Lagrange Multipliers in Portfolio Optimization with Mixed Constraints

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Abstract

Portfolio optimization is a critical and complex task in finance, aiming to achieve an optimal asset allocation that balances risk and return while considering investor objectives and constraints. In this research, we employ a methodology utilizing Lagrange multipliers with the incorporation of slack variables to construct an investment portfolio consisting of two stocks, denoted as A and B, with known expected returns and standard deviations. The main problem addresses a scenario with uncorrelated stocks and a fixed investment budget, with the objective of maximizing the portfolio's expected return while adhering to a specified risk threshold. Through comprehensive quantitative, numerical, and statistical analysis, we conduct a detailed case study to thoroughly investigate the trade-off between risk and return. To capture the benefits of diversification and enhance risk management, our research extends to incorporate correlation coefficients that reflect the inter-dependencies among stocks. By examining various correlation coefficients, we rigorously assess their impact on optimal allocation proportions and expected portfolio returns. We employ visualizations and provide Python code examples to facilitate comprehension and reproducibility. To broaden the applicability of our analysis, we derive analytical solutions for both the main problem and its extension to the general case, where all variables are treated as arbitrary. We then introduce a minimum risk threshold that ensures stock proportions remain within a realistic range, promoting the adoption of rational allocation strategies. Moreover, we acknowledge the limitations inherent in model simplicity and the challenges associated with accurately estimating expected returns. We also emphasize the influence of real-world events on stock price fluctuations and portfolio uncertainties. To address these complexities and enhance portfolio management strategies, we discuss potential future directions. Overall, this research significantly contributes to the field of portfolio optimization by incorporating correlation coefficients and conducting a thorough analysis of their effects on allocation proportions and expected returns. Supported by rigorous quantitative, numerical, and statistical analysis, our findings emphasize the importance of considering diversification benefits, implementing effective risk management strategies, and understanding the impact of correlation coefficients when optimizing investment portfolios.

Contents

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1 Introduction

1.1 Main Problem and Its Extension

Portfolio optimization is a fundamental problem in finance that seeks to determine the optimal asset combination for a portfolio, taking into account investor objectives and constraints. In this research project, we leverage the method of Lagrange multipliers to construct an optimal investment portfolio consisting of two stocks, denoted as A and B, with known expected returns and standard deviations.

The main problem can be summarized as follows: We consider a scenario with a fixed investment budget, where stocks A and B are uncorrelated and exhibit distinct and positive expected returns, indicating their potential profitability. Additionally, both stocks have positive standard deviations, which serve as measures of their individual risk levels. The primary objective is to find the optimal allocation of investments between stocks A and B that maximizes the expected return while satisfying specific constraints. These constraints ensure that all available funds are allocated to the two stocks, maintaining non-negative proportions, while also meeting a predefined threshold for portfolio risk.

In addition to addressing the main problem, we extend our analysis by introducing the consideration of a correlation coefficient while keeping all other assumptions intact. This extension is essential for a comprehensive portfolio management analysis, as real-world financial markets exhibit inter-dependencies among stocks. Through the examination of various correlation coefficients, we gain valuable insights into their effects on optimal allocation proportions and expected portfolio returns.

1.2 Methodology

The research paper is structured as follows:

- In Section 2, we present the main techniques employed to address the problem of portfolio optimization and its extension. These techniques include:
  1. Utilizing the method of Lagrange multipliers, a mathematical tool that optimizes an objective function while considering multiple equality constraints.
  2. Integrating slack variables to convert inequality constraints into equivalent equality constraints, allowing for a unified treatment of both constraint types and efficient application of the Lagrange multipliers method.
  3. Applying the Sequential Least Squares Programming (SLSQP) algorithm to validate the analytical solutions derived from the proposed problems, ensuring their reliability and accuracy.
- In Section 3, we conduct a comprehensive case study that focuses on the main problem of portfolio optimization. We select specific values for variables, including expected returns, standard deviations of each stock, and a predefined portfolio risk limit. The case study explores two different levels of portfolio risk limits: 15% and 10%. We obtain analytical solutions under these assumptions, validate their accuracy using numerical solutions derived from the SLSQP algorithm, and perform a statistical analysis to gain deeper insights into their implications.
- In Section 4, we address the general case where all variables are treated as arbitrary, allowing for scalable solutions. We derive analytical solutions representing the optimal allocation proportions of each stock and establish a formula to determine the minimum portfolio risk limit required for feasible solutions. This formula guides investment strategies by enabling the selection of suitable stocks based on desired risk levels.
- In Section 5, we extend the main problem by incorporating correlation coefficients while maintaining other assumptions. We obtain analytical solutions using the Lagrange multipliers method and conduct a statistical analysis using a 15% risk limit from the previous case study. By varying the correlation coefficient within the range of -1 to 1, we investigate its impact on optimal allocation proportions and maximum expected portfolio return, enhancing portfolio management strategies.
- In Section 6, we highlight the limitations of the proposed problems, focusing on two main aspects. Firstly, we acknowledge that the simplicity of the model architecture alone may not provide sufficient guidance for robust investment strategies. Secondly, we address the impact of real-world events on stock price fluctuations and uncertainties.
- In Section 7, we draw conclusions and insights based on our research findings. We discuss potential future directions to address the limitations identified and further improve the proposed methodologies.

2 Main Techniques

2.1 Method of Lagrange Multipliers

The Lagrange multipliers method [3] is a powerful mathematical technique used to optimize a function while considering multiple equality constraints. It provides a means to determine the maximum or minimum values of an objective function subject to these specific constraints.

Consider a scenario where we aim to find the maximum or minimum values of a function of n variables, denoted by $f(x_1, x_2, \dots, x_n)$, subject to m equality constraints of the form

$$g_i(x_1, x_2, \dots, x_n) = c_i, \tag{1}$$

where $i = 1, \dots, m$. To incorporate these constraints, we introduce m Lagrange multipliers λ_i, and form the equation:

$$\begin{aligned} &\nabla f(x_1, x_2, \dots, x_n) \\ &= \lambda_1 \nabla g_1(x_1, x_2, \dots, x_n) + \lambda_2 \nabla g_2(x_1, x_2, \dots, x_n) + \cdots + \lambda_m \nabla g_m(x_1, x_2, \dots, x_n). \end{aligned} \tag{2}$$

Here, $\nabla f(x_1, x_2, \dots, x_n)$ represents the gradient of the objective function, and $\nabla g_i(x_1, x_2, \dots, x_n)$ represents the gradient of each constraint function $g_i(x_1, x_2, \dots, x_n)$.

To solve the optimization problem using Lagrange multipliers, our objective is to determine the values of $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m$ that simultaneously satisfy the gradient equation (2) and the m equality constraint equations (1). These equations constitute a system of $n + m$ equations with $n + m$ unknowns as follows:

$$\begin{aligned} f_{x_1} &= \lambda_1 g_{1_{x_1}} + \lambda_2 g_{2_{x_1}} + \cdots + \lambda_m g_{m_{x_1}} \\ f_{x_2} &= \lambda_1 g_{1_{x_2}} + \lambda_2 g_{2_{x_2}} + \cdots + \lambda_m g_{m_{x_2}} \\ &\vdots \\ f_{x_n} &= \lambda_1 g_{1_{x_n}} + \lambda_2 g_{2_{x_n}} + \cdots + \lambda_m g_{m_{x_n}} \\ g_1(x_1, x_2, \dots, x_n) &= c_1 \\ g_2(x_1, x_2, \dots, x_n) &= c_2 \\ &\vdots \\ g_m(x_1, x_2, \dots, x_n) &= c_m. \end{aligned}$$

2.2 Slack Variables

Slack variables [1] play a vital role in optimization problems by converting inequality constraints into equivalent equality constraints. This conversion enables the application of the Lagrange multipliers method, facilitating the optimization of the objective function subject to a combination of inequality and equality constraints.

Let us consider a function denoted as $f(x_1, x_2, \dots, x_n)$, with n variables, for which we seek to determine the maximum or minimum values. However, this function is subject to m inequality constraints of the form

$$g_i(x_1, x_2, \dots, x_n) \le c_i, \tag{3}$$

where $i = 1, \dots, m$. To address these inequality constraints, we introduce m non-negative slack variables, denoted as s_i^2, associated with each inequality constraint. The purpose of these slack variables is to transform the inequality constraints into equivalent equality constraints, allowing for a unified treatment of both constraint types.

