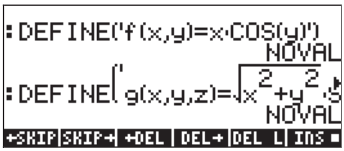
Chapter 14
Multi-variate Calculus Applications

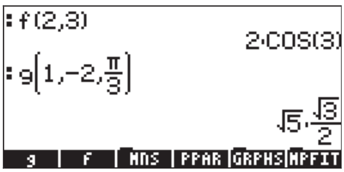
Multi-variate calculus refers to functions of two or more variables. In this Chapter we discuss the basic concepts of multi-variate calculus including partial derivatives and multiple integrals.

Multi-variate functions

A function of two or more variables can be defined in the calculator by using the DEFINE function (← DEF). To illustrate the concept of partial derivative, we will define a couple of multi-variate functions, f(x,y) = x cos(y), and g(x,y,z) = $(x^2+y^2)^{1/2}\sin(z)$, as follows:



We can evaluate the functions as we would evaluate any other calculator function, e.g.,



Graphics of two-dimensional functions are possible using Fast3D, Wireframe, Ps-Contour, Y-Slice, Gridmap, and Pr-Surface plots as described in Chapter 12.

Partial derivatives

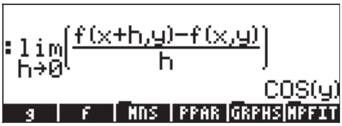
Consider the function of two variables z = f(x,y), the partial derivative of the function with respect to x is defined by the limit

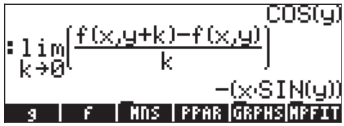
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad .$$

Similarly,

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k} \, .$$

We will use the multi-variate functions defined earlier to calculate partial derivatives using these definitions. Here are the derivatives of f(x,y) with respect to x and y, respectively:





Notice that the definition of partial derivative with respect to x, for example, requires that we keep y fixed while taking the limit as h→0. This suggest a way to quickly calculate partial derivatives of multi-variate functions: use the rules of ordinary derivatives with respect to the variable of interest, while considering all other variables as constant. Thus, for example,

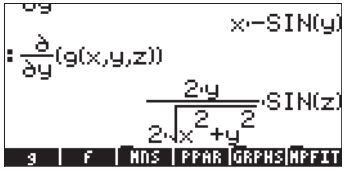
$$\frac{\partial}{\partial x}\left(x\cos(y)\right) = \cos(y), \; \frac{\partial}{\partial y}\left(x\cos(y)\right) = -x\sin(y) \, ,$$

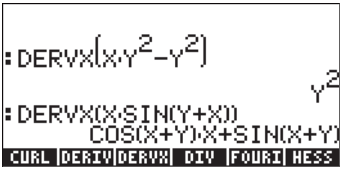
which are the same results as found with the limits calculated earlier.
Consider another example,

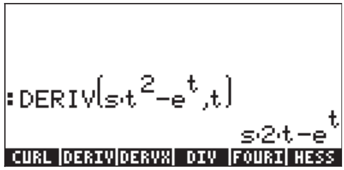
$$\frac{\partial}{\partial x}\left(yx^2 + y^2\right) = 2yx + 0 = 2xy$$

In this calculation we treat y as a constant and take derivatives of the expression with respect to x.

Similarly, you can use the derivative functions in the calculator, e.g., DERVX, DERIV, ∂ (described in detail in Chapter 13) to calculate partial derivatives. Recall that function DERVX uses the CAS default variable VX (typically, 'X'), therefore, with DERVX you can only calculate derivatives with respect to X. Some examples of first-order partial derivatives are shown next:







Higher-order derivatives

The following second-order derivatives can be defined

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

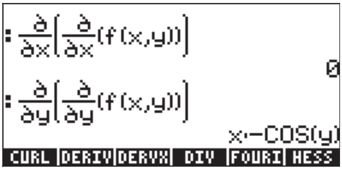
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

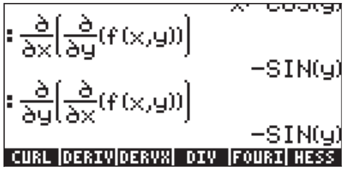
The last two expressions represent cross-derivatives, the partial derivatives signs in the denominator shows the order of derivation. In the left-hand side, the derivation is taking first with respect to x and then with respect to y, and in the right-hand side, the opposite is true. It is important to indicate that, if a function is continuous and differentiable, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Third-, fourth-, and higher order derivatives are defined in a similar manner.

