OSM Boot Camp **Math Problem Set 4**

Harrison Beard

16 July 2018

Exercise 6.6.

Find and identify all critical points of

$$f(x, y) = 3x^2 y + 4xy^2 + xy.$$

Determine whether they are the locations of local maxima, minima, or saddle points.

Solution. Differentiating, we have

$$\begin{aligned} \mathbf{D} f(x, y) &= \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 6xy + 4y^2 + y & 2x^2 + 8xy + x \end{bmatrix}, \end{aligned}$$

which equals $\mathbf{0}$ at $(x, y) = \begin{cases} (0, 0) \\ \left(0, -\frac{1}{4}\right) \\ \left(\frac{1}{3}, 0\right) \\ \left(-\frac{1}{9}, -\frac{1}{12}\right) \end{cases}$. Thus our Hessian is

$$\mathbf{D}^2 f(x, y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix},$$

which evaluates to a negative definite matrix only for $(x, y) = \left(-\frac{1}{9}, -\frac{1}{12}\right)$, so $(x, y) = \left(-\frac{1}{9}, -\frac{1}{12}\right)$ is a local maximum. For the points $(x, y) = \begin{cases} (0, 0) \\ \left(0, -\frac{1}{4}\right) \\ \left(\frac{1}{3}, 0\right) \end{cases}$, the Hessian is neither positive nor negative definite, so they are saddle points. □

Exercise 6.7.

An unconstrained quadratic optimization problem is an optimization problem with no constraints where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is quadratic, meaning that it can be written in the form

$$f(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c, \tag{6.17}$$

for some square matrix $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$ and some vector $\mathbf{b} \in \mathbb{R}^n$.

(i). Prove that for any square matrix $\mathbf{A}$ the matrix $\mathbf{Q} = \mathbf{A}^\mathsf{T} + \mathbf{A}$ is symmetric, and $\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} = \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 2\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$, so (6.17) is equal to

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c \qquad (6.18).$$

Thus we may always assume that quadratic functions are of the form (6.18) with $\mathbf{Q}$ symmetric.

(ii). Prove that any minimizer $\mathbf{x}^*$ of f is a solution of the equation

$$\mathbf{Q}^\mathsf{T}\mathbf{x}^* = \mathbf{b}. \tag{6.19}$$

(iii). Prove that the quadratic minimization problem (6.17) will have a solution if and only if $\mathbf{Q}$ is positive definite, and in that case, the minimizer is the solution of the linear system (6.19). Explain why this shows that solving the system (6.19) with positive definite $\mathbf{Q}$ is equivalent to solving the quadratic optimization problem (6.18).

Depending on $\mathbf{Q}$, the best way to solve the linear system (6.19) is often to use optimization algorithms on the quadratic problem (6.17) instead of using linear solvers on (6.19).

Solution.

(i). If $\mathbf{A}$ is square, then $\mathbf{A}^\mathsf{T} + \mathbf{A}$ is

$$\begin{aligned}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} &= \begin{bmatrix} 2 \cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} + a_{1n} & \cdots & 2 \cdot a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + a_{1n} & \cdots & 2 \cdot a_{nn} \end{bmatrix},\end{aligned}$$

so $(\mathbf{A}^\mathsf{T} + \mathbf{A})^\mathsf{T}$ is

$$\begin{bmatrix} 2 \cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + a_{1n} & \cdots & 2 \cdot a_{nn} \end{bmatrix},$$

which is itself, so it is symmetric. So, we have that

$$\begin{aligned} \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} &= \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T} + \mathbf{A})\mathbf{x} \\ &= \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} \\ &= \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} \\ &= 2 \cdot \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}. \end{aligned}$$

So, we can see that

$$\begin{aligned}
\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c &= \frac{1}{2}\cdot\left(2\cdot\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \frac{1}{2}\cdot\left(\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \frac{1}{2}\cdot\left(\mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T} + \mathbf{A})\mathbf{x} \\
&= \frac{1}{2}\cdot\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c.
\end{aligned}$$

