Chapter 4
Calculations with complex numbers

This chapter shows examples of calculations and application of functions to complex numbers.

Definitions

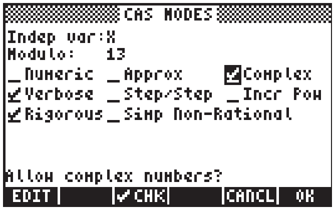
A *complex number z* is a number written as $z = x + iy$, where x and y are real numbers, and i is the *imaginary unit* defined by $i^2 = -1$. The complex number $x+iy$ has a *real part*, $x = Re(z)$, and an *imaginary part*, $y = Im(z)$. We can think of a complex number as a point $P(x,y)$ in the x-y plane, with the x-axis referred to as the *real axis*, and the y-axis referred to as the *imaginary axis*. Thus, a complex number represented in the form $x+iy$ is said to be in its *Cartesian representation*. An alternative Cartesian representation is the ordered pair $z = (x,y)$. A complex number can also be represented in polar coordinates (*polar representation*) as $z = re^{i\theta} = r\cdot\cos\theta + i\, r\cdot\sin\theta$, where $r = |z| = \sqrt{x^2 + y^2}$ is the *magnitude* of the complex number z, and $\theta = Arg(z) = \arctan(y/x)$ is the *argument* of the complex number z. The relationship between the Cartesian and polar representation of complex numbers is given by the *Euler formula*: $e^{i\theta} = \cos\theta + i \sin\theta$. The *complex conjugate* of a complex number $z = x + iy = re^{i\theta}$, is $\bar{z} = x - iy = re^{-i\theta}$. The complex conjugate of *i* can be thought of as the reflection of z about the real (*x*) axis. Similarly, the *negative* of z, $-z = -x-iy = -re^{i\theta}$, can be thought of as the reflection of *z* about the origin.

Setting the calculator to COMPLEX mode

When working with complex numbers it is a good idea to set the calculator to complex mode, using the following keystrokes:

MODE CAS ▼ ▼ ▶ ✓CHK

The COMPLEX mode will be selected if the CAS MODES screen shows the option _Complex checked, i.e.,



Press OK, twice, to return to the stack.

Entering complex numbers

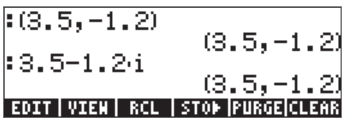
Complex numbers in the calculator can be entered in either of the two Cartesian representations, namely, *x+iy*, or *(x,y)*. The results in the calculator will be shown in the ordered-pair format, i.e., *(x,y)*. For example, with the calculator in ALG mode, the complex number *(3.5,-1.2)*, is entered as:

← () 3 . 5 → , +/- 1 . 2 ENTER

A complex number can also be entered in the form *x+iy*. For example, in ALG mode, *3.5-1.2i* is entered as:

3 . 5 − 1 . 2 × ← i ENTER

The following screen results after entering these complex numbers:



In RPN mode, these numbers will be entered using the following keystrokes:

← () 3 . 5 → , 1 . 2 +/- ENTER

(Notice that the change-sign keystroke is entered after the number 1.2 has been entered, in the opposite order as the ALG mode exercise).

The resulting RPN screen will be:

Notice that the last entry shows a complex number in the form *x+iy*. This is so because the number was entered between single quotes, which represents an algebraic expression. To evaluate this number use the EVAL key(EVAL).

Once the algebraic expression is evaluated, you recover the complex number (3.5,1.2).

Polar representation of a complex number

The result shown above represents a Cartesian (rectangular) representation of the complex number 3.5-1.2i. A polar representation is possible if we change the coordinate system to cylindrical or polar, by using function CYLIN. You can find this function in the catalog (↱ CAT). Changing to polar shows the result in RPN mode:

For this result, it is in standard notation and the angular measure is set to radians (you can always change to radians by using function RAD). The result shown above represents a magnitude, 3.7, and an angle 0.33029…. The angle symbol (∠) is shown in front of the angle measure.

Return to Cartesian or rectangular coordinates by using function RECT (available in the catalog, ↱ CAT). A complex number in polar representation is written as $z = r \cdot e^{i\theta}$. You can enter this complex number into the calculator by using an ordered pair of the form (r, $\angle\theta$). The angle symbol (∠) can be entered as ALPHA ↱ 6 . For example, the complex number $z = 5.2e^{1.5i}$, can be entered as follows (the figures show the stack, before and after entering the number):

Because the coordinate system is set to rectangular (or Cartesian), the calculator automatically converts the number entered to Cartesian coordinates, i.e., $x = r \cos \theta$, $y = r \sin \theta$, resulting, for this case, in (0.3678…, 5.18…).