To calculate higher order derivatives in the calculator, simply repeat the derivative function as many times as needed. Some examples are shown below:



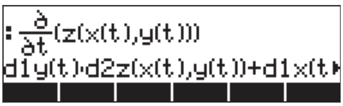


The chain rule for partial derivatives

Consider the function z = f(x,y), such that x = x(t), y = y(t). The function z actually represents a composite function of *t* if we write it as z = f[x(t),y(t)]. The chain rule for the derivative dz/dt for this case is written as

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

To see the expression that the calculator produces for this version of the chain rule use:



The result is given by d1y(t)·d2z(x(t),y(t))+d1x(t)·d1z(x(y),y(t)). The term d1y(t) is to be interpreted as "the derivative of y(t) with respect to the 1st independent variable, i.e., t", or d1y(t) = dy/dt. Similarly, d1x(t) = dx/dt. On the other hand, d1z(x(t),y(t)) means "the first derivative of z(x,y) with respect to the first independent variable, i.e., x", or d1z(x(t),y(t)) = ∂z/∂x. Similarly, d2z(x(t),y(t)) = ∂z/∂y. Thus, the expression above is to be interpreted as:

dz/dt = (dy/dt)·(∂z/∂y) + (dx/dt)· (∂z/∂x).

Total differential of a function z = z(x,y)

From the last equation, if we multiply by dt, we get the total differential of the function $z = z(x,y)$, i.e., $dz = (\partial z/\partial x)\cdot dx + (\partial z/\partial y)\cdot dy$.

A different version of the chain rule applies to the case in which $z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$, so that $z = f[x(u,v), y(u,v)]$. The following formulas represent the chain rule for this situation:

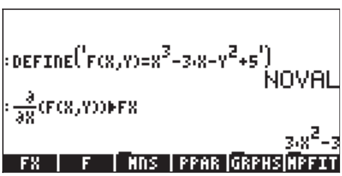
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

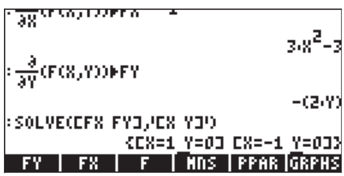
Determining extrema in functions of two variables

In order for the function $z = f(x,y)$ to have an extreme point (extrema) at (x_o,y_o), its derivatives $\partial f/\partial x$ and $\partial f/\partial y$ must vanish at that point. These are *necessary* conditions. The *sufficient conditions* for the function to have an extreme at point (x_o,y_o) are $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, and $\Delta = (\partial^2 f/\partial x^2)\cdot(\partial^2 f/\partial y^2)-[\partial^2 f/\partial x\partial y]^2 > 0$. The point (x_o,y_o) is a relative maximum if $\partial^2 f/\partial x^2 < 0$, or a relative minimum if $\partial^2 f/\partial x^2 > 0$. The value Δ is referred to as the discriminant.

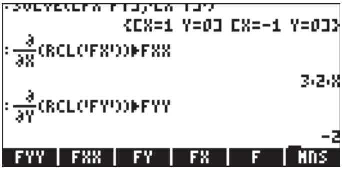
If $\Delta = (\partial^2 f/\partial x^2)\cdot(\partial^2 f/\partial y^2)-[\partial^2 f/\partial x\partial y]^2 < 0$, we have a condition known as a *saddle point*, where the function would attain a maximum in x if we were to hold y constant, while, at the same time, attaining a minimum if we were to hold x constant, or vice versa.