□

(ii). This is a direct corollary of the FONC—$\mathbf{Q}^\mathsf{T}\mathbf{x}^* = \mathbf{b}$ because $f'(\mathbf{x}) = \mathbf{Q}^\mathsf{T}\mathbf{x} - \mathbf{b}$. □

(iii). If $\mathbf{Q}$ PSD, then $f''(\mathbf{x}) > 0$ $\forall \mathbf{x} \in \mathbb{R}^n$. By (6.19) and the fact that $\mathbf{Q}$ is invertible, we know that $\mathbf{x}^* = \mathbf{Q}^{-1}\mathbf{b}$, where $\mathbf{x}^*$ is the value where $f'(\mathbf{x}^*) = 0$. So, by SOSC, $\mathbf{x}^*$ is a *unique* minimizer of f. If $\mathbf{x}^*$ is the unique minimizer, then by SONC we know that $\mathbf{Q}$ is PSD and $\mathbf{Q}^\mathsf{T}\mathbf{x}^* = \mathbf{b}$. If $\mathbf{Q}$ has at least one e-value equal to zero, then $\mathbf{x}^*$ is *not* unique, so $\mathbf{Q}$ *must* be positive definite. □

Exercise 6.11.

Consider a quadratic function $f(x) = ax^2 + bx + c$, here $a > 0$, and $b, c \in \mathbb{R}$. Show that for any initial guess $x_0 \in \mathbb{R}$, one iteration of Newton's method lands at the unique minimizer of f.

Solution. f has a minimum at $x = -\frac{b}{2a}$ where $f(x) = a\cdot\left(x + \frac{b}{2a}\right)^2 - a\cdot\left(\frac{b}{2a}\right)^2$, so the Newton's method always yields

$$\begin{aligned}
x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0} \\
&= x_0 - \frac{2ax_0 + b}{2a} \\
&= -\frac{b}{2a}.
\end{aligned}$$

□

Exercise 6.15.

Code up (in Python/NumPy) the secant method for finding a minimizer of a function. Your code should accept two initial guesses x_0 and x_1, a desired level of accuracy ε, and a callable function $f'(x)$. It should return an approximation to a minimizer of f, provided the algorithm converges for the initial conditions. For the stopping criterion, use $|x_{k+1} - x_k| < |x_k|\,\varepsilon$. Be sure your code has methods for identifying and handling cases where the sequence does not converge.

Write your code in a notebook and test it with the function $f(x) = x^4 - 14x^3 + 60x^2 + 70x$ and initial guess $x_0 = 0$. Plot the function and the points and function values at each iteration in an appropriate graph.

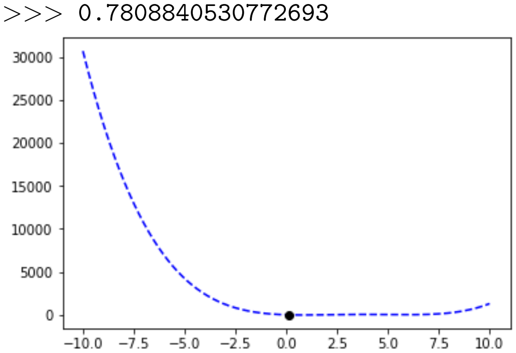
Solution.

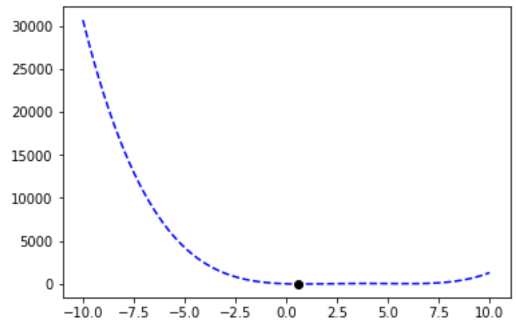
Algorithm 6.15.