By incorporating the slack variables, each original inequality constraint (3) is expressed as an equality constraint

$$g_i(x_1, x_2, \dots, x_n) + s_i^2 = c_i.$$

We represent these newly formed equality constraints as $h_i(x_1, x_2, \dots, x_n, s_i)$, i.e., as functions involving the variables $x_1, x_2, \dots, x_n$, and the slack variable s_i.

$$h_i(x_1, x_2, \dots, x_n, s_i) = g_i(x_1, x_2, \dots, x_n) + s_i^2. \tag{4}$$

Introducing slack variables ensures that the original inequality constraints are satisfied as equality constraints. By incorporating these derived equality constraints, along with any pre-existing equality constraints, we can now apply the Lagrange multipliers method to optimize the objective function.

Similar to the treatment of equality constraints, we employ m Lagrange multipliers, denoted as λ_i, and formulate the following equation:

$$\begin{aligned}&\nabla f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)\\ &= \lambda_1 \nabla h_1(x_1, x_2, \ldots, x_n, s_1) + \lambda_2 \nabla h_2(x_1, x_2, \ldots, x_n, s_2) + \cdots + \lambda_m \nabla h_m(x_1, x_2, \ldots, x_n, s_m). \quad (5)\end{aligned}$$

Here, $\nabla f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ represents the gradient of the objective function, and $\nabla h_i(x_1, x_2, \ldots, x_n, s_i)$ represents the gradient of each newly formed equality constraint function, $h_i(x_1, x_2, \ldots, x_n, s_i)$.

To solve the optimization problem using the Lagrange multipliers method, our objective is to find the values of $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m, s_1, s_2, \ldots, s_m$ that simultaneously satisfy the gradient equation (5) and the m equality constraint equations (4). This leads to a system of $n+2m$ equations and $n + 2m$ unknown variables:

$$\begin{aligned}
f_{x_1} &= \lambda_1 g_{1_{x_1}} + \lambda_2 g_{2_{x_1}} + \cdots + \lambda_m g_{m_{x_1}}\\
f_{x_2} &= \lambda_1 g_{1_{x_2}} + \lambda_2 g_{2_{x_2}} + \cdots + \lambda_m g_{m_{x_2}}\\
&\vdots\\
f_{x_n} &= \lambda_1 g_{1_{x_n}} + \lambda_2 g_{2_{x_n}} + \cdots + \lambda_m g_{m_{x_n}}\\
f_{s_1} &= \lambda_1 g_{1_{s_1}} + \lambda_2 g_{2_{s_1}} + \cdots + \lambda_m g_{m_{s_1}}\\
f_{s_2} &= \lambda_1 g_{1_{x_2}} + \lambda_2 g_{2_{s_2}} + \cdots + \lambda_m g_{m_{s_2}}\\
&\vdots\\
f_{s_m} &= \lambda_1 g_{1_{s_m}} + \lambda_2 g_{2_{s_m}} + \cdots + \lambda_m g_{m_{s_m}}\\
h_1(x_1, x_2, \ldots, x_n, s_1) &= c_1\\
h_2(x_1, x_2, \ldots, x_n, s_2) &= c_2\\
&\vdots\\
h_m(x_1, x_2, \ldots, x_n, s_m) &= c_m.
\end{aligned}$$

2.3 Sequential Least Squares Programming (SLSQP)

The SLSQP algorithm [5] is a robust optimization method widely acclaimed for its versatility in solving diverse real-world problems. Built on rigorous mathematical principles and iterative techniques, SLSQP ensures precise and efficient optimization.

At its core, SLSQP starts with an initial guess for the variables and iteratively improves them to enhance the objective function, specifically tailored to the problem and variables involved. The algorithm incorporates equality constraints to enforce specific variable equalities and inequality constraints to impose variable inequalities. These constraints establish the feasible solution space, ensuring the satisfaction of all necessary conditions. SLSQP utilizes mathematical techniques such as calculus and least squares regression to compute gradients. These gradients capture the rate of change of the objective function and constraints with respect to the variables. They play a crucial role in determining the appropriate adjustments to the variables, facilitating the optimization process within the defined constraints. Least squares regression is employed to approximate the objective function and constraints using quadratic equations, simplifying the analysis and estimation of necessary variable changes. By fitting quadratic models to the functions using least squares regression, SLSQP identifies the best-fitting quadratic equations that minimize the squared differences between the original functions and the approximations. These quadratic approximations enable the algorithm to estimate the curvature and behavior of the functions in a local region around the current solution, facilitating a deeper understanding of the objective function and constraints. By combining calculus, gradients, and least squares regression, SLSQP optimizes the objective function while satisfying the defined constraints.

The algorithm iterates until specific convergence criteria are met, indicating a satisfactory solution. These criteria may involve observing a small change in the objective function, reaching a maximum number of iterations, or satisfying a predefined tolerance level. Once the convergence criteria are fulfilled, SLSQP provides the final optimized values for the variables, representing the best solution obtained by the algorithm for the given problem.

3 A Case Study of the Main Problem

Consider a scenario where we have a fixed budget of $100,000 to invest in two uncorrelated stocks, denoted as A and B. The expected returns and standard deviations for these stocks are as follows:

- Stock A: Expected return of 20% and standard deviation of 10%
- Stock B: Expected return of 30% and standard deviation of 20%

Our objective is to determine the optimal allocation of our investment between these two stocks in order to maximize the expected return, while satisfying the constraints imposed by our budget and risk considerations. The budget constraint requires us to allocate all of the funds to the two stocks, with non-negative proportions. Additionally, we will explore two different values for the portfolio risk limit: 15% and 10%.

3.1 Problem Formulation

Let ω_A denote the proportion invested in Stock A, and ω_B represent the proportion invested in Stock B. According to portfolio theory [2], the expected return on a portfolio is determined by taking a weighted average of the expected returns of its constituent stocks, with the portfolio proportions serving as the weights. This relationship can be expressed as

$$E(r_p) = E(r_A)\omega_A + E(r_B)\omega_B. \tag{6}$$

Here, $E(r_p)$ represents the expected return of the portfolio, $E(r_A)$ denotes the expected return of Stock A, and $E(r_B)$ represents the expected return of Stock B. As per the assumption, we have $E(r_A) \neq E(r_B)$ and $E(r_A), E(r_B) > 0$, indicating that the expected returns of Stocks A and B are distinct and positive.

Given that Stocks A and B are uncorrelated, with a correlation coefficient $\rho_{AB} = 0$, the portfolio variance, denoted as σ_p^2, is computed as follows:

$$\begin{aligned}
\sigma_p^2 &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\mathrm{Cov}(r_A, r_B) \\
&= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} \\
&= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2.
\end{aligned}$$

Here, σ_A and σ_B denote the standard deviations of Stocks A and B, respectively. Based on the assumption, we have $\sigma_A, \sigma_B > 0$, indicating that both σ_A and σ_B are positive values. In order to limit the portfolio risk to a predetermined level R, we impose the risk constraint as follows:

$$\sigma_p \leq R,$$

which can also be expressed as

$$\sigma_p^2 = \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 \leq R^2.$$

The objective is to maximize the expected return while adhering to the constraints imposed by budget and risk considerations. The budget constraint requires investing all available funds into the two stocks, with non-negative proportions. Therefore, the optimization problem for the general case, using arbitrary variables, can be formulated as follows:

$$\max \quad E(r_p) = E(r_A)\omega_A + E(r_B)\omega_B$$

subject to the following constraints:

$$\omega_A + \omega_B = 1 \tag{7}$$
$$\omega_A, \omega_B \geq 0 \tag{8}$$
$$\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 \leq R^2. \tag{9}$$

These constraints ensure that the proportions invested in Stocks A and B sum to 1, and both proportions are non-negative. Additionally, the last constraint limits the portfolio risk by constraining the weighted sum of the squared standard deviations of the stocks.