On the other hand, if the coordinate system is set to cylindrical coordinates (use CYLIN), entering a complex number (x,y), where x and y are real numbers, will produce a polar representation. For example, in cylindrical coordinates, enter the number (3.,2.). The figure below shows the RPN stack, before and after entering this number:

Simple operations with complex numbers

Complex numbers can be combined using the four fundamental operations (+ − × ÷). The results follow the rules of algebra with the caveat that $i^2 = -1$. Operations with complex numbers are similar to those with real numbers. For example, with the calculator in ALG mode and the CAS set to *Complex*, we'll attempt the following sum: (3+5i) + (6-3i):

Notice that the real parts (3+6) and imaginary parts (5-3) are combined together and the result given as an ordered pair with real part 9 and imaginary part 2. Try the following operations on your own:

(5-2i) - (3+4i) = (2,-6)
(3-i)·(2-4i) = (2,-14)
(5-2i)/(3+4i) = (0.28,-1.04)
1/(3+4i) = (0.12, -0.16)

Notes:
The product of two numbers is represented by: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$.
The division of two complex numbers is accomplished by multiplying both numerator and denominator by the complex conjugate of the denominator, i.e.,

$$\frac{x_1+iy_1}{x_2+iy_2}=\frac{x_1+iy_1}{x_2+iy_2}\cdot\frac{x_2-iy_2}{x_2-iy_2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\cdot\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

Thus, the inverse function INV (activated with the 1/x key) is defined as

$$\frac{1}{x+iy}=\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}=\frac{x}{x^2+y^2}+i\cdot\frac{y}{x^2+y^2}$$

Changing sign of a complex number

Changing the sign of a complex number can be accomplished by using the +/- key, e.g., -(5-3i) = -5 + 3i

Entering the unit imaginary number

To enter the unit imaginary number type : ← i

Notice that the number *i* is entered as the ordered pair *(0,1)* if the CAS is set to APPROX mode. In EXACT mode, the unit imaginary number is entered as *i*.

Other operations

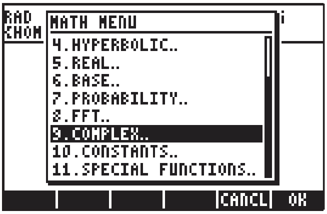
Operations such as magnitude, argument, real and imaginary parts, and complex conjugate are available through the CMPLX menus detailed later.

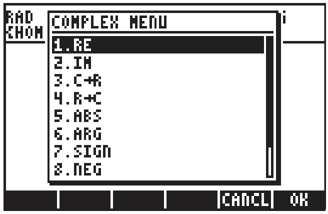
The CMPLX menus

There are two CMPLX (CoMPLeX numbers) menus available in the calculator. One is available through the MTH menu (introduced in Chapter 3) and one directly into the keyboard (→ CMPLX). The two CMPLX menus are presented next.

CMPLX menu through the MTH menu

Assuming that system flag 117 is set to **CHOOSE boxes** (see Chapter 2), the CMPLX sub-menu within the MTH menu is accessed by using: ← MTH 9 OK . The following sequence of screen shots illustrates these steps:

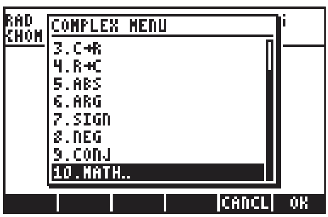




The first menu (options 1 through 6) shows the following functions:

RE(z) : Real part of a complex number
IM(z) : Imaginary part of a complex number
C→R(z) : Takes a complex number (x,y) and separates it into its real and imaginary parts
R→C(x,y): Forms the complex number *(x,y)* out of real numbers x and y
ABS(z) : Calculates the magnitude of a complex number or the absolute value of a real number.
ARG(z): Calculates the argument of a complex number.

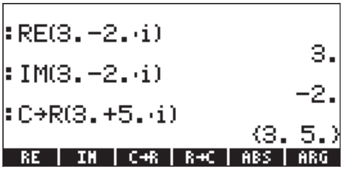
The remaining options (options 7 through 10) are the following:



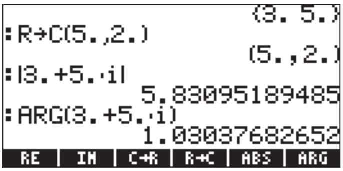
SIGN(z) : Calculates a complex number of unit magnitude as z/|z|.
NEG : Changes the sign of z
CONJ(z): Produces the complex conjugate of z

Examples of applications of these functions are shown next. Recall that, for ALG mode, the function must precede the argument, while in RPN mode, you enter the argument first, and then select the function. Also, recall that you can get these functions as soft menus by changing the setting of system flag 117 (See Chapter 3).

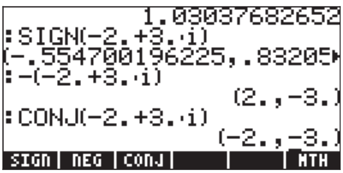
This first screen shows functions RE, IM, and C→R. Notice that the last function returns a list {3. 5.} representing the real and imaginary components of the complex number:



The following screen shows functions R→C, ABS, and ARG. Notice that the ABS function gets translated to |3.+5.·i|, the notation of the absolute value. Also, the result of function ARG, which represents an angle, will be given in the units of angle measure currently selected. In this example, ARG(3.+5.·i) = 1.0303… is given in radians.

