Lesson 8: Adding and Subtracting Polynomials

Student Outcomes

- Students understand that the sum or difference of two polynomials produces another polynomial and relate polynomials to the system of integers; students add and subtract polynomials.

Classwork

Exercise 1 (7 minutes)

Have students complete Exercise 1 part (a) and use it for a brief discussion on the notion of base. Then have students continue with the remainder of the exercise.

Exercise 1

a. **How many quarters, nickels, and pennies are needed to make $\$1.13$?**

Answers will vary.

4 quarters, 2 nickels, 3 pennies

b. **Fill in the blanks:**

$$8,943 = \underline{8} \times 1000 + \underline{9} \times 100 + \underline{4} \times 10 + \underline{3} \times 1$$
$$= \underline{8} \times 10^3 + \underline{9} \times 10^2 + \underline{4} \times 10 + \underline{3} \times 1$$

c. **Fill in the blanks:**

$$8,943 = \underline{1} \times 20^3 + \underline{2} \times 20^2 + \underline{7} \times 20 + \underline{3} \times 1$$

d. **Fill in the blanks:**

$$113 = \underline{4} \times 5^2 + \underline{2} \times 5 + \underline{3} \times 1$$

Scaffolding:

- Mayan, Aztec, and Celtic cultures all used base 20. The word *score* (which means 20) originated from the Celtic language.
- Students could be asked to research more on this and on the cultures who use or used base 5 and base 60.

Next ask:

- Why do we use base 10? Why do we humans have a predilection for the number 10?
- Why do some cultures have base 20?
- How do you say 87 in French? How does the Gettysburg Address begin?
  - *Quatre-vingt-sept: 4-20s and 7; Four score and seven years ago…*
- Computers use which base system?
  - *Base 2*

Exercise 2 (5 minutes)

In Exercise 2, we are laying the foundation that polynomials written in standard form are simply base x *numbers*. The practice of filling in specific values for x and finding the resulting values lays a foundation for connecting this algebra of polynomial expressions with the later lessons on polynomial functions (and other functions) and their inputs and outputs.

Work through Exercise 2 with the class.

Exercise 2

Now let's be as general as possible by not identifying which base we are in. Just call the base x.

Consider the expression $1 \cdot x^3 + 2 \cdot x^2 + 7 \cdot x + 3 \cdot 1$, or equivalently $x^3 + 2x^2 + 7x + 3$.

a. **What is the value of this expression if $x = 10$?**

$1,273$

b. **What is the value of this expression if $x = 20$?**

$8,943$

Point out that the expression we see here is just the generalized form of their answer from part (b) of Exercise 1. However, as we change x, we get a different number each time.

Exercise 3 (10 minutes)

Allow students time to complete Exercise 3 individually. Then elicit responses from the class.

Exercise 3

a. **When writing numbers in base 10, we only allow coefficients of 0 through 9. Why is that?**

Once you get ten of a given unit, you also have one of the unit to the left of that.

b. **What is the value of $22x + 3$ when $x = 5$? How much money is 22 nickels and 3 pennies?**

113

$\$1.13$

c. **What number is represented by $4x^2 + 17x + 2$ if $x = 10$?**

572

d. **What number is represented by $4x^2 + 17x + 2$ if $x = -2$ or if $x = \frac{2}{3}$?**

-16

$\frac{136}{9}$

e. **What number is represented by $-3x^2 + \sqrt{2}x + \frac{1}{2}$ when $x = \sqrt{2}$?**

$-\frac{7}{2}$

Point out, as highlighted by Exercises 1 and 3, that carrying is not necessary in this type of expression (polynomial expressions). For example, $4x^2 + 17x + 2$ is a valid expression. However, in base ten arithmetic, coefficients of value ten or greater are not conventional notation. Setting $x = 10$ in $4x^2 + 17x + 2$ yields 4 hundreds, 17 tens, and 2 ones, which is to be expressed as 5 hundreds, 7 tens, and 2 ones.

Discussion (11 minutes)

- The next item in your student materials is a definition for a polynomial expression. Read the definition carefully, and then create 3 polynomial expressions using the given definition.

POLYNOMIAL EXPRESSION: A *polynomial expression* is either

1. **A numerical expression or a variable symbol, or**
2. **The result of placing two previously generated polynomial expressions into the blanks of the addition operator (__+__) or the multiplication operator (__×__).**

- Compare your polynomial expressions with a neighbor's. Do your neighbor's expressions fall into the category of polynomial expressions?

Resolve any debates as to whether a given expression is indeed a polynomial expression by referring back to the definition and discussing as a class.

