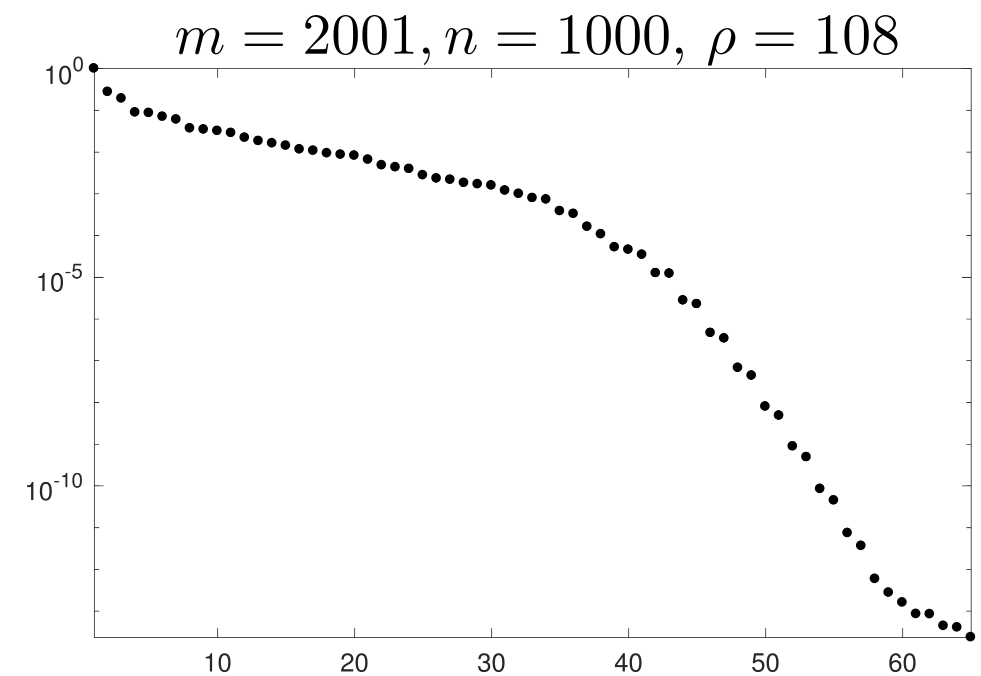
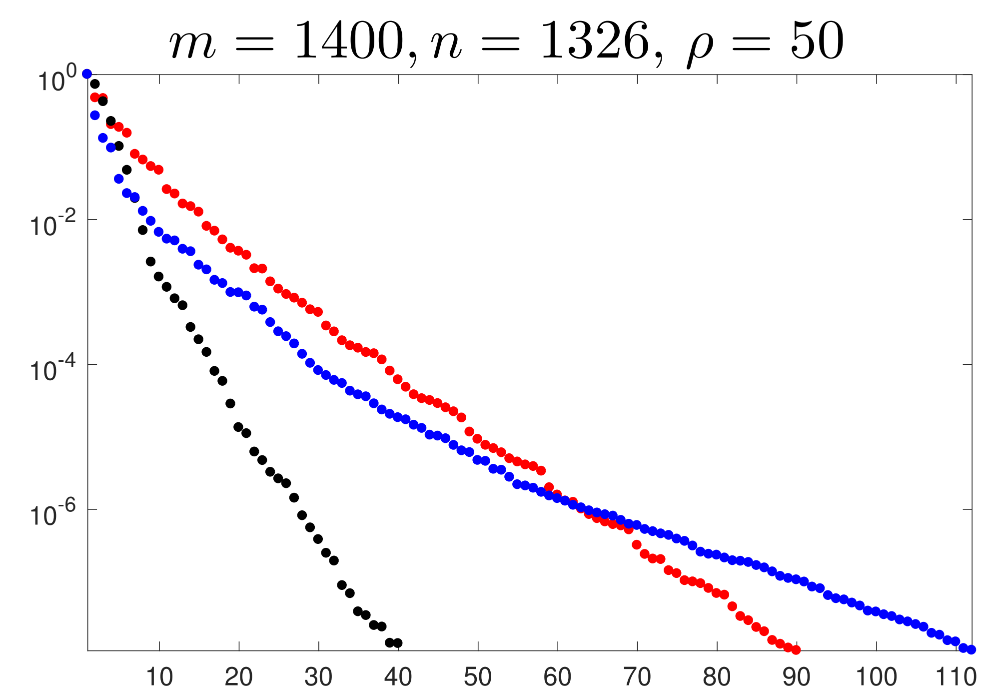


2D (and d-dimensional) Vandermonde

$$\left[\ 1 \mid x \mid y \mid x^2 \mid xy \mid y^2 \mid x^3 \mid x^2y \mid xy^2 \mid y^3 \mid \cdots \mid x^{n-1} \mid \cdots \mid y^{n-1}\ \right]$$

