Lagrange Multipliers in Portfolio Optimization with Mixed Constraints

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Main Problem and Its Extension

Table: Comparison of Main Problem and Its Extension

Aspect	**Main Problem**	**Extension**
Scenario	Uncorrelated Stocks A and B	Correlated Stocks A and B
Assumptions	Positive and distinct expected returns, along with positive standard deviations	
Constraints	Full allocation, non-negative proportions, predefined risk threshold	
Objective	Maximize the expected return while satisfying specific constraints	
Optimization Method	**Lagrange multipliers with the incorporation of slack variables**	
Interpretation: - Risk-return trade-offs - The benefits of diversification - The impact of correlation coefficients		

Methodology

The research paper is structured as follows:

- Comprehensive case study that focuses on the main problem of portfolio optimization.
- General case where all variables are treated as arbitrary, allowing for scalable solutions.
- Incorporation of correlation coefficients while maintaining other assumptions.
- Limitations of the proposed problems.
- Conclusions and insights based on our research findings.

Techniques employed in the paper:

1. Lagrange multipliers [3]
2. Integration of slack variables [1]
3. Sequential Least Squares Programming (SLSQP) [5]

Method of Lagrange Multipliers

Consider a scenario where we aim to find the maximum or minimum values of a function of n **variables**, denoted by $f(x_1, x_2, \ldots, x_n)$, subject to m **equality constraints** of the form

$$g_i(x_1, x_2, \ldots, x_n) = c_i, \tag{1}$$

where $i = 1, \ldots, m$. To incorporate these constraints, we introduce m **Lagrange multipliers** λ_i, and form the equation:

$$\begin{gathered}\nabla f(x_1, x_2, \ldots, x_n) \\ = \lambda_1 \nabla g_1(x_1, x_2, \ldots, x_n) + \cdots + \lambda_m \nabla g_m(x_1, x_2, \ldots, x_n).\end{gathered} \tag{2}$$

Goal: Determine the values of $x_1, x_2, \ldots, x_n$,$\lambda_1, \lambda_2, \ldots, \lambda_m$ that simultaneously satisfy the gradient equation (2) and the m equality constraint equations (1).

Slack Variables

Let us consider a function denoted as $f(x_1, x_2, \ldots, x_n)$, with n **variables**, for which we seek to determine the maximum or minimum values. However, this function is subject to m **inequality constraints** of the form

$$g_i(x_1, x_2, \ldots, x_n) \leq c_i, \tag{3}$$

where $i = 1, \ldots, m$.

To address these inequality constraints, we introduce m **non-negative slack variables**, denoted as s_i^2, associated with each inequality constraint.

$\Rightarrow$ UNIFIED approach to handling both types of constraints

By incorporating the **slack variables**, each original inequality constraint (3) is expressed as an equivalent equality constraint:

$$g_i(x_1, x_2, \ldots, x_n) + s_i^2 = c_i.$$

We represent these newly formed equality constraints as

$$h_i(x_1, x_2, \ldots, x_n, s_i) = g_i(x_1, x_2, \ldots, x_n) + s_i^2. \tag{4}$$

Similar to the treatment of equality constraints, we employ m **Lagrange multipliers**, denoted as λ_i, and formulate the following equation:

$$\begin{aligned}&\nabla f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)\\ &= \lambda_1 \nabla h_1(x_1, x_2, \ldots, x_n, s_1) + \cdots + \lambda_m \nabla h_m(x_1, x_2, \ldots, x_n, s_m).\end{aligned} \tag{5}$$

Goal: Find the values of $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m, s_1, s_2, \ldots, s_m$ that simultaneously satisfy the gradient equation (5) and the m equality constraint equations (4).