Considering the assumptions in the case study, we assign the expected returns of Stocks A and B as $E(r_A) = 0.2$ (or 20%) and $E(r_B) = 0.3$ (or 30%), respectively. The corresponding standard deviations are $\sigma_A = 0.1$ (or 10%) and $\sigma_B = 0.2$ (or 20%). To formulate the optimization problem, our objective is to maximize the expected return $E(r_p)$, which is given by:

$$\max \quad E(r_p) = 0.2\omega_A + 0.3\omega_B$$

subject to the following constraints:

$$\omega_A + \omega_B = 1 \tag{10}$$
$$\omega_A, \omega_B \geq 0 \tag{11}$$
$$0.01\omega_A^2 + 0.04\omega_B^2 \leq R^2. \tag{12}$$

3.2 Analytical Solutions

3.2.1 $R = 15\%$

Lemma 1. *The optimal allocation proportions to maximize the portfolio's expected return, subject to a portfolio risk limit of 15%, are \$26,148 invested in Stock A and \$73,852 invested in Stock B. By implementing this allocation strategy, we achieve a maximum expected return of 27.385% while managing the portfolio risk at 15%.*

Proof. Given the portfolio risk limit $R = 0.15$ (or 15%), our third constraint (12) can be expressed as

$$0.01\omega_A^2 + 0.04\omega_B^2 \leq 0.0225.$$

Consider the optimization problem where we aim to maximize the expected return of the portfolio, represented by $f(\omega_A, \omega_B) = E(r_p)$. Subject to this objective, we represent the budget constraint (10) as

$$g(\omega_A, \omega_B) = \omega_A + \omega_B - 1 = 0.$$

To handle the inequalities (11) and (12), we introduce slack variables, denoted as s_1^2, s_2^2, and s_3^2, to convert these inequalities into equivalent equality constraints as follows:

$$\begin{aligned} h(\omega_A, s_1) &= \omega_A - s_1^2 = 0 \\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0 \\ k(\omega_A, \omega_B, s_3) &= 0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 + s_3^2 = 0. \end{aligned}$$

These transformed constraints, along with the equality constraint $g(\omega_A, \omega_B) = 0$, form a unified set of four equality constraints. To incorporate these constraints, we introduce four Lagrange multipliers denoted as λ_i, where $i = 1, \ldots, 4$. The gradient equation is then formulated as:

$$\begin{aligned} &\nabla f(\omega_A, \omega_B, s_1, s_2, s_3) \\ &= \lambda_1 \nabla g(\omega_A, \omega_B) + \lambda_2 \nabla h(\omega_A, s_1) + \lambda_3 \nabla j(\omega_B, s_2) + \lambda_4 \nabla k(\omega_A, \omega_B, s_3). \end{aligned} \tag{13}$$

To solve the optimization problem using the Lagrange multipliers method, we seek the values of the nine variables $\omega_A, \omega_B, \lambda_1, \lambda_2, \lambda_3, \lambda_4, s_1, s_2, s_3$ that satisfy both the gradient equation (13) and the four equality constraint equations. This leads to a system of nine equations with nine unknown variables:

$$\begin{aligned} f_{\omega_A} &= \lambda_1 g_{\omega_A} + \lambda_2 h_{\omega_A} + \lambda_3 j_{\omega_A} + \lambda_4 k_{\omega_A} \\ f_{\omega_B} &= \lambda_1 g_{\omega_B} + \lambda_2 h_{\omega_B} + \lambda_3 j_{\omega_B} + \lambda_4 k_{\omega_B} \\ f_{s_1} &= \lambda_1 g_{s_1} + \lambda_2 h_{s_1} + \lambda_3 j_{s_1} + \lambda_4 k_{s_1} \\ f_{s_2} &= \lambda_1 g_{s_2} + \lambda_2 h_{s_2} + \lambda_3 j_{s_2} + \lambda_4 k_{s_2} \\ f_{s_3} &= \lambda_1 g_{s_3} + \lambda_2 h_{s_3} + \lambda_3 j_{s_3} + \lambda_4 k_{s_3} \\ g(\omega_A, \omega_B) &= 0 \\ h(\omega_A, s_1) &= 0 \\ j(\omega_B, s_2) &= 0 \\ k(\omega_A, \omega_B, s_3) &= 0. \end{aligned} \tag{14}$$

Upon evaluating the partial derivatives and substituting them into the respective functions, the resulting expressions are as follows:

$$0.2 = \lambda_1 + \lambda_2 + 0.02\lambda_4 \tag{15}$$

$$0.3 = \lambda_1 + \lambda_3 + 0.08\lambda_4 \tag{16}$$

$$0 = -2s_1\lambda_2 \tag{17}$$

$$0 = -2s_2\lambda_3 \tag{18}$$

$$0 = 2s_3\lambda_4 \tag{19}$$

$$\omega_A + \omega_B - 1 = 0 \tag{20}$$

$$\omega_A - s_1^2 = 0 \tag{21}$$

$$\omega_B - s_2^2 = 0 \tag{22}$$

$$0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 + s_3^2 = 0. \tag{23}$$

Considering equation (19), two cases arise: $s_3 = 0$ or $\lambda_4 = 0$.

- Case 1: $s_3 = 0$

  From equations (20) and (23), we can establish a system of two equations:

  $$\begin{aligned} \omega_A + \omega_B - 1 &= 0 \\ 0.01\omega_A^2 + 0.04\omega_B^2 - 0.0225 &= 0. \end{aligned}$$

  By solving the first equation for ω_B and substituting it into the second equation, we derive a quadratic equation in terms of ω_A as follows:

  $$0.05\omega_A^2 - 0.08\omega_A + 0.0175 = 0.$$

  Using the quadratic formula, the proportion of Stock A is calculated as

  $$\omega_A = 0.8 \pm 10\sqrt{0.0029}.$$

If $\omega_A = 0.8 + 10\sqrt{0.0029}$, then $\omega_A > 1$. However, since both $\omega_A, \omega_B \in [0, 1]$, we conclude that ω_A must be approximately 0.26148 (when choosing the negative square root). From equation (20), we find that the proportion of Stock B, denoted as ω_B, is approximately 0.73852.

- Case 2: $\lambda_4 = 0$
  By substituting $\lambda_4 = 0$ into equations (15) and (16), we obtain the following equations:

$$0.2 = \lambda_1 + \lambda_2 \tag{24}$$

$$0.3 = \lambda_1 + \lambda_3. \tag{25}$$

  Subtracting equation (24) from equation (25), we find

$$0.1 = \lambda_3 - \lambda_2.$$

  Thus, it follows that $\lambda_2 \neq \lambda_3$. From equations (17) and (18), we can deduce that exactly one of λ_3 and λ_2 must be equal to 0.

  a. $\lambda_3 = 0$
     If $\lambda_3 = 0$, then $\lambda_2 \neq 0$. Considering equation (17), we find that $s_1 = 0$. Consequently, utilizing equation (21), we obtain the proportion of Stock A as $\omega_A = 0$. Substituting this value into equation (20), we deduce that the proportion of Stock B, denoted by ω_B, equals 1. However, upon evaluating these values using equation (23), we find that s_3^2 would equal -0.0175, which is an infeasible result. Therefore, there are no solutions for this particular case.

  b. $\lambda_2 = 0$
     If $\lambda_2 = 0$, then $\lambda_3 \neq 0$. Referring to equation (18), we find that $s_2 = 0$. Consequently, utilizing equation (22), we determine that the proportion of Stock B, denoted by ω_B, is $\omega_B = 0$. Substituting this value into equation (20), we deduce that the proportion of Stock A, denoted by ω_A, equals 1. Upon evaluating these values using equation (23), we find that s_3^2 would equal 0.0125, which is a valid result. So, there is one solution for this case: $\omega_A = 1$, $\omega_B = 0$.

By evaluating the solutions (ω_A, ω_B) obtained from cases 1 and 2b on the objective function $f(\omega_A, \omega_B)$, we derive the following results:

- For the approximate values $(\omega_A, \omega_B) \approx (0.26148, 0.73852)$, the expected return of the portfolio is $f(0.26148, 0.73852) \approx 0.27385$ (or 27.385%), and the portfolio risk is $\sigma_p = 0.15$ (or 15%).
- For $(\omega_A, \omega_B) = (1, 0)$, the expected return of the portfolio is $f(1, 0) = 0.2$ (or 20%), and the portfolio risk is $\sigma_p = 0.1$ (or 10%).

Consequently, by allocating $0.26148 \cdot 100,000 = \$26,148$ to Stock A and $0.73852 \cdot 100,000 = \$73,852$ to Stock B, we achieve the maximum expected return of 27.385% with a portfolio risk of 15%. □

3.2.2 $R = 10\%$

Lemma 2. *The optimal allocation proportions to maximize the portfolio's expected return, subject to a portfolio risk limit of 10%, are \$60,000 invested in Stock A and \$40,000 invested in Stock B. By implementing this allocation strategy, we achieve a maximum expected return of 24% while managing the portfolio risk at 10%.*

Proof. To address the case study with a portfolio risk limit of 10%, we follow the same methodology as described in Section 3.2.1 for solving the case study with a risk limit of $R = 15\%$. The only modification lies in the portfolio risk constraint (12), which can now be adjusted to:

$$0.01\omega_A^2 + 0.04\omega_B^2 \leq 0.01.$$

Consequently, the equality constraint from Section 3.2.1, incorporating the slack variable s_3^2 denoted as $k(\omega_A, \omega_B, s_3)$, transforms into:

$$k(\omega_A, \omega_B, s_3) = 0.01\omega_A^2 + 0.04\omega_B^2 - 0.01 + s_3^2 = 0.$$

Solving system (14) after evaluating its partial derivatives and replacing the new equality constraint $k(\omega_A, \omega_B, s_3) = 0$ also gives us two cases: $s_3 = 0$ or $\lambda_4 = 0$.