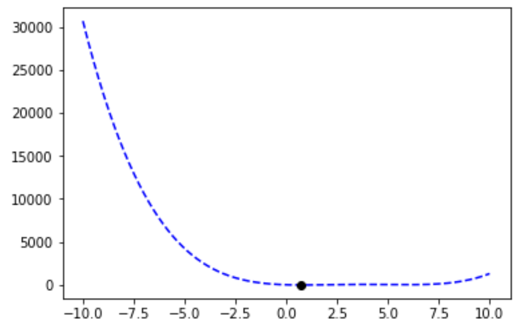
```
import matplotlib.pyplot as plt
import numpy as np
from scipy import linalg as la

def f(x):
        return x**4 - 14*x**3 + 60*x**2 - 70*x

def f_pr(x):
        return 4 * x**3 - 14 * 3 * x**2 + 60 * 2 * x - 70


def problem6_15(x_0, x_1, ε, f_pr):
        MAX_IT = 1_000
        k = 0
        x_k = x_0
        x_kp1 = x_1
        X = np.linspace(-10, 10, 1_000)
        while k < MAX_IT:
                x_km1 = x_k
                x_k = x_kp1
                x_kp1 = x_k - f_pr(x_k) * (x_k - x_km1)/(f_pr(x_k) - f_pr(x_km1))
                k += 1
                plt.plot(X, f(X), "b--")
                plt.plot(x_k, f(x_k), "ko")
                plt.show()
                if la.norm(x_kp1 - x_k) < ε * la.norm(x_k):
                        break
        return x_k
```

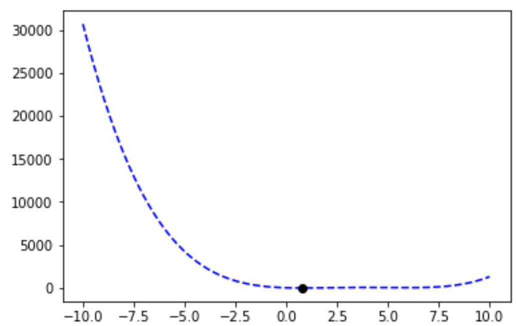
Test.

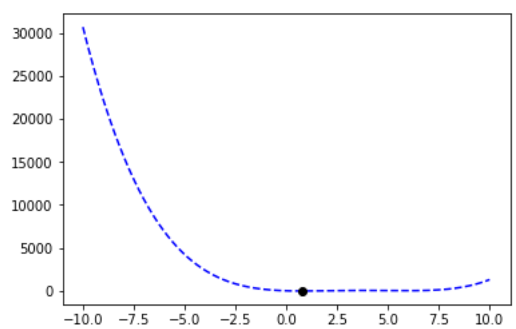
```
>>> problem6_15(0, 0.1, 1e-10, f_pr)
>>> 0.7808840530772693
```

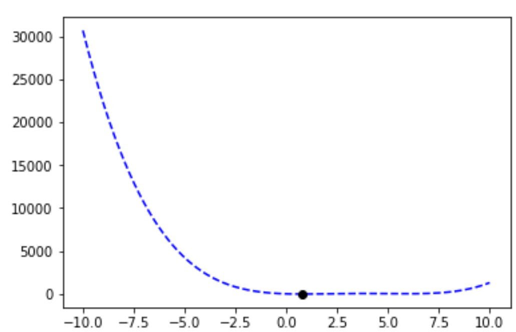


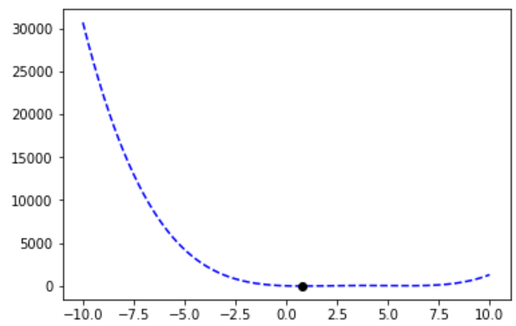

30000
25000
20000
15000
10000
5000
0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


30000
25000
20000
15000
10000
5000
0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


30000
25000
20000
15000
10000
5000
0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


30000
25000
20000
15000
10000
5000
0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


30000
25000
20000
15000
10000
5000
0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0



Exercise 7.1.