A Case Study of the Main Problem

Consider a scenario where we have a fixed budget of \$100,000 to invest in 2 uncorrelated stocks, denoted as A and B. The expected returns and standard deviations for these stocks are as follows:

- Stock A: Expected return of 20% and standard deviation of 10%
- Stock B: Expected return of 30% and standard deviation of 20%

Goal: Determine the optimal allocation between stocks A and B to maximize the expected return, while satisfying the constraints imposed by our budget and risk considerations.

- **Budget Constraint**: Allocate all of the funds to the 2 stocks, with non-negative proportions.
- **Portfolio Risk Constraint**: Explore 2 different values for the risk limit: 15% and 10%.

Problem Formulation

By [2], the **expected return on a portfolio** $E(r_p)$ is determined by taking a weighted average of the expected returns of its constituent stocks. This relationship can be expressed as

$$E(r_p) = E(r_A)\omega_A + E(r_B)\omega_B. \tag{6}$$

As per the assumption, $E(r_A) \neq E(r_B)$ and $E(r_A), E(r_B) > 0$.

Since the correlation coefficient $\rho_{AB} = 0$, the **portfolio variance** σ_p^2 is computed as follows:

$$\begin{aligned}\sigma_p^2 &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B \mathrm{Cov}(r_A, r_B) \\ &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} \\ &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2.\end{aligned}$$

Here, $\sigma_A, \sigma_B > 0$. In order to limit the portfolio risk to a predetermined level R, we impose the **risk constraint** as follows:

$$\sigma_p \leq R \Leftrightarrow \sigma_p^2 \leq R^2.$$

The **general optimization problem**, using arbitrary variables, can be formulated as follows:

$$\max \quad E(r_p) = E(r_A)\omega_A + E(r_B)\omega_B$$

subject to the following constraints:

$$\omega_A + \omega_B = 1 \tag{7}$$

$$\omega_A, \omega_B \geq 0 \tag{8}$$

$$\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 \leq R^2. \tag{9}$$

Considering the assumptions in the case study, we assign the expected returns of Stocks A and B as $E(r_A) = 0.2$ (or 20%) and $E(r_B) = 0.3$ (or 30%), respectively. The corresponding standard deviations are $\sigma_A = 0.1$ (or 10%) and $\sigma_B = 0.2$ (or 20%).

So, our objective is to **maximize the expected return** $E(r_p)$, which is given by:

$$\max \quad E(r_p) = 0.2\omega_A + 0.3\omega_B$$

subject to the mixed constraints:

$$\omega_A + \omega_B = 1 \tag{10}$$

$$\omega_A, \omega_B \geq 0 \tag{11}$$

$$0.01\omega_A^2 + 0.04\omega_B^2 \leq R^2. \tag{12}$$

Analytical Solution: $R = 15\%$

Lemma (1)

The optimal allocation proportions to maximize the portfolio's expected return, subject to a portfolio risk limit of 15%, are \$26,148 invested in Stock A and \$73,852 invested in Stock B. By implementing this allocation strategy, we achieve a maximum expected return of 27.385% while managing the portfolio risk at 15%.

Proof: Given the portfolio risk limit $R = 0.15$ (or 15%), the portfolio risk constraint (12) can be expressed as

$$0.01\omega_A^2 + 0.04\omega_B^2 \leq 0.0225.$$

We will represent the **expected return of the portfolio** $E(r_p)$ as $f(\omega_A, \omega_B) = 0.2\omega_A + 0.3\omega_B$ and the **budget constraint** (10) as

$$g(\omega_A, \omega_B) = \omega_A + \omega_B - 1 = 0.$$

To handle the **inequality constraints** $\omega_A, \omega_B \geq 0$ and $0.01\omega_A^2 + 0.04\omega_B^2 \leq 0.0225$, we introduce 3 **slack variables**, denoted as s_1^2, s_2^2, and s_3^2, to convert these inequalities into equivalent equality constraints as follows:

$$\begin{aligned} h(\omega_A, s_1) &= \omega_A - s_1^2 = 0 \\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0 \\ k(\omega_A, \omega_B, s_3) &= 0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 + s_3^2 = 0. \end{aligned}$$

These transformed constraints, along with the equality constraint $g(\omega_A, \omega_B) = 0$, form a unified set of 4 equality constraints.