- Case 1: $s_3 = 0$

  This leads to a new system of two equations:

  $$\begin{aligned} \omega_A + \omega_B - 1 &= 0 \\ 0.01\omega_A^2 + 0.04\omega_B^2 - 0.01 &= 0. \end{aligned}$$

  By solving the first equation for ω_B and substituting it into the second equation, we obtain a quadratic equation in terms of ω_A:

  $$0.05\omega_A^2 - 0.08\omega_A + 0.03 = 0.$$

  Using the quadratic formula, we find the values of ω_A to be:

  $$\omega_A = \{1, 0.6\}.$$

  For $\omega_A = 0.6$, we have $\omega_B = 0.4$. As a result, the expected return of the portfolio is $f(0.6, 0.4) = 0.24$ (or 24%), while the portfolio risk σ_p remains at the prescribed level of 10%.

  In the case where $\omega_A = 1$, we find $\omega_B = 0$. This allocation solution is consistent with the findings presented in Section 3.2.1. Consequently, the expected return of the portfolio remains unchanged at $f(1, 0) = 0.2$ (or 20%), while the portfolio risk σ_p continues to be at 10%.

- Case 2: $\lambda_4 = 0$

  We can deduce from equations (15) and (16) that $\lambda_2 \neq \lambda_3$. This implies that exactly one of λ_2 and λ_3 must be equal to 0. Considering these two cases, we find two solutions for $(\omega_A, \omega_B) = \{(1, 0), (0, 1)\}$.

  If $(\omega_A, \omega_B) = (0, 1)$, we obtain $s_3^2 = -0.03$, which leads to a contradiction. So, for this case, we only retain the solution $(\omega_A, \omega_B) = (1, 0)$, which coincides with one of the solutions in Case 1.

Therefore, by allocating \$60,000 to Stock A and \$40,000 to Stock B, we attain the maximum expected return of 24% while maintaining a portfolio risk of 10%. □

3.2.3 Portfolio Allocation Strategies

In this section, we delve into the analysis of portfolio allocation strategies and emphasize the fundamental trade-off between risk and return in portfolio management. We explore two different allocation scenarios, $(\omega_A, \omega_B) = (0.6, 0.4)$ and $(\omega_A, \omega_B) = (0.26148, 0.73852)$, as presented in Table 1, to investigate the impact of different allocations on expected return and portfolio risk.

For the allocation of 60% of the portfolio to Stock A and 40% to Stock B, the portfolio achieves an expected return of 24% with a corresponding risk level of 10%. This allocation strategy aims for a balanced approach to risk and return, as reflected by the moderate level of portfolio risk relative to the expected return. It offers growth potential while maintaining a relatively lower level of risk. On the other hand, with an allocation of approximately 26.148% to Stock A and 73.852% to Stock B, the portfolio demonstrates a higher expected return of 27.385%. However, this allocation also

Table 1: Portfolio Performance

(ω_A, ω_B)	Portfolio Expected Return (%)	Portfolio Risk (%)
$(0.6, 0.4)$	24	10
$(0.26148, 0.73852)$	27.385	15

leads to an increased portfolio risk of 15%. The higher expected return in this scenario comes with elevated risk, indicating a potential for greater returns but with a higher likelihood of fluctuations in the portfolio value. These results emphasize the fundamental trade-off between risk and return in portfolio management. Investors seeking higher returns must be willing to accept higher levels of risk, while those inclined towards risk mitigation may opt for more conservative allocations.

It is important to note that the presented allocations are specific examples based on the hypothetical data. The ideal allocation proportions can vary significantly based on factors such as investors' risk preferences, expected returns of individual stocks, and correlations among stock returns.

3.3 Numerical Solutions

To validate the analytical solutions for the main problem case study presented in Section 3.2, we utilize the SLSQP Algorithm 1 implemented in Python code 1 (see Appendix 8).

Algorithm 1 SLSQP Algorithm for the Case Study

1: Import the necessary libraries, including **"numpy"** for mathematical operations and **"scipy.optimize"** for optimization methods.
2: Define the objective function, which represents the quantity we want to maximize. In this case, the objective function is defined as the negative sum of the products of allocation percentages and corresponding expected returns. The negative sign is used to convert the maximization problem into a minimization problem.
3: Define the equality constraint, which ensures that the allocation percentages sum up to 1. This constraint enforces the condition that all the available budget must be invested in the two stocks.
4: Define the inequality constraint, which limits the portfolio risk. It is formulated as an inequality expression based on the quadratic equation representing the portfolio risk. The inequality ensures that the risk does not exceed the predefined value.
5: Set the initial guess for the allocation percentages. Here, both allocations are initialized as 0.
6: Set the bounds for the allocation percentages. The bounds restrict the range within which the allocation percentages can vary. In this case, the bounds are set between 0 and 1, indicating that the proportions must be non-negative and cannot exceed 100%.
7: Define the constraints by specifying their type (equality or inequality) and the corresponding constraint functions.
8: Solve the optimization problem using the SLSQP algorithm. The algorithm aims to minimize the objective function while satisfying the defined constraints. The initial guess, bounds, and constraints are provided as inputs to the algorithm.
9: Obtain the optimal solution from the optimization result. The optimal solution represents the allocation percentages that maximize the expected return while satisfying the constraints.
10: Print the optimal solution, displaying the calculated allocation percentages for the two stocks.
11: Calculate the maximum value of the objective function, which corresponds to the maximum expected return of the portfolio. The negative sign of the objective function is inverted to obtain the positive value.
12: Print the maximum value of the objective function, indicating the maximum expected return achievable with the given constraints.

3.4 Statistical Analysis

In this section, we present a comprehensive statistical analysis of the case study, focusing on the relationship between stock allocation and portfolio risk. We begin by assuming uncorrelated stocks with a correlation coefficient of 0. Under this assumption, the portfolio variance (σ_p^2) can be derived as

$$\sigma_p^2 = \omega_A^2 \sigma_A^2 + \omega_B^2 \sigma_B^2,$$

where ω_A and ω_B represent the proportions of Stocks A and B, respectively, and σ_A^2 and σ_B^2 represent their variances.

To express the proportion of Stock B in terms of the proportion of Stock A, we substitute ω_B with $1 - \omega_A$, which allows us to compute the portfolio standard deviation (σ_p) as

$$\sigma_p = \sqrt{\omega_A^2 \sigma_A^2 + (1 - \omega_A)^2 \sigma_B^2}.$$

Figure 1 (generated using Python code 2) visually illustrates the relationship between portfolio risk (σ_p) and the proportion of Stock A.

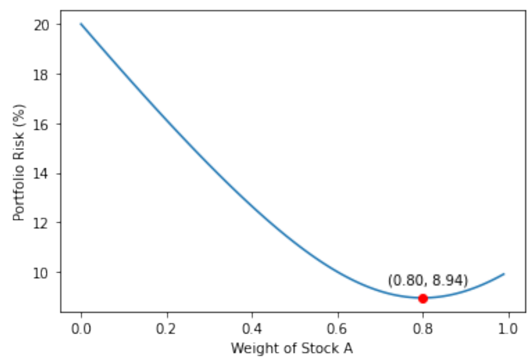

Figure 1: Portfolio Risk as a Function of Proportion of Stock A.

When considering the standard deviations of Stocks A (10%) and B (20%), we observe that Stock A exhibits lower investment risk. Figure 1 highlights an interesting finding: minimizing portfolio risk does not require investing solely in the less risky stock. By reducing the weight of Stock A from 1 to 0.8, the expected return of the portfolio increases while the portfolio risk falls below Stock A's risk level (10%), reaching 8.94%. This finding emphasizes the importance of diversification, spreading investments across multiple assets. Allocating 20% of the initially Stock A-dominated portfolio to Stock B leads to a risk reduction of over 1% (Figure 1), while enhancing the expected portfolio return by 2%.

To further analyze the impact of different stock allocations on portfolio risk, we introduce Figure 2 that is generated using Python code 3. This graph demonstrates the optimal weighting of ω_A and ω_B as a function of portfolio risk tolerance. Analyzing Figure 2, we observe that as the level of portfolio risk tolerance increases, the optimal investment in Stock B rises. At a tolerance level of 20%, Stock B becomes the sole constituent of the portfolio. This behavior arises from the inclusion of stocks with higher standard deviation when there is a greater risk tolerance. In addition, allocating a larger proportion to the more volatile Stock B results in an increased expected return due to its higher expected return compared to Stock A. These findings align with our previous analysis in Section 3.2.3, highlighting the trade-off between risk and return in portfolio management. Investors seeking higher returns must accept higher levels of risk, while those prioritizing risk mitigation may prefer more conservative allocations. The optimal allocation proportions depend on individual goals, risk tolerance, and investment strategies.

