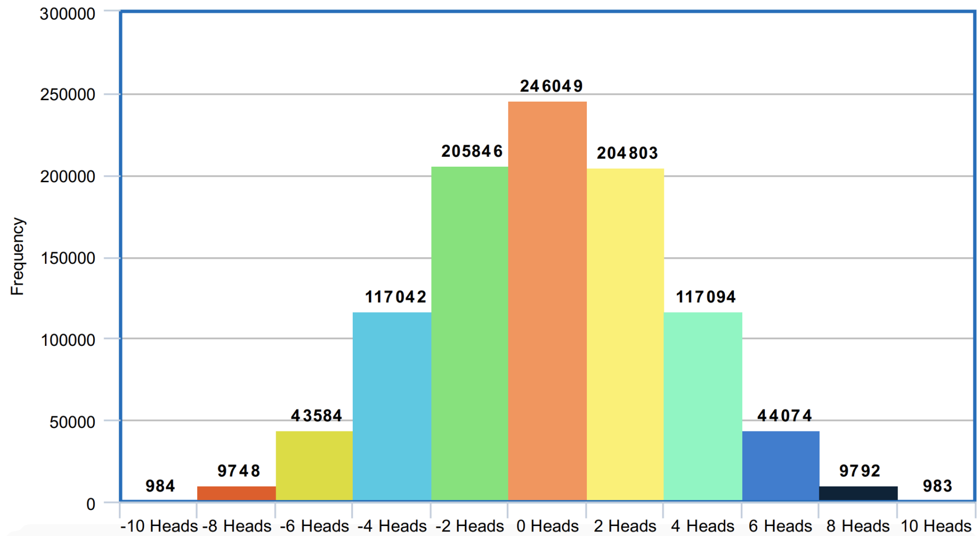
# Discrete Mathematics

DD **1**. Prove that, for an $n$-step random walk on the integers starting at 0, the ending position is odd if $n$ is odd, and even if $n$ is even.

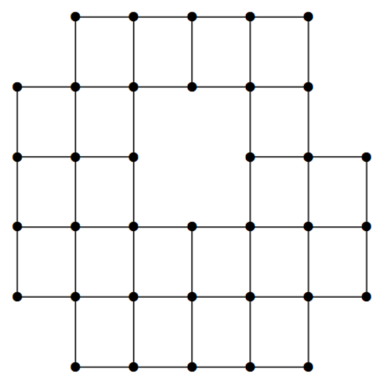
DD 2. (Continuation) Compute the probabilities of landing on $-2, -1, 0, 1$ and $2$, for a 2-step random walk. What are the probabilities of landing points for a 3-step random walk? What about for a 10-step random walk?




For this last part, check that your answers agree with the histogram to the right, for a million trials of a 10-step random walk (picture by Sam Yan).

AMS **3**. Prove that a connected graph has an Eulerian circuit if and only if every valence is even.

PEA 4. If a connected graph has vertices of odd valence, then an Eulerian circuit is out of the question. There is still the possibility of an Eulerian path, however (which finishes at a vertex different from the starting vertex). Suppose that a graph has two vertices of odd valence. Introducing an extra edge that joins these two vertices is called *Eulerizing the graph*. The graph shown at right does not have an Eulerian circuit. If it is *Eulerized*, however, a circuit can be found. When an extra edge duplicates an existing edge (joining edges already adjacent), think of it as a *reused edge*. Show that this graph can be Eulerized by means of reused edges, and do it using as few duplicates as possible.



AMS **5**. As discussed in Page 13 # 5, the *Four-Color Theorem* was very difficult to prove, and so far, only a computer-aided proof is known. Instead of proving that, we'll prove the *Six-Color Theorem.* To do so, we'll need the following result:

**Planar graph fact:** Every planar graph has at least one vertex of valence 5 or less.

Prove this. *Hint*: use the Graph Fact and the Euler Characteristic.

PEA 6. *A word graph.* Consider a graph whose vertices are English words, with an edge connecting two vertices if it is possible to transform one into the other by changing one letter, as in MATH to PATH. Create the graph for the words EAT, PAN, PAT, RAN, RAT, RUN, RUT. Is it connected? Is it planar?

DD *Arrow's Impossibility Theorem.* Consider the following requirements for a *voting system*, which is any algorithm that takes as input the voters' ranked lists of preferences of candidates, and outputs a ranked group preference list (possibly with ties) of all of the candidates:

(A) *Universal admissibility:* Each voter can rank the candidates however they like.

(B) *Unanimity:* If every voter prefers P to Q, the voting system ranks P over Q.

(C) *Irrelevant alternatives:* If, when the voting systems ranks only candidates P and Q (as in the Condorcet method), P is preferred to Q, then introducing a third option, R, must not make Q ranked above P.

(D) *No cycles:* The system always produces a ranked list, with no cycles.

