Chapter 15
Vector Analysis Applications

In this Chapter we present a number of functions from the CALC menu that apply to the analysis of scalar and vector fields. The CALC menu was presented in detail in Chapter 13. In particular, in the DERIV&INTEG menu we identified a number of functions that have applications in vector analysis, namely, CURL, DIV, HESS, LAPL. For the exercises in this Chapter, change your angle measure to radians.

Definitions

A function defined in a region of space such as $\phi(x,y,z)$ is known as a scalar field, examples are temperature, density, and voltage near a charge. If the function is defined by a vector, i.e., $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i}+g(x,y,z)\mathbf{j}+h(x,y,z)\mathbf{k}$, it is referred to as a vector field.

The following operator, referred to as the 'del' or 'nabla' operator, is a vector-based operator that can be applied to a scalar or vector function:

$$\nabla[\ ] = i\cdot\frac{\partial}{\partial x}[\ ] + j\cdot\frac{\partial}{\partial y}[\ ] + k\cdot\frac{\partial}{\partial z}[\ ]$$

When this operator is applied to a scalar function we can obtain the gradient of the function, and when applied to a vector function we can obtain the divergence and the curl of that function. A combination of gradient and divergence produces another operator, called the Laplacian of a scalar function. These operations are presented next.

Gradient and directional derivative

The gradient of a scalar function $\phi(x,y,z)$ is a vector function defined by

$$grad\phi = \nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + j\cdot\frac{\partial\phi}{\partial y} + k\cdot\frac{\partial\phi}{\partial z}$$

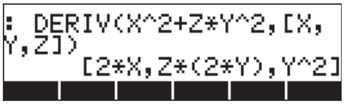
The dot product of the gradient of a function with a given unit vector represents the rate of change of the function along that particular vector. This rate of change is called the directional derivative of the function, $D_u\phi(x,y,z) = \mathbf{u}\bullet\nabla\phi$.

At any particular point, the maximum rate of change of the function occurs in the direction of the gradient, i.e., along a unit vector **u** = $\nabla\phi/|\nabla\phi|$.

The value of that directional derivative is equal to the magnitude of the gradient at any point $D_{max}\phi(x,y,z) = \nabla\phi \bullet \nabla\phi/|\nabla\phi| = |\nabla\phi|$

The equation $\phi(x,y,z) = 0$ represents a surface in space. It turns out that the gradient of the function at any point on this surface is normal to the surface. Thus, the equation of a plane tangent to the curve at that point can be found by using a technique presented in Chapter 9.

The simplest way to obtain the gradient is by using function DERIV, available in the CALC menu, e.g.,

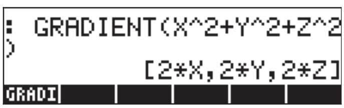


A program to calculate the gradient

The following program, which you can store into variable GRADIENT, uses function DERIV to calculate the gradient of a scalar function of X,Y,Z. Calculations for other base variables will not work. If you work frequently in the (X,Y,Z) system, however, this function will facilitate calculations:

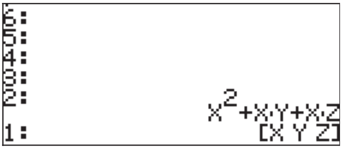
<< X Y Z 3 →ARRY DERIV >>

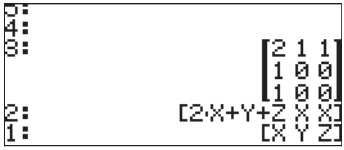
Type the program while in RPN mode. After switching to ALG mode, you can call the function GRADIENT as in the following example:



Using function HESS to obtain the gradient

The function HESS can be used to obtain the gradient of a function as shown next. As indicated in Chapter 14, function HESS takes as input a function of n independent variables $\phi(x_1, x_2, \ldots, x_n)$, and a vector of the functions ['x_1' 'x_2'…'x_n']. Function HESS returns the <u>Hessian matrix</u> of the function ϕ, defined as the matrix **H** = $[h_{ij}] = [\partial\phi/\partial x_i \partial x_j]$, the gradient of the function with respect to the n-variables, **grad** f = [$\partial\phi/\partial x_1$, $\partial\phi/\partial x_2$, … $\partial\phi/\partial x_n$], and the list of variables ['x_1' 'x_2'…'x_n']. Consider as an example the function $\phi(X,Y,Z) = X^2 + XY + XZ$, we'll apply function HESS to this scalar field in the following example in RPN mode:



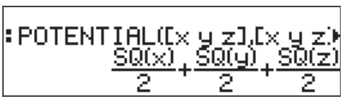


Thus, the gradient is [2X+Y+Z, X, X]. Alternatively, one can use function DERIV as follows: DERIV(X^2+X*Y+X*Z,[X,Y,Z]), to obtain the same result.