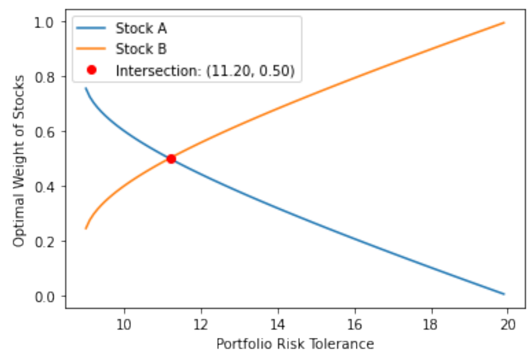

Figure 2: Optimal Proportions of Stocks A and B as Functions of Portfolio Risk Tolerance.

4 Main Problem using Arbitrary Variables

4.1 Analytical Solution

Proposition 1. *Assuming positive proportions for both Stocks A and B, the optimal allocation proportions that maximize the expected return of the portfolio are*

$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2} \quad \textit{and} \quad \omega_B = 1 - \omega_A.$$

Proof. The general case using arbitrary variables follows a similar approach to the previous case study. Our objective is to maximize the expected return of the portfolio, represented by equation (6), subject to the mixed constraints (7), (8), and (9).

Let $f(\omega_A, \omega_B) = E(r_p)$ denote the expected return of the portfolio to be maximized, and

$$g(\omega_A, \omega_B) = \omega_A + \omega_B - 1 = 0$$

represent the budget constraint. To convert the inequality constraints (8) and (9) into equivalent equality constraints, we introduce three non-negative slack variables: s_1^2, s_2^2, and s_3^2. The inequality constraints are then expressed as follows:

$$\begin{aligned} h(\omega_A, s_1) &= \omega_A - s_1^2 = 0 \\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0 \\ k(\omega_A, \omega_B, s_3) &= \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 + s_3^2 = 0. \end{aligned}$$

We introduce Lagrange multipliers λ_i for $i = 1, \ldots, 4$ and formulate the gradient equation as in equation (13).

To solve the optimization problem using the Lagrange multipliers method, we seek the values of the nine variables: $\omega_A, \omega_B, \lambda_1, \lambda_2, \lambda_3, \lambda_4, s_1, s_2, s_3$. These values must satisfy the gradient equation (13) and the four equality constraint equations, resulting in system (14) of nine equations. Solving this system yields the values of the nine unknown variables. By evaluating the partial derivatives and performing the necessary substitutions, we derive the following system of equations:

$$\begin{align}
E(r_A) &= \lambda_1 + \lambda_2 + 2\sigma_A^2\omega_A\lambda_4 \tag{26}\\
E(r_B) &= \lambda_1 + \lambda_3 + 2\sigma_B^2\omega_B\lambda_4 \tag{27}\\
0 &= -2s_1\lambda_2 \tag{28}\\
0 &= -2s_2\lambda_3 \tag{29}\\
0 &= 2s_3\lambda_4 \tag{30}\\
\omega_A + \omega_B - 1 &= 0 \tag{31}\\
\omega_A - s_1^2 &= 0 \tag{32}\\
\omega_B - s_2^2 &= 0 \tag{33}\\
\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 + s_3^2 &= 0. \tag{34}
\end{align}$$

By examining equation (30), we observe two possible scenarios: either $s_3 = 0$ or $\lambda_4 = 0$.

- Case 1: $s_3 = 0$
  From equations (31) and (34), we establish a system of two equations:

  $$\begin{aligned}
  \omega_A + \omega_B - 1 &= 0\\
  \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 &= 0.
  \end{aligned}$$

  By solving the first equation for ω_B and substituting the result into the second equation, we derive the following quadratic equation:

  $$(\sigma_A^2 + \sigma_B^2)\omega_A^2 - (2\sigma_B^2)\omega_A + (\sigma_B^2 - R^2) = 0. \tag{35}$$

  By utilizing the quadratic formula on equation (35), we obtain the proportion for Stock A as follows:

  $$\begin{aligned}
  \omega_A &= \frac{2\sigma_B^2 \pm \sqrt{4\sigma_B^4 - 4(\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{2(\sigma_A^2 + \sigma_B^2)}\\
  &= \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2}.
  \end{aligned} \tag{36}$$

  Referring to equation (31), the proportion for Stock B can be determined as below:

  $$\begin{aligned}
  \omega_B &= 1 - \omega_A\\
  &= 1 - \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2}.
  \end{aligned}$$

- Case 2: $\lambda_4 = 0$
  By substituting $\lambda_4 = 0$ into equations (26) and (27), we obtain a system:

  $$\begin{align}
  E(r_A) &= \lambda_1 + \lambda_2 \tag{37}\\
  E(r_B) &= \lambda_1 + \lambda_3. \tag{38}
  \end{align}$$

  By subtracting equation (37) from equation (38), we get

  $$E(r_B) - E(r_A) = \lambda_3 - \lambda_2. \tag{39}$$

  Given the assumption that the expected return of Stock A and Stock B are distinct, it follows that λ_3 and λ_2 are not equal. By analyzing equations (28), (29), and (39), it becomes evident that precisely one of λ_3 and λ_2 must be equal to zero.

a. $\lambda_3 = 0$
If $\lambda_3 = 0$, we can conclude from equation (39) that λ_2 is non-zero. Given that $\lambda_2 \neq 0$, equation (28) implies that $s_1 = 0$. Substituting $s_1 = 0$ into equation (32) allows us to determine the proportion for Stock A, which is $\omega_A = 0$. Furthermore, considering that the sum of proportions should be 1, according to equation (31), the proportion for Stock B is $\omega_B = 1$.

b. $\lambda_2 = 0$
If $\lambda_2 = 0$, we can deduce from equation (39) that λ_3 is non-zero. Given that $\lambda_3 \neq 0$, equation (29) implies that $s_2 = 0$. Substituting $s_2 = 0$ into equation (33) allows us to determine that the proportion for Stock B is $\omega_B = 0$. Based on equation (31), the proportion for Stock A is $\omega_A = 1$.

Considering the scenarios outlined in cases 1, 2a, and 2b, we evaluate the pairs of solutions (ω_A, ω_B) using the objective function (6). Consequently, we obtain three corresponding values of the expected return of the portfolio as follows:

- For the case where
$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2} \quad \text{and} \quad \omega_B = 1 - \omega_A,$$
we consider only the scenario where ω_A is non-negative. In this case, the expected return of the portfolio is given by $f(\omega_A, \omega_B) = E(r_A)\omega_A + E(r_B)\omega_B$, and the portfolio risk is determined by $\sigma_p = \sqrt{\omega_A^2 \sigma_A^2 + \omega_B^2 \sigma_B^2}$.
- When $(\omega_A, \omega_B) = (0, 1)$, the expected return of the portfolio is $f(0, 1) = E(r_B)$, and the portfolio risk is σ_B.
- When $(\omega_A, \omega_B) = (1, 0)$, the expected return of the portfolio is $f(1, 0) = E(r_A)$, and the portfolio risk is σ_A.

Given the provided values for expected returns and standard deviations of each stock, along with the predefined portfolio risk, we can compare the values of $f(\omega_A, \omega_B)$ obtained from the three solutions (ω_A, ω_B) described earlier. Through this comparison, we can identify the optimal allocation proportions that maximize the expected return of the portfolio. □

4.2 Lower Bound of the Portfolio Risk

Lemma 3. *The minimum portfolio risk limit required to obtain feasible solutions for the main problem involving two uncorrelated stocks A and B is*

$$R_{\min} = \sqrt{\frac{\sigma_A^2 \sigma_B^2}{\sigma_A^2 + \sigma_B^2}}.$$

Proof. By examining equation (36), which determines the proportion of Stock A, a square root is present. Since we must ensure that the proportions result in real-valued solutions, it is necessary to maintain a non-negative discriminant, given by $\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)$. Consequently, our objective is to establish the condition under which this discriminant is non-negative, guaranteeing the validity of the solutions for the stock proportions.

$$\begin{aligned} &\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2) \geq 0 \\ \Leftrightarrow\ &\sigma_B^4 \geq (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2) \\ \Leftrightarrow\ &\sigma_B^4 \geq \sigma_B^2 \sigma_A^2 + \sigma_B^4 - R^2(\sigma_A^2 + \sigma_B^2) \\ \Leftrightarrow\ &R^2 \geq \frac{\sigma_A^2 \sigma_B^2}{\sigma_A^2 + \sigma_B^2}. \end{aligned} \tag{40}$$

Therefore, in order to ensure the feasibility of finding the proportions of stocks, there exists a minimum portfolio risk limit of $R_{\min} = \sqrt{\frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2}}$ under the given assumption.