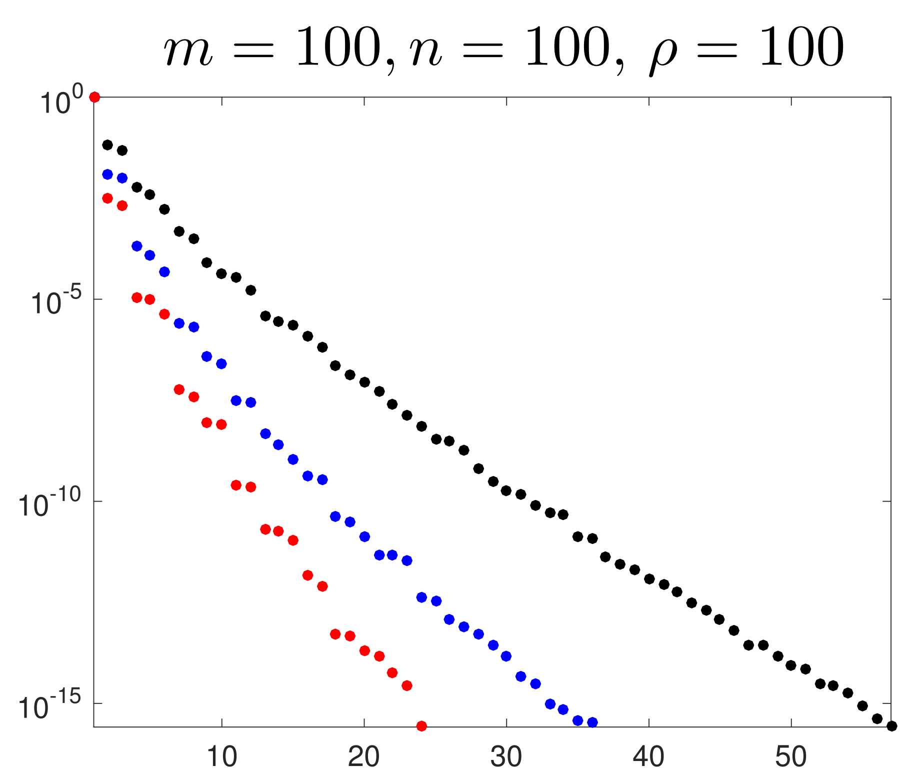


J. Sethna

K. Quinn

Structured matrices

$$(C_2)_{j,k} = \frac{1}{|z_j - w_k|^2}$$



Matrices with high (A,B)-displacement rank

$$AX - XB = F$$

Standard assumption: $\mathrm{rank}(F)$ is small. $\Rightarrow$ $\boxed{\sigma_{j\rho+1}(X) \leq C\mu^{-j}\|X\|_2}$

Our assumption: the singular values of F decay rapidly. $\Rightarrow$?

I. efficient construction of low rank approximations to X

II. Bounds on singular values of X

III. elliptic PDE solvers:
Spectral accuracy, optimal complexity + low rank approximation

The factored ADI algorithm

$AX - XB = F \qquad A({\color{red}ZDY^*}) - ({\color{red}ZDY^*})B = USV^* \qquad (\text{S of size } \rho \times \rho)$

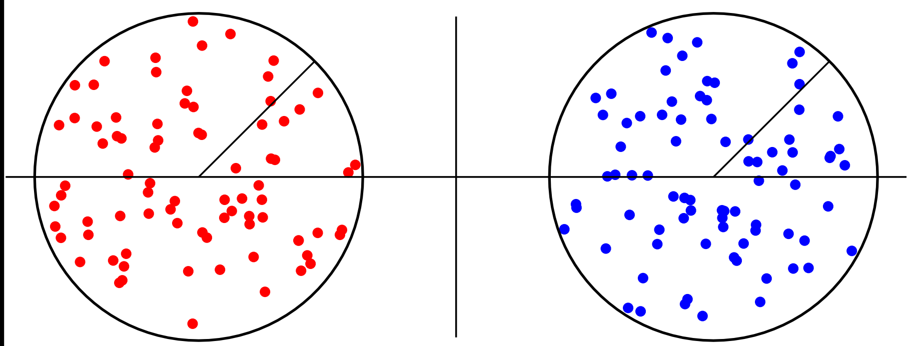
$$Z^{(k)} = \left[\ \hat{Z}^{(1)}\ \middle|\ \hat{Z}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Z}^{(k)}\ \right], \quad \begin{cases} \hat{Z}^{(1)} = (A - \beta_1 I)^{-1} US, \\ \hat{Z}^{(i+1)} = (A - \alpha_i I)(A - \beta_{i+1} I)^{-1} Z^{(i)} \end{cases}$$

$$Y^{(k)} = \left[\ \hat{Y}^{(1)}\ \middle|\ \hat{Y}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Y}^{(k)}\ \right], \quad \begin{cases} \hat{Y}^{(1)} = (B^* - \alpha_1 I)^{-1} V, \\ \hat{Y}^{(i+1)} = (B^* - \beta_i I)(B^* - \alpha_{i+1} I)^{-1} Y^{(i)} \end{cases}$$

$$D^{(k)} = \mathrm{diag}\left((\beta_1 - \alpha_1) I_\rho, \cdots, (\beta_k - \alpha_k) I_\rho\right)$$

$$X^{(k)} = Z^{(k)} D^{(k)} {Y^{(k)}}^*$$

"ADI-friendly" spectra



$\lambda(A)$ $\qquad$ $\lambda(B)$

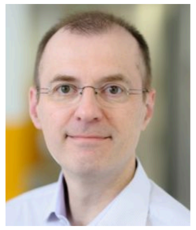
(1) Can choose k, $\{\alpha_j\}_{j=1}^k$, $\{\beta_j\}_{j=1}^k$ so that

$$\|X - X^{(k)}\|_2 \leq C\mu^{-k} \|X\|_2$$

(2) $X^{(k)}$ is of rank at most $k\rho$

$$(1) + (2) \implies \sigma_{k\rho+1}(X) \leq C\mu^{-k}\|X\|_2$$

(Y. I. Zolotarev)



(G. Starke)

Key Observation:

$X^{(k)}$ is just one of many possible low rank approximations generated by fADI.