To incorporate the constraints, we introduce 4 **Lagrange multipliers** denoted as λ_i. The gradient equation is then formulated as:

$$\nabla f(\omega_A, \omega_B, s_1, s_2, s_3)$$

$$= \lambda_1 \nabla g(\omega_A, \omega_B) + \lambda_2 \nabla h(\omega_A, s_1) + \lambda_3 \nabla j(\omega_B, s_2) + \lambda_4 \nabla k(\omega_A, \omega_B, s_3). \tag{13}$$

We seek the values of the 9 variables $\omega_A, \omega_B, \lambda_1, \lambda_2, \lambda_3, \lambda_4, s_1, s_2, s_3$ that satisfy both the gradient equation (13) and the 4 equality constraint equations. This leads to a system of 9 equations with 9 unknown variables:

$$
\begin{aligned}
f_{\omega_A} &= \lambda_1 g_{\omega_A} + \lambda_2 h_{\omega_A} + \lambda_3 j_{\omega_A} + \lambda_4 k_{\omega_A} \\
f_{\omega_B} &= \lambda_1 g_{\omega_B} + \lambda_2 h_{\omega_B} + \lambda_3 j_{\omega_B} + \lambda_4 k_{\omega_B} \\
f_{s_1} &= \lambda_1 g_{s_1} + \lambda_2 h_{s_1} + \lambda_3 j_{s_1} + \lambda_4 k_{s_1} \\
f_{s_2} &= \lambda_1 g_{s_2} + \lambda_2 h_{s_2} + \lambda_3 j_{s_2} + \lambda_4 k_{s_2} \\
f_{s_3} &= \lambda_1 g_{s_3} + \lambda_2 h_{s_3} + \lambda_3 j_{s_3} + \lambda_4 k_{s_3} \\
g(\omega_A, \omega_B) &= 0 \\
h(\omega_A, s_1) &= 0 \\
j(\omega_B, s_2) &= 0 \\
k(\omega_A, \omega_B, s_3) &= 0.
\end{aligned} \tag{14}
$$

Upon evaluating the partial derivatives and substituting them into the respective functions, the resulting expressions are as follows:

$$0.2 = \lambda_1 + \lambda_2 + 0.02\lambda_4 \tag{15}$$

$$0.3 = \lambda_1 + \lambda_3 + 0.08\lambda_4 \tag{16}$$

$$0 = -2s_1\lambda_2 \tag{17}$$

$$0 = -2s_2\lambda_3 \tag{18}$$

$$0 = 2s_3\lambda_4 \tag{19}$$

$$\omega_A + \omega_B - 1 = 0 \tag{20}$$

$$\omega_A - s_1^2 = 0 \tag{21}$$

$$\omega_B - s_2^2 = 0 \tag{22}$$

$$0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 + s_3^2 = 0. \tag{23}$$

Considering equation (19), 2 cases arise: $s_3 = 0$ or $\lambda_4 = 0$.

• Case 1: $s_3 = 0$

From equations (20) and (23), we can establish a system of 2 equations:

$$\omega_A + \omega_B - 1 = 0$$
$$0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 = 0.$$

By solving the first equation for ω_B and substituting it into the second equation, we derive a quadratic equation as follows:

$$0.05\omega_A^2 - 0.08\omega_A + 0.0175 = 0.$$

Using the quadratic formula, the **proportion of Stock** A is calculated:

$$\omega_A = 0.8 \pm 10\sqrt{0.0029}.$$

If $\omega_A = 0.8 + 10\sqrt{0.0029}$, then $\omega_A > 1$. However, since both $\omega_A, \omega_B \in [0, 1]$, we obtain $\omega_A \approx 0.26148$ (when choosing the negative square root). From the budget constraint (20), the **proportion of Stock** B is $\omega_B \approx 0.73852$.