(E) There are no dictators.

**Arrow's Impossibility Theorem:** No voting system exists that satisfies all five requirements. In particular, insisting on requirements A, B, C and D necessitates a dictatorship.

DD 7. Show by means of an example that the Condorcet method violates the *no cycles* criterion.

DD **8**. Show that the Condorcet method satisfies the *irrelevant alternatives* criterion.

DD 9. In the 1992 U.S. presidential election, the candidates were Bill Clinton (C), George H.W. Bush (B), and Ross Perot (P), who won 43, 38, and 19 percent of the popular vote, respectively. Suppose that voters' actual preference lists were as follows: CBP (43%), BPC (38%), PBC (19%).

(a) Show that, in a race between Clinton and Bush (removing Perot), Bush wins.

(b) Show that with the *Plurality* method (as was actually used), Clinton wins.

(c) Explain why this violates the *irrelevant alternatives* criterion.

THIS ASSIGNMENT HAS A THIRD PAGE
DON'T MISS PROBLEM 10
IT IS SUPER

DD Geometric ways of measuring partisan gerrymandering, such as the Skew, Isoperimetric and Square Reock measures (Page 13 # 2), tend to actually measure districts' *compactness*. It is arguably more relevant to directly assess whether a districting plan favors one political party over the other. Many ways of measuring this have been proposed; we will study two of them: *partisan symmetry* and *efficiency gap*.

*Partisan symmetry.* This measure uses the *seats-votes* curve, which has the *vote share* along the $x$-axis and the *seat share* along the $y$-axis. We'll follow convention, and use the Republican vote share. One election gives you exactly one data point on this curve. To construct the rest of it, political scientists use the assumption of *uniform partisan swing.*

DD 10. In 2016, the percentages of Republican voters in New Mexico's three congressional districts were 34.9%, 62.7%, and 37.6%. Refer to the picture at the bottom of the page.[3]

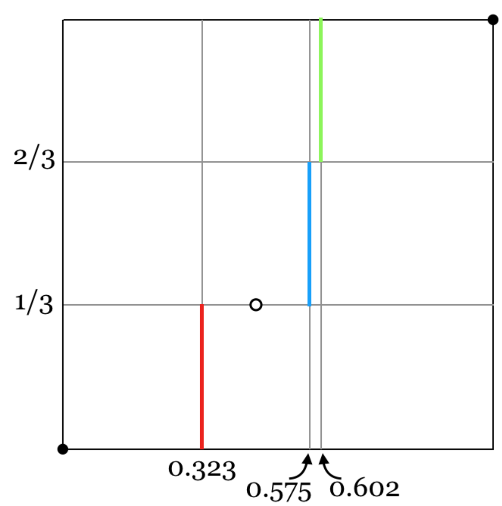
- Assuming an equal number of voters in each district, the Republican vote share was $(34.9 + 62.7 + 37.6)/3 = 45\%$, and the Republican seat share was 1/3, so the outcome of this election was $(0.45, 0.33)$. This is the white dot in the picture.
- Assuming "uniform partisan swing" means that we add an equal number of percentage points to each district until one of them goes over (or under) 50%. First, we add 12.4% so that the third district hits 50%. This yields the simulated election $(47.3, 75.1, 50)$ for a vote share of $(47.3 + 75.1 + 50)/3 = 57.5\%$ and a seat share of 2/3, meaning that the seat share jumps from 1/3 to 2/3 at a vote share of 0.575 (the blue line).
- Adding 2.7 more percentage points so the first district goes over 50% yields the simulated election $(50, 77.8, 52.7)$ for a vote share of 60.2% and a seat share of 3/3, meaning that the seat share jumps from 2/3 to 1 at a vote share of 0.602 (the green line).
- Going the other way from the true election results until all of the districts are under 50% requires subtracting 12.7 percentage points, yielding the simulated election results $(22.2, 50, 24.7)$, for vote share 32.3% and seat share 0/3, meaning that the vote share jumps from 0 to 1/3 at a vote share of 0.323 (the red line).

(a) Explain why the points $(0, 0)$ and $(1, 1)$ must both be on *every* seats-votes curve.

(b) Explain why the seats-votes curve is horizontal with 0 seat share between votes shares of 0 and 0.323, and horizontal with 1/3 seat share between vote shares of 0.323 and 0.575. Fill in and explain the other two horizontal parts of the seats-votes curve.

(c) Based on the seats-votes curve, do you think that New Mexico's districting plan favors Republicans, Democrats, or neither? Explain.

(d) Do you think that the assumption of uniform partisan swing is plausible? Explain.

