Computing with rational approximations (with applications in signal processing)

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Joint work with

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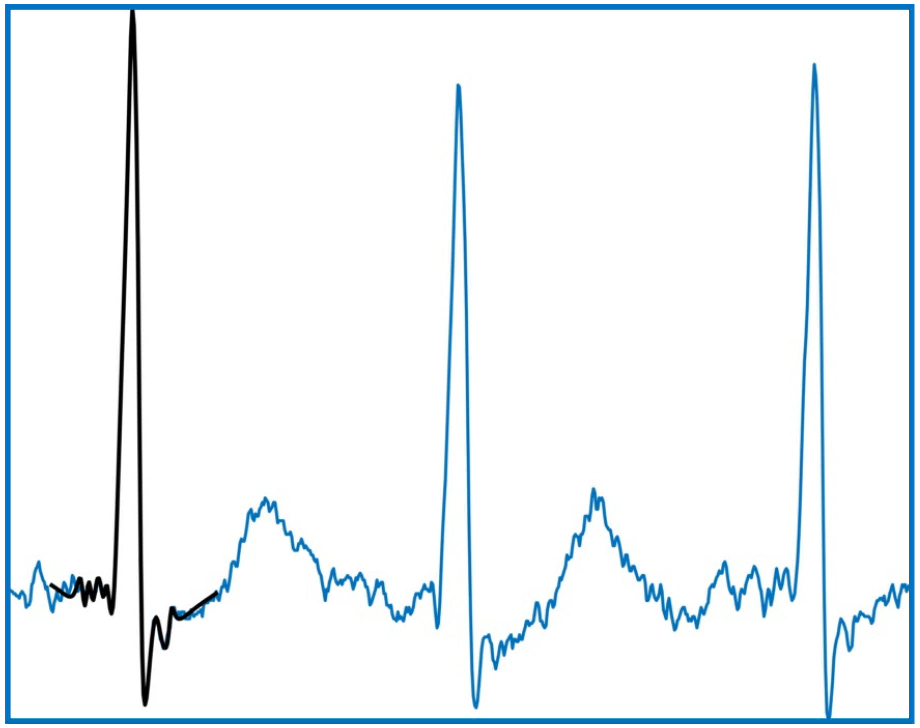
Computing with rational functions

<u>Signal reconstruction:</u> geophone/hydrophone/ECG monitoring,extrapolation, filtering

[Belykin & Monzón (2009), Moitra (2018), Fridli, Lósci & Schipp (2012), Vetterli, Marziliano, & Blu (2002)]



<u>Feature extraction:</u> abnormality detection, classification, parameter recovery

[Gilián (2016), Moitra (2018) , Peter & Plonka (2013), Potts & Tasche (2013)]

<u>Modeling/PDEs/simulation:</u> advection in discontinuous media, shock or cusp development, Painlevé equations

[Tee & Trefethen (2006), Beylkin, Haut & Monzon (2013) , Fasondini, Fornberg & Weideman (2017)]

<u>Numerical linear algebra:</u> nonlinear eigenvalue problems, evaluating matrix functions, solving matrix equations, reduced order modeling

[Lietaert, Pèrez, Vandereycken & Meerbergen (2018), Berljafa & Güttel (2015, 2017), Nakatsukasa & Gawlik (2018), Antoulus, Sorensen & Gugercin (2000), many, many more…]

Computing with trigonometric rationals

<u>GOAL:</u> Develop software tools for working adaptively with trigonometric rational approximations to periodic functions.

- "Near-optimal" rational approximations
- No guiding parameters required
- Basic tools: algebraic operations (sum, product), calculus, filtering, rootfinding/polefinding, visualization

- Works with noisy data, undersampled/resolved data, missing data.

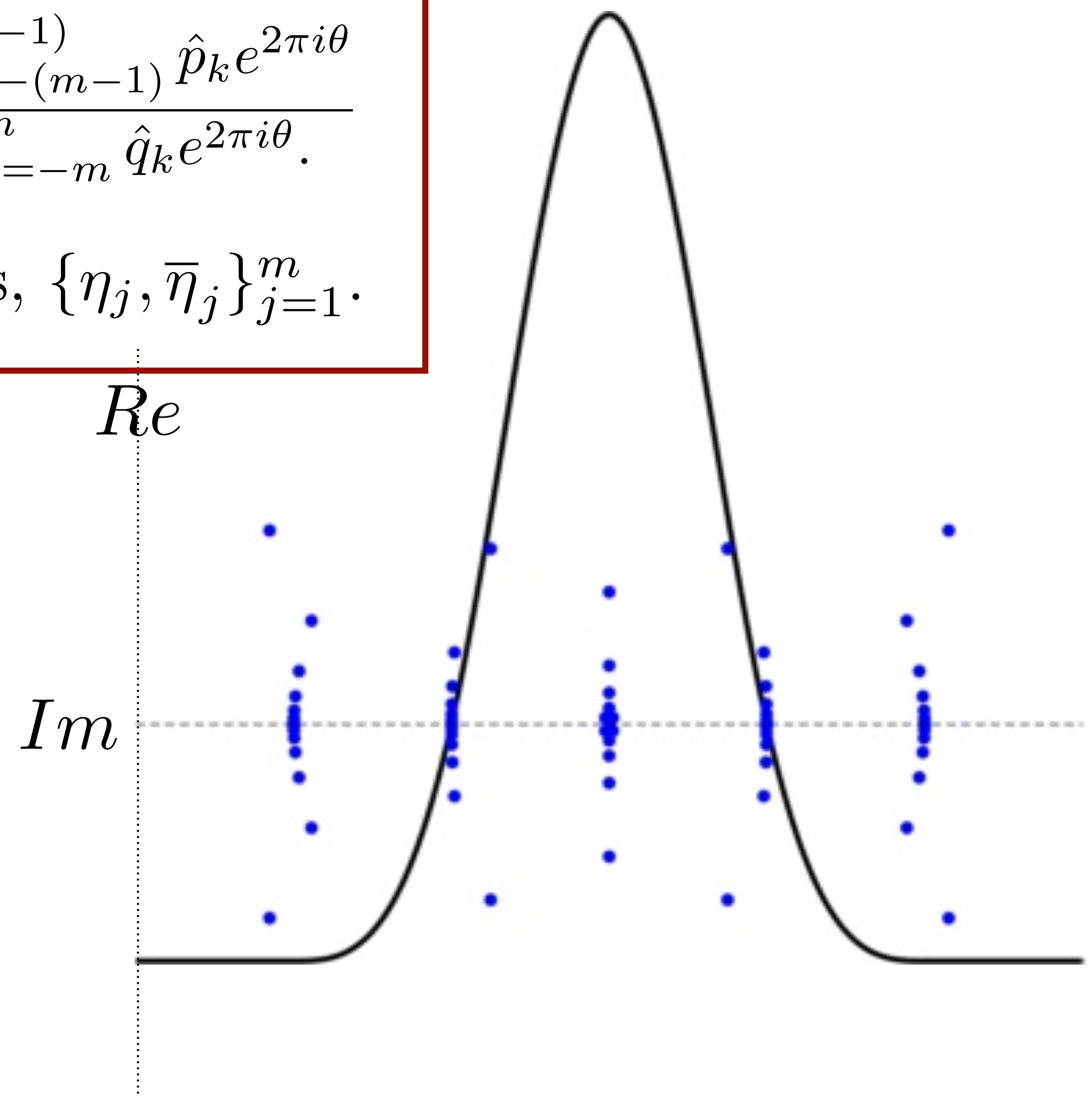
Approximate Prony (Fourier space)	$\longrightarrow$ + $\longleftarrow$	Trigonometric AAA (value space)

Trigonometric rational functions in value space

f is periodic, real-valued, continuous on $[0,1)$, $\int_0^1 f(\theta)\mathrm{d}\theta = 0$.

We seek $r_m \approx f$, where

- $r_m(\theta) = \dfrac{p_{m-1}(\theta)}{q_m(\theta)} = \dfrac{\sum_{k=-(m-1)}^{(m-1)} \hat{p}_k e^{2\pi i\theta}}{\sum_{k=-m}^{m} \hat{q}_k e^{2\pi i\theta}}.$
- r_m has $2m$ simple poles, $\{\eta_j, \overline{\eta}_j\}_{j=1}^m$.



Barycentric rational interpolants

(P. Henrici)

(J.P. Berrut)

Trigonometric Barycentric rational interpolants:

$$r(\theta) = \frac{n(\theta)}{d(\theta)} = \frac{\sum_{k=1}^{2M} u_k f_k \mathrm{cst}\,(\pi\theta - \pi\theta_k)}{\sum_{k=1}^{2M} u_k \mathrm{cst}\,(\pi\theta - \pi\theta_k)}, \qquad \mathrm{cst}\theta = \begin{cases} \csc\theta, & M \text{ is odd,} \\ \cot\theta, & M \text{ is even.} \end{cases}$$

Key properties

- interpolant at θ_k : $r(\theta_k) = f_k$
- degree of numerator, denominator controlled by choice of u_k.
- numerically stable evaluation, excellent compression when θ_k, u_k chosen appropriately.

? Barycentric nodes: $(\theta_1, \ldots, \theta_{2M})$
Barycentric weights: $(u_1, \ldots, u_{2M})$
Interpolation values: $(f_1, \ldots, f_{2M})$

➡ Trigonometric AAA

[Berrut (2005) , Berrut & Trefethen (2004), Henrici (1979), Higham (2004), Austin & Xu (2017)]

Trigonometric rational functions in Fourier space

Key observation 1: The Fourier series of r_m can be efficiently represented by a short sum of complex, decreasing exponentials.

If $r_m(\theta) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{2\pi i \ell \theta}$, then for $\ell \geq 0$,

$$c_\ell = s_m(\ell) := \sum_{j=1}^{M} w_j e^{-\lambda_j \ell},$$

where $\lambda_j = 2\pi i \eta_j$, $w_j \neq 0$.