(Beckermann & Townsend, 2017), (Lu & Wachspress, 1991), (Benner, Truhar & Li, 2009), (Li and White, 2002), (Starke, 1992), (Zolotarev, 1877)

A modification of fADI

$$A(ZDY^*) - (ZDY^*)B = USV^* \qquad \text{(S of size } \rho \times \rho)$$

X can be written as a sum of rank 1 ADI terms:

$$Z^{(k)} = \left[\begin{array}{ccc|ccc|c|ccc} \mathbf{z}_{11} & \cdots & \mathbf{z}_{\rho 1} & \mathbf{z}_{12} & \cdots & \mathbf{z}_{\rho 2} & \cdots & \mathbf{z}_{1k} & \cdots & \mathbf{z}_{\rho k} \end{array}\right]$$

$$Y^{(k)} = \left[\begin{array}{ccc|ccc|c|ccc} \mathbf{y}_{11} & \cdots & \mathbf{y}_{\rho 1} & \mathbf{y}_{12} & \cdots & \mathbf{y}_{\rho 2} & \cdots & \mathbf{y}_{1k} & \cdots & \mathbf{y}_{\rho k} \end{array}\right]$$

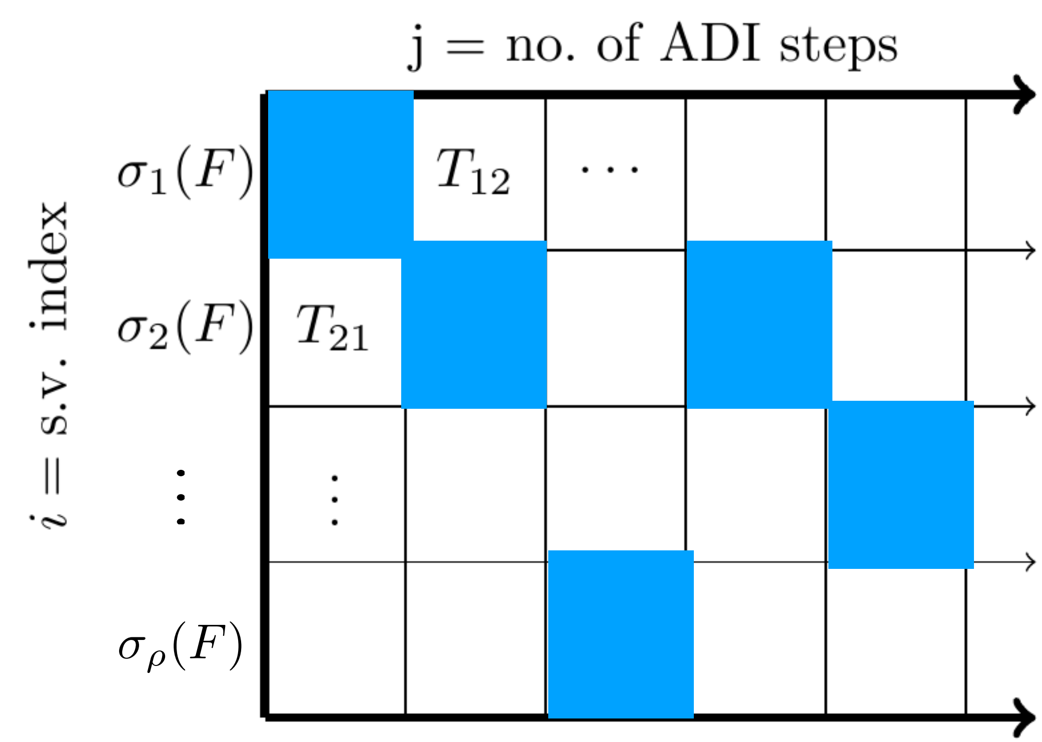
$$D^{(k)} = \operatorname{diag}\left((\beta_1 - \alpha_1)I_\rho, \cdots, (\beta_k - \alpha_k)I_\rho\right)$$

$$\boxed{ZDY^* = \sum_{i=1}^{\rho} \sum_{j=1}^{\infty} \underbrace{d_{ij}\mathbf{z}_{ij}\mathbf{y}_{ij}^*}_{T_{ij}}}$$

Choose any K terms to construct a rank K approximation to X.