Potential of a gradient

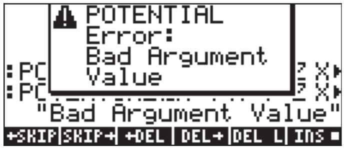
Given the vector field, **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j**+h(x,y,z)**k**, if there exists a function $\phi(x,y,z)$, such that $f = \partial\phi/\partial x$, $g = \partial\phi/\partial y$, and $h = \partial\phi/\partial z$, then $\phi(x,y,z)$ is referred to as the potential function for the vector field **F**. It follows that **F** = grad $\phi = \nabla\phi$.

The calculator provides function POTENTIAL, available through the command catalog (→ CAT), to calculate the potential function of a vector field, if it exists. For example, if **F**(x,y,z) = **xi** + y**j** + z**k**, applying function POTENTIAL we find:



Since function SQ(x) represents x^2, this results indicates that the potential function for the vector field **F**(x,y,z) = x**i** + y**j** + z**k**, is $\phi(x,y,z) = (x^2+y^2+z^2)/2$.

Notice that the conditions for the existence of $\phi(x,y,z)$, namely, $f = \partial\phi/\partial x$, $g = \partial\phi/\partial y$, and $h = \partial\phi/\partial z$, are equivalent to the conditions: $\partial f/\partial y = \partial g/\partial x$, $\partial f/\partial z = \partial h/\partial x$, and $\partial g/\partial z = \partial h/\partial y$. These conditions provide a quick way to determine if the vector field has an associated potential function. If one of the conditions $\partial f/\partial y = \partial g/\partial x$, $\partial f/\partial z = \partial h/\partial x$, $\partial g/\partial z = \partial h/\partial y$, fails, a potential function $\phi(x,y,z)$ does not exist. In such case, function POTENTIAL returns an error message. For example, the vector field **F**(x,y,z) = (x+y)**i** + (x-y+z)**j** + xz**k**, does not have a potential function associated with it, since, ∂f/∂z ≠ ∂h/∂x. The calculator response in this case is shown below:

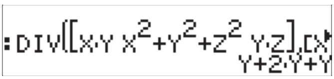


Divergence

The divergence of a vector function, **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j**+h(x,y,z)**k**, is defined by taking a "dot-product" of the del operator with the function, i.e.,

$$divF = \nabla \bullet F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Function DIV can be used to calculate the divergence of a vector field. For example, for **F**(X,Y,Z) = [XY,X^2+Y^2+Z^2,YZ], the divergence is calculated, in ALG mode, as follows:

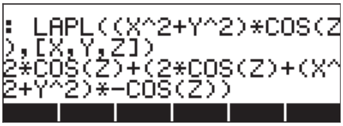


Laplacian

The divergence of the gradient of a scalar function produces an operator called the Laplacian operator. Thus, the Laplacian of a scalar function φ(x,y,z) is given by

$$\nabla^2 \phi = \nabla \bullet \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}$$

The partial differential equation $\nabla^2\phi = 0$ is known as Laplace's equation. Function LAPL can be used to calculate the Laplacian of a scalar function. For example, to calculate the Laplacian of the function φ(X,Y,Z) = (X^2+Y^2)cos(Z), use:



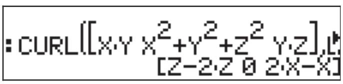
Curl

The curl of a vector field **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j**+h(x,y,z)**k**, is defined by a "cross-product" of the del operator with the vector field, i.e.,

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x}[\ ] & \dfrac{\partial}{\partial y}[\ ] & \dfrac{\partial}{\partial z}[\ ] \\ f(x,y,z) & g(x,y,z) & h(x,y,z) \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \mathbf{j}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \mathbf{k}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)$$

The curl of vector field can be calculated with function CURL. For example, for the function **F**(X,Y,Z) = [XY,X^2+Y^2+Z^2,YZ], the curl is calculated as follows:



Irrotational fields and potential function

In an earlier section in this chapter we introduced function POTENTIAL to calculate the potential function φ(x,y,z) for a vector field, **F**(x,y,z) = f(x,y,z)**i**+ g(x,y,z)**j**+ h(x,y,z)**k**, such that **F** = grad φ = ∇φ. We also indicated that the conditions for the existence of φ, were: ∂f/∂y = ∂g/∂x, ∂f/∂z = ∂h/∂x, and ∂g/∂z = ∂h/∂y. These conditions are equivalent to the vector expression

curl **F** = ∇×**F** = 0.