- Case 2: $\lambda_4 = 0$

By substituting $\lambda_4 = 0$ into equations (15) and (16) and then combining these equations, we find

$$0.1 = \lambda_3 - \lambda_2.$$

Thus, it follows that $\lambda_2 \neq \lambda_3$. From equations $0 = -2s_1\lambda_2$ and $0 = -2s_2\lambda_3$, we can deduce that exactly one of λ_3 and λ_2 must be equal to 0.

- If $\lambda_3 = 0$, then $\lambda_2 \neq 0$. So, $s_1 = 0$. As $\omega_A - s_1^2 = 0$, we obtain $\omega_A = 0$. By the budget constraint, $\omega_B = 1$. However, upon evaluating these values using the risk constraint $0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 + s_3^2 = 0$, we find that $s_3^2 = -0.0175$, which is an **infeasible result**.
- If $\lambda_2 = 0$, then $\lambda_3 \neq 0$, resulting in $s_2 = 0$. Since $\omega_B - s_2^2 = 0$, we obtain $\omega_B = 0$. So, $\omega_A = 1$. Upon verifying this solution, we find that $s_3^2 = 0.0125$, demonstrating its validity.

By evaluating the solutions (ω_A, ω_B) on the objective function $f(\omega_A, \omega_B)$, we derive the following results:

- For the approximate values $(\omega_A, \omega_B) \approx (0.26148, 0.73852)$, the expected return of the portfolio is $f(0.26148, 0.73852) \approx 0.27385$ (or 27.385%), and the portfolio risk is $\sigma_p = 0.15$ (or 15%).
- For $(\omega_A, \omega_B) = (1, 0)$, the expected return of the portfolio is $f(1, 0) = 0.2$ (or 20%), and the portfolio risk is $\sigma_p = 0.1$ (or 10%).

Consequently, by allocating $0.26148 \cdot 100,000 = \$26,148$ to Stock A and $0.73852 \cdot 100,000 = \$73,852$ to Stock B, we achieve the **maximum expected return** of 27.385% with a **portfolio risk** of 15%.

Analytical Solution: $R = 10\%$

Lemma (2)

The optimal allocation proportions to maximize the portfolio's expected return, subject to a portfolio risk limit of 10%, are \$60,000 invested in Stock A and \$40,000 invested in Stock B. By implementing this allocation strategy, we achieve a maximum expected return of 24% while managing the portfolio risk at 10%.

Proof: We follow the same methodology as described for when $R = 15\%$. The only modification is in the **portfolio risk constraint** (12), which can be adjusted to:

$$0.01\omega_A^2 + 0.04\omega_B^2 \leq 0.01.$$

Consequently, the equality constraint incorporating the slack variable s_3^2 denoted as $k(\omega_A, \omega_B, s_3)$, transforms into:

$$k(\omega_A, \omega_B, s_3) = 0.01\omega_A^2 + 0.04\omega_B^2 - 0.01 + s_3^2 = 0.$$

Similar to Lemma 1, solving the gradient system gives us 2 cases:

$$s_3 = 0 \quad \text{or} \quad \lambda_4 = 0.$$

- Case 1: $s_3 = 0$

This leads to a new system of 2 equations:

$$\omega_A + \omega_B - 1 = 0$$
$$0.01\omega_A^2 + 0.04\omega_B^2 - 0.01 = 0.$$

By rewriting $\omega_B = 1 - \omega_A$, we get a quadratic equation in terms of ω_A as follows:

$$0.05\omega_A^2 - 0.08\omega_A + 0.03 = 0.$$

Using the quadratic formula, we find the values of ω_A to be:

$$\omega_A = \{1, 0.6\}.$$

When $\omega_A = 0.6$, the corresponding value of $\omega_B = 0.4$. Consequently, the portfolio's expected return is 0.24 (or 24%). Similarly, when $\omega_A = 1$, we find $\omega_B = 0$, resulting in an expected return of 0.2 (or 20%). In both cases, the portfolio risk σ_p remains at 10%.