Which choice is best?

$$\|X - \tilde{X}^{(K)}\|_2 \leq \sum_{T_{ij} \notin \tilde{X}^{(K)}} \|T_{ij}\|_2$$



Key Observation:

$$\|T_{ij}\|_2 \approx C\sigma_i(F)\mu^{-(j-1)}$$

$$\begin{bmatrix} \mathbf{z}_{11} & \cdots & \mathbf{z}_{\rho 1} \end{bmatrix} = (A - \beta_1 I)^{-1} \begin{bmatrix} \sigma_1(F)\mathbf{u}_1 & \cdots & \sigma_\rho(F)\mathbf{u}_\rho \end{bmatrix}$$

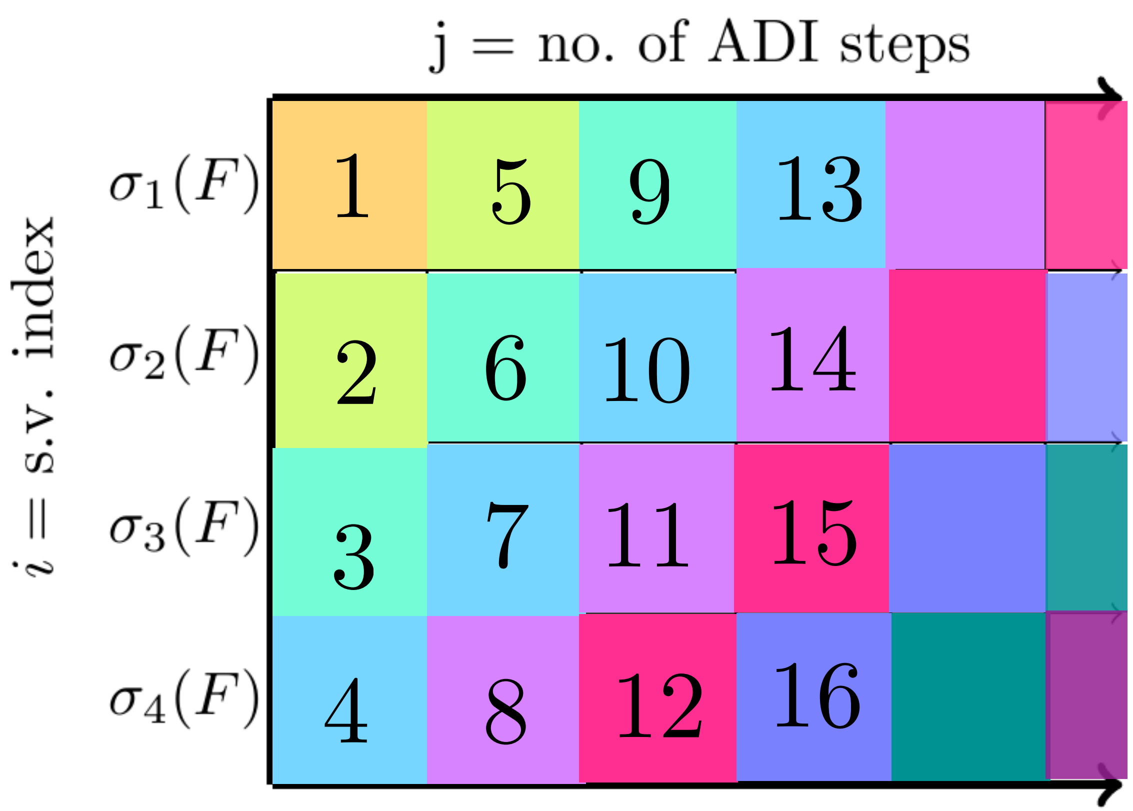
$$\begin{bmatrix} \mathbf{z}_{12} & \cdots & \mathbf{z}_{\rho 2} \end{bmatrix} = (A - \alpha_1 I)^{-1}(A - \beta_2 I) \begin{bmatrix} \mathbf{z}_{11} & \cdots & \mathbf{z}_{\rho 1} \end{bmatrix}$$

A modification of fADI

$$A(ZDY^*) - (ZDY^*)B = USV^* \qquad \text{(S of size } \rho \times \rho)$$

<u>Example:</u>

Suppose that $\sigma_{i+1}(F) \le C\mu^{-i}$, so that $\|T_{ij}\|_2 \approx C\mu^{-(i+j-2)}$



fADI approximant:

$$\|X - X^{(k)}\|_2 \le \varepsilon \|X\|_2$$

$$\mathrm{rank}(X^{(k)}) \le k^2$$

A modification of fADI

$$A(ZDY^*) - (ZDY^*)B = USV^* \qquad \text{(S of size } \rho \times \rho\text{)}$$