To validate condition (40) for determining the lower bound of the portfolio risk, we can employ an alternative approach by analyzing the portfolio risk constraint (9) in the main problem. Let us define the function for portfolio variance as:

$$r(\omega_A, \omega_B) = \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2.$$

Given that variance is a metric for evaluating portfolio risk, our objective is to minimize this function, $r(\omega_A, \omega_B)$. By rearranging the budget constraint (7), we can express the proportion of Stock A in terms of the proportion of Stock B as $\omega_A = 1 - \omega_B$. Substituting this relationship into the function r, we obtain a simplified expression:

$$\begin{aligned} r(\omega_A) &= \sigma_A^2\omega_A^2 + \sigma_B^2(1-\omega_A)^2 \\ &= (\sigma_A^2+\sigma_B^2)\omega_A^2 - (2\sigma_B^2)\omega_A + \sigma_B^2. \end{aligned} \tag{41}$$

To determine the critical point(s) of r, we can proceed by setting the derivative of r equal to zero.

$$\begin{aligned} r'(\omega_A^*) = 0 &\Leftrightarrow 2(\sigma_A^2+\sigma_B^2)\omega_A^* - (2\sigma_B^2) = 0 \\ &\Leftrightarrow \omega_A^* = \frac{\sigma_B^2}{\sigma_A^2+\sigma_B^2}. \end{aligned}$$

By substituting this value of ω_A^* into equation (41), we obtain:

$$\begin{aligned} r(\omega_A^*) &= (\sigma_A^2+\sigma_B^2)\left(\frac{\sigma_B^2}{\sigma_A^2+\sigma_B^2}\right)^2 - (2\sigma_B^2)\left(\frac{\sigma_B^2}{\sigma_A^2+\sigma_B^2}\right) + \sigma_B^2 \\ &= \frac{\sigma_B^4}{\sigma_A^2+\sigma_B^2} - \frac{2\sigma_B^4}{\sigma_A^2+\sigma_B^2} + \frac{\sigma_B^2(\sigma_A^2+\sigma_B^2)}{\sigma_A^2+\sigma_B^2} \\ &= \frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2}. \end{aligned}$$

According to the Second Derivative Test, since

$$r''(\omega_A) = 2(\sigma_A^2+\sigma_B^2) > 0,$$

the minimum limit for the portfolio risk is

$$R_{\min} = \sqrt{r\left(\frac{\sigma_B^2}{\sigma_A^2+\sigma_B^2}\right)} = \sqrt{\frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2}},$$

which coincides with the result obtained from the first approach using the discriminant of ω_A. □

Setting a lower bound for the portfolio risk is necessary to ensure the feasibility and practicality of the problem. Choosing an extremely small portfolio risk can lead to impractical or unattainable solutions. In addition, by introducing a minimum portfolio risk threshold, we enforce a constraint that promotes the development of a balanced and diversified portfolio.

5 Extension of the Main Problem

In this expanded section of the main problem, we extend our analysis by incorporating a correlation coefficient while maintaining all other assumptions outlined in Section 1.1. Our focus shifts towards examining a scenario involving two correlated stocks. In real-world financial markets, stocks are rarely uncorrelated, as there are inter-dependencies among companies and industries. By including correlated stocks, we enhance the comprehensiveness of our portfolio management analysis based on a thorough evaluation of risk, return, and diversification considerations.

5.1 Analytical Solution

Proposition 2. *Assuming that both stocks A and B must have positive proportions, the optimal proportion for Stock A that maximizes the expected return of the portfolio in the extended scenario is*

$$\omega_A = \frac{(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}) \pm \sqrt{(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB})^2 - (\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB})(\sigma_B^2 - R^2)}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}}.$$

The proportion for Stock B can be derived by subtracting the proportion for Stock A from 1.

Proof. We now proceed to formulate the problem for the extended scenario, which retains the same setup as described in Section 3.1. In this case, we consider the correlation between two stocks, denoted as stocks A and B, with a correlation coefficient of ρ_{AB}. The portfolio variance can then be calculated using the following formula:

$$\begin{aligned}\sigma_p^2 &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\mathrm{Cov}(r_A, r_B)\\ &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB}.\end{aligned}$$

Here, the correlation coefficient ρ_{AB} may take a non-zero value. As a result, the portfolio risk constraint, previously expressed as (9), will be modified to the following inequality constraint:

$$\omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} \le R^2. \tag{42}$$

Despite the modification to the portfolio risk constraint, we can apply the same process outlined in Section 4.1 to solve this optimization problem.

Our objective is to maximize the expected return of the portfolio, as represented by equation (6), subject to the mixed constraints (7), (8), and (42). Following a similar approach as in Section 4.1, we define $f(\omega_A, \omega_B) = E(r_p)$ to denote the expected return of the portfolio to be maximized, and $g(\omega_A, \omega_B) = \omega_A + \omega_B - 1 = 0$ as the budget constraint. In order to convert the inequality constraints (8) and (42) into equality constraints, we introduce slack variables: s_1^2, s_2^2, and s_3^2. The inequality constraints can then be expressed as follows:

$$\begin{aligned}h(\omega_A, s_1) &= \omega_A - s_1^2 = 0\\ j(\omega_B, s_2) &= \omega_B - s_2^2 = 0\\ k(\omega_A, \omega_B, s_3) &= \omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} - R^2 + s_3^2 = 0.\end{aligned}$$

Using Lagrange multipliers λ_i for $i = 1, \ldots, 4$, we formulate the gradient equation as shown in equation (13). To solve the optimization problem with the Lagrange multipliers method, we determine the values of the variables: $\omega_A, \omega_B, \lambda_1, \lambda_2, \lambda_3, \lambda_4, s_1, s_2, s_3$. These values must satisfy the gradient equation (13) and the four equality constraint equations. Upon evaluating the partial derivatives and making the necessary substitutions, we derive a system of nine equations, which aligns with the equations presented in Section 4.1. However, it is important to note that the final equation (34), pertaining to the portfolio risk constraints following the conversion, will undergo modifications and can be expressed as follows:

$$\omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} - R^2 + s_3^2 = 0. \tag{43}$$

After solving the system, we obtain two solutions $(\omega_A, \omega_B) = \{(0, 1), (1, 0)\}$ that are identical to those presented in Section 4.1. To determine the remaining solution, we set s_3 to zero in equation (43) and solve the following system of two equations:

$$\begin{aligned}\omega_A + \omega_B - 1 &= 0\\ \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{AB} - R^2 &= 0.\end{aligned}$$

By solving the first equation for ω_B and substituting the result into the second equation, we derive the following expression:

$$\sigma_A^2\omega_A^2 + \sigma_B^2(1-\omega_A)^2 + 2\omega_A(1-\omega_A)\sigma_A\sigma_B\rho_{AB} - R^2 = 0$$

$$\Leftrightarrow \sigma_A^2\omega_A^2 + \sigma_B^2(1-2\omega_A+\omega_A^2) + (2\omega_A - 2\omega_A^2)\sigma_A\sigma_B\rho_{AB} - R^2 = 0$$

$$\Leftrightarrow (\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB})\omega_A^2 - 2(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB})\omega_A + (\sigma_B^2 - R^2) = 0. \tag{44}$$

Applying the quadratic formula on equation (44), we get the proportion for Stock A as follows:

$$\begin{aligned}\omega_A &= \frac{2(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}) \pm \sqrt{4(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB})^2 - 4(\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB})(\sigma_B^2 - R^2)}}{2(\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB})} \\ &= \frac{(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}) \pm \sqrt{(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB})^2 - (\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB})(\sigma_B^2 - R^2)}}{\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB}}. \end{aligned} \tag{45}$$

So, the proportion for Stock B can be determined by subtracting the proportion for Stock A from 1, as given by the formula (45). Note that we only consider the root(s) where ω_A is non-negative. $\square$

5.2 Statistical Analysis

In the realm of portfolio management, the correlation coefficient ρ_{AB} assumes a critical role in elucidating the relationship between the returns of stocks A and B. Ranging from -1 to +1, the correlation coefficient quantifies the extent of linear association between their returns. A value of +1 denotes a perfect positive linear relationship, signifying that as the return of Stock A increases, the return of Stock B also rises proportionally, and vice versa. Conversely, a correlation coefficient of -1 indicates a perfect negative linear relationship, where the returns of Stocks A and B move in opposite directions with a constant proportional change. A correlation coefficient of 0 implies no linear relationship, indicating the independence of Stock A and Stock B in terms of their returns.