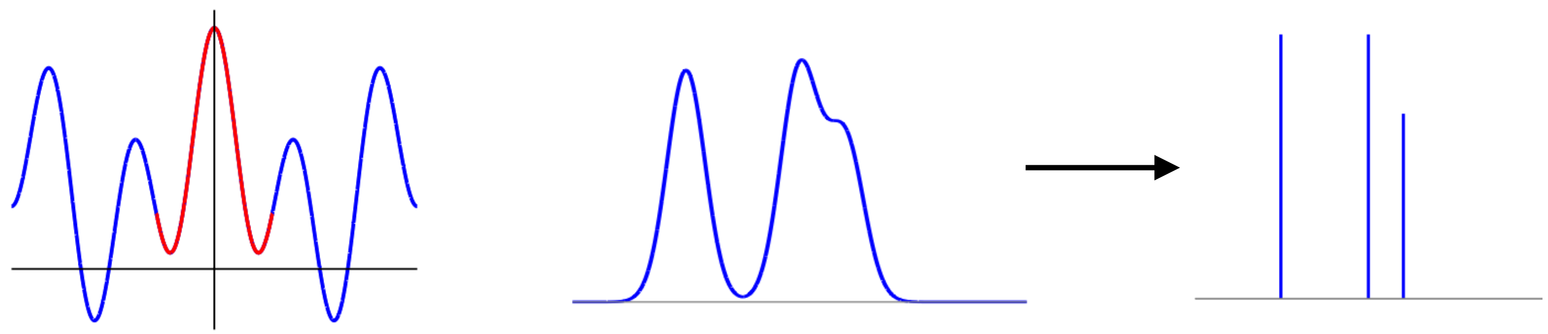
Key observation 2:

- $s_M(\ell)$ can be exactly recovered by observing $(c_0, \ldots, c_{2M+1})$. (Prony's method.)

- $r_m \approx f$ can be constructed by solving the approximate interpolation problem $|\hat{f}_\ell - s_M(\ell)| < \epsilon$, $0 \leq \ell \leq \hat{f}_{N_\epsilon}$. (Approximate Prony's method.)

[Belykin & Monzon (2005, 2009) , Peter & Plonka (2013), Potts & Tasche (2013)]

The superresolution property



$$\boxed{\text{Extrapolation in Fourier space} \iff \text{superresolution in value space}}$$

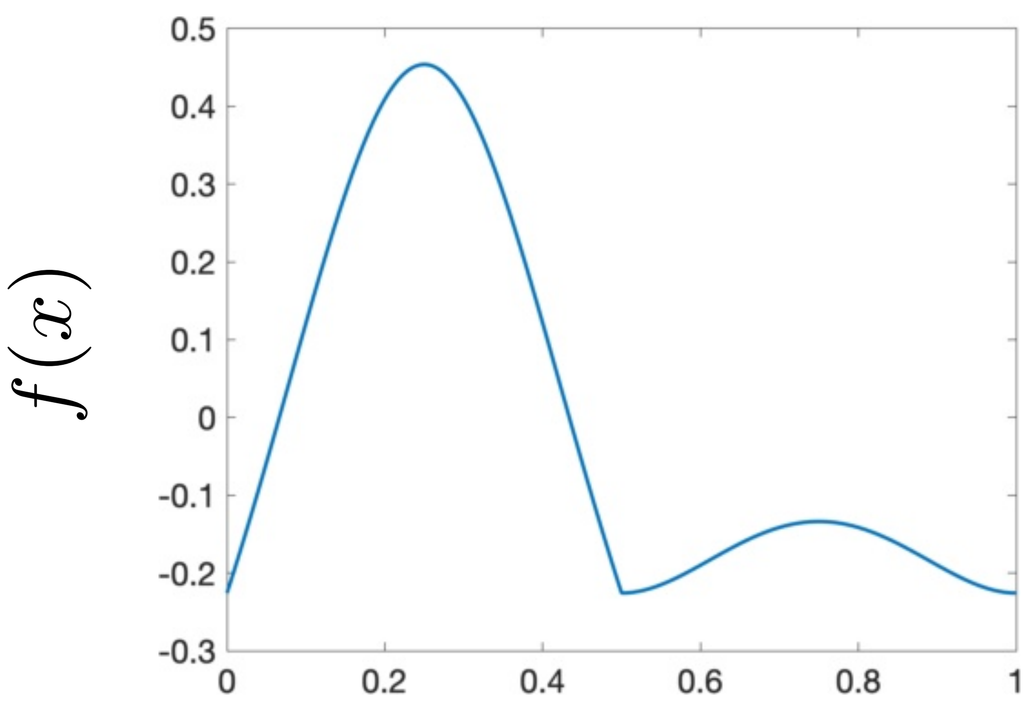
$$\text{Consider } r_M(z) = \sum_{j=1}^{M} \frac{\omega_j z_j}{z - z_j} + \frac{-\overline{\omega_j}/\overline{z}_j}{z - 1/\overline{z}_j}, \text{ where } z_j = e^{-\lambda_j}.$$

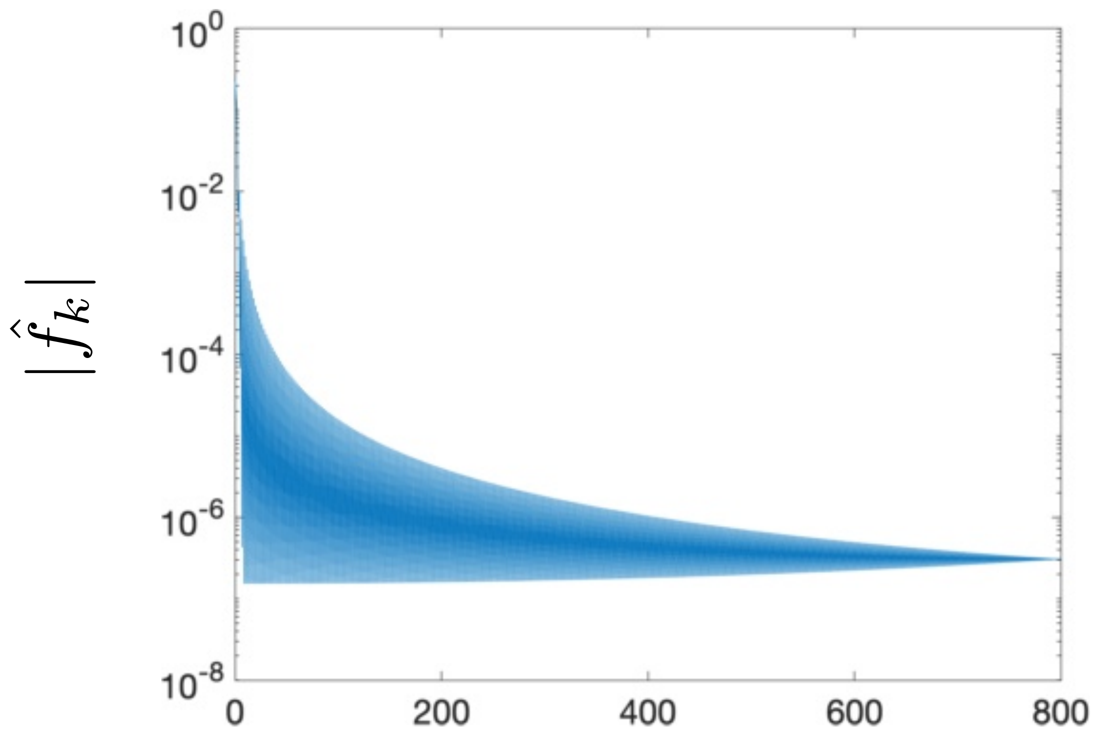
$$2M + 1 \text{ Fourier coefficients of } r_M \to s_M(\ell) = \sum_{j=1}^{M} \omega_j e^{-\lambda_j \ell}.$$

[Image from Candès & Fernandez-Granda (2012)]

AAA+Prony: Rational approximation in practice

Problem : Fourier coefficients decay slowly, sample is underresolved...
How can I construct an exponential sum representation of $r_m \approx f$?





(Fourier space)