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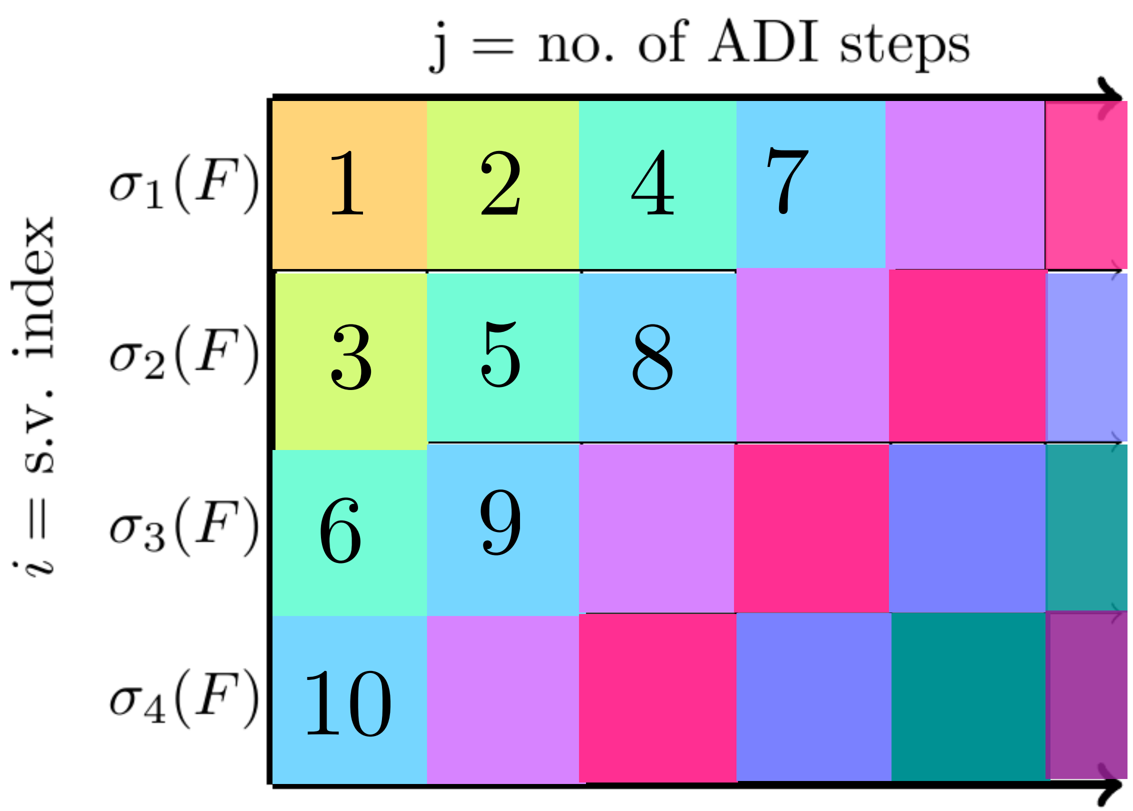


fADI approximant:

$$\|X - X^{(k)}\|_2 \leq \varepsilon \|X\|_2$$

$$\mathrm{rank}(X^{(k)}) \leq k^2$$

best choice:

$$\|X - \tilde{X}^{(k)}\|_2 \leq \varepsilon \|X\|_2$$

$$\mathrm{rank}(\tilde{X}^{(k)}) \leq k(k+1)/2$$

Generalization : Explicit bounds when singular values of F decay geometrically

Automation : FI-ADI algorithm (works with any approximate SVD of F)

Example: Bounding singular values

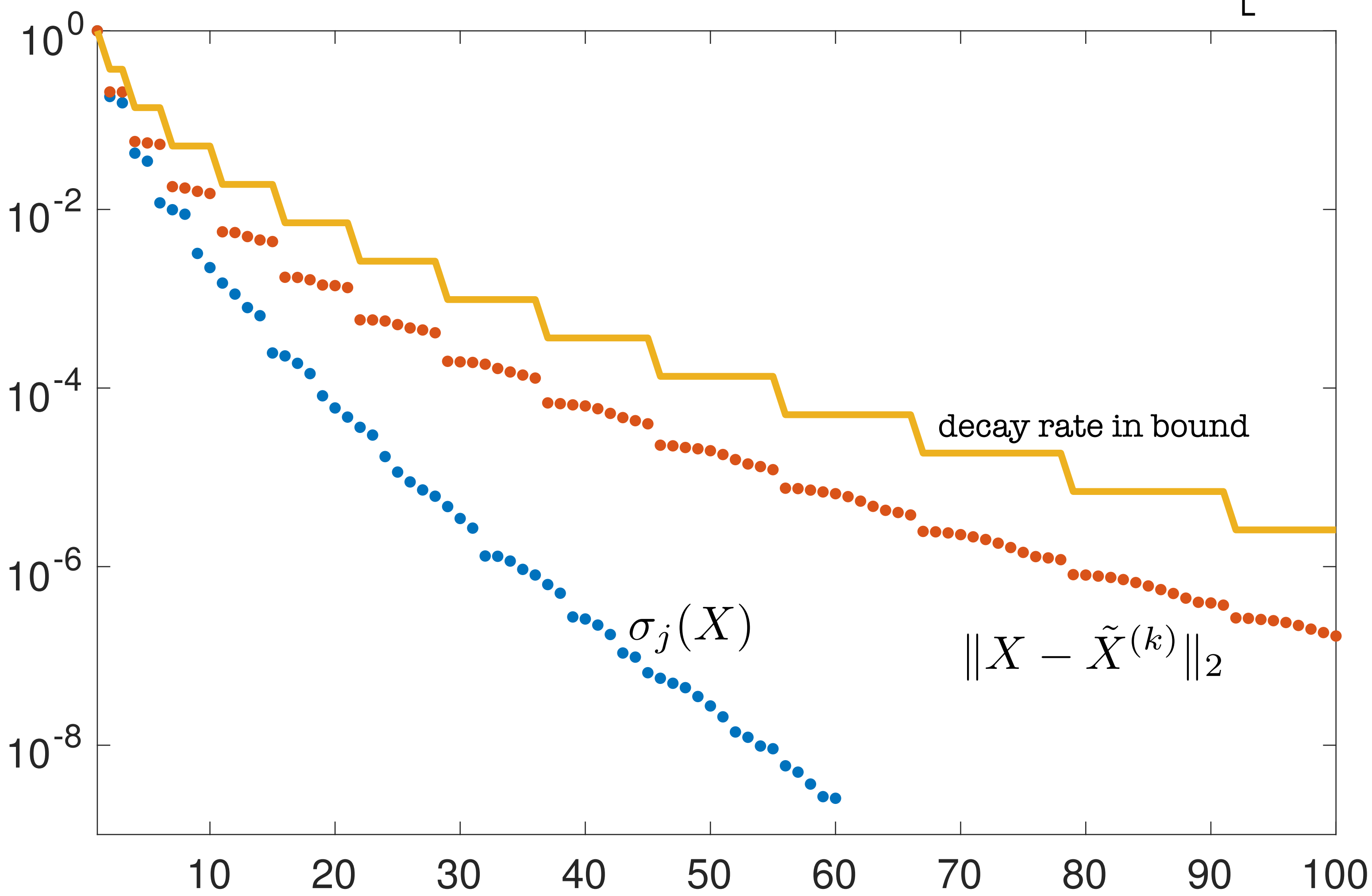
Theorem: (Townsend & W.) Let $\lambda(A) \subset E, \lambda(B) \subset -E$, with $E = \{z \in \mathbb{C} : |z - z_0| \leq \eta\}$, and A, B normal matrices. If $AX - XB = F$ and the singular values of F decay at the same rate as the ADI error, then