• Case 2: $\lambda_4 = 0$

Similar to Lemma 1, we obtain 2 solutions for

$$(\omega_A, \omega_B) = \{(1,0), (0,1)\}.$$

If $(\omega_A, \omega_B) = (0,1)$, we obtain $s_3^2 = -0.03$, which is **infeasible**. So, for this case, we only retain the solution $(\omega_A, \omega_B) = (1,0)$, which coincides with one of the solutions in Case 1 where $s_3 = 0$.

$\Rightarrow$ We attain the **maximum expected return** of 24% while maintaining a **portfolio risk** of 10%, by allocating \$60,000 to Stock A and \$40,000 to Stock B

Portfolio Allocation Strategies

Table: Portfolio Performance

(ω_A, ω_B)	Expected Return (%)	Risk (%)
$(0.6, 0.4)$	24	10
$(0.26148, 0.73852)$	27.385	15

- For the allocation of 60% to Stock A and 40% to Stock B, the portfolio achieves an expected return of 24% with a corresponding risk level of 10%. This **balanced approach** offers growth potential while maintaining a moderate level of risk.
- With an allocation of approximately 26.148% to Stock A and 73.852% to Stock B, the portfolio demonstrates a higher expected return of 27.385% but also an increased risk level of 15%.
- These findings emphasize the **trade-off between risk and return** in portfolio management, where higher returns are associated with higher levels of risk.

Numerical Solutions

To **validate the analytical solutions** for the main problem case study, we utilize the SLSQP algorithm as follows:

1. Import the necessary libraries.
2. Define the objective function, which represents the quantity we want to maximize.
3. Define the equality constraint, which ensures that the allocation percentages sum up to 1.
4. Define the inequality constraint, which limits the portfolio risk.
5. Set the initial guess for the allocation percentages.
6. Set the bounds for the allocation percentages.
7. Define the constraints by specifying their type (equality or inequality) and the corresponding constraint functions.
8. Solve the optimization problem using the SLSQP algorithm.
9. Calculate the maximum value of the objective function by obtaining the optimal solution from the optimization result.

Statistical Analysis

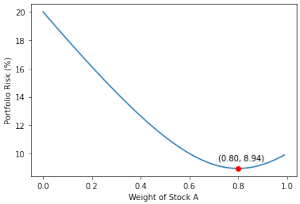

Figure: Portfolio Risk as a Function of Proportion of Stock A.

- Stock A has a lower investment risk compared to Stock B.
- **Minimizing portfolio risk does not require investing solely in the less risky stock**.
- By reducing the weight of Stock A from 1 to 0.8, the portfolio's expected return increases while the portfolio risk falls below Stock A's risk level (10%), reaching 8.94%.
- Diversification plays a crucial role in portfolio management.

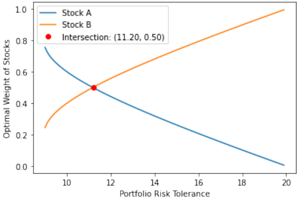

Figure: Optimal Proportions of Stocks A and B as Functions of Portfolio Risk Tolerance.

- As the level of portfolio risk tolerance increases, the optimal investment in Stock B also increases.
- At a tolerance level of 20%, Stock B becomes the sole constituent of the portfolio.
- Allocating a larger proportion to the more volatile Stock B leads to an increased expected return due to its higher expected return compared to Stock A.
- These findings once again emphasize the **trade-off between risk and return** in portfolio management.