Prove Proposition 7.1.5.
Prop 7.1.5. *If S is a nonempty subset of V, then* $\operatorname{conv}(S)$ *is convex.*

Solution. Start by rewriting $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{conv}(S)$ as a parameterized linear combination of $\mathbf{x}_i$'s in S:

$$\mathbf{v}_1 = \sum_{j=1}^{n} \lambda_{i_j} \cdot \mathbf{x}_{i_j}, \qquad \mathbf{v}_2 = \sum_{k=1}^{m} \lambda_{i_k} \cdot \mathbf{x}_{i_k},$$

with the conditions that

$$\sum_{j=1}^{n} \lambda_{i_j} = \sum_{k=1}^{m} \lambda_{i_k} = 1.$$

It follows that

$$\begin{aligned} \lambda \cdot \mathbf{v}_1 + (1-\lambda) \cdot \mathbf{v}_2 &= \lambda \cdot \sum_{j=1}^{n} \lambda_{i_j} \cdot \mathbf{x}_{i_j} + (1-\lambda) \cdot \sum_{k=1}^{m} \lambda_{i_k} \cdot \mathbf{x}_{i_k} \\ &= \sum_{j=1}^{n} \lambda \cdot \lambda_{i_j} \cdot \mathbf{x}_{i_j} + \sum_{k=1}^{m} (1-\lambda) \cdot \lambda_{i_k} \cdot \mathbf{x}_{i_k} \\ &= \sum_{i=1}^{n+m} \left(\chi\left(\{i \leq n\}\right) \cdot \lambda \cdot \lambda_{i_j} + \chi\left(\{i > n\}\right) \cdot (1-\lambda) \cdot \lambda_{i_k} \right) \cdot \mathbf{x}_i, \end{aligned}$$

where

$$\sum_{i=1}^{n+m} \left(\chi\left(\{i \leq n\}\right) \cdot \lambda \cdot \lambda_{i_j} + \chi\left(\{i > n\}\right) \cdot (1-\lambda) \cdot \lambda_{i_k} \right) = \sum_{j=1}^{n} \lambda \cdot \lambda_{i_j} + \sum_{i=1}^{m} (1-\lambda) \cdot \lambda_{i_k},$$

so

$$\lambda \cdot \sum_{j=1}^{n} \lambda_{i_j} + (1-\lambda) \cdot \sum_{k=1}^{m} \lambda_{i_k} = 1,$$

so $\lambda \cdot \mathbf{v}_1 + (1-\lambda) \cdot \mathbf{v}_2 \in \operatorname{conv}(S)$ for any $\lambda \in (0,1)$. $\square$

Exercise 7.2.

Prove that:

(i) . A hyperplane is convex.

(ii) . A half space is convex.

Solution.

(i) . For hyperplane $H = \{\mathbf{x} \in V | \langle \mathbf{a}, \mathbf{x} \rangle = b\}$, for $\mathbf{a} \neq \mathbf{0} \in V$ and $b \in \mathbb{R}$, consider that for arbitrary $\mathbf{v}, \mathbf{u} \in H$, we have

$$\begin{aligned}\langle \mathbf{a}, \lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \rangle &= \langle \mathbf{a}, \lambda \cdot \mathbf{v} \rangle + \langle \mathbf{a}, (1-\lambda) \cdot \mathbf{u} \rangle \\ &= \lambda \cdot \langle \mathbf{a}, \mathbf{v} \rangle + (1-\lambda) \cdot \langle \mathbf{a}, \mathbf{u} \rangle \\ &= \lambda \cdot b + (1-\lambda) \cdot b \\ &= b,\end{aligned}$$