$$\sigma_{t+1}(X) \leq C_E \mu^{-(\sqrt{8t+1}-1)/2} \|X\|_2, \qquad t = k(k+1)/2 < n.$$

Example (1):

$$(C_2)_{j,k} = \frac{1}{|z_j - w_k|^2}$$

$$\begin{bmatrix} \bar{z}_1 & & & \\ & \bar{z}_2 & & \\ & & \ddots & \\ & & & \bar{z}_m \end{bmatrix} C_2 - C_2 \begin{bmatrix} \bar{w}_1 & & & \\ & \bar{w}_2 & & \\ & & \ddots & \\ & & & \bar{w}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{z_1 - w_1} & \frac{1}{z_1 - w_2} & \cdots \\ \frac{1}{z_2 - w_1} & \ddots & \\ \vdots & & \end{bmatrix}$$



Example: Bounding singular values

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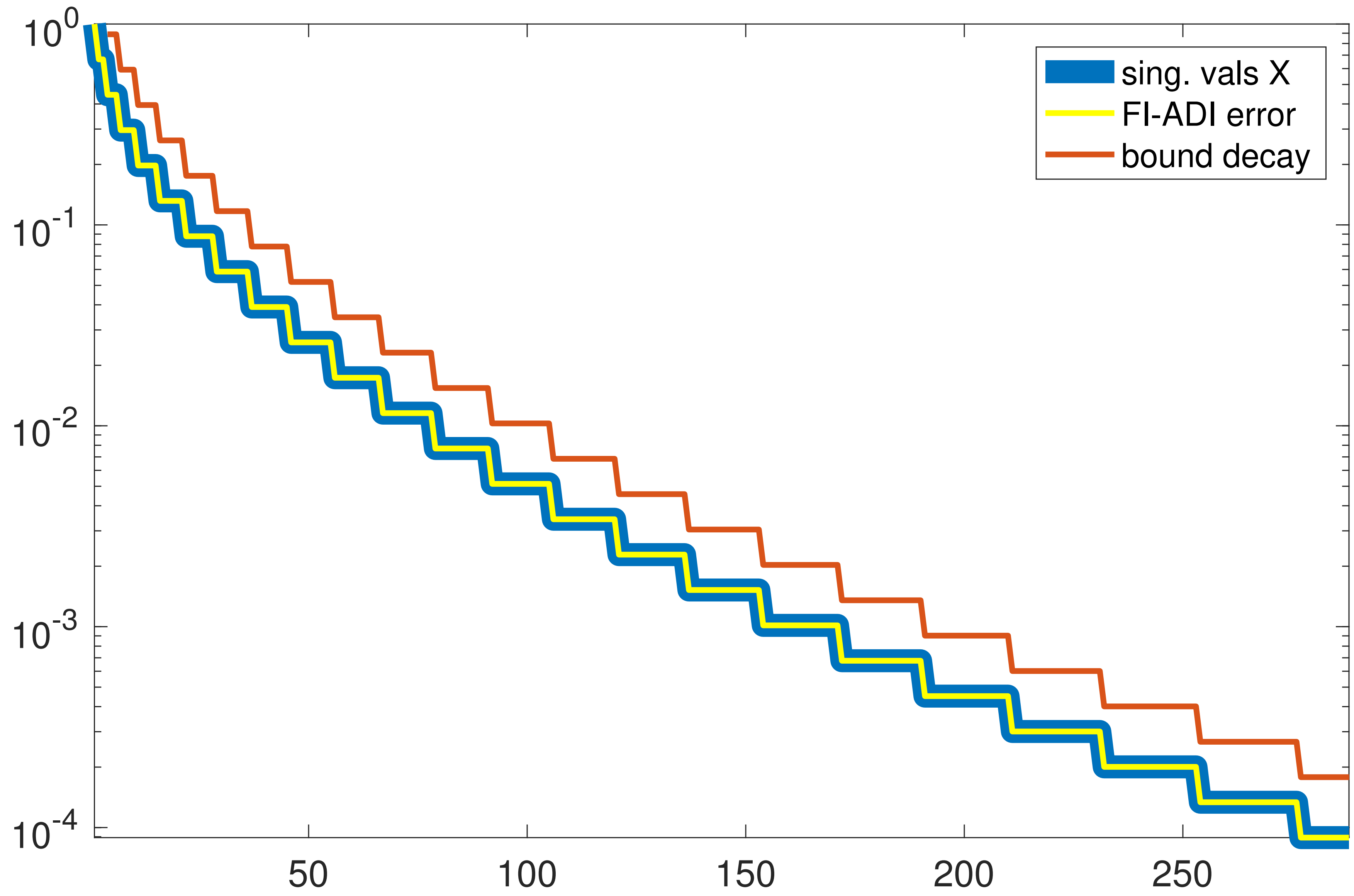
$$\sigma_{t+1}(X) \leq C_E \mu^{-(\sqrt{8t+1}-1)/2} \, \|X\|_2, \qquad t = k(k+1)/2 < n.$$

