CSC208 0314 Lab: Introduction to Algorithmic Verification

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1 Problem: Constructive Thoughts

Claim 1. *There exists a knight's walk for any chessboard of size* $n \geq 4$

Claim 2. *There exists a rook's tour for any chessboard of size* $n \geq 1$

1.1 Proof the Knight's Walks claim by induction on the size of the chessboard.

Proof. Let us perform induction on the claim. We assume that the chessboard's rows and columns are displayed as x,y axis. For the case $n = 4$, the order of positions for traversing across a row are as follows:

$*(1,1)->(2,3)->(4,2)->*(2,1)->(3,3)->(1,2)->*(3,1)->(4,3)->(2,2)->*(4,1)$

Since we can perform this order of moves forwards to the right, or backwards to the left, and since we can stop at any point along the row, we have shown that row traversal is possible.

The order of positions for traversing across a column are as follows:

$*(1,1)->(3,2)->(2,4)->*(1,2)->(3,3)->(2,1)->*(1,3)->(3,4)->(2,2)->*(1,4)$

Since we can perform this order of moves forwards up, or backwards down, and since we can stop at any point along the column, we have shown that column traversal is possible.

Therefore, since we have shown that row and column traversal is possible, we can perform the alternating set of row and column traversals as follows:
$(1,1)->*(4,1)->*(4,2)->*(1,2)->*(1,3)->*(4,3)->*(4,4)->*(1,4)$
Thus, for an $n = 4$ board, a knight's walk exists.

For our inductive case, let us show that there exists a knights walk for $n - 1 \geq 4$
Let $n - 1$ be k. We can perform the following order of row and column traversals to visit every square on the k x k board.

$(1,1)->*(k,1)->*(k,2)->*(1,2)->*(1,3)->*(k,3)->*(k,4)->*...(k,k)$

□

1.2 From your proof, describe an algorithm for performing a Knight's walk on an chessboard with. Because your proof is inductive, your algorithm should be recursive, following the structure of your proof.

An algorithm is apparent from our proof. You can use groups of 3 moves to get to a neighboring square, and since you can always move to a neighboring square you can use a zig-zag pattern, described as such:

$(1,1)- > *(n,1)- > *(n,2)- > *(1,2)- > *(1,3)- > *(n,3)- > *(n,4)- > *...(n,n)$

1.3 Next, describe an algorithm for performing a Rook's tour on an n x n chessboard of size $n \geq 1$. Try to describe it recursively, similarly to your proof/algorithm for the Knight's walk.

Since a rook can always move to a neighboring square in 1 move, you can use a similar zig-zag pattern of row and column traversals, stopping at every square only once:

$(1,1)- > *(n,1)- > *(n,2)- > *(1,2)- > *(1,3)- > *(n,3)- > *(n,4)- > *...(n,n)$

1.4 Finally, translate your Rook's tour algorithm into a proof of the Rook's Tour claim by induction on the dimensions of the chessboard.

Let us prove the claim by induction:

The first case, $n = 1$
The rook starts on square (1, 1). The tour is complete.

The inductive case: Let us prove a rook tour exists for any board $n - 1 \geq 1$.

Let $k = n - 1$ and x, y be the initial x, y coordinates.
The following ordering of row and column traversals, where the rook visits every square in the traversal, is a tour of the board with size k x k:

$(x,y)- > *(k,y)- > *(k,y+1)- > *(x,y+1)- > *...- > *(k,k)$

Thus we have shown a rook tour for any n x n board where $n \geq 1$