CSC341 Lab 1B

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1 Problem 1 : Formal Proof

Claim 1

i

Let $x \in \Sigma^*$ be arbitrary. By the definition of Σ^*, we can fix $a_1, a_2, ..., a_n \in \Sigma$ such that $x = a_1a_2...a_n$. Then $x\epsilon = a_1a_2...a_n = x$, and $\epsilon x = a_1a_2...a_n = x$. Thus ϵ is the identity under concatenation.

ii

Let $x, y, z \in \Sigma^*$ be arbitrary. By the definition of Σ^*, we can fix $a_1, a_2, ..., a_m, b_1, b_2, ..., b_n, c_1, c_2, ..., c_k$ such that $x = a_1a_2...a_m$, $y = b_1b_2...b_n$, and $z = c_1c_2...c_k$. Then $xy = a_1a_2...a_mb_1b_2...b_n$, so $(xy)z = a_1a_2...a_mb_1b_2...b_nc_1c_2...c_k$. Similarly, we have $yz = b_1b_2...b_nc_1c_2...c_k$, so $x(yz) = a_1a_2...a_mb_1b_2...b_nc_1c_2...c_k$, so $x(yz) = (xy)z$. Thus concatenation is associative.

Claim 2

Let u, z be arbitrary identities of Σ^*. Because u is an identity, $u + z = z$. Furthermore, because z is an identity, $u + z = u$. Thus $u = z$ because equality is transitive.

Claim 3

We first prove the following statement:

Lemma 1 *For all strings a, b, we have $|a| + |b| = |ab|$.*

Let $a = a_1a_2...a_m$ and $b = b_1b_2...b_n$ be arbitrary strings. Unpacking the definition of concatenation, $ab = a_1a_2...a_mb_1b_2...b_n$, so $|ab| = m + n = |a| + |b|$.

We now prove Claim 3. Let x, y be arbitrary strings. Because x is a prefix of y, we can fix a string z_1 such that $xz_1 = y$. Thus $|xz_1| = |y|$. By Lemma 1, we have $|x| + |z_1| = |y|$, so $|x| \leq |y|$ because $|z_1| \geq 0$ by the definition of length. Similarly, because y is a prefix of x, we can fix a string z_2 such that $yz_2 = x$, so $|yz_2| = |x|$ and $|y| \leq |x|$. Therefore $|x| = |y| = |xz_1| = |x| + |z_1|$ by Lemma 1, so $|z_1| = 0$. Thus z_1 must be the empty string, so $xz_1 = x$. Because $xz_1 = y$, this implies $x = y$ as desired.

1.1 Claim 4

First, for the base case when $n = 0$, $2^0 = 1$ at node level 0 for the tree holds true because there is always a single node at level 0.

Assuming there is an arbitrary level, $k + 1$ and k, at level $k + 1$, there will be exactly two children for each node at level k since the tree is complete. By the inductive hypothesis, there are 2^k nodes at level k, so this makes the number of nodes at level $k + 1$ to be $2 * 2^k = 2^{k+1}$.

Through structural induction, since we proved the base case and the inductive process, the claim is true for all levels of the complete and perfect binary tree.