Example (2): FI-ADI error $\|X - \tilde{X}^{(k)}\|_2$ is provably a sharp bound.



Application: low rank Poisson solver

Example (1): The "ADI model problem"

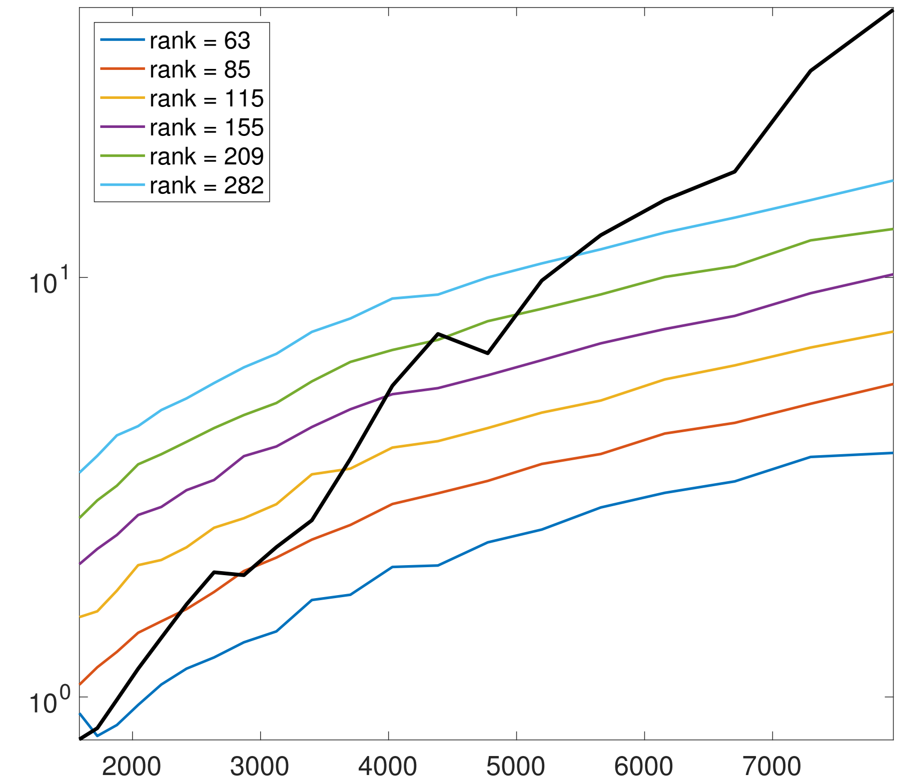
FD discretization on $\Omega = [-1,1]^2$

$$\Delta u = f, \quad u(\pm 1, 0) = u(0, \pm 1) = 0$$


D. Fortunato

Example (2):