s_m

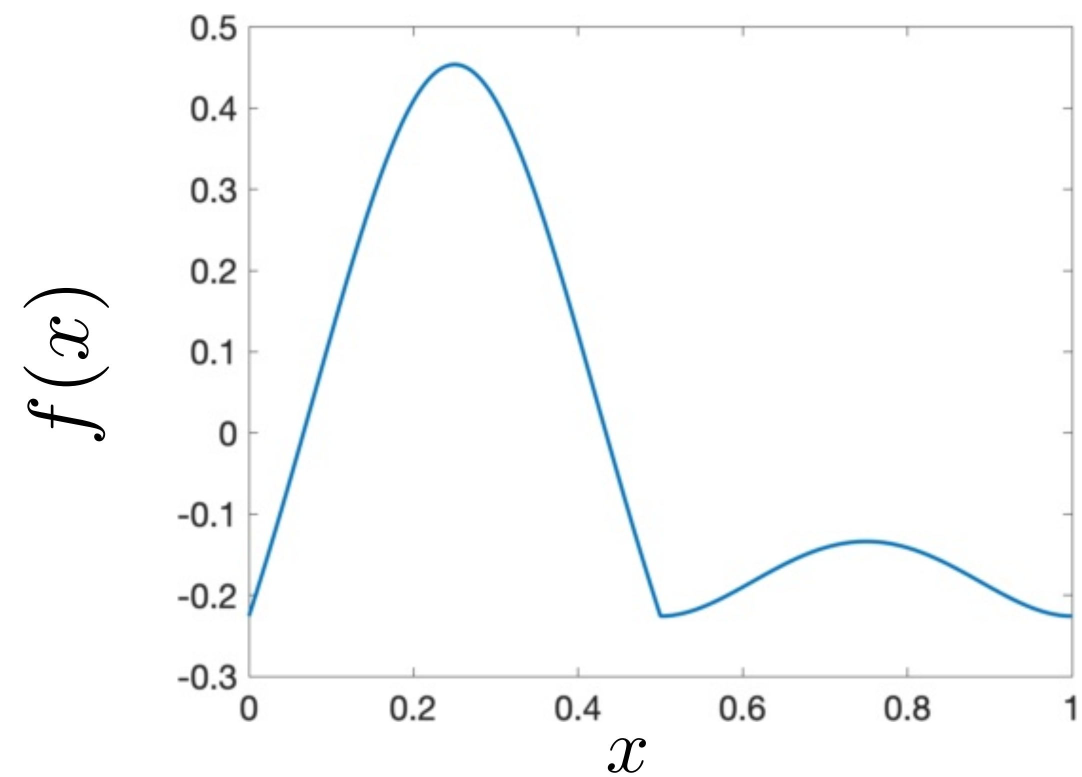
$\mathcal{F}(r_m)$

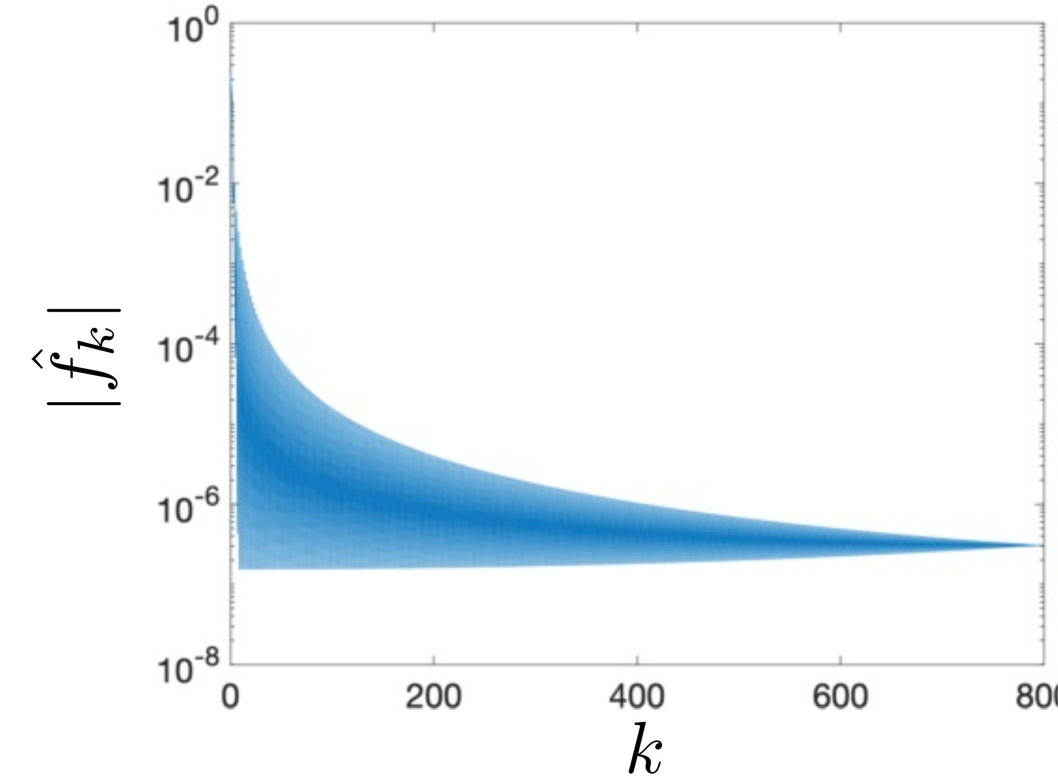
+

(value space)

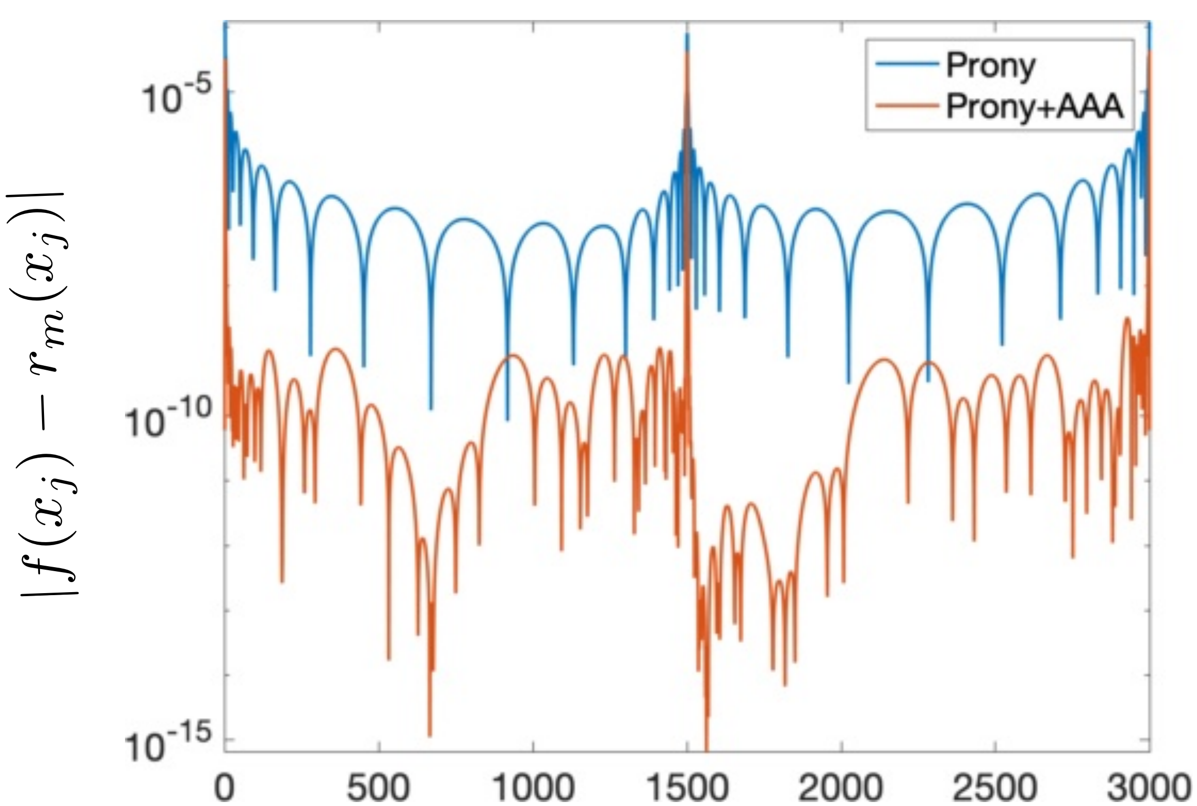
Trigonometric
AAA
r_m

AAA+Prony: Rational approximation in practice

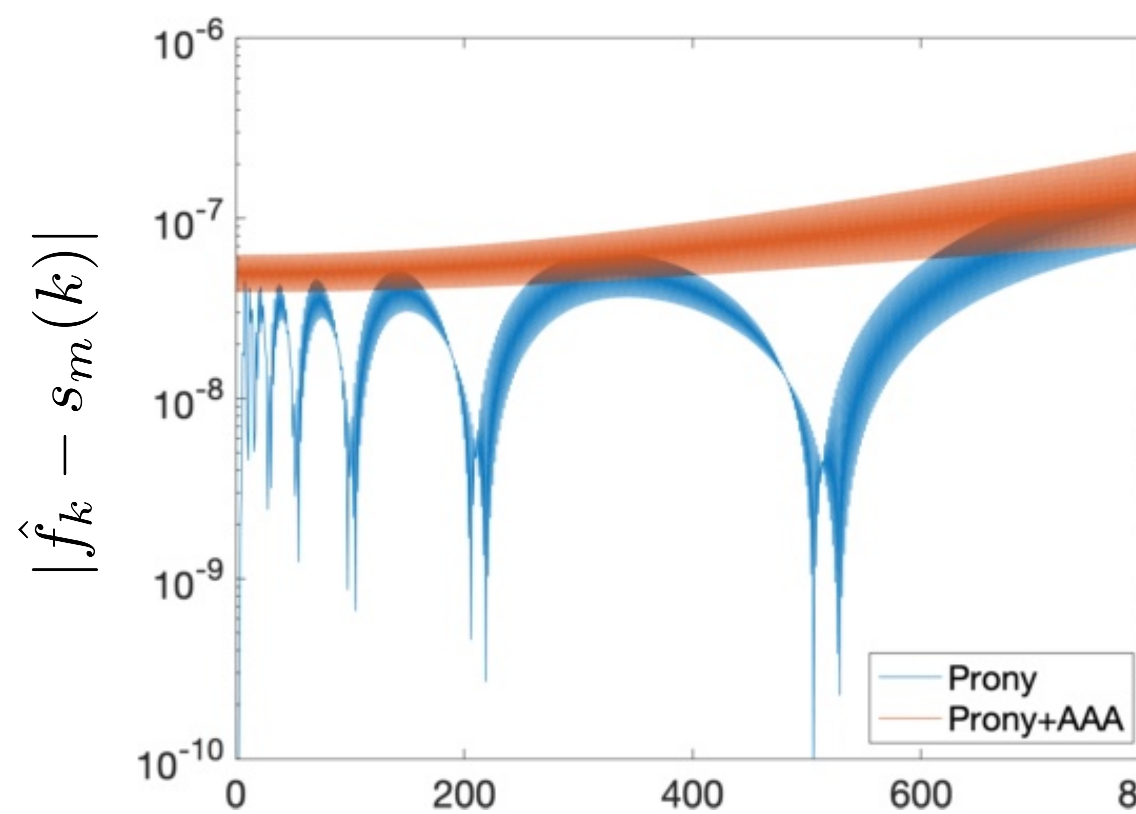




Exp. sum format: value space error



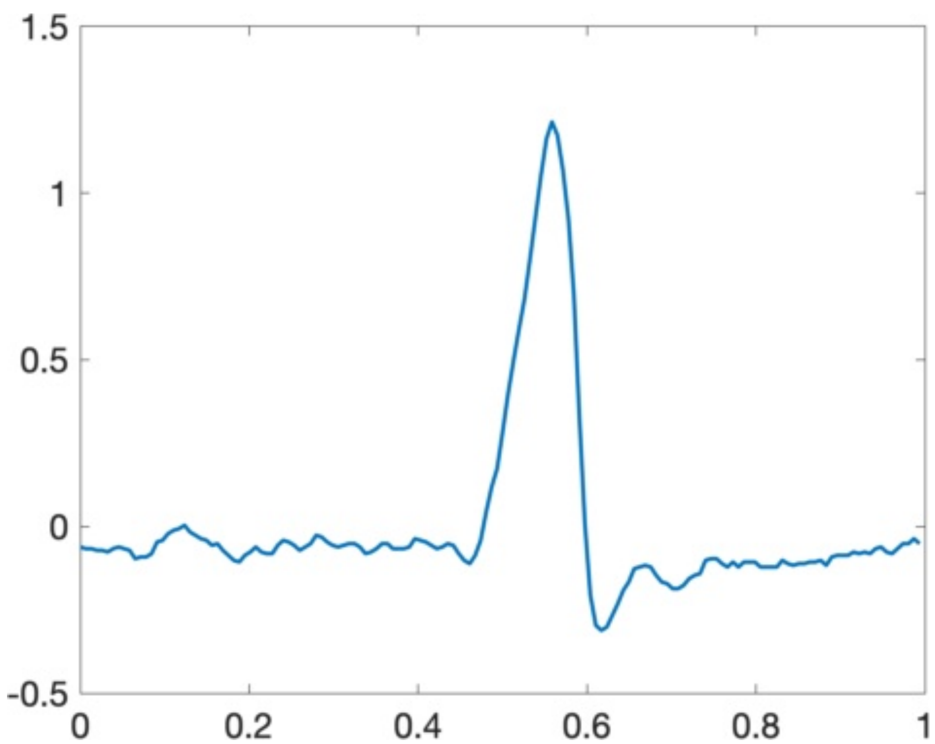
Exp. sum format: Fourier space error



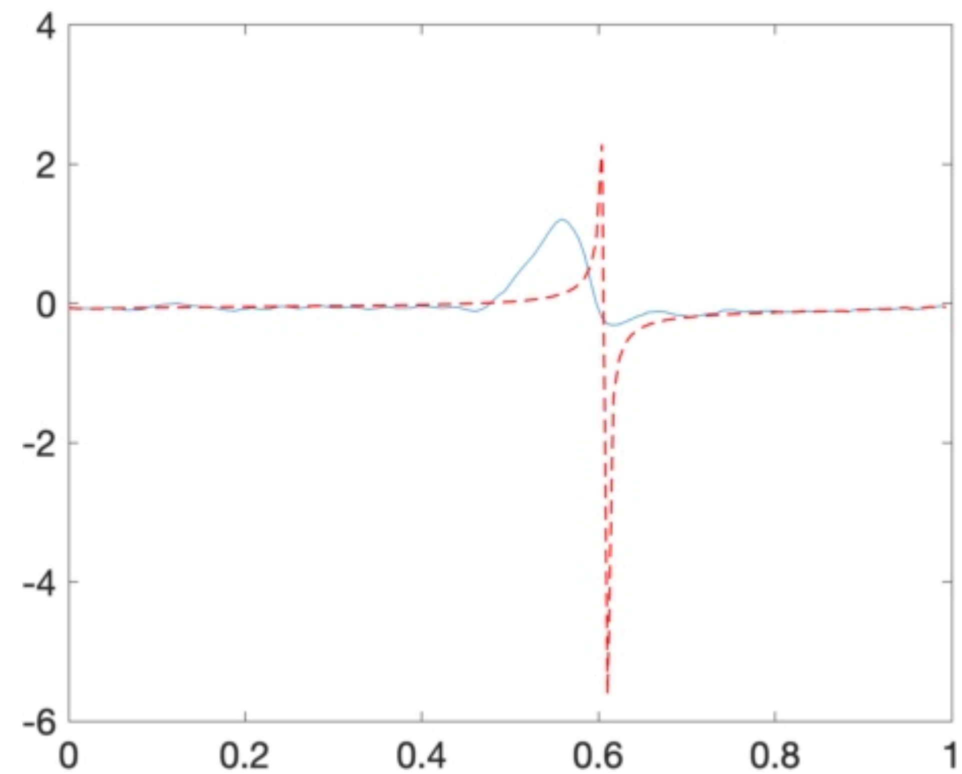
AAA+Prony: Rational approximation in practice

<u>Problem:</u> Noisy data, limited spatial resolution...

How can I construct a barycentric representation of $r_m \approx f$?



Only 154 noisy data points!



<u>Try to find r_m, with trig-aaa, tol $:= 1e^{-2}$</u>.

[Data from <u>http://www.physionet.org/physiobank/database/MIT</u>]

AAA+Prony: Rational approximation in practice

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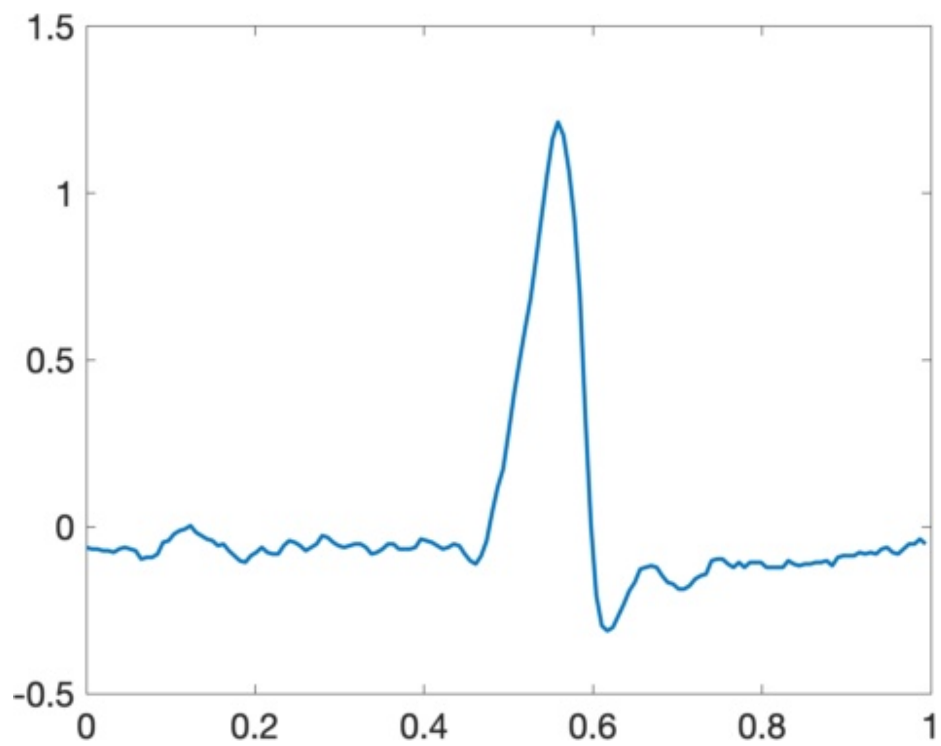

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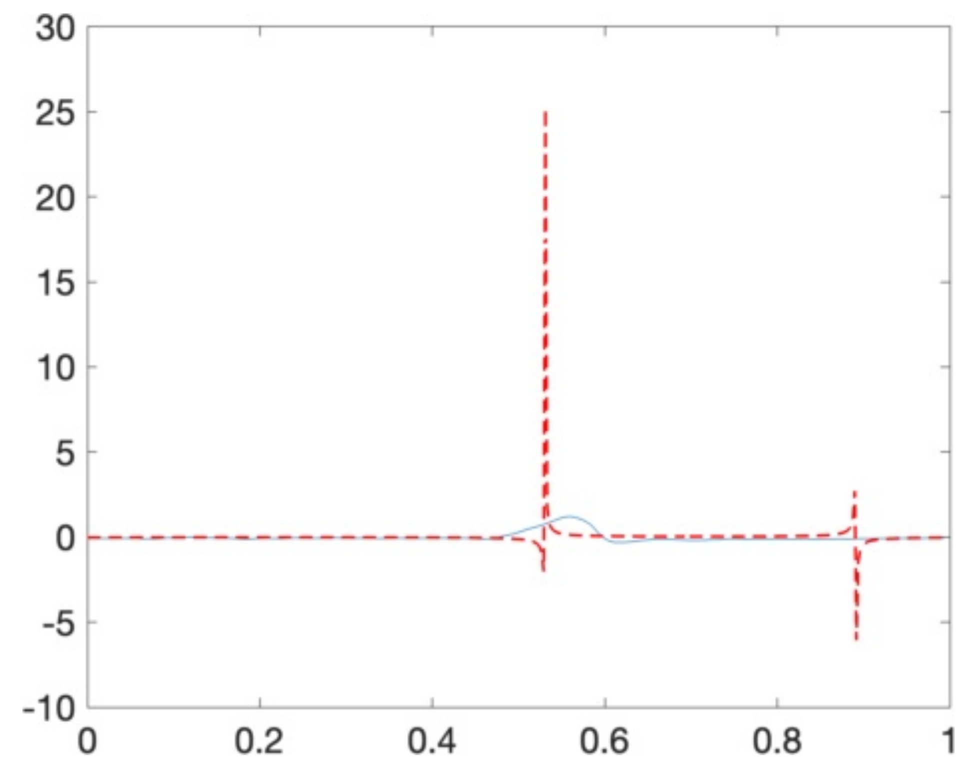

Try to find r_m, with trig-aaa, tol $:= 1e^{-1}$.