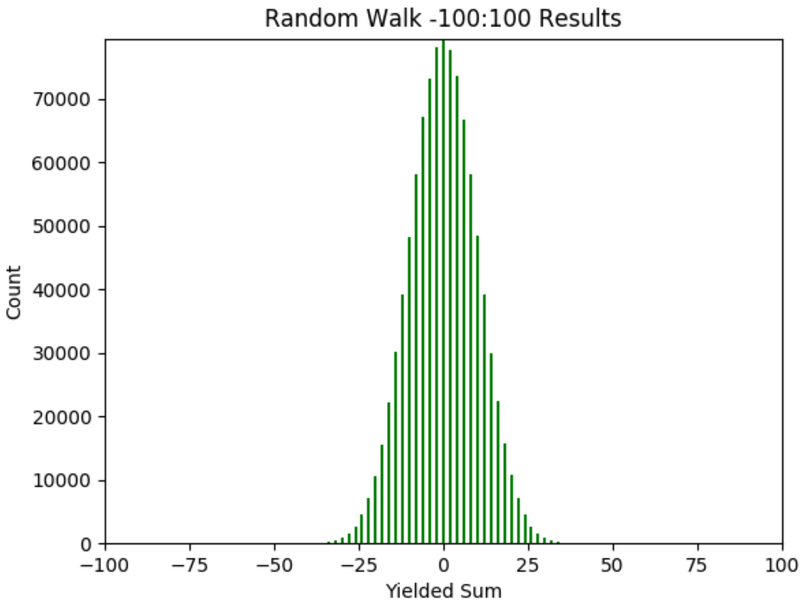



[3]This problem is from Moon Duchin's course Math of Social Choice, Math 19-02 at Tufts University.

# Discrete Mathematics

DD **1**. Prove that a connected graph has an Eulerian path if and only if it has 0 or 2 vertices of odd valence.

PEA 2. Eulerizing a graph requires that the vertices of odd valence be paired up. It therefore seems that this technique will not work on a graph that has an *odd* number of odd vertices. What do you think?




DD 3. The results of a million 100-step random walks are at right (picture by Salima Bourguiba). Using the graph (or your data, if you have it), estimate the probability of a result *greater than*
(a) 0 (b) 10

(c) 25 (d) 50.

DD 4. Explain what it might mean to go for a random walk on a *graph*. Choose your favorite graph, devise a method of random walking on it, and record the outcome of a 10-step random walk.

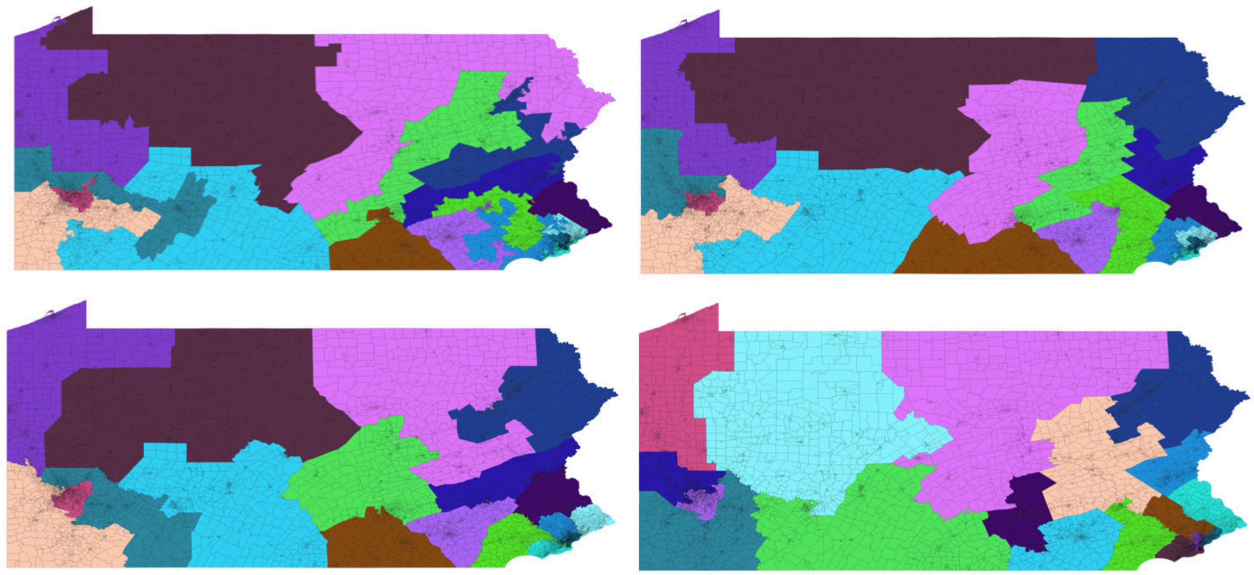
DD 5. *Shape isn't everything.* Historically, in order to create districting plans that favored a particular political party, the people who drew the lines used unusual shapes, such as "Goofy kicking Donald Duck." With modern mapping software, such geometric gymnastics have become unnecessary, and it is possible to create maps with "compact" districts that nonetheless strongly favor one political party.