A vector field **F**(x,y,z), with zero curl, is known as an <u>irrotational</u> field. Thus, we conclude that a potential function φ(x,y,z) always exists for an irrotational field **F**(x,y,z).

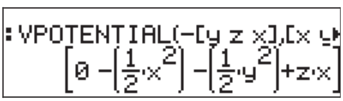
As an example, in an earlier example we attempted to find a potential function for the vector field **F**(x,y,z) = (x+y)**i** + (x-y+z)**j** + xz**k**, and got an error message back from function POTENTIAL. To verify that this is a rotational field (i.e., $\nabla\times$**F** $\neq$ 0), we use function CURL on this field:

On the other hand, the vector field **F**(x,y,z) = **xi** + y**j** + z**k**, is indeed irrotational as shown below:

Vector potential

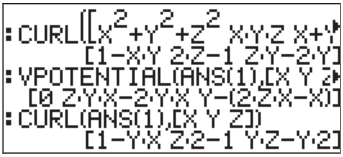
Given a vector field **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j**+h(x,y,z)**k**, if there exists a vector function Φ(x,y,z) = ϕ(x,y,z)**i**+ψ(x,y,z)**j**+η(x,y,z)**k**, such that **F** = curl Φ = $\nabla \times \Phi$, then function Φ(x,y,z) is referred to as the <u>vector potential</u> of **F**(x,y,z).

The calculator provides function VPOTENTIAL, available through the command catalog (CAT), to calculate the vector potential, Φ(x,y,z), given the vector field, **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j**+h(x,y,z)**k**. For example, given the vector field, **F**(x,y,z) = -(y**i**+z**j**+x**k**), function VPOTENTIAL produces



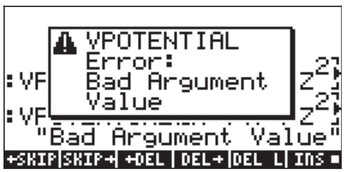
i.e., Φ(x,y,z) = $-x^2/2$**j** + $(-y^2/2+zx)$**k**.

It should be indicated that there is more than one possible vector potential functions Φ for a given vector field **F**. For example, the following screen shot shows that the curl of the vector function Φ_1 = [$X^2+Y^2+Z^2$,XYZ,X+Y+Z] is the vector **F** = $\nabla\times\Phi_2$ = [1-XY,2Z-1,ZY-2Y]. Application of function VPOTENTIAL produces the vector potential function Φ_2 = [0, ZYX-2YX, Y-(2ZX-X)], which is different from Φ_1. The last command in the screen shot shows that indeed **F** = $\nabla\times\Phi_2$. Thus, a vector potential function is not uniquely determined.



The components of the given vector field, **F**(x,y,z) = f(x,y,z)**i**+g(x,y,z)**j** +h(x,y,z)**k**, and those of the vector potential function, Φ(x,y,z) = ϕ(x,y,z)**i**+ψ(x,y,z)**j**+η(x,y,z)**k**, are related by f = $\partial\eta/\partial y$ - $\partial\psi/\partial x$, g = $\partial\phi/\partial z$ - $\partial\eta/\partial x$, and h = $\partial\psi/\partial x$ - $\partial\phi/\partial y$.

A condition for function Φ(x,y,z) to exists is that div **F** = $\nabla\bullet$**F** = 0, i.e., $\partial f/\partial x$ + $\partial g/\partial y$ + $\partial f/\partial z$ = 0. Thus, if this condition is not satisfied, the vector potential function Φ(x,y,z) does not exist. For example, given **F** = [X+Y,X-Y,Z^2], function VPOTENTIAL returns an error message, since function F does not satisfy the condition $\nabla\bullet$**F** = 0:



The condition $\nabla\bullet$**F** $\neq$ 0 is verified in the following screen shot: