Counting and Probability

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August 2023

1 The Basics

1.1 What is probability?

Probability is a way to represent how likely something is to happen. Denote $P(A)$ as the probability that event A happens.

$$P(A) = \frac{\#\text{ of outcomes where A is met}}{\#\text{ of total outcomes}}$$

For example, the probability of flipping heads on a coin is $\frac{1}{2}$ because each time we flip the coin, there is only 1 way to get a head and there are two possible outcomes: heads or tails. What does this number truly mean? Say I were to flip a coin a lot of times. This means that approximately half of the time, I would get heads on the coin.

1.2 Adding probabilities

Let's say I have a bag of marbles. In this bag, there are 4 red marbles, 3 orange marbles, 2 yellow marbles, and 1 green marble. The probability of drawing a red marble is $\frac{4}{10} = \frac{2}{5}$ because there are 4 red marbles out of 10 total marbles. Similarly we have

$$P(\text{Orange}) = \frac{3}{10}$$

$$P(\text{Yellow}) = \frac{2}{10}$$

$$P(\text{Green}) = \frac{1}{10}$$

What if I wanted to find the probability that when I draw a marble, it is either a red or green one? Well, there are 4 red marbles and 1 green marble so the probability of getting one of these is $\frac{4+1}{10} = \frac{1}{2}$

1.3 Multiplying probabilities

Using the same situation as the section above, what if I want to draw 2 marbles? Let's say I want to draw one marble, put it back in the bag, and then draw another marble. What's the probability that I first draw a red marble and then a green marble? We have to multiply the probability of drawing an orange marble by the probability of drawing a green marble since this is the probability of both events happening. In other words:

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$$

so the probability of getting a yellow marble and then a green marble is the probability of getting a yellow marble multiplied by the probability of getting a green marble, which is $\frac{2}{5} \cdot \frac{1}{10} = \frac{1}{25}$

1.4 Exercises

1. What is the probability of rolling an odd number on a dice?
2. What is the probability of rolling a prime number on a dice?
3. What is the probability of rolling 3 or 4 on a standard 6-sided dice?
4. When 2 dice are rolled, what is the probability that the sum of the faces that show up on the dice is 4?
5. When 3 coins are flipped, what is the probability that you get exactly 3 heads?
6. What is the expected value of a single dice roll?
7. What is the expected value of two dice rolls?
8. Jack is playing a game. If he wins, which occurs with probability $\frac{2}{5}$, he wins 1000, but if he loses, then he loses 500. If Jack plays the game once, what is his expected winnings?
9. What is the probability of drawing 3 aces in a row from the top of a standard deck of 54 cards.
10. An ordered pair (m, n), where m and n are integers, $1 \leq m \leq 4$ and $1 \leq n \leq 8$, is selected at random. Express as a common fraction the probability that both m and n are even.
11. A point (x, y) is randomly selected such that $0 \leq x \leq 8$ and $0 \leq y \leq 4$. What is the probability that both $x \leq 2$ and $y \leq 2$? Express your answer as a common fraction.
12. On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

    (A) $\frac{14}{85}$ **(B)** $\frac{7}{25}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{7}{10}$
13. The faces of each of two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

    (A) $\frac{4}{9}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$
14. A box contains 5 chips, numbered 1, 2, 3, 4, and 5. Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds 4. What is the probability that 3 draws are required?

    (A) $\frac{1}{15}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{4}$
15. One fair die has faces 1, 1, 2, 2, 3, 3 and another has faces 4, 4, 5, 5, 6, 6. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum will be odd?

    (A) $\frac{1}{3}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{2}{3}$