Advanced Geometry

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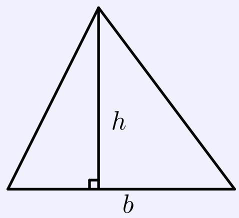
1 Formulas

Some formulas you probably already know so I won't write them down in this (ex. the sum of the angles in a triangle is 180°. We will go over some of the formulas that are worth memorizing.
Here's a few formulas for triangles

Theorem 1.1 (Area of a triangle)
The area of a triangle with base b and height h is $\frac{bh}{2}$



Sometimes you won't know the length of the height or the length of the height of the triangle. Here's a formula to find the area of a triangle given it's side lengths.

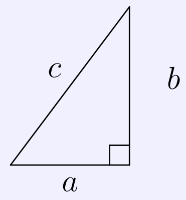
Theorem 1.2 (Heron's Formula)
In a triangle with side lengths a, b, and c, let s be the semiperimeter of the triangle. The area of the triangle will be $\sqrt{s(s-a)(s-b)(s-c)}$

The semiperimeter is half the perimeter, or $\frac{a+b+c}{2}$ in this case.

Theorem 1.3 (Area of an Equilateral Triangle)
The area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, where s is the side length of the triangle.

An equilateral is a triangle with all side lengths equal to each other.

Theorem 1.4 (Pythagorean Theorem)
In a right triangle, the sum of the squares of the legs is equal to the square of the length of the hypotenuse.



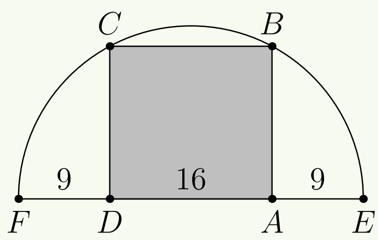
In other words, $a^2 + b^2 = c^2$

Theorem 1.5
The area of a rectangle with base length l and height (width) h is lh.

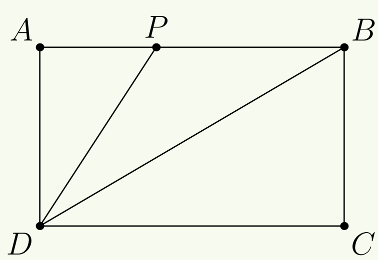
Theorem 1.6 (Area of a Circle)
The area of a circle with radius r is πr^2

Here's some example problems:

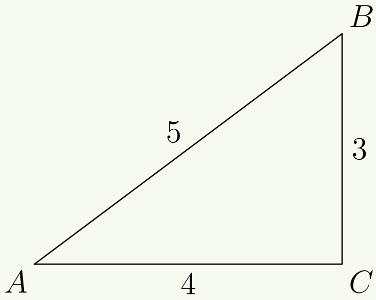
Example 1.7 (AMC 8)
Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



Example 1.8 (AMC 10)
In rectangle $ABCD$, $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?



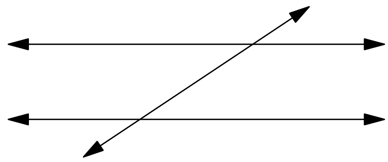
Example 1.9 (AMC 10)
A paper triangle with sides of lengths $3, 4$, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



2 Angles

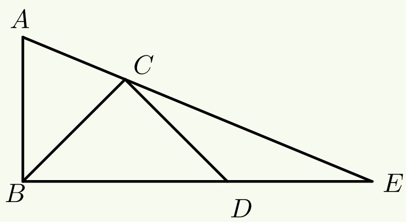
Here are some facts about angles:

- The sum of the angles in a triangle is $180°$
- When two lines are intersected by a transversal, all the acute angles are equal and all the obtuse angles are equal.



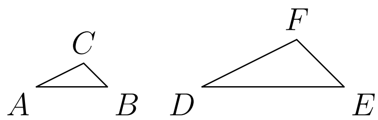
- The sum of the measures of angles that form a line is $180°$
- The sum of the degrees in a polygon with n sides is $180(n-2)$
- The measure of an interior angle in a regular polygon with n sides is $\frac{(180)(n-2)}{n}$ or $180 - \frac{360}{n}$

Example 2.1 (2015 MATHCOUNTS State Sprint)
Points C and D are chosen on the sides of right triangle ABE, as shown, such that the four segments AB, BC, CD and DE each have length 1 inch. What is the measure of angle BAE, in degrees? Express your answer as a decimal to the nearest tenth.



3 Similar Triangles

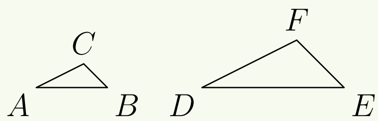
Two triangles are similar if they share the same angles. If triangles ABC and DEF are similar, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$



In other words, two triangles are similar if they have the exact same shape, but different (or even if the same) size. Similar triangles are very helpful when solving problems. Similar triangles appear often when there are parallel lines so be sure to look for similar triangles when there are parallel lines. If triangle ABC and DEF are similar, we write $ABC \sim DEF$ Here are a few example problems:

Example 3.1

In the diagram shown, $ABC \sim DEF$. Given that $AB = 7, BC = 3, DE = 14$, and $DF = 10$, find FE and AC.



Example 3.2 (AMC 10)

An equilateral triangle of side length 10 is completely filled in by non-overlapping equilateral triangles of side length 1. How many small triangles are required?

Example 3.3 (AMC10)

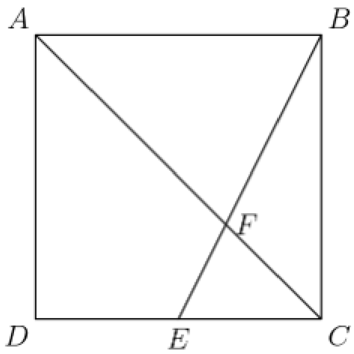
Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal $\overline{AC}$, and E is on AB with $\overline{ME} \perp \overline{AC}$. What is the area of $\triangle AME$?

4 Practice Problems

Problem 1 (MATHCOUNTS). Two similar right triangles have areas of 30 cm^2 and 270 cm^2. The length of the hypotenuse of the smaller triangle is 13 cm. What is the length of the hypotenuse of the larger triangle?

Problem 2 (AIME). Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$.

Problem 3 (AMC 8). Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



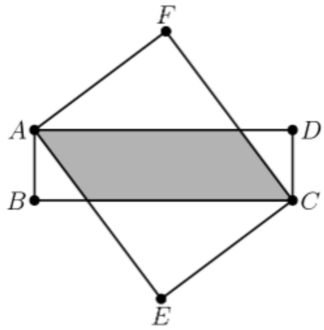
Problem 4 (AMC 10). In regular hexagon $ABCDEF$, points W, X, Y, and Z are chosen on sides $\overline{BC}$, $\overline{CD}$, $\overline{EF}$, and $\overline{FA}$ respectively, so lines AB, ZW, YX, and ED are parallel and equally spaced. What is the ratio of the area of hexagon $WCXYFZ$ to the area of hexagon $ABCDEF$?

Problem 5 (AMC 10). A regular octagon is formed by cutting an isosceles right triangle from each of the corners of a square with sides of length 2000. What is the length of each side of the octagon?

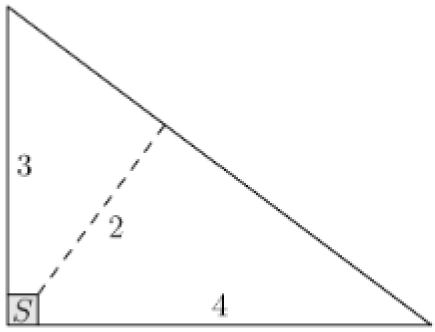
Problem 6 (AIME). In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 7 (AMC 10). A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

Problem 8 (AIME). In the diagram below, $ABCD$ is a rectangle with side lengths $AB = 3$ and $BC = 11$, and $AECF$ is a rectangle with side lengths $AF = 7$ and $FC = 9$, as shown. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 9 (AMC 10). Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



5 Answer Key

1. 39
2. 25
3. 108
4. $\frac{11}{27}$
5. $2000(\sqrt{2}-1)$ 547
6. 547
7. $\frac{37}{35}$
8. 109
9. $\frac{145}{147}$