so $\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \in H$. □

(ii) . For halfspace $H = \{\mathbf{x} \in V | \langle \mathbf{a}, \mathbf{x} \rangle \leq b\}$, for $\mathbf{a} \neq \mathbf{0} \in V$ and $b \in \mathbb{R}$, consider that for arbitrary $\mathbf{v}, \mathbf{u} \in H$, we have

$$\begin{aligned}\langle \mathbf{a}, \lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \rangle &= \langle \mathbf{a}, \lambda \cdot \mathbf{v} \rangle + \langle \mathbf{a}, (1-\lambda) \cdot \mathbf{u} \rangle \\ &= \lambda \cdot \langle \mathbf{a}, \mathbf{v} \rangle + (1-\lambda) \cdot \langle \mathbf{a}, \mathbf{u} \rangle \\ &\leq \lambda \cdot b + (1-\lambda) \cdot b \\ &= b,\end{aligned}$$

so $\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \in H$. □

Exercise 7.4.

Prove the following theorem: Let $C \subset \mathbb{R}^n$ be nonempty, closed, and convex. A point $\mathbf{p} \in C$ is the projection of $\mathbf{x}$ onto C if and only if

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0, \qquad \forall \mathbf{y} \in C. \tag{7.14}$$

Prove the statements below and then write a complete proof of the theorem:

(i) . $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2 + \|\mathbf{p} - \mathbf{y}\|^2 + \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle$.

(ii) . If (7.14) holds, then $\|\mathbf{x} - \mathbf{y}\| > \|\mathbf{x} - \mathbf{p}\|$ for all $\mathbf{y} \in C$, $\mathbf{y} \neq \mathbf{p}$. *Hint*: Use the identity in *(i)* .

(iii) . If $\mathbf{z} = \lambda \mathbf{y} + (1-\lambda)\mathbf{p}$, where $0 \leq \lambda \leq 1$, then

$$\|\mathbf{x} - \mathbf{z}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2 + 2\lambda \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle + \lambda^2 \|\mathbf{y} - \mathbf{p}\|^2. \tag{7.15}$$

(iv) . If $\mathbf{p}$ is a projection of $\mathbf{x}$ onto the convex set C, then $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$ for all $\mathbf{y} \in C$. *Hint*: Use (7.15) to show that

$$0 \leq 2 \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle + \lambda \|\mathbf{y} - \mathbf{p}\|^2, \qquad \forall \mathbf{y} \in C, \ \lambda \in [0, 1].$$

Solution.

(i).

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{p}-\mathbf{y}\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle.\end{aligned}$$

□

(ii). $\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \geq 0\ \forall \mathbf{y}\in C$ and $\langle \mathbf{x},\mathbf{x}\rangle \geq 0$ and $\langle \mathbf{x},\mathbf{x}\rangle = 0 \iff \mathbf{x}=\mathbf{0}$ and $\mathbf{y}\neq\mathbf{p}$ so $\|\mathbf{y}-\mathbf{p}\|^2 > 0$, and thus

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \\ &> \|\mathbf{x}-\mathbf{p}\|^2,\end{aligned}$$

where $\langle\cdot,\cdot\rangle \geq 0$ by definition, so

$$\|\mathbf{x}-\mathbf{y}\|^2 > \|\mathbf{x}-\mathbf{p}\|^2.$$

□

(iii).

$$\begin{aligned}\mathbf{p}-\mathbf{z} &= \mathbf{p}-\lambda\cdot\mathbf{y}-(1-\lambda)\cdot\mathbf{p} \\ &= \lambda\cdot(\mathbf{p}-\mathbf{y}),\end{aligned}$$

so by (i),

$$\begin{aligned}\|\mathbf{x}-\mathbf{z}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{z}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{z}\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\lambda(\mathbf{p}-\mathbf{y})\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \lambda\cdot(\mathbf{p}-\mathbf{y})\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + 2\cdot\lambda\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \lambda^2\cdot\|\mathbf{p}-\mathbf{y}\|^2.\end{aligned}$$

□

(iv). For $\mathbf{y}\in C$, note that $\lambda\cdot\mathbf{y}+(1-\lambda)\cdot\mathbf{p}$ is also in C for $\lambda\in[0,1]$. From (7.15), if $\mathbf{p}$ is a projection of $\mathbf{x}$ onto C, then $\|\mathbf{x}-\mathbf{p}\| < \|\mathbf{x}-\mathbf{v}\|$ for $\mathbf{v}\neq\mathbf{p}$, $\mathbf{v}\in C$. From (iii), we have

$$\begin{aligned}0 &\leq \|\mathbf{x}-\mathbf{z}\|^2 - \|\mathbf{x}-\mathbf{p}\|^2 \\ &= 2\cdot\lambda\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \lambda^2\cdot\|\mathbf{y}-\mathbf{p}\|^2,\end{aligned}$$

so

$$0 \leq \langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \frac{1}{2}\cdot\lambda\cdot\|\mathbf{y}-\mathbf{p}\|^2,$$

and for $\lambda = 0$ we have

$$0 \leq \langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle.$$

□

For the proof: From (ii), $\|\mathbf{x}-\mathbf{y}\| > \|\mathbf{x}-\mathbf{p}\|$ for all $\mathbf{y}\in C$ where $\mathbf{y}\neq\mathbf{p}$, which is simply the definition of projection. From (iv), we can see the converse is true. □

Exercise 7.8.

Prove that if $f:\mathbb{R}^m\to\mathbb{R}$ is convex, if $\mathbf{A}\in \mathrm{M}_{m\times n}(\mathbb{R})$, and if $\mathbf{b}\in\mathbb{R}^m$, then the function $g:\mathbb{R}^m\to\mathbb{R}$ given by $g(\mathbf{x}) = f(\mathbf{A}\mathbf{x}+\mathbf{b})$ is convex.

Solution. For $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have that

$$\begin{aligned} g(\lambda \cdot \mathbf{x}_1 + (1-\lambda)\cdot \mathbf{x}_2) &= f(\lambda \cdot \mathbf{A}\mathbf{x}_1 + (1-\lambda)\cdot \mathbf{A}\mathbf{x}_2 + \mathbf{b}) \\ &= f(\lambda \cdot (\mathbf{A}\mathbf{x}_1 + \mathbf{b}) + (1-\lambda)\cdot(\mathbf{A}\mathbf{x}_2 + \mathbf{b})) \\ &\le \lambda \cdot f(\mathbf{A}\mathbf{x}_1 + \mathbf{b}) + (1-\lambda)\cdot f(\mathbf{A}\mathbf{x}_2 + \mathbf{b}) \\ &= \lambda \cdot g(\mathbf{x}_1) + (1-\lambda)\cdot g(\mathbf{x}_2). \end{aligned}$$

□

Exercise 7.12.

Prove the following:

(i). The set $\mathrm{PD}_n(\mathbb{R})$ of positive-definite matrices in $\mathrm{M}_n(\mathbb{R})$ is convex.

(ii). The function $f(X) = \log(\det(\mathbf{X}))$ is convex on $\mathrm{PD}_n(\mathbb{R})$. To prove this, show the following:

(a). The function f is convex if for every $\mathbf{A}, \mathbf{B} \in \mathrm{PD}_n(\mathbb{R})$ the function $g(t) : [0,1] \to \mathbb{R}$ given by $g(t) = f(t\mathbf{A} + (1-t)\mathbf{B})$ is convex.

(b). Use the fact that positive definite matrices are normal to show that there is an $\mathbf{S}$ such that $\mathbf{S}^{\mathsf{H}}\mathbf{S} = \mathbf{A}$ and

$$\begin{aligned} g(t) &= -\log(\det(\mathbf{S}^{\mathsf{H}}(t\mathbf{I} + (1-t)(\mathbf{S}^{\mathsf{H}})^{-1}\mathbf{B}\mathbf{S}^{-1})\mathbf{S})) \\ &= -\log(\det(\mathbf{A})) - \log(\det(t\mathbf{I} + (1-t)(\mathbf{S}^{\mathsf{H}})^{-1}\mathbf{B}\mathbf{S}^{-1})). \end{aligned}$$

(c). Show that

$$g(t) = -\sum_{i=1}^{n} \log(t + (1-t)\lambda_i) - \log(\det(\mathbf{A})),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $(\mathbf{S}^{\mathsf{H}})^{-1}\mathbf{B}\mathbf{S}^{-1}$.

(d). Prove that $g''(t) \ge 0$ for all $t \in [0,1]$.

Solution.

(i). For $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{PD}_n(\mathbb{R})$, $\lambda \in [0,1]$, and $\mathbf{x} \in \mathbb{R}^n$, note that by positive definiteness it follows that

$$\begin{aligned} \mathbf{x}^{\mathsf{T}}(\lambda \cdot \mathbf{A}_1 + (1-\lambda)\cdot \mathbf{A}_2)\mathbf{x} &= \lambda \cdot \left(\mathbf{x}^{\mathsf{T}}\mathbf{A}_1\mathbf{x}\right) + (1-\lambda)\cdot\left(\mathbf{x}^{\mathsf{T}}\mathbf{A}_2\mathbf{x}\right) \\ &> 0. \end{aligned}$$

□

(ii).

(a). Fix $t_{\mathbf{A}}, t_{\mathbf{B}} \in \mathbb{R}$, $\mathbf{A}, \mathbf{B} \in \mathrm{PD}_n(\mathbb{R})$, $\lambda \in [0,1]$ arbitrarily. Then

$$\lambda \cdot g(t_{\mathbf{A}}) + (1-\lambda)\cdot g(t_{\mathbf{B}}) = \lambda \cdot f(t_{\mathbf{A}} \cdot \mathbf{A} + (1-t_{\mathbf{A}})\cdot \mathbf{B}) + (1-\lambda)\cdot f(t_{\mathbf{B}} \cdot \mathbf{A} + (1 - t_{\mathbf{B}} \cdot \mathbf{B})$$

and

$$\begin{aligned} g(\lambda \cdot t_{\mathbf{A}} + (1-\lambda)\cdot t_{\mathbf{B}} &= f((\lambda \cdot t_{\mathbf{A}} + (1-\lambda)\cdot t_{\mathbf{B}})\cdot \mathbf{A} + (1 - \lambda \cdot t_{\mathbf{A}} + (1-\lambda)\cdot t_{\mathbf{B}})\cdot \mathbf{B}) \\ &= f(\lambda \cdot (t_{\mathbf{A}} \cdot \mathbf{A} + (1-t_{\mathbf{A}})\cdot \mathbf{B}) + (1-\lambda)\cdot(t_{\mathbf{B}} \cdot \mathbf{A} + (1-t_{\mathbf{B}})\cdot \mathbf{B})) \end{aligned}$$

so

$$\begin{aligned} & f\left(\lambda \cdot \left(t_{\mathbf{A}} \cdot \mathbf{A} + (1-t_{\mathbf{A}}) \cdot \mathbf{B}\right) + (1-\lambda) \cdot \left(t_{\mathbf{B}} \cdot \mathbf{A} + (1-t_{\mathbf{B}}) \cdot \mathbf{B}\right)\right) \\ \leq \; & \lambda \cdot f\left(t_{\mathbf{A}} \cdot \mathbf{A} + (1-t_{\mathbf{A}}) \cdot \mathbf{B}\right) + (1-\lambda) \cdot f\left(t_{\mathbf{B}} \cdot \mathbf{A} + (1-t_{\mathbf{B}}) \cdot \mathbf{B}\right). \end{aligned}$$

□

(b). $\mathbf{A} \in \mathrm{PD}_n(\mathbb{R})$ so $\exists \mathbf{S}$, nonsingular, such that $\mathbf{A} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$:

$$t \cdot \mathbf{