AAA+Prony: Rational approximation in practice

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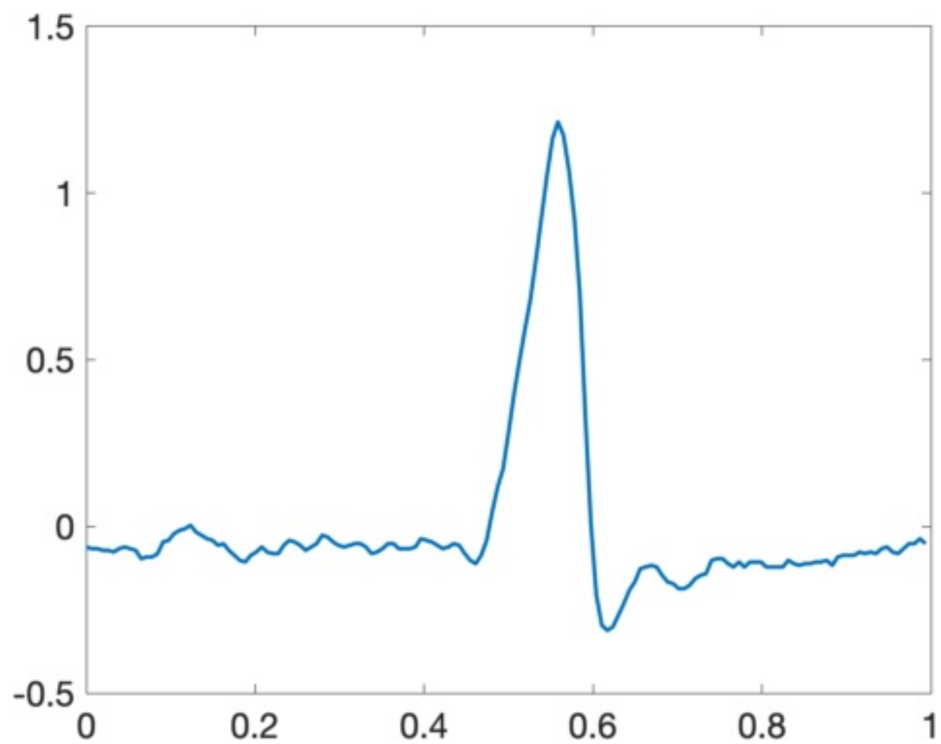

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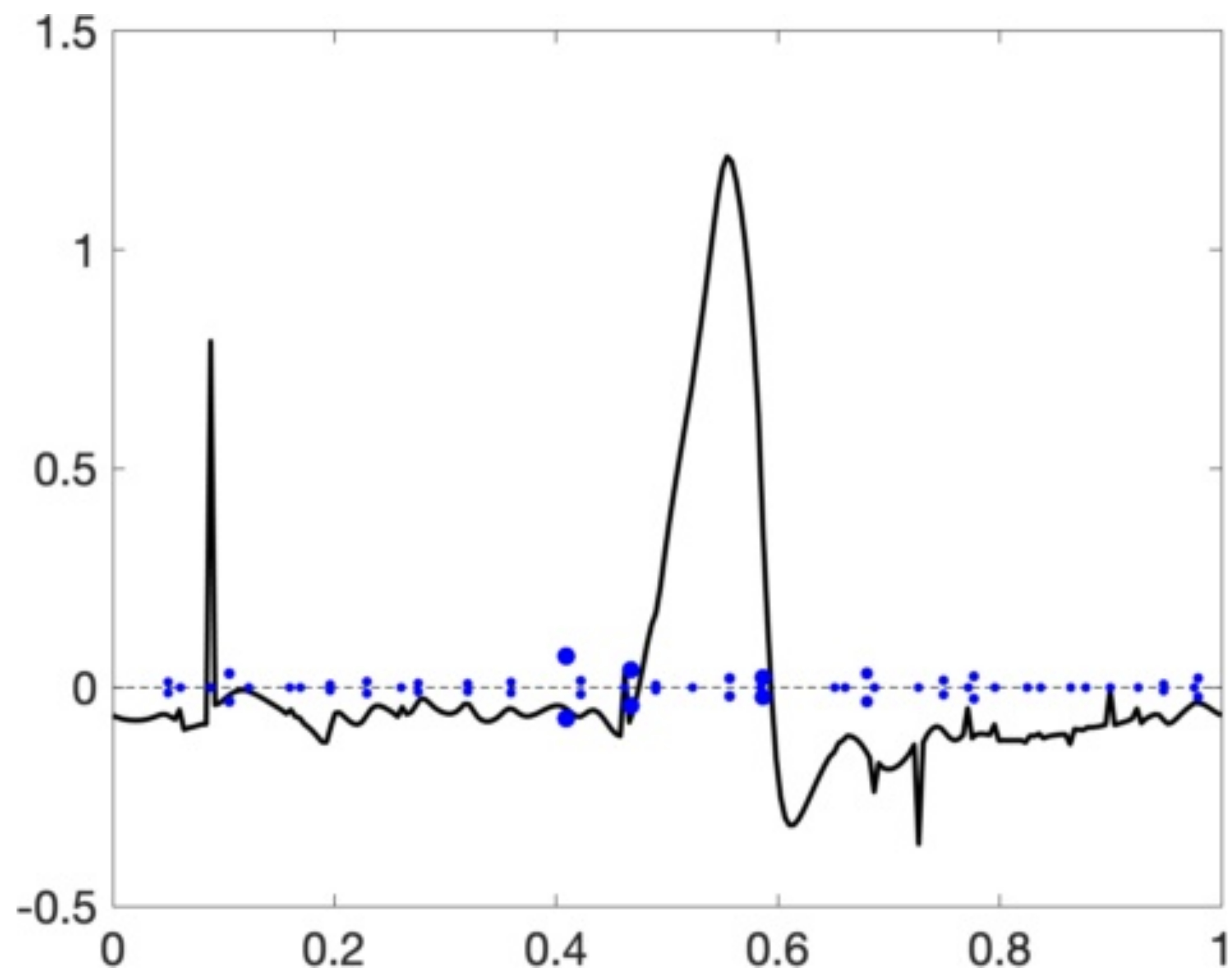

$m = 58$. The poles on the complex plane are superimposed over a plot of the interpolant.

AAA+Prony: Rational approximation in practice

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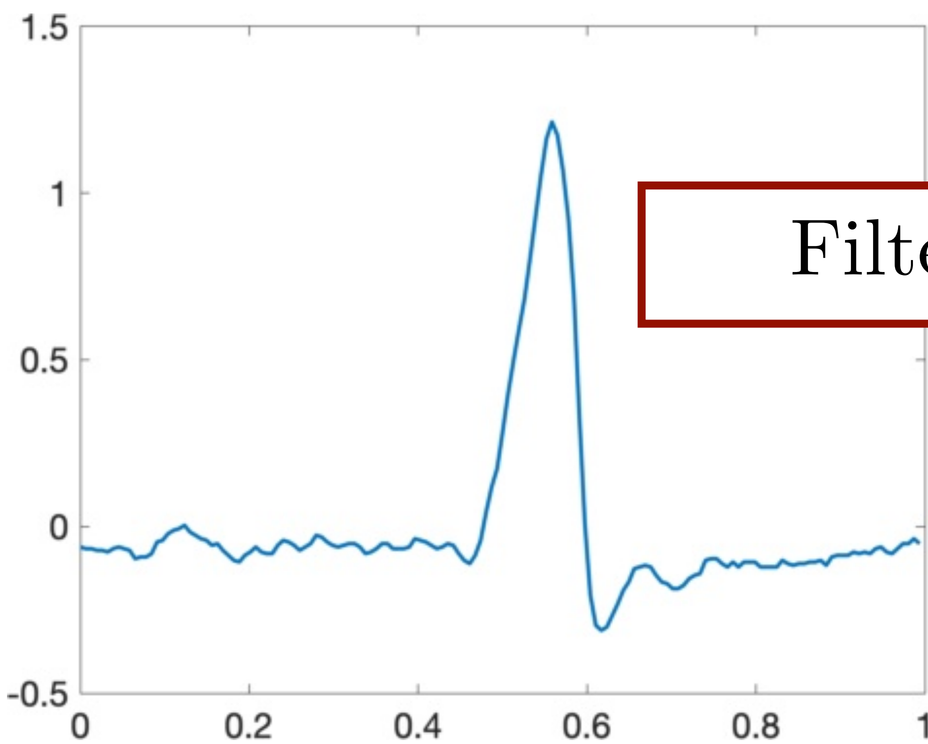
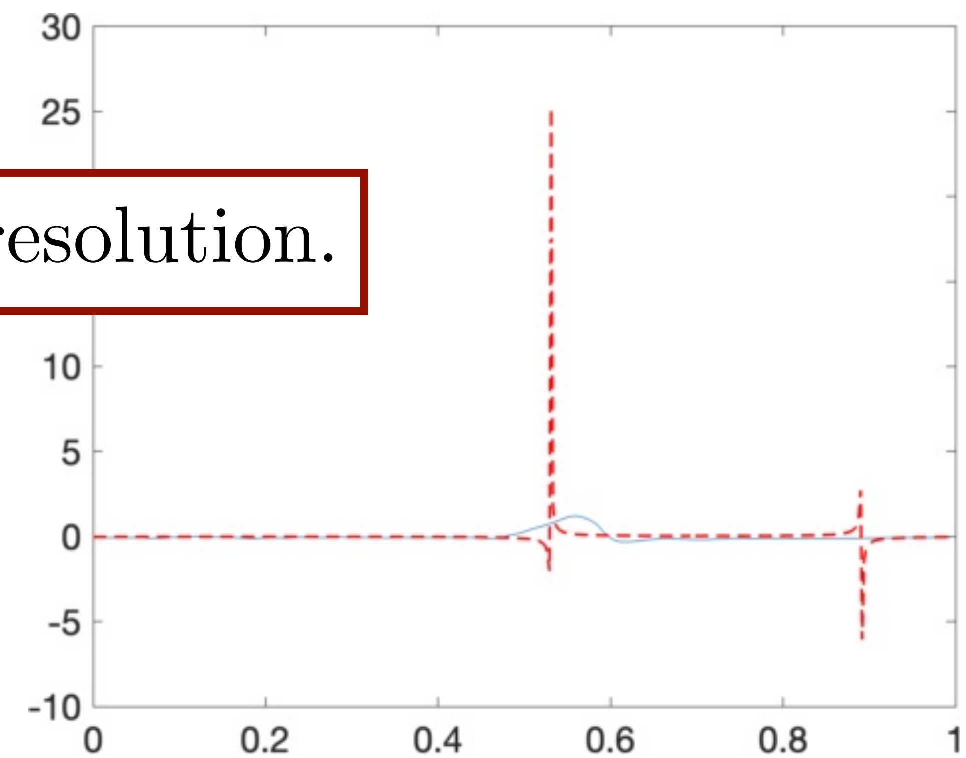


Only 154 noisy data points!