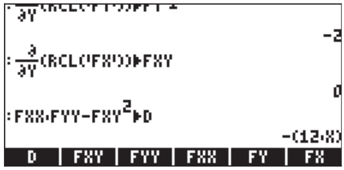
Example 1 – Determine the extreme points (if any) of the function $f(X,Y) = X^3-3X-Y^2+5$. First, we define the function f(X,Y), and its derivatives $fX(X,Y) = \partial f/\partial X$, $fY(X,Y) = \partial f/\partial Y$. Then, we solve the equations $fX(X,Y) = 0$ and $fY(X,Y) = 0$, simultaneously:



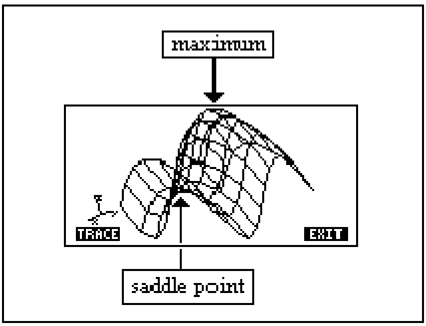


We find critical points at (X,Y) = (1,0), and (X,Y) = (-1,0). To calculate the discriminant, we proceed to calculate the second derivatives, $fXX(X,Y) = \partial^2 f/\partial X^2$, $fXY(X,Y) = \partial^2 f/\partial X/\partial Y$, and $fYY(X,Y) = \partial^2 f/\partial Y^2$.





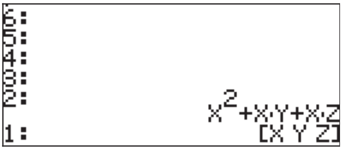
The last result indicates that the discriminant is $\Delta = -12X$, thus, for (X,Y) = (1,0), $\Delta < 0$ (saddle point), and for (X,Y) = (-1,0), $\Delta > 0$ and $\partial^2 f/\partial X^2 < 0$ (relative maximum). The figure below, produced in the calculator, and edited in the computer, illustrates the existence of these two points:

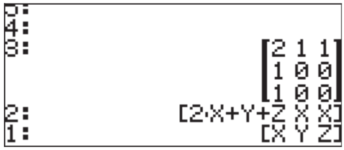


Using function HESS to analyze extrema

Function HESS can be used to analyze extrema of a function of two variables as shown next. Function HESS, in general, takes as input a function of n independent variables $\phi(x_1, x_2, \ldots, x_n)$, and a vector of the functions [‘x_1’ ‘x_2’…’x_n’]. Function HESS returns the Hessian matrix of the function ϕ, defined as the matrix $\mathbf{H} = [h_{ij}] = [\partial^2\phi/\partial x_i \partial x_j]$, the gradient of the function with respect to the n-variables, **grad** $f = [\ \partial\phi/\partial x_1, \partial\phi/\partial x_2\ , \ldots\ \partial\phi/\partial x_n]$, and the list of variables [‘x_1’ ‘x_2’…’x_n’].

Applications of function HESS are easier to visualize in the RPN mode. Consider as an example the function $\phi(X,Y,Z) = X^2 + XY + XZ$, we'll apply function HESS to function ϕ in the following example. The screen shots show the RPN stack before and after applying function HESS.





When applied to a function of two variables, the gradient in level 2, when made equal to zero, represents the equations for critical points, i.e., $\partial\phi/\partial x_i = 0$, while the matrix in level 3 represent second derivatives. Thus, the results from the HESS function can be used to analyze extrema in functions of two variables. For example, for the function $f(X,Y) = X^3-3X-Y^2+5$, proceed as follows in RPN mode:

'X^3-3*X-Y^2+5' ENTER ['X','Y'] ENTER	Enter function and variables
HESS	Apply function HESS
SOLVE	Find critical points
EVAL	Decompose vector
's1' STO▶ 's2' STO▶	Store critical points

The variables s1 and s2, at this point, contain the vectors ['X=-1','Y=0] and ['X=1','Y=0], respect

'H' STO▶	Store Hessian matrix
VAR H S1 SUBST ↱ →NUM	Substitute s1 into H

The resulting matrix **A** has a_{11} elements $a_{11} = \partial^2\phi/\partial X^2 =$ -6., $a_{22} = \partial^2\phi/\partial X^2 =$ -2., and $a_{12} = a_{21} = \partial^2\phi/\partial X\partial Y = 0$. The discriminant, for this critical point s1(-1,0) is $\Delta = (\partial^2 f/\partial x^2)\cdot(\partial^2 f/\partial y^2)-[\partial^2 f/\partial x\partial y]^2 = (-6.)(-2.) = 12.0 > 0$. Since $\partial^2\phi/\partial X^2 < 0$, point s1 represents a relative maximum.

