# 2023 Mission Math Summer Camp Advanced Algebra

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## 1 Basics

Algebra is a very broad topic. Here are some algebraic formulas you should know.

### 1.1 Basics

- $(x+y)^2 = x^2 + 2xy + y^2$
- $x^2 - y^2 = (x-y)(x+y)$
- $x^{2n+1} + y^{2n+1} = (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - x^{2n-3}y^3 + x^{2n-4}y^4 + \cdots + x^2y^{2n-2} - xy^{2n-1} + y^{2n})$
- $x^{2n+1} - y^{2n+1} = (x-y)(x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + x^{2n-3}y^3 + x^{2n-4}y^4 + \cdots + x^2y^{2n-2} + xy^{2n-1} + y^{2n})$
- $1 + 2 + 3 + \cdots + n = \frac{(n)(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{(n)(n+1)}{2}\right)^2$
- Binomial theorem: $(x+y)^n =$
$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
- Sophie-Germain Identity: $a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$
- $x^3 + y^3 + z^3 = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$

### 1.2 Problems

1. $\frac{1000^2}{252^2 - 248^2}$ equals

   **(A)** $62,500$ **(B)** $1,000$ **(C)** $500$ **(D)** $250$ **(E)** $\frac{1}{2}$ [1985 AHSME #1]

2. Compute $\sqrt{(31)(30)(29)(28)+1}$. [1989 AIME #1]

3. Factor completely the expression $(a-b)^3 + (b-c)^3 + (c-a)^3$. [2009 SMT Team #2].

4. Find $x^2 + y^2$ if $x$ and $y$ are positive integers such that

$$\begin{aligned} xy + x + y &= 71 \\ x^2y + xy^2 &= 880. \end{aligned}$$

   [1991 AIME #1 ]

# 2 Sequences and Series

## 2.1 Arithmetic sequences

An arithmetic sequence is a finite sequence such that consecutive terms differ by a constant, which we call the common difference. An example of an arithmetic sequence is $1, 3, 5, 7, 9$ and so on. Let $a$ be the first term in an arithmetic sequence and $d$ be the common difference. The $n^{th}$ term of the sequence is $a + d(n-1)$ and the sum of the first $n$ terms is $(n)\left(\frac{2a+d(n-1)}{2}\right)$, which is the average of the first and last terms in the sequence multiplied by the number of terms.

## 2.2 Geometric Sequences

Like an arithmetic sequence, there is a special relationship between consecutive terms in a geometric sequence. Instead of adding a number to get the next, we multiply by a common ratio. For example, $1, 2, 4, 8, 16$ is a geometric sequence because each term is equal to the previous term times two. A geometric sequence can be finite or infinite. Let's say we have a geometric sequence with first term $a$ and common ratio of $r$. Therefore, our sequence is $a, ar, ar^2, ....$ The $n^{th}$ term in a geometric sequence is $ar^{n-1}$. For a finite geometric sequence, the sum is $\frac{a(r^n-1)}{r-1}$. For an infinite geometric sequence, if $0 < r < 1$ the sum is $\frac{a}{1-r}$.

## 2.3 Problems

1. What is the arithmetic mean of the squares of the first ten positive integers? Express your answer as a decimal to the nearest tenth. [2018 MATHCOUNTS State Team # 1]

2. $90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 =$

   **(A)** 845 **(B)** 945 **(C)** 1005 **(D)** 1025 **(E)** 1045 [1985 AHSME #2]

3. Find the sum of the infinite geometric series

   $$6 + 3.6 + 2.16 + 1.296 + \cdots.$$

   [1996 MATHCOUNTS National Target # 2]

4. At each basketball practice last week, Jenny made twice as many free throws as she made at the previous practice. At her fifth practice she made 48 free throws. How many free throws did she make at the first practice?

   **(A)** 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 15

   [2004 AMC 10B # 3]

5. The third and fourth terms of a geometric sequence are 100 and 200. What is the first term of the sequence? [2019 MATHCOUNTS State Team #3]

6. The first term of an arithmetic sequence is $-37$, and the 2nd term is $-30$. What is the smallest positive term of the sequence? [MATHCOUNTS]

7. The third term of an arithmetic sequence is $-1$, and the ninth term is 20. What is the thirteenth term?

8. How many terms are there in the arithmetic sequence 13, 16, 19, $\ldots$, 70, 73?

   **(A)** 20 **(B)** 21 **(C)** 24 **(D)** 60 **(E)** 61

9. If the mean of a list of 1000 consecutive odd integers is 74, what is the least number in the list? [2022 MATHCOUNTS National Sprint # 12]

10. The sum of the first 20 positive even integers is also the sum of four consecutive even integers. What is the largest of these four integers? [MATHCOUNTS]

# 3 Polynomials

A polynomial is written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

for numbers $a_i$. If $f(a) = 0$, then we call $a$ a root of $P$. The degree of a polynomial, or deg(P) is equal to the highest power in the polynomial, so deg(P) $= n$.

The Fundamental Theorem of Algebra says that a polynomial with degree $n$ has $n$ not necessarily distinct roots.

## 3.1 Polynomial manipulation

1. If $f(x) = ax^2 + bx + c$, $f(1) = 0$, $f(2) = 1$ and $f(3) = 8$, what is the value of $c$? [2014 MATHCOUNTS National Target # 1]
2. If the polynomial $ax^3 + bc^2 + cx + d$ is graphed in the plane, what will its y-intercept be?
3. If the polynomial $3x^2 - 6x + 1$ has roots $r$ and $s$, construct a polynomial that has roots $-r$ and $-s$
4. The solutions of the equation $2x^2 + 5x - 12 = 0$ are $m$ and $n$. What is the value of $(m-1)(n-1)$? Express your answer as a common fraction. [2018 MATHCOUNTS Chapter Target # 2]
5. Let $P(x) = (x-2)(x-3)(x-4)(x-5)$. What is the sum of the coefficients of $P$?

## 3.2 Vieta's Formulas

If $ax^2 + bx + c$ can be factored as $a(x-r)(x-s)$ for its roots r and s, then we can expand $a(x-r)(x-s)$ algebraically to get $ax^2 - arx - asx + ars$. Matching the coefficients of this quadratic to the original and dividing by a, we get two equations:

$$r + s = -\frac{b}{a}$$
$$rs = \frac{c}{a}.$$

In other words, the sum of the roots of a quadratic is the negative ratio between its second and first coefficient, and their product is the ratio of their first and third coefficient. Note that this can be generalized to a polynomial of any degree by writing it as a product of linear terms, expanding, and equating coefficients.

## 3.3 Problems

1. The polynomial $2x^2 + 4x + 3$ has roots $a$ and $b$. What is $a^2 + b^2$?
2. Suppose that $a$ and $b$ are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions $a$ and $b$. Evaluate $a$ and $b$.
3. What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$$

# 4 Exercises

This section contains some more challenging problems that apply concepts talked about in this handout. These applications are a bit harder to find and sometimes require a combination of concepts:

1. Compute $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 1998^2 + 1999^2$. [1983 NYSMAL T-1]

2. If $a_0, a_1, a_2, \ldots$ is an arithmetic sequence and $18a_{20} + 20 = 20a_{18} + 18$, compute $a_0$. [2018 ARML Local # 2]

3. What is the sum of the solutions to the equation $\sqrt[4]{x} = \frac{12}{7-\sqrt[4]{x}}$? [1986 AIME # 1]

4. Find the value of $a_2 + a_4 + a_6 + a_8 + \cdots + a_{98}$ if $a_1, a_2, a_3 \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$. [1984 AIME # 1]

5. Two arithmetic sequences $A$ and $B$ both begin with 30 and have common differences of absolute value 10, with sequence $A$ increasing and sequence $B$ decreasing. What is the absolute value of the difference between the 51st term of sequence $A$ and the 51st term of sequence $B$? [2009 MATHCOUNTS State Target # 2]

6. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series
$$12 + 12r + 12r^2 + 12r^3 + \cdots$$
. Let $a$ between $-1$ and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.[2016 AIME I # 1]

7. Let $x_1 = 97$, and for $n > 1$ let $x_n = \frac{n}{x_{n-1}}$. Calculate the product $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$. [1985 AIME # 1]

8. Set $A$ consists of $m$ consecutive integers whose sum is $2m$, and set $B$ consists of $2m$ consecutive integers whose sum is $m$. The absolute value of the difference between the greatest element of $A$ and the greatest element of $B$ is 99. Find $m$. [2004 AIME I # 2]

9. If $a$, $b$, $c$, $d$, and $e$ are constants such that every $x > 0$ satisfies
$$\frac{5x^4 - 8x^3 + 2x^2 + 4x + 7}{(x+2)^4} = a + \frac{b}{x+2} + \frac{c}{(x+2)^2} + \frac{d}{(x+2)^3} + \frac{e}{(x+2)^4},$$
then what is the value of $a + b + c + d + e$? [2009 Math Prize for Girls # 2]

10. The sum of an infinite geometric progression is 20. If each term had been squared, the sum would have been 80. Compute the first term. [1997 NYSMAL T-3]

11. The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.[2011 AIME II # 5]

12. Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots. [2001 AIME I # 3]

13. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are $a, b$, and $c$, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find $t$. [1996 AIME # 5]

14. The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where $a$, $b$, and $c$ are positive integers. Find $a + b + c$. [2013 AIME I # 5]

15. If the integer $k$ is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find $k$. [1989 AIME # 7]

16. Let $r, s$, and $t$ be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r+s)^3 + (s+t)^3 + (t+r)^3$. [2008 AIME II # 7]

17. Find the eighth term of the sequence $1440, 1716, 1848, \ldots$, whose terms are formed by multiplying the corresponding terms of two arithmetic sequences. [2003 AIME II # 8]

18. Find $A^2$, where $A$ is the sum of the absolute values of all roots of the following equation:

$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{x}}}}}$$

[1991 AIME # 7]

19. It is possible to place positive integers into the vacant twenty-one squares of the $5 \times 5$ square shown below so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*).

| | | | | |
|---|---|---|---|---|
| | | | $*$ | |
| | 74 | | | |
| | | | | 186 |
| | | 103 | | |
| 0 | | | | |

[1988 AIME # 6]

20. Find the number of ordered pairs of integers $(a, b)$ such that the sequence

$$3, 4, 5, a, b, 30, 40, 50$$

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression. [2022 AIME I # 6]