The pictures on the next page show four plans for dividing Pennsylvania into 18 districts with equal population.[4] Pennsylvania has about 50% Democratic and 50% Republican voters. The plan that contains "Goofy kicking Donald Duck" (GKDD), which was enacted by the (Republican) legislature in 2011, has resulted in electing 13 Republican representatives and 5 Democrats. Of the plans below, two of them result in (actual or predicted) results of 13R-5D, and the other two are close to 9R-9D. Which ones do you think are which?

To be more specific, the maps are (in chronological order of invention) GKDD, the map proposed by the (Republican) legislature to replace GKDD, the map proposed by the (Democratic) governor to replace GDKK, and the map created by an impartial "special master."

DD 6. (Continuation) Indeed, one of the maps on the next page is the current districting plan, enacted earlier this year, under which Tuesday's election (November 6, 2018) was conducted. Look up the results of the election: After Tuesday's elections, how many of PA's 18 representatives are Republicans and how many are Democrats? Who was elected as the congressional representative for district 7, which contains Swarthmore?

[4]Pictures from Moon Duchin's talk "Random Walks and Gerrymandering," available at `https://vimeo.com/293465324`



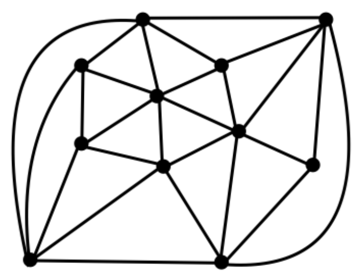
AMS 7. **The Six-Color Theorem.** Every planar graph can be colored with at most six colors.

*Proof.* To 6-color any planar graph, follow these four steps:

1. Locate a vertex of order 5 or less. This exists because (a) ________________________.
2. Delete that vertex and all edges connected to it.

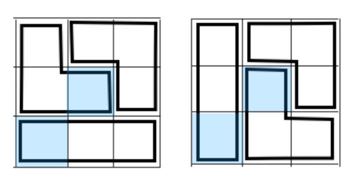
   Keep repeating steps 1 and 2 until only 5 vertices are left. Keep track of the order in which you deleted the vertices.
3. Color the five remaining vertices with the colors 1, 2, 3, 4, 5.
4. Put back the last vertex and edges you deleted. Color that vertex a different color than the vertices adjacent to it. This is possible because (b) ________________________.

Repeat step 4, replacing vertices in the reverse of the order they were deleted. There is always an available color for the replaced vertex, because (c) ________________________. After you put all the vertices back, you will have reconstructed the original graph, and each vertex will be colored with one of the 6 colors.



(d) Run the algorithm from the theorem to color the graph to the right. Record all the steps of the process.

DD 8. Consider a state consisting of 9 towns in a $3 \times 3$ block. Your job is to divide them into 3 districts, each consisting of 3 towns, where, as usual, towns within a district must meet along an edge. Two possible ways of doing this are shown. Draw pictures of all possible ways of doing this. How many are there?



DD 9. (Continuation) By flipping which districts the two blue squares are assigned to, we can turn the districting plan on the left into the one on the right. Create a graph, where the vertices are districting plans of this $3 \times 3$ block, and an edge connects two plans if, as in this example, a swap of two squares (any two) transforms one into the other. Is it connected? Is it trivalent? Is it planar? DON'T MISS THE 10TH PROBLEM

DD

10. The table below summarizes the calculations from Page 17 # 10. The actual election results are in bold, and the "tipping points" for each district are listed.

**New Mexico's 2012 Congressional election**

| R seats | District 1 | District 2 | District 3 | R vote share |
|---|---|---|---|---|
| 0 or 1 | 0.222 | 0.5 | 0.249 | 0.323 |
| **1** | **0.349** | **0.627** | **0.376** | **0.45** |
| 1 or 2 | 0.473 | 0.751 | 0.5 | 0.575 |
| 2 or 3 | 0.5 | 0.778 | 0.527 | 0.602 |

(a) Fill in the table below for Nevada's 2012 Congressional election results, and then produce a seats-votes curve. Do you think that New Mexico's districting plan favors Republicans, Democrats, or neither? Explain.

**Nevada's 2012 Congressional election**

| R seats | District 1 | District 2 | District 3 | District 4 | R vote share |
|---|---|---|---|---|---|
| 0 or 1 | | | | | |
| 1 or 2 | | | | | |
| **2** | **0.332** | **0.612** | **0.540** | **0.457** | **0.485** |
| 2 or 3 | | | | | |
| 3 or 4 | | | | | |

(b) Explain why the point $(0.5, 0.5)$ should be on every seats-votes curve (for a state with an even number of representatives), if the map favors neither Republicans nor Democrats.