Next, we substitute the second point, s2, into H:

VAR H S2 SUBST ↱ →NUM	Substitute s2 into H

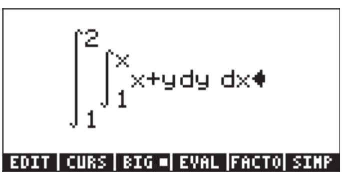
The resulting matrix has elements $a_{11} = \partial^2\phi/\partial X^2 = 6.$, $a_{22} = \partial^2\phi/\partial X^2 = -2.$, and $a_{12} = a_{21} = \partial^2\phi/\partial X\partial Y = 0$. The discriminant, for this critical point s2(1,0) is $\Delta = (\partial^2 f/\partial x^2)\cdot(\partial^2 f/\partial y^2)-[\partial^2 f/\partial x\partial y]^2 = (6.)(-2.) = -12.0 < 0$, indicating a saddle point.

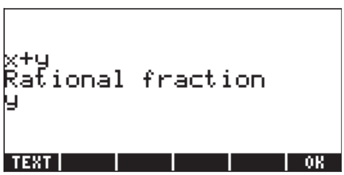
Multiple integrals

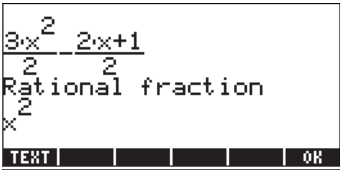
A physical interpretation of an ordinary integral, $\int_a^b f(x)dx$, is the area under the curve y = f(x) and abscissas x = a and x = b. The generalization to three dimensions of an ordinary integral is a double integral of a function f(x,y) over a region R on the x-y plane representing the volume of the solid body contained under the surface f(x,y) above the region R. The region R can be described as R = {a<x<b, f(x)<y<g(x)} or as R = {c<y<d, r(y)<x<s(y)}. Thus, the double integral can be written as

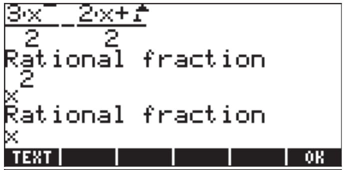
$$\iint_R \phi(x,y)dA = \int_a^b \int_{f(x)}^{g(x)} \phi(x,y)dydx = \int_c^d \int_{r(y)}^{s(y)} \phi(x,y)dydx$$

Calculating a double integral in the calculator is straightforward. A double integral can be built in the Equation Writer (see example in Chapter 2). An example follows. This double integral is calculated directly in the Equation Writer by selecting the entire expression and using function EVAL. The result is 3/2. Step-by-step output is possible by setting the Step/Step option in the CAS MODES screen.









Jacobian of coordinate transformation

Consider the coordinate transformation x = x(u,v), y = y(u,v). The Jacobian of this transformation is defined as

$$|J| = \det(J) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

When calculating an integral using such transformation, the expression to use is $\iint_R \phi(x,y)dydx = \iint_{R'} \phi[x(u,v), y(u,v)]\,|J|\,dudv$, where R' is the region R expressed in (u,v) coordinates.

Double integral in polar coordinates

To transform from polar to Cartesian coordinates we use x(r,θ) = r cos θ, and y(r, θ) = r sin θ. Thus, the Jacobian of the transformation is

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\cdot\sin(\theta) \\ \sin(\theta) & r\cdot\cos(\theta) \end{vmatrix} = r$$

With this result, integrals in polar coordinates are written as

$$\iint_{R'} \phi(r,\theta)dA = \int_{\alpha}^{\beta} \int_{f(\theta)}^{g(\theta)} \phi(r,\theta) r dr d\theta$$

where the region R’ in polar coordinates is R’ = {$\alpha < \theta < \beta$, $f(\theta) < r < g(\theta)$}.

Double integrals in polar coordinates can be entered in the calculator, making sure that the Jacobian |J| = r is included in the integrand. The following is an example of a double integral calculated in polar coordinates, shown step-by-step: