

# OSM Boot Camp: Econ ProbSet1

Harrison Beard

25 June 2018

## Exercise 1.

**Question.** Consider the problem of the owner of an oil field. The owner has  $B$  barrels of oil. She can sell these barrels at price  $p$ , at time  $t$ . Her objective is to maximize the discounted present value of sales of oil—we'll assume there are no extraction costs. The owner discounts the future at a rate given by  $\frac{1}{1+r}$  (where  $r$  is the real interest rate and assumed to be constant). Answer the following:

1. What are the state variables?
2. What are the control variables?
3. What does the transition equation look like?
4. Write down the sequence problem of the owner. Write down the Bellman equation.
5. What does the owner's Euler equation look like?
6. What would the solution of the problem look like if  $p_{t+1} = p_t$  for all  $t$ ? What would the solution look like if  $p_{t+1} > (1+r)p_t$  for all  $t$ ? What is the condition on the path of prices necessary for an interior solution (where the owner will extract some, but not all, of the oil)?

*Tips:*

1. *No need to use a computer here—This equation wants you to apply your theory of dynamic programming.*
2. *Pay attention to binding constraints.*

**Answer.**

1. The state variable is  $B_t$  (how many barrels of oil left at time  $t$ ) and the parameter is  $r$  (the constant interest rate).
2. The control variables are the price  $p_t$  (and  $t$ , the time at which to sell the barrel). And, say that  $b_t$  is the amount of barrels sold in period  $t$ .
3. Transition equation:  $B' = B - b$ . The amount of barrels of oil left tomorrow is the amount left today, minus the amount of barrels sold.
4. The sequence problem:

$$V_T(B) = \max_{\{(p_1, \dots, p_T), (b_1, \dots, b_T)\}} \sum_{t=1}^T \left( \frac{1}{1+r} \right)^t \pi(b_t, p_t),$$

where  $\pi(b_t, p_t)$  is our profit function, which we can represent as  $p_t b_t$ . So we have

$$V_T(B) = \max_{\{(p_1, \dots, p_T), (b_1, \dots, b_T)\}} \sum_{t=1}^T \frac{p_t b_t}{(1+r)^t},$$

such that  $B = \sum_{t=1}^T b_t$ . The Bellman Equation would be

$$V_{T+1}(B) = \max_{\{p_1, b_1\}} \{p_1 b_1 + V_T(B - b_1)\}.$$

5. Setting up the Lagrangian:

$$\mathcal{L} = \max_{\{(p_1, \dots, p_T), (b_1, \dots, b_T)\}} \sum_{t=1}^T \frac{p_t b_t}{(1+r)^t} - \lambda_1 \left( B - \sum_{t=1}^T b_t \right) - \lambda_2(p_t)$$

Taking some FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial b_1} &= \frac{p_1}{(1+r)} + \lambda_1 \implies -\lambda_1 = \frac{p_1}{(1+r)} \\ \frac{\partial \mathcal{L}}{\partial b_2} &= \frac{p_2}{(1+r)^2} + \lambda_1 \implies -\lambda_1 = \frac{p_2}{(1+r)^2} \\ &\vdots \end{aligned}$$

From here, we can find the Euler equation:

$$\frac{p_1}{(1+r)} = \frac{p_2}{(1+r)^2} \implies p_2 = (1+r)p_1.$$

Note that this relationship holds for all  $p_t$  and  $p_{t+1}$  for  $t \in \{1, \dots, T-1\}$ . So in general, our Euler equation is

$$p_{t+1} = (1+r)p_t.$$

6. What would the solution of the problem look like if  $p_{t+1} = p_t$  for all  $t$ ? What would the solution look like if  $p_{t+1} > (1+r)p_t$  for all  $t$ ? What is the condition on the path of prices necessary for an interior solution (where the owner will extract some, but not all, of the oil)?
- If  $p_{t+1} = p_t$  for all  $t$ , the solution would be sell a larger amount at the beginning than later on (in fact, sell all of them in the first period, since there's no intertemporal utility constraint to even out the selling of oil), since compounded interest would erode away the present-value price of barrels for large  $t$ .
  - If  $p_{t+1} > (1+r)p_t$ , then the solution would be to sell more barrels in the future (in fact, sell all of them in the last period, for similar reasons as mentioned above), since the price mark-up effect would exceed the compounded interest effect when discounting back to present-value prices.
  - The owner would extract some, but not all, of the oil (achieve an *interior solution*) if the Inada condition on the path of prices was put in place.

## Exercise 2.

**Question.** The Neoclassical Growth Model is a workhorse model in macroeconomics. The problem for the social planner is to maximize the discounted expected utility for agents in the economy:

$$\max_{\{c\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t).$$

The resource constraint is given as:

$$y_t = c_t + i_t.$$

The law of motion for the capital stock is:

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

Output is determined by the aggregate production function:

$$y_t = z_t k_t^\alpha.$$

Assume that  $z_t$  is stochastic. In particular, it is an i.i.d. process distributed as  $\log(z) \sim \mathcal{N}(0, \sigma_z)$ .

1. What is (are) the state variables(s)?
2. What are the control variables?
3. Write down the Bellman Equation that represents this sequence problem.
4. Solve the growth model given the following parameterization (you may use VFI or PFI):

Parameter	Description	Value
$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$	CRRA utility	
$\gamma$	Coefficient of Relative Risk Aversion	0.5
$\beta$	Discount factor	0.96
$\delta$	Rate of physical depreciation	0.05
$\alpha$	Curvature of production function	0.4
$\sigma_z$	SD of productivity shocks	0.2

- Plot the value function.
- Plot the policy function for the choice of consumption.
- Plot the policy function for the choice of capital next period.

*Tips:*

1. *The fact that the shocks are i.i.d. makes the computation simpler. Consider integrating over a Monte Carlo simulation of the shocks to find the expected values needed.*
2. *Write functions for the utility function and the production function—and depending on your solution method, variants on these as well such as the marginal utility function.*
3. *Be careful when it's possible that infeasible values of  $c_t$  or  $k_t$  may be chosen in your solution method.*

**Answer.**

1. The state variable is  $k_t$ , the capital stock; the parameters are  $\beta, \delta, \alpha, \sigma_z, k_t$ : the discount rate, the

depreciation rate, the capital intensity, and the SD of productivity shocks.

2. The control variables are  $c_t$  and  $i_t$ : the consumption and investment.
3. The Bellman equation can be represented as

$$V_{T+1}(k_0) = \max_{c_0, i_0} \{u(c_0) + V_T(c_1)\},$$

where  $c_1 = y_1 - i_1$  and  $y_1 = z_1 k_1^\alpha$ , and  $k_1 = (1 - \delta)k_0 + i_0$ , so we have that

$$\begin{aligned} c &= y - i \\ &= z k^\alpha - i \\ &= z k^\alpha - (k' - (1 - \delta)k). \end{aligned}$$

$$V_{T+1}(k_0) = \max_{c_0, i_0} \left\{ u(c_0) + E_0 \sum_{t=0}^{\infty} \beta^t u(z_1 ((a - \delta)k_0 + i_0)^\alpha - i_1) \right\};$$

note that  $V_T(c_1) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ .

4. See attached.

### Exercise 3.

**Question.** Use the same neoclassical growth model as above, but consider the case where there is serial correlation in the productivity shock. In particular, assume that  $z_t$  is given by:

$$\log(z_t) = \rho \log(z_{t-1}) + v_t$$

where  $v_t \sim \mathcal{N}(0, \sigma_v)$ . Let  $\rho = 0.8$  and  $v = 0.1$ .

1. Write down the Bellman Equation that represents the planner's problem in this case.
2. Approximate the AR(1) process with a Markov chain and solve the model:
  - Plot the value function for at least 3 values of the productivity shock.
  - Plot the policy function for the choice of consumption for at least 3 values of the productivity shock.
  - Plot the policy function for the choice of capital next period for at least 3 values of the productivity shock.

*Tips:*

1. Use `quantecon.markov.approximate` or the `ar1.approximate.py` module to approximate the AR(1) process.

**Answer.**

1. Similar to the planner's Bellman equation can be represented as (now replacing in the formulas for  $u(\cdot)$  and  $V_T(\cdot)$ ):

$$V_{T+1}(k_0) = \max_{c_0} \left\{ \frac{c_0^{1-\gamma}}{1-\gamma} + \sum_{t=1}^T \beta^t \frac{c_1^{1-\gamma}}{1-\gamma} \right\}.$$

2. See attached.

## Exercise 4.

**Question.** The search and matching model of labor markets is a key model in the macroeconomic labor literature. In one version of this model, potential workers receive wage offers from a distribution of wages in each period. Potential workers must decide whether to accept and begin work at this age (and work at this age forever) or decline the offer and continue to “search” (i.e., receive wage offers from some exogenous distribution).

The potential workers seek to maximize the expected, discounted sum of earnings:

$$E_0 \sum_{t=0}^{\infty} \beta^t y_t.$$

Income,  $y_t$ , is equal to  $w_t$  if employed. If unemployed, agents receive unemployment benefits  $b$ . Assume that wage offers are distributed as  $\log(w_t) \sim \mathcal{N}(\mu, \sigma)$ .

1. Write down the Bellman equation representing this optimal stopping problem.
2. Solve the model, using the following parameterization:

Parameter	Description	Value
$\beta$	Rate of time preference	0.96
$b$	Unemployment benefits	0.05
$\mu$	Mean of log wages	0.0
$\sigma$	SD of wage draws	0.15

- Plot the value function.
- Find the “reservation wage” for the unemployed worker (i.e., the wage that makes her indifferent between accepting the job offer and not)
- Vary  $b$  from 0.5 to 1.0 and plot the reservation wage for each value of  $b$ . How do unemployment benefits affect the reservation wage?

### Answer.

1. The Bellman equation is

$$V(w) = \max \left\{ \frac{w}{1 - \beta}, b + \beta E_{w'} V(w') \right\}.$$

2. See attached.