In the current analysis of the main problem case study, we incorporate a risk constraint of $R = 15\%$ and explore the impact of the correlation coefficient ρ_{AB} on the optimal allocation proportions of Stocks A and B as well as the expected portfolio returns. To facilitate this analysis, we construct two graphs. Figure 3 visualizes the maximum expected return of the portfolio as a function of the correlation coefficient, providing valuable insights into the effect of changes in correlation on portfolio performance (generated using Python code 4). Additionally, Figure 4 illustrates the optimal stock weights, specifically the proportions of Stocks A and B, as a function of the correlation coefficient, highlighting the shifting allocation strategies with varying correlations (generated using Python code 5).

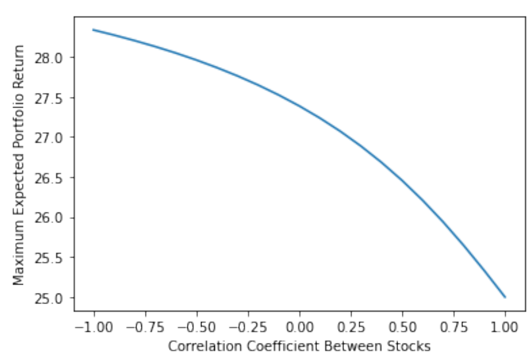

Figure 3: Maximum Portfolio Expected Return as a Function of Correlation Coefficient.

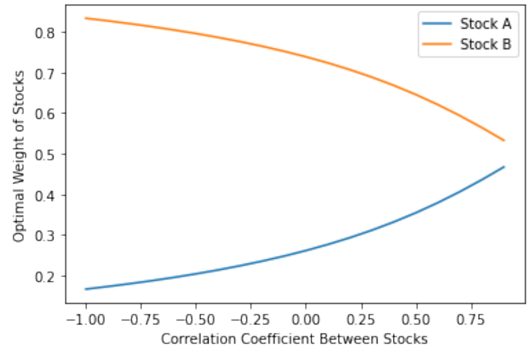

Figure 4: Optimal Stock Weights as a Function of Correlation Coefficient.

As the correlation between Stocks A and B increases, the portfolio's risk tends to rise. This can be attributed to the term $2\omega_A\omega_B\text{Cov}(r_A, r_B)$ in the portfolio risk calculation, assuming constant stock weights. To maintain the risk within the specified constraint, it becomes necessary to reduce the weight of the riskier stock (Stock B) in the portfolio. Consequently, the proportion of Stock B decreases while the proportion of Stock A increases. As Stock B typically offers higher profitability, the reduction in its weight leads to a decrease in the maximum expected return of the portfolio, as depicted in Figure 3.

It is crucial to emphasize that as the correlation coefficient increases, the diversification benefit derived from allocating both stocks diminishes. A strong positive correlation between Stocks A and B implies a heavy reliance on their individual performance, rendering the portfolio vulnerable to systematic risks. Conversely, a strong negative correlation signifies that Stocks A and B tend to move in opposite directions, providing some hedging benefits but potentially limiting the overall return potential of the portfolio.

The findings from Figures 3 and 4 underscore the intricate relationship between the correlation coefficient, optimal allocation proportions, and expected portfolio returns. Investors must carefully consider the trade-off between risk and return when making investment decisions. A higher correlation coefficient implies greater risk and potentially lower expected returns. Conversely, a lower correlation coefficient offers greater diversification benefits and potential risk reduction, leading to improved risk-adjusted returns. Ultimately, determining the optimal allocation proportions relies on individual risk tolerance, investment objectives, and preferences. It is crucial to consider factors beyond the correlation coefficient, such as the characteristics of individual stocks, sector diversification, and non-linear dependencies, to make well-informed investment decisions.

6 Limitations

In this section, we acknowledge the limitations inherent in the proposed research problems, with a focus on two pivotal aspects. Firstly, while the model architecture serves as a valuable analytical tool, we recognize that its simplicity may not provide adequate guidance for robust investment strategies. Real-world investment scenarios entail intricate factors, including market dynamics, investor behavior, and evolving economic conditions, which introduce additional sources of risk and uncertainty not explicitly captured by the model. The model's ability to comprehend the complexities of the investment landscape may be limited, highlighting the need for more sophisticated approaches.

Secondly, the impact of real-world events on stock price fluctuations and uncertainties presents a substantial challenge. The proposed problems optimize stock allocations based on hypothetical data, which by its nature cannot account for unforeseen events that can significantly influence investment outcomes. For instance, the COVID-19 pandemic and recent advancements in artificial intelligence

(AI) have showcased how unexpected events can disrupt markets and lead to unpredictable shifts in stock prices [4, 6, 8]. These events defy reliable prediction on a regular basis, posing a challenge for incorporating them into the optimization process. Furthermore, macroeconomic conditions play a vital and influential role in asset returns, introducing volatility and shaping market dynamics. The decisions made by central banks, such as changes in interest rates, hold particular significance in determining stock valuations [7, 9]. It is worth emphasizing that the consequences of these decisions manifest non-uniformly across markets, given the disparate responses observed among various sectors and industries. This intricate interplay between economic conditions and monetary policies introduces an additional layer of complexity and non-linearity, thereby underscoring the limitations inherent in our mathematical analysis when it comes to comprehensively capturing the complete picture.

7 Conclusions

Portfolio optimization is a complex and crucial task in finance, aiming to achieve an optimal asset allocation that balances risk and return while considering investor objectives and constraints. In this research, we employ the methodology utilizing Lagrange multipliers with the incorporation of slack variables to construct an investment portfolio consisting of two stocks, characterized by known expected returns and standard deviations. Through rigorous quantitative, numerical, and statistical analysis, we have derived significant and robust findings, emphasizing the fundamental trade-off between risk and return.

Our study highlights that investors seeking higher returns must be willing to accept higher levels of risk, whereas those prioritizing risk mitigation may opt for more conservative allocations. We showcase various portfolio allocation strategies and their corresponding risk-return profiles, emphasizing the critical role of diversification in reducing portfolio risk and enhancing expected returns. Notably, our analysis demonstrates that optimal allocation proportions depend on the correlation between stocks. Higher correlations correspond to increased risk and potentially lower expected returns, as the portfolio becomes more reliant on individual stock performance. Conversely, lower correlation coefficients offer better diversification and potential risk reduction, leading to improved risk-adjusted returns.

To extend the applicability of our analysis, we move beyond the specific case study and address the general case, treating all variables as arbitrary, thus enabling scalable solutions. Through rigorous mathematical derivations, we obtain analytical solutions that precisely determine the optimal allocation proportions for each stock in the portfolio. Additionally, we present a formula that identifies the minimum portfolio risk limit required for feasible solutions. By introducing this minimum risk threshold, our methodology promotes the development of a balanced and diversified portfolio, ensuring that stock proportions remain within a realistic range and encouraging the adoption of rational allocation strategies.

While our research provides valuable insights, we acknowledge certain limitations and propose avenues for future improvement. The simplicity of our model architecture may not fully capture the complexities of real-world investment scenarios. To address this, future research could explore the incorporation of additional factors and more sophisticated models to enhance the accuracy of expected return estimation and risk management strategies. Furthermore, the impact of external events and market uncertainties on stock price fluctuations presents a challenge within our analysis. Developing dynamic risk management techniques that account for unforeseen market shocks and unexpected events would bolster the robustness and adaptability of our methodologies.

In conclusion, our comprehensive case study on portfolio optimization, supported by rigorous quantitative, numerical, and statistical analysis, offers valuable insights into risk-return trade-offs, the importance of diversification, and the influence of correlation coefficients. By extending our analysis to the general case, we present scalable solutions applicable to diverse portfolios. However, to further enhance the practicality and effectiveness of portfolio optimization, it is essential to address the identified limitations and advance our methodologies. By incorporating additional factors, sophisticated models, and dynamic risk management techniques, future research can develop more robust, accurate, and adaptable investment strategies.

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8 Appendix

This section contains all the Python codes used to generate the graphs presented in the main paper, as well as numerical solutions to validate the analytical solutions for the case study of the main problem.

1. Code 1 is used to implement the generalized SLSQP Algorithm 1. The algorithm takes into account a budget constraint, requiring that all funds be invested in these n stocks, and ensures that the proportion allocated to each stock is non-negative. Additionally, it considers a portfolio risk constraint. By providing input parameters such as the number of stocks to consider, their respective expected returns, standard deviations, and a predetermined risk tolerance, the algorithm can accurately determine the optimal allocation of funds.