- Note that the definition of a polynomial expression includes subtraction (add the additive inverse instead), dividing by a nonzero number (multiply by the multiplicative inverse instead), and even exponentiation by a nonnegative integer (use the multiplication operator repeatedly on the same numerical or variable symbol).

List several of the student-generated polynomials on the board. Include some that contain more than one variable.

Initiate the following discussion, presenting expressions on the board when relevant.

- Just as the expression $(3 + 4) \cdot 5$ is a numerical expression but not a number, $(x + 5) + (2x^2 - x)(3x + 1)$ is a polynomial expression but not technically a *polynomial*. We reserve the word *polynomial* for polynomial expressions that are written simply as a sum of monomial terms. This begs the question: What is a monomial?
- A *monomial* is a polynomial expression generated using only the multiplication operator (__×__). Thus, it does not contain $+$ or $-$ operators.
- Just as we would not typically write a number in factored form and still refer to it as a number (we might call it a number in factored form), similarly, we do not write a monomial in factored form and still refer to it as a monomial. We multiply any numerical factors together and condense multiple instances of a variable factor using (whole number) exponents.
- Try creating a monomial.
- Compare the monomial you created with your neighbor's. Is your neighbor's expression really a monomial? Is it written in the standard form we use for monomials?

- There are also such things as binomials and trinomials. Can anyone make a conjecture about what a binomial is and what a trinomial is and how they are the same or different from a polynomial?

Students may conjecture that a binomial has two of something and that a trinomial three of something. Further, they might conjecture that a polynomial has many of something. Allow for discussion and then state the following:

- A binomial is the sum (or difference) of two monomials. A trinomial is the sum (or difference) of three monomials. A polynomial, as stated earlier, is the sum of one or more monomials.
- The *degree of a monomial* is the sum of the exponents of the variable symbols that appear in the monomial.
- The *degree of a polynomial* is the degree of the monomial term with the highest degree.
- While polynomials can contain multiple variable symbols, most of our work with polynomials will be with *polynomials in one variable*.
- What do polynomial expressions in one variable look like? Create a polynomial expression in one variable, and compare with your neighbor.

Post some of the student-generated polynomials in one variable on the board.

- Let's relate polynomials to the work we did at the beginning of the lesson.
- Is this expression an integer in base 10? $10(100 + 22 - 2) + 3(10) + 8 - 2(2)$
- Is the expression equivalent to the integer $1{,}234$?
- How did we find out?
- We rewrote the first expression in our standard form, right?
- Polynomials in one variable have a standard form as well. Use your intuition of what standard form of a polynomial might be to write this polynomial expression as a polynomial in standard form: $2x(x^2 - 3x + 1) - (x^3 + 2)$, and compare your result with your neighbor.
  - *Students should arrive at the answer* $x^3 - 6x^2 + 2x - 2$*.*

Confirm that in standard form, we start with the highest degreed monomial and continue in descending order.

- The *leading term* of a polynomial is the term of highest degree that would be written first if the polynomial is put into standard form. The *leading coefficient* is the coefficient of the leading term.
- What would you imagine we mean when we refer to the *constant term* of the polynomial?
  - *A constant term is any term with no variables. To find the constant term of a polynomial, be sure you have combined any and all constant terms into one single numerical term, written last if the polynomial is put into standard form. Note that a polynomial does not have to have a constant term (or could be said to have a constant term of* 0*).*

As an extension for advanced students, assign the task of writing a formal definition for *standard form* of a polynomial. The formal definition is provided below for your reference:

> A polynomial expression with one variable symbol x is in *standard form* if it is expressed as $a_n x^n + a_{n-1} x^{n-1} + a_1 x + a_0$, where n is a non negative integer, and $a_0, a_1, a_2, \ldots, a_n$ are constant coefficients with $a_n \neq 0$. A polynomial expression in x that is in standard form is often called a *polynomial in* x.