Filter + superresolution.



Try to find r_m, with trig-aaa, tol $:= 1e^{-1}$.

(Fourier space)

Approximate Prony s_m

$\mathcal{F}^{-1}(s_m)$ $+$

(value space)

r_m

AAA+Prony: Rational approximation in practice

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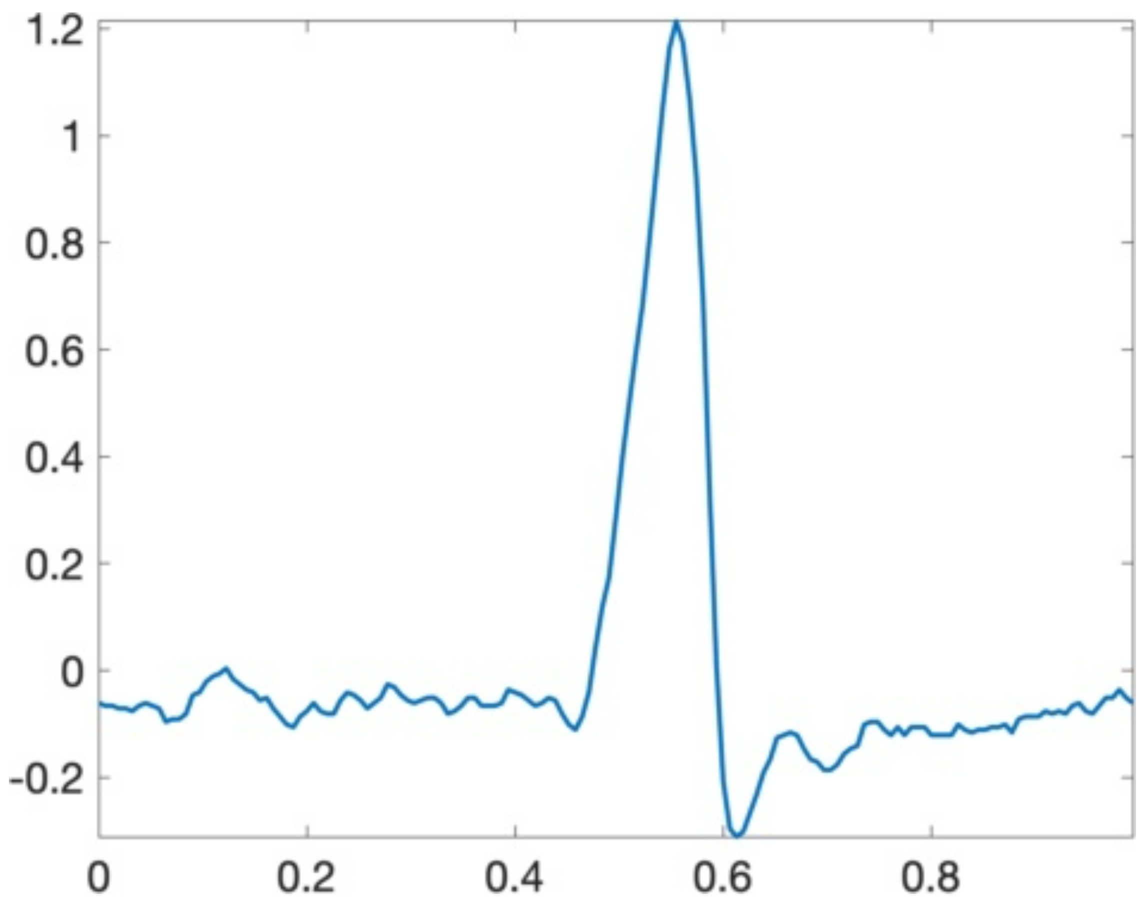

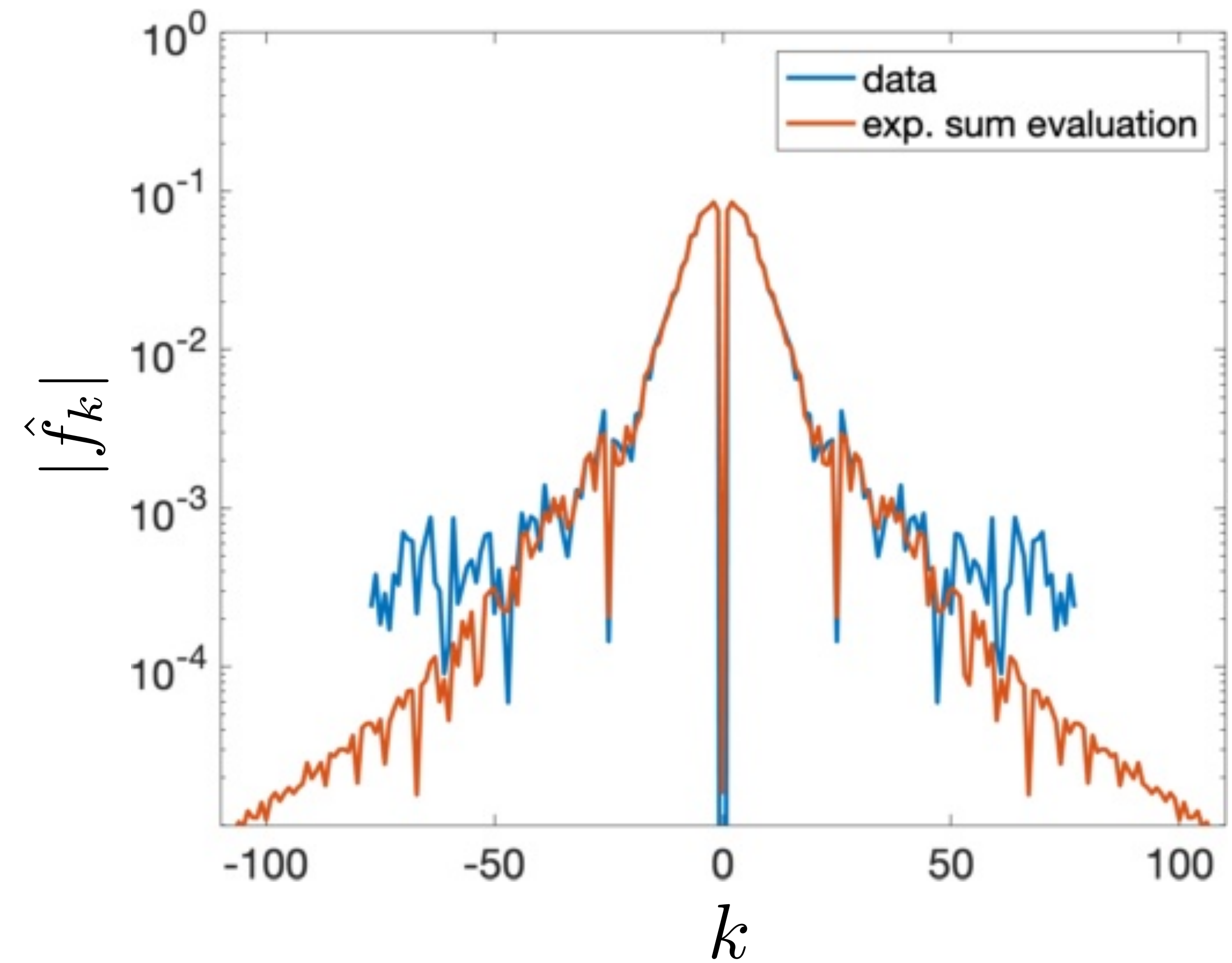

(Fourier space)

Approximate Prony

s_m

$\mathcal{F}^{-1}(s_m)$

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(value space)

r_m

AAA+Prony: Rational approximation in practice

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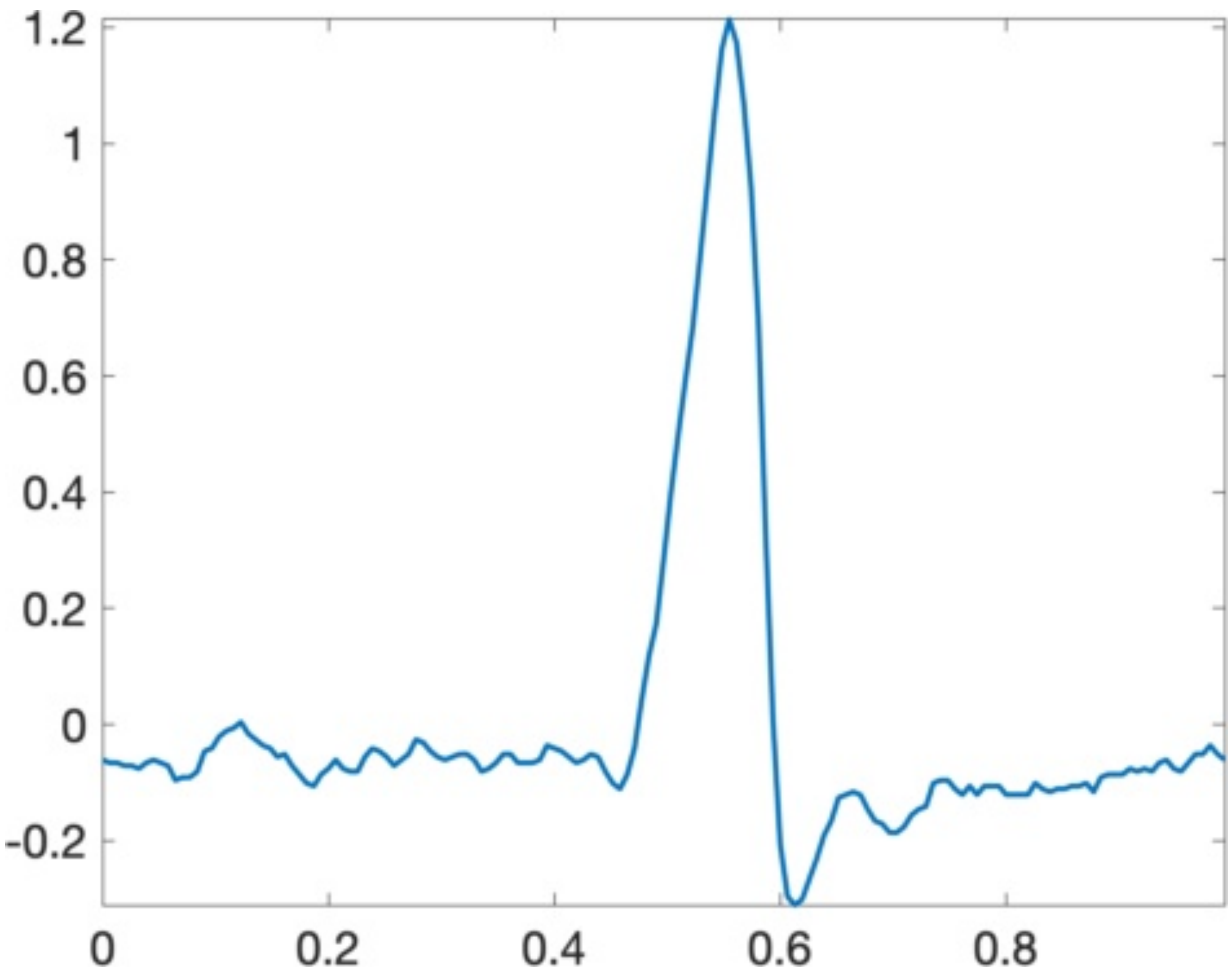
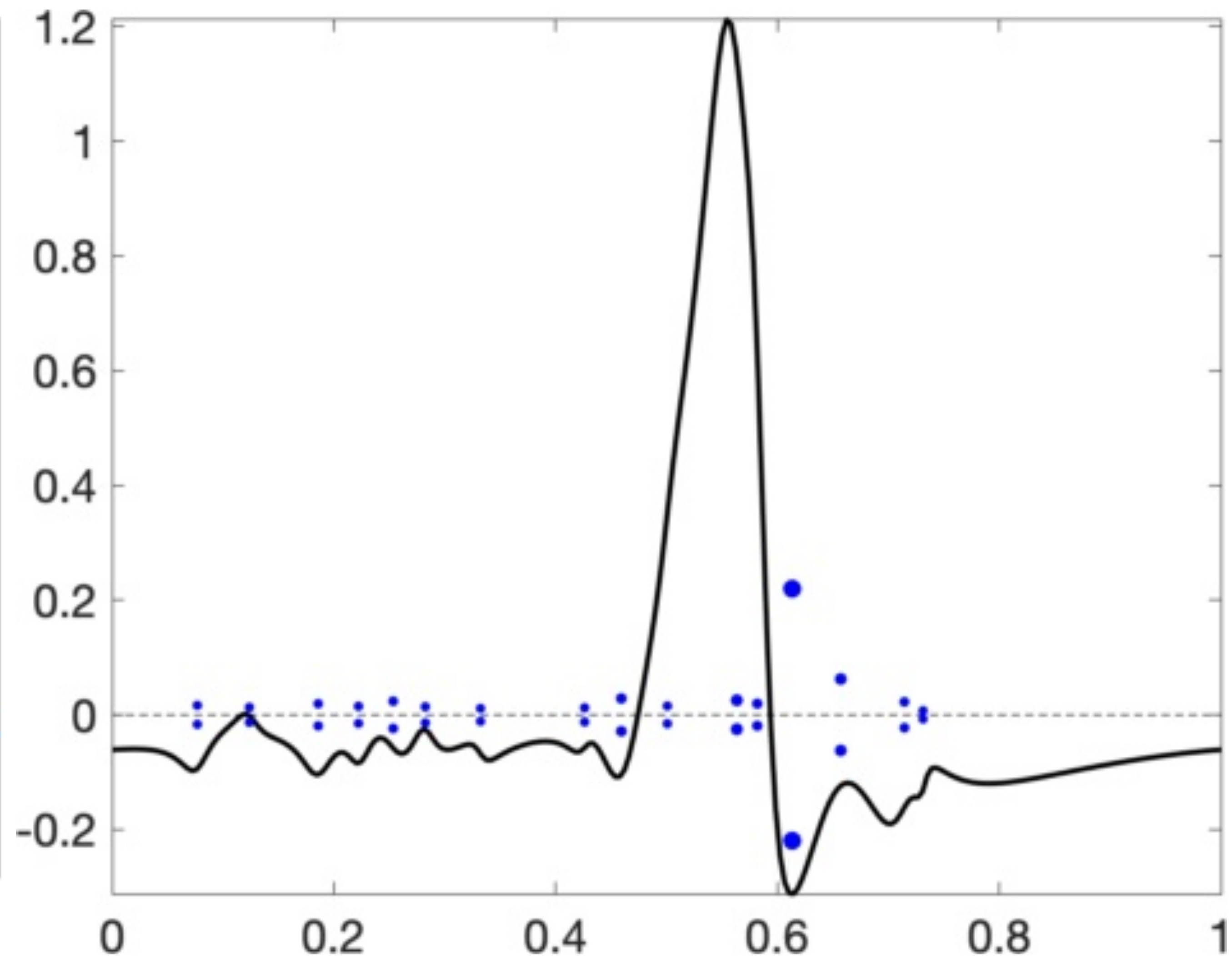


(Fourier space)



(value space)

Approximate Prony s_m

$\mathcal{F}^{-1}(s_m)$ $+$

r_m

Computing with barycentric rationals

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Differentiation:

- closed form derivative formula. [Berrut, Baltensperger, Mittelmann (2005)]
- rational spectral collocation methods.[Tee & Trefethen (2006)]

Rootfinding/Polefinding:

- $(2m+1) \times (2m+1)$ generalized eigenvalue problem.

Missing data, nonequispaced samples: [Nakatsukasa, Trefethen, & Sète (2018),

- extremely flexible, works automatically with nonuniform samples.

Stable evaluation on the interval: [Higham (2004), Austin & Xu (2017)]

- stability if there are no spurious poles.

Computing with exponential sums

$$\boxed{s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}} \rightleftarrows \boxed{r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}}$$

Filters, convolution, sums, differentiation, etc: Easy to apply to Fourier coefficients.

$$s_{M+L}(k) = s_M(k) + s_L(k) \to s_{\tilde{M}}(k)$$

- optimal recompression based on Hankel operator theory. [Beylkin, Haut & Monzon (2013)]
- simple, less theoretically grounded recompression based on Prony's method/rectangular Hankel matrices (at worst $\mathcal{O}(N_\epsilon m^2)$, often $\mathcal{O}(m^3)$).

Preserves pole symmetry:

- The poles of $r_M = \mathcal{F}(\sum_{j=1}^{M} w_j e^{-\lambda_j k})$ are $\eta_j = -\lambda_j/2\pi i, \overline{\eta}_j$

Robustness to noise: [Peter & Plonka (2013), Potts & Tasche (2013), M. Vetterli, P. Marziliano,

- Equivalent to the "annihilating filter" for denoising.

barycentric to sums of exponentials

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell} \xleftarrow{\texttt{s = ft(r)}} r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Key Idea: Each λ_j can be expressed explicitly using the poles of r.

The poles of $r(\theta)$ are $\log(\mu_j)/(2\pi i)$, where $(\mu_1, \ldots, \mu_{2m})$ are the finite generalized eigenvalues of the pencil $Ev = \mu Bv$, and

$$E = \begin{pmatrix} z_1 & & & & iu_1 z_1 \\ & z_2 & & & iu_2 z_2 \\ & & \ddots & & \vdots \\ & & & z_{2M} & iu_{2M} z_{2M} \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & & & & iu_1 z_1 \\ & 1 & & & iu_2 z_2 \\ & & \ddots & & \vdots \\ & & & 1 & iu_{2M} z_{2M} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad z_j = \exp(2\pi i \theta_j).$$

barycentric to sums of exponentials

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

$\longleftarrow$ `s = ft(r)`

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Key Idea: Each λ_j can be expressed explicitly using the poles of r.

- Find the poles of $r \to$ compute each λ_j.
- Evaluate r at $2N$ points $\to$ N Fourier coefficient
- Solve $Vw = s$, where s is an $\mathcal{O}(M)$ subset of coefficients.

and $V_{\ell k} = e^{-\lambda_k(\ell-1)}$.

barycentric to sums of exponentials

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

$$\xrightarrow{\texttt{r = ift(s)}}$$

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Naive idea:

- Evaluate $\mathcal{F}^{-1}(s)$ using pole-residue or FFT.
- Apply trig-aaa to data.

Results:

- best case scenario: we find a great approximation without increasing M.
- acceptable scenario: M is increased, but a good approximation is found.
- worst case scenario: spurious poles become unavoidable.

Accuracy or stability must suffer! (unacceptable)

sums of exponentials to barycentric

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

`r = ift(s)`

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \mathrm{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Idea: Try to preserve the poles of $\mathcal{F}^{-1}(s)$.