2. Code 2 is utilized to generate Figure 1 that illustrates the relationship between the portfolio risk and the proportion of Stock A. This graph visually represents the impact of varying the allocation of Stock A on the resulting portfolio risk.

```python
import matplotlib.pyplot as plt
import numpy as np

# Set the x-coordinates from 0 to 1 with a step size of 0.01
x = np.arange(0, 1, 0.01)

# Calculate the corresponding y-coordinates
y = np.sqrt(100 * x ** 2 + 400 * (1 - x) ** 2)

# Find the lowest point on the graph
lowest_point_idx = np.argmin(y)
lowest_point_x = x[lowest_point_idx]
lowest_point_y = y[lowest_point_idx]

# Set labels for the plot
plt.xlabel('Weight of Stock A')
plt.ylabel('Portfolio Risk (%)')

# Plot the points
plt.plot(x, y)

# Mark the lowest point on the graph
plt.plot(lowest_point_x, lowest_point_y, 'ro')

# Annotate the coordinates of the lowest point
plt.annotate(f'({lowest_point_x:.2f}, {lowest_point_y:.2f})', (lowest_point_x, lowest_point_y),
             textcoords="offset points", xytext=(4, 10), ha='center')

# Display the plot
plt.show()
```

Code 2: Graph of Portfolio Risk as a Function of Proportion of Stock A.

3. Code 3 is employed to generate Figure 2 that visually represents the optimal proportions of Stocks A and B as a function of the portfolio risk tolerance. This graph provides valuable insights into the allocation strategy based on risk tolerance.

```python
import matplotlib.pyplot as plt
import numpy as np

# Generate x-coordinates from 9 to 20 with a step size of 0.1
x = np.arange(9, 20, 0.1)

# Calculate y-coordinates
y1 = (800 - np.sqrt(800 ** 2 - 4 * 500 * (400 - x ** 2))) / 1000
y2 = 1 - y1

# Find the coordinates of the intersection point
intersection_x = x[np.argmin(np.abs(y1 - y2))]
intersection_y = y1[np.argmin(np.abs(y1 - y2))]

# Plot the line for Stock A
plt.plot(x, y1, label='Stock A')

# Plot the line for Stock B
plt.plot(x, y2, label='Stock B')

# Plot the intersection point
plt.plot(intersection_x, intersection_y, 'ro')

# Set the labels for the x-axis and y-axis
plt.xlabel('Portfolio Risk Tolerance')
plt.ylabel('Optimal Weight of Stocks')

# Add a legend with the intersection point label
legend_label = f'Intersection: ({intersection_x:.2f}, {intersection_y:.2f})'
plt.legend(labels=['Stock A', 'Stock B', legend_label], loc='best')

# Display the plot
plt.show()
```

Code 3: Graph of Optimal Proportions of Stocks A and B Based on Portfolio Risk Tolerance.

4. Code 4 is employed to generate Figure 3 depicting the maximum expected return of the portfolio as a function of the correlation coefficient. This graph offers valuable insights into the influence of the correlation coefficient on the portfolio's performance.

```python
import numpy as np
from scipy.optimize import minimize
import matplotlib.pyplot as plt

def maximize_return(correlation):
    # Define the objective function to maximize (expected return)
    def expected_return(weights):
        weight_a, weight_b = weights
        return -1 * (20 * weight_a + 30 * weight_b)

    # Define the budget constraint
    def budget_constraint(weights):
        weight_a, weight_b = weights
        return weight_a + weight_b - 1

    # Define the risk constraint
    def risk_constraint(weights):
        weight_a, weight_b = weights
        return -(100 * weight_a ** 2 + 400 * weight_b ** 2 + \
                 2 * weight_a * weight_b * 10 * 20 * correlation - 225)

    # Specify the initial guess and bounds for weights
    initial_weights = [0, 0]
    weight_bounds = [[0, 1], [0, 1]]

    # Define the constraints as a list of dictionaries
    constraints = [
        {'type': 'eq', 'fun': budget_constraint},
        {'type': 'ineq', 'fun': risk_constraint}
    ]

    # Execute the optimization to find the optimal weight for Stock A
    result = minimize(expected_return, initial_weights, method='SLSQP',
                      constraints=constraints, bounds=weight_bounds,
                      options={'ftol': 1e-6})

    return -1 * result.fun

# Generate correlation coefficients from -1 to 1 with a step size of 0.1
x = np.round(np.arange(-1, 1.1, 0.1), 1)

# Calculate the corresponding maximum expected return
y = [round(maximize_return(coeff), 5) for coeff in x]

# Plotting the results
plt.xlabel('Correlation Coefficient Between Stocks')
plt.ylabel('Maximum Expected Portfolio Return')

plt.plot(x, y)
plt.show()
```

Code 4: Graph of Maximum Portfolio Expected Return as a Function of Correlation Coefficient.

5. Code 5 is utilized to generate Figure 4 depicting the optimal proportions of Stocks A and B as a function of the correlation coefficient. This graph showcases the dynamic allocation strategies that evolve as the correlation coefficient changes.

```python
import matplotlib.pyplot as plt
import numpy as np
from scipy.optimize import minimize

def optimize_portfolio(correlation):
    # Define the objective function to maximize (expected return)
    def expected_return(weights):
        weight_a, weight_b = weights
        return -1 * (20 * weight_a + 30 * weight_b)

    # Define the budget constraint
    def budget_constraint(weights):
        weight_a, weight_b = weights
        return weight_a + weight_b - 1

    # Define the risk constraint
    def risk_constraint(weights):
        weight_a, weight_b = weights
        return -(100 * weight_a ** 2 + 400 * weight_b ** 2 + \
                 2 * weight_a * weight_b * 10 * 20 * correlation - 225)

    # Specify the initial guess and bounds for weights
    initial_weights = [0, 0]
    weight_bounds = [[0, 1], [0, 1]]

    # Define the constraints as a list of dictionaries
    constraints = [
        {'type': 'eq', 'fun': budget_constraint},
        {'type': 'ineq', 'fun': risk_constraint}
    ]

    # Execute the optimization to find the optimal weight for Stock A
    result = minimize(expected_return, initial_weights, method='SLSQP',
                      constraints=constraints, bounds=weight_bounds,
                      options={'ftol': 1e-6})

    return result.x[0]

# Generate correlation coefficients from -1 to 1 with a step size of 0.1
x = np.arange(-1, 1, 0.1)

# Calculate optimal weights for Stock A and B
weights_a = [round(optimize_portfolio(coeff), 5) for coeff in x]
weights_b = [round(1 - weight, 5) for weight in weights_a]

# Plotting the results
plt.xlabel('Correlation Coefficient Between Stocks')
plt.ylabel('Optimal Weight of Stocks')

plt.plot(x, weights_a, label="Stock A")
plt.plot(x, weights_b, label="Stock B")

plt.legend()
plt.show()
```

Code 5: Graph of Optimal Stock Weights as a Function of the Correlation Coefficient.

```python
import numpy as np
from scipy.optimize import minimize

# Input the number of stocks in the portfolio
num_stocks = int(input("Enter the number of stocks in the portfolio: "))

# Create a list of expected stock returns
returns = [float(input(f"Enter the expected return of Stock {i + 1}: "))
           for i in range(num_stocks)]

# Create a list of stock standard deviations
std_devs = [float(input(f"Enter the standard deviation of Stock {i + 1}: "))
            for i in range(num_stocks)]

# Input the portfolio risk tolerance
risk_tolerance = float(input("Enter the risk tolerance: "))

# Define the objective function to maximize (expected return)
def maximize_expected_return(weights):
    return -np.dot(weights, returns)

# Define the budget constraint
def budget_constraint(weights):
    return np.sum(weights) - 1

# Define the risk constraint
def risk_constraint(weights):
    portfolio_risk = np.dot(np.square(weights), np.square(std_devs))
    return - (portfolio_risk - risk_tolerance ** 2)

# Specify the initial guess for weights
initial_weights = np.zeros(num_stocks)

# Specify bounds for weights
weight_bounds = [[0, 1]] * num_stocks

# Define the constraints as a list of dictionaries
constraints = [{'type': 'eq', 'fun': budget_constraint},
               {'type': 'ineq', 'fun': risk_constraint}]

# Execute the optimization
result = minimize(maximize_expected_return, initial_weights, method='SLSQP',
                  constraints=constraints, bounds=weight_bounds,
                  options={'ftol': 1e-6})

# Print results
print("\nOptimal stock weighting:")
print([round(weight, 5) for weight in result.x])
print("\nMaximum expected return, as a percentage:")
print(round(-result.fun, 5))
```

Code 1: Generalized SLSQP Algorithm for n Uncorrelated Stocks.