Exercise 4 (5 minutes)

Exercise 4

Find each sum or difference by combining the parts that are alike.

a. $417 + 231 =$ __4__ hundreds + __1__ tens + __7__ ones + __2__ hundreds + __3__ tens + __1__ ones
$=$ __6__ hundreds + __4__ tens + __8__ ones

b. $(4x^2 + x + 7) + (2x^2 + 3x + 1)$

$6x^2 + 4x + 8$

c. $(3x^3 - x^2 + 8) - (x^3 + 5x^2 + 4x - 7)$

$2x^3 - 6x^2 - 4x + 15$

d. $3(x^3 + 8x) - 2(x^3 + 12)$

$x^3 + 24x - 24$

e. $(5 - t - t^2) + (9t + t^2)$

$8t + 5$

f. $(3p + 1) + 6(p - 8) - (p + 2)$

$8p - 49$

Closing (3 minutes)

- How are polynomials analogous to integers?
  - *While integers are in base 10, polynomials are in base x.*
- If two polynomials are added together, is the result sure to be another polynomial? The difference of two polynomials?
  - *Students will likely reply, "yes," based on the few examples and their intuition.*
- Are you sure? Can you think of an example where adding or subtracting two polynomials does not result in a polynomial?
  - *Students thinking about $x^2 - x^2 = 0$ could suggest not. At this point, review the definition of a polynomial. Constant symbols are polynomials.*

Lesson Summary

A *monomial* is a polynomial expression generated using only the multiplication operator (__×__). Thus, it does not contain $+$ or $-$ operators. *Monomials* are written with numerical factors multiplied together and variable or other symbols each occurring one time (using exponents to condense multiple instances of the same variable).

A *polynomial* is the sum (or difference) of monomials.

The *degree of a monomial* is the sum of the exponents of the variable symbols that appear in the monomial.

The *degree of a polynomial* is the degree of the monomial term with the highest degree.

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 8: Adding and Subtracting Polynomials

Exit Ticket

1. Must the sum of three polynomials again be a polynomial?

2. Find $(w^2 - w + 1) + (w^3 - 2w^2 + 99)$.

Exit Ticket Sample Solutions

1. Must the sum of three polynomials again be a polynomial?

   Yes.

2. Find $(w^2 - w + 1) + (w^3 - 2w^2 + 99)$.

   $w^3 - w^2 - w + 100$

Problem Set Sample Solutions

1. Celina says that each of the following expressions is actually a binomial in disguise:
   - i. $5abc - 2a^2 + 6abc$
   - ii. $5x^3 \cdot 2x^2 - 10x^4 + 3x^5 + 3x \cdot (-2)x^4$
   - iii. $(t+2)^2 - 4t$
   - iv. $5(a-1) - 10(a-1) + 100(a-1)$
   - v. $(2\pi r - \pi r^2)r - (2\pi r - \pi r^2) \cdot 2r$

   For example, she sees that the expression in (i) is algebraically equivalent to $11abc - 2a^2$, which is indeed a binomial. (She is happy to write this as $11abc + (-2)a^2$, if you prefer.)

   Is she right about the remaining four expressions?

   She is right about the remaining four expressions. They all can be expressed as binomials.

2. Janie writes a polynomial expression using only one variable, x, with degree 3. Max writes a polynomial expression using only one variable, x, with degree 7.
   a. What can you determine about the degree of the sum of Janie's and Max's polynomials?

      The degree would be 7.

   b. What can you determine about the degree of the difference of Janie's and Max's polynomials?

      The degree would be 7.

3. Suppose Janie writes a polynomial expression using only one variable, x, with degree of 5, and Max writes a polynomial expression using only one variable, x, with degree of 5.
   a. What can you determine about the degree of the sum of Janie's and Max's polynomials?

      The maximum degree could be 5, but it could also be anything less than that. For example, if Janie's polynomial were $x^5 + 3x - 1$, and Max's were $-x^5 + 2x^2 + 1$, the degree of the sum is only 2.

   b. What can you determine about the degree of the difference of Janie's and Max's polynomials?

      The maximum degree could be 5, but it could also be anything less than that.

4. **Find each sum or difference by combining the parts that are alike.**

   a. $(2p + 4) + 5(p - 1) - (p + 7)$

      $6p - 8$

   b. $(7x^4 + 9x) - 2(x^4 + 13)$

      $5x^4 + 9x - 26$

   c. $(6 - t - t^4) + (9t + t^4)$

      $8t + 6$

   d. $(5 - t^2) + 6(t^2 - 8) - (t^2 + 12)$

      $4t^2 - 55$

   e. $(8x^3 + 5x) - 3(x^3 + 2)$

      $5x^3 + 5x - 6$

   f. $(12x + 1) + 2(x - 4) - (x - 15)$

      $13x + 8$

   g. $(13x^2 + 5x) - 2(x^2 + 1)$

      $11x^2 + 5x - 2$

   h. $(9 - t - t^2) - \frac{3}{2}(8t + 2t^2)$

      $-4t^2 - 13t + 9$

   i. $(4m + 6) - 12(m - 3) + (m + 2)$

      $-7m + 44$

   j. $(15x^4 + 10x) - 12(x^4 + 4x)$

      $3x^4 - 38x$