If $f = \mathcal{F}^{-1}(s)$, then for any set of distinct points $(t_1, \ldots, t_{2M})$, there are weights $(u_1, \ldots, u_{2M})$ so that $\mathcal{F}^{-1}(s) = r$.

It is numerically unstable to compute them directly!

sums of exponentials to barycentric

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

$$\xrightarrow{\texttt{r = ift(s)}}$$

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Idea: Try to preserve the poles of $\mathcal{F}^{-1}(s)$.

Use the fact that $D_t u = 0$, where

$$D_t = \begin{bmatrix} \cot(\pi p_1 - \pi t_1) & \cdots & \cot(\pi p_1 - \pi t_{2m}) \\ \vdots & & \vdots \\ \cot(\pi p_{2m} - \pi t_1)) & \cdots & \cot(\pi p_{2m} - \pi t_{2m}) \end{bmatrix}.$$

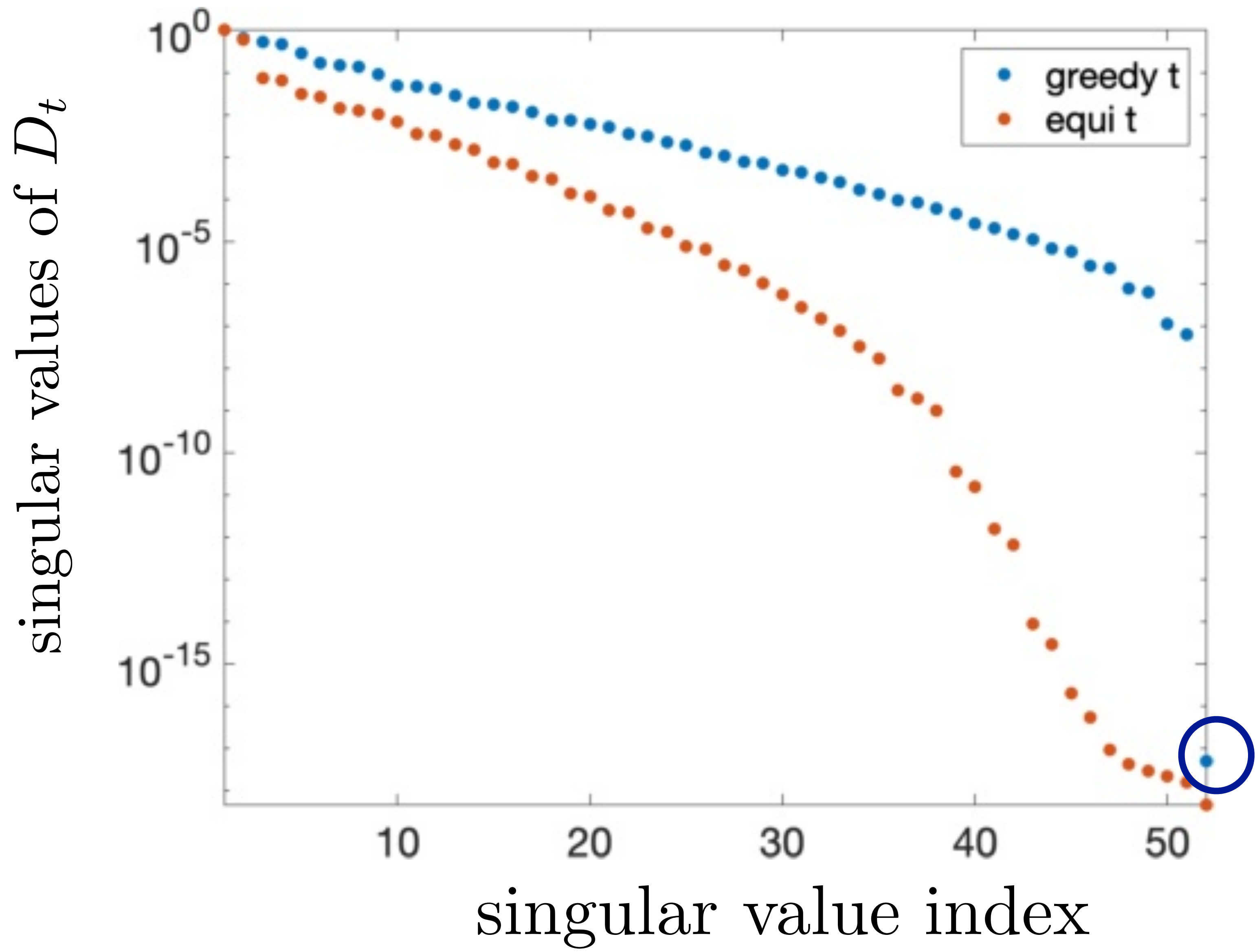
Pole-guided trigonometric AAA

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

`r = ift(s)`

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}{\sum_{k=1}^{2M} u_k \text{cst}\left(\pi\theta - \pi\theta_k\right)}$$

Can I choose any points $(t_0, \ldots, t_{2M})$?



Pole-guided trigonometric AAA

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

`r = ift(s)`

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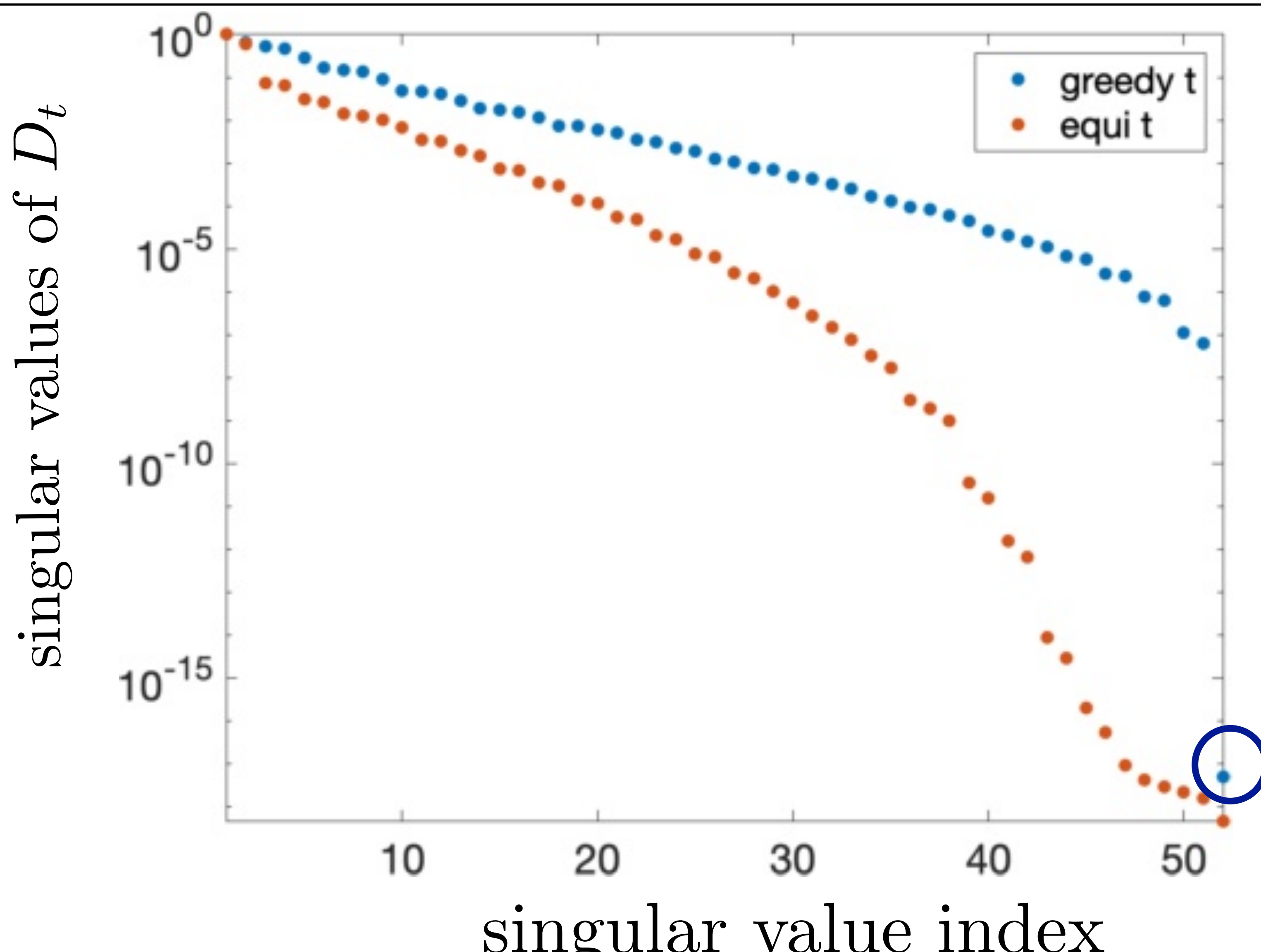
Idea: select interpolating points by trying to maximize the gap.



Pole-guided trigonometric AAA

$$s(\ell) = \sum_{j=1}^{M} w_j e^{-\lambda_j \ell}$$

`r = ift(s)`

$$r(\theta) = \frac{\sum_{k=1}^{2M} u_k f_k \mathrm{cst}\,(\pi\theta - \pi\theta_k)}{\sum_{k=1}^{2M} u_k \mathrm{cst}\,(\pi\theta - \pi\theta_k)}$$

Idea: select interpolating points by trying to maximize the gap.

- Given sample locations $T \in \mathbb{R}^{N\times 1}$, and poles $p = (p_1, \dots p_{2M})$, construct the $2M \times N$ matrix $(D_T)_{jk} = \cot(\pi p_j - \pi T_k)$.
- Apply CPQR algorithm to choose a well-conditioned column subset of D_T. If column k is chosen, then T_k is chosen as an interpolating point.
- (Additional details incorporate a LS fit to samples)