Main Problem using Arbitrary Variables

Proposition (1)

Assuming positive proportions for both Stocks A and B, the optimal allocation proportions that maximize the expected return of the portfolio are

$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2} \quad \textit{and} \quad \omega_B = 1 - \omega_A.$$

Analytical Solution

Proof: Let $f(\omega_A, \omega_B) = E(r_p)$ denote the expected return of the portfolio to be maximized, and

$$g(\omega_A, \omega_B) = \omega_A + \omega_B - 1 = 0$$

represent the budget constraint. Introducing 3 non-negative **slack variables** $\{s_1^2, s_2^2, s_3^2\}$, the inequality constraints can be converted into equivalent equality constraints as follows:

$$\begin{aligned} h(\omega_A, s_1) &= \omega_A - s_1^2 = 0 \\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0 \\ k(\omega_A, \omega_B, s_3) &= \sigma_A^2 \omega_A^2 + \sigma_B^2 \omega_B^2 - R^2 + s_3^2 = 0. \end{aligned}$$

By utilizing **Lagrange multipliers** λ_i for $i = 1, \ldots, 4$ and formulating the gradient equation, we can derive a system of 9 equations by evaluating the partial derivatives of this gradient and performing the necessary substitutions.

$$E(r_A) = \lambda_1 + \lambda_2 + 2\sigma_A^2\omega_A\lambda_4 \tag{24}$$

$$E(r_B) = \lambda_1 + \lambda_3 + 2\sigma_B^2\omega_B\lambda_4 \tag{25}$$

$$0 = -2s_1\lambda_2 \tag{26}$$

$$0 = -2s_2\lambda_3 \tag{27}$$

$$0 = 2s_3\lambda_4 \tag{28}$$

$$\omega_A + \omega_B - 1 = 0 \tag{29}$$

$$\omega_A - s_1^2 = 0 \tag{30}$$

$$\omega_B - s_2^2 = 0 \tag{31}$$

$$\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 + s_3^2 = 0. \tag{32}$$

Similar to the case study, we observe 2 possible scenarios:

$$s_3 = 0 \text{ or } \lambda_4 = 0.$$

- Case 1: $s_3 = 0$

We can establish a system of 2 equations:

$$\omega_A + \omega_B - 1 = 0$$
$$\sigma_A^2 \omega_A^2 + \sigma_B^2 \omega_B^2 - R^2 = 0.$$

By solving the first equation for ω_B and substituting the result into the second equation, we derive the following quadratic equation:

$$(\sigma_A^2 + \sigma_B^2)\omega_A^2 - (2\sigma_B^2)\omega_A + (\sigma_B^2 - R^2) = 0. \tag{33}$$

Using the quadratic formula on equation (33), we get the **proportion for Stock** A as follows:

$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2}. \tag{34}$$

- Case 2: $\lambda_4 = 0$

We obtain a system of 2 equations:

$$E(r_A) = \lambda_1 + \lambda_2 \tag{35}$$

$$E(r_B) = \lambda_1 + \lambda_3. \tag{36}$$

By subtracting equation (35) from equation (36), we get

$$E(r_B) - E(r_A) = \lambda_3 - \lambda_2. \tag{37}$$

Given the assumption that the expected return of Stock A and Stock B are distinct, it follows that $\lambda_3 \neq \lambda_2$. By analyzing equations $-2s_1\lambda_2 = 0$ and $-2s_2\lambda_3 = 0$, it becomes evident that precisely one of λ_3 and λ_2 must be equal to zero.

- Case 2a: $\lambda_3 = 0$
  If $\lambda_3 = 0$, then $\lambda_2 \neq 0$. As $0 = -2s_1\lambda_2$, the slack variable $s_1 = 0$. Since $\omega_A - s_1^2 = 0$, it allows us to determine the **proportion for Stock** A, which is $\omega_A = 0$. Furthermore, considering that the sum of proportions should be 1, the **proportion for Stock** B is $\omega_B = 1$.
- Case 2b: $\lambda_2 = 0$
  If $\lambda_2 = 0$, then $\lambda_3 \neq 0$. So, equation $0 = -2s_2\lambda_3$ implies that $s_2 = 0$. Substituting $s_2 = 0$ into equation $\omega_B - s_2^2 = 0$ allows us to determine the **proportion for Stock** B, which is $\omega_B = 0$. Thus, the **proportion for Stock** A is $\omega_A = 1$.

Consequently, we obtain 3 corresponding values of the expected return of the portfolio as follows:

- For the case where

$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2} \quad \text{and} \quad \omega_B = 1 - \omega_A,$$

  we consider only the scenario where ω_A is non-negative. In this case, the expected return of the portfolio is given by $f(\omega_A, \omega_B) = E(r_A)\omega_A + E(r_B)\omega_B$, and the portfolio risk is determined by $\sigma_p = \sqrt{\omega_A^2 \sigma_A^2 + \omega_B^2 \sigma_B^2}$.
- When $(\omega_A, \omega_B) = (0, 1)$, the expected return of the portfolio is $f(0, 1) = E(r_B)$, and the portfolio risk is σ_B.
- When $(\omega_A, \omega_B) = (1, 0)$, the expected return of the portfolio is $f(1, 0) = E(r_A)$, and the portfolio risk is σ_A.

Lower Bound of the Portfolio Risk

Lemma (3)

The minimum portfolio risk limit required to obtain feasible solutions for the main problem involving 2 uncorrelated stocks A and B is

$$R_{\min} = \sqrt{\frac{\sigma_A^2 \sigma_B^2}{\sigma_A^2 + \sigma_B^2}}.$$

Proof.

We must ensure that the **discriminant** in our proportions for ω_A and ω_B, given by $\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)$, is non-negative.

$$\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2) \geq 0$$

$$\begin{aligned}
&\Leftrightarrow \sigma_B^4 \geq (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2) \\
&\Leftrightarrow \sigma_B^4 \geq \sigma_B^2\sigma_A^2 + \sigma_B^4 - R^2(\sigma_A^2 + \sigma_B^2) \\
&\Leftrightarrow R^2 \geq \frac{\sigma_A^2\sigma_B^2}{\sigma_A^2 + \sigma_B^2}. \qquad (38)
\end{aligned}$$

Therefore, in order to ensure the feasibility of finding the proportions of stocks, there exists a **minimum portfolio risk limit** of $R_{\min} = \sqrt{\frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2}}$ under the given assumption. □

Extension of the Main Problem

Proposition (2)

Let $\sigma = \sigma_B^2 - \sigma_A \sigma_B \rho_{AB}$. Assuming that both stocks A and B must have positive proportions, the optimal proportion for Stock A that maximizes the expected return of the portfolio in the extended scenario is

$$\omega_A = \frac{\sigma \pm \sqrt{\sigma^2 - (\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB})(\sigma_B^2 - R^2)}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}}.$$

The proportion for Stock B can be derived by subtracting the proportion for Stock A from 1.

Analytical Solution

Proof: The portfolio variance can then be calculated using the following formula:

$$\begin{aligned}\sigma_p^2 &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B \mathrm{Cov}(r_A, r_B)\\ &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB}.\end{aligned}$$

Here, the correlation coefficient ρ_{AB} may take a non-zero value. As a result, the **portfolio risk constraint** will be modified to the following inequality constraint:

$$\omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} \leq R^2. \qquad (39)$$

The inequality constraints can be converted to equivalent equality constraints as follows:

$$\begin{aligned} h(\omega_A, s_1) &= \omega_A - s_1^2 = 0 \\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0 \\ k(\omega_A, \omega_B, s_3) &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} - R^2 + s_3^2 = 0. \end{aligned}$$

We are able to solve this problem with **Lagrange multipliers method** in the same way as the main problem. Upon solving the partial derivatives of the gradient equation and making the necessary substitutions, we arrive at a system of 9 equations.

After solving this system, we obtain 2 solutions

$$(\omega_A, \omega_B) = \{(0,1), (1,0)\}$$

that are identical to those in the main problem. To determine the remaining solution, we set $s_3 = 0$ and solve the following system of 2 equations:

$$\begin{aligned} \omega_A + \omega_B - 1 &= 0 \\ \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} - R^2 &= 0. \end{aligned}$$

By solving the first equation for ω_B and substituting the result into the second equation, we derive the following expression:

$$\sigma_A^2\omega_A^2 + \sigma_B^2(1-\omega_A)^2 + 2\omega_A(1-\omega_A)\sigma_A\sigma_B\rho_{AB} - R^2 = 0$$

$$\Leftrightarrow \sigma_A^2\omega_A^2 + \sigma_B^2(1-2\omega_A+\omega_A^2) + (2\omega_A - 2\omega_A^2)\sigma_A\sigma_B\rho_{AB} - R^2 = 0$$
$$\Leftrightarrow (\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB})\omega_A^2 - (2\sigma)\omega_A + (\sigma_B^2 - R^2) = 0.$$

Applying the quadratic formula, we get the **proportion for Stock** A as follows:

$$\omega_A = \frac{\sigma \pm \sqrt{\sigma^2 - (\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB})(\sigma_B^2 - R^2)}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}}.$$

The **proportion for Stock** B can be determined by subtracting the proportion for Stock A from 1. Note that we only consider the root(s) where ω_A is non-negative.

Statistical Analysis

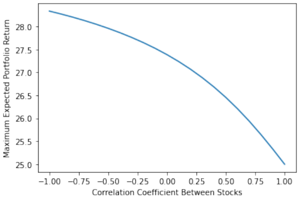

Figure: Maximum Expected Return of the Portfolio as a Function of Correlation Coefficient.

- As the correlation between Stocks A and B increases, the portfolio's risk tends to rise.
- To maintain risk within the specified constraint, the weight of the riskier stock (Stock B) needs to be reduced in the portfolio.
- As Stock B typically offers higher profitability, the reduction in its weight leads to a decrease in the maximum expected return of the portfolio.

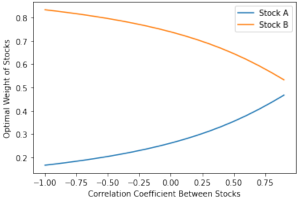

Figure: Optimal Stock Weights as a Function of Correlation Coefficient.

- As the correlation coefficient increases, the diversification benefit derived from allocating both stocks diminishes.
- **Higher correlation coefficients** imply greater risk and potentially lower expected returns.
- **Lower correlation coefficients** offer greater diversification benefits and potential risk reduction, leading to improved risk-adjusted returns.

Limitations

Technological, **micro and macroeconomic**, and **political conditions** can substantially impact investment return, limiting the scope of our analysis [4, 6, 7, 8, 9].

- The COVID-19 pandemic and recent advancements in artificial intelligence have showcased how unexpected events can disrupt markets and lead to unpredictable shifts in stock prices.
- The decisions made by central banks, such as changes in interest rates, hold particular significance in determining stock valuations.

$\Rightarrow$ The model's ability to comprehend the complexities of the investment landscape may be limited, highlighting the need for more sophisticated approaches.

Conclusions

- Our research employs rigorous quantitative, numerical, and statistical analysis to provide valuable insights into the **risk-return trade-offs**, the **importance of diversification**, and the **influence of correlation coefficients**.
  1. Investors seeking higher returns must accept higher levels of risk, while risk mitigation favors conservative allocations.
  2. Diversification emerges as a critical mechanism for reducing portfolio risk.
  3. The correlation between stocks plays a pivotal role in determining optimal allocation proportions: higher correlations lead to heightened risk and potentially lower returns, whereas lower correlation coefficients offer better diversification and improved risk-adjusted returns.
- Future research can explore more sophisticated models and factors to enhance accuracy and risk management.

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