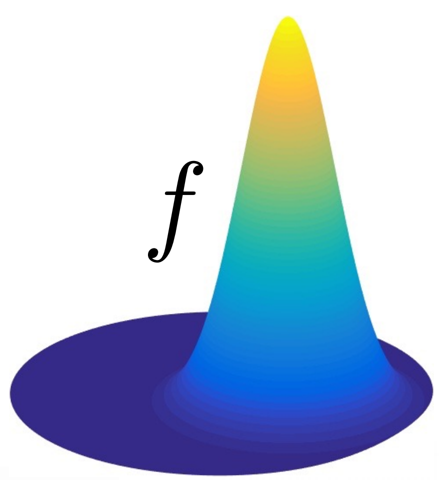
ADI-friendly spectral discretizations

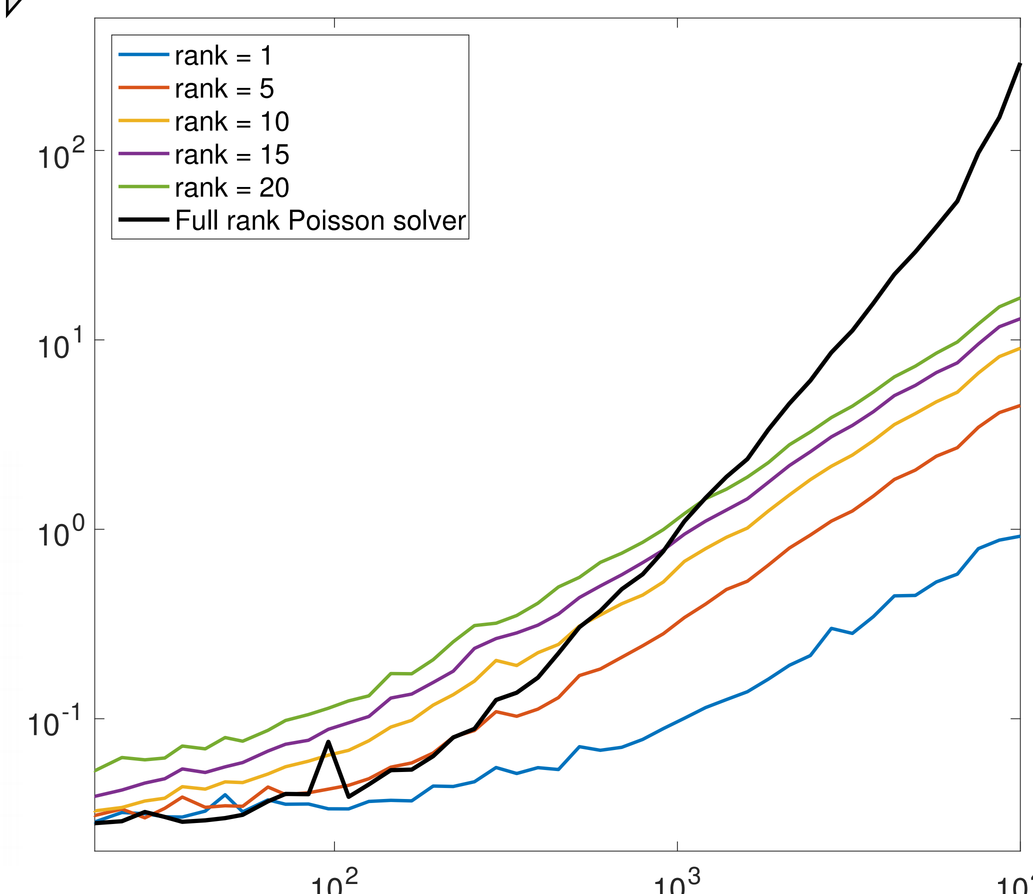


Optimal complexity solvers

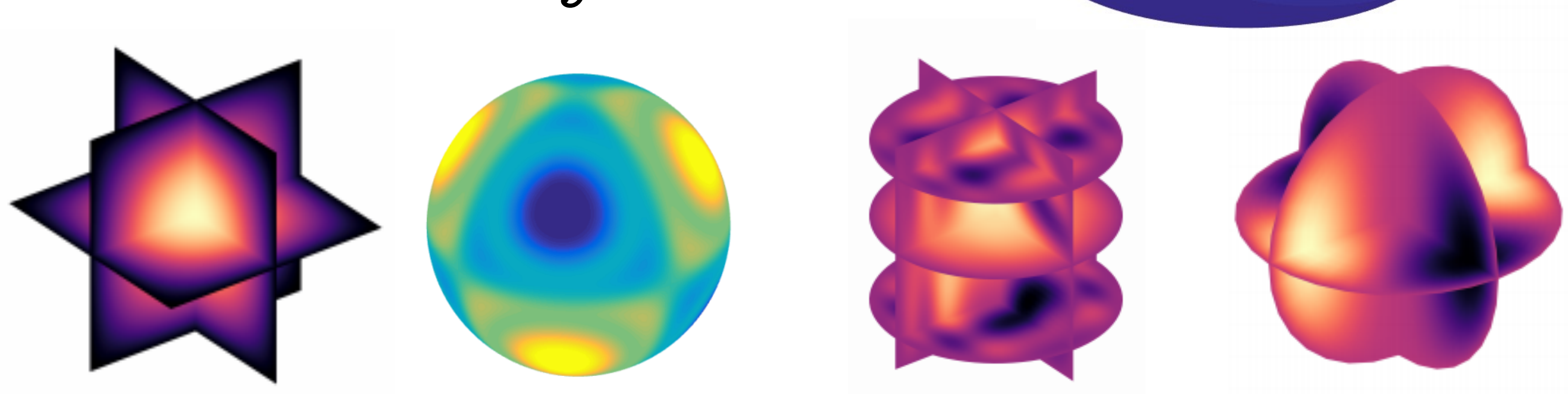
Ultraspherical spectral discretization on unit disk

$$\Delta u = f, \quad u(\theta, \rho)\Big|_{\rho=1} = 0$$





works on a variety of domains:



(Fortunato & Townsend, 2017), (Olver & Townsend, 2013), (W., Townsend & Wright, 2017)