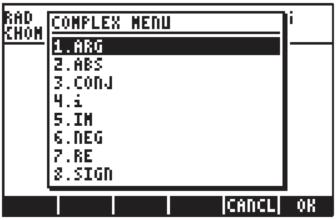


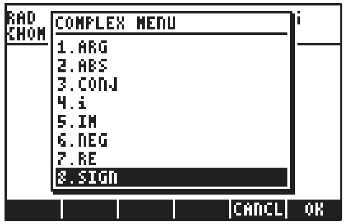
In the next screen we present examples of functions SIGN, NEG (which shows up as the negative sign -), and CONJ.



CMPLX menu in keyboard

A second CMPLX menu is accessible by using the right-shift option associated with the / key, i.e., → CMPLX . With system flag 117 set to CHOOSE boxes, the keyboard CMPLX menu shows up as the following screens:



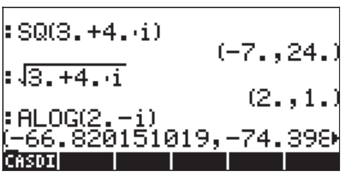


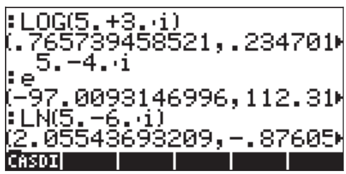
The resulting menu include some of the functions already introduced in the previous section, namely, ARG, ABS, CONJ, IM, NEG, RE, and SIGN. It also includes function *i* which serves the same purpose as the keystroke combination ⬅ *i*, i.e., to enter the unit imaginary number *i* in an expression.

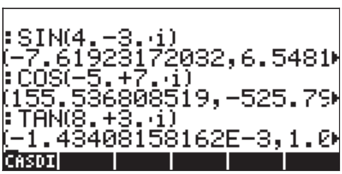
The keyboard-base CMPLX menu is an alternative to the MTH-based CMPLX menu containing the basic complex number functions. Try the examples shown earlier using the keyboard-based CMPLX menu for practice.

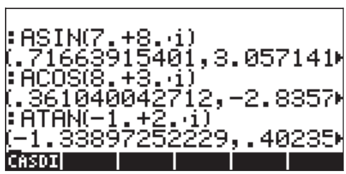
Functions applied to complex numbers

Many of the keyboard-based functions defined in Chapter 3 for real numbers, e.g., SQ, ,LN, e^x, LOG, 10^X, SIN, COS, TAN, ASIN, ACOS, ATAN, can be applied to complex numbers. The result is another complex number, as illustrated in the following examples. To apply this functions use the same procedure as presented for real numbers (see Chapter 3).





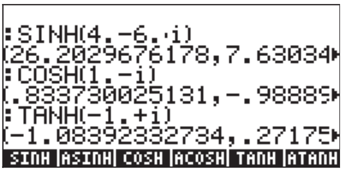


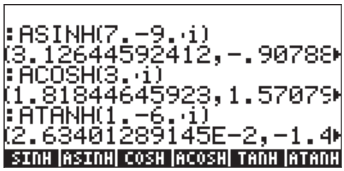


Note: When using trigonometric functions and their inverses with complex numbers the arguments are no longer angles. Therefore, the angular measure selected for the calculator has no bearing in the calculation of these functions with complex arguments. To understand the way that trigonometric functions, and other functions, are defined for complex numbers consult a book on complex variables.

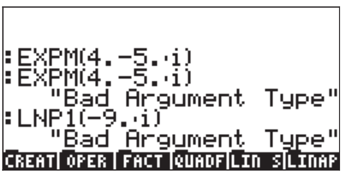
Functions from the MTH menu

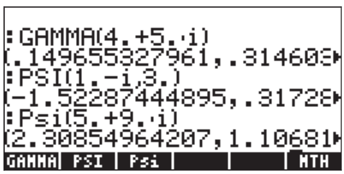
The hyperbolic functions and their inverses, as well as the Gamma, PSI, and Psi functions (special functions) were introduced and applied to real numbers in Chapter 3. These functions can also be applied to complex numbers by following the procedures presented in Chapter 3. Some examples are shown below:





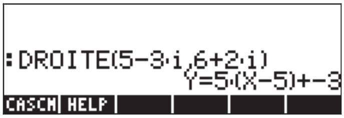
The following screen shows that functions EXPM and LNP1 do not apply to complex numbers. However, functions GAMMA, PSI, and Psi accept complex numbers:





Function DROITE: equation of a straight line

Function DROITE takes as argument two complex numbers, say, x_1+iy_1 and x_2+iy_2, and returns the equation of the straight line, say, y = a+bx, that contains the points (x_1,y_1) and (x_2,y_2). For example, the line between points A(5,-3) and B(6,2) can be found as follows (example in Algebraic mode):



Function DROITE is found in the command catalog (CAT).

Using EVAL(ANS(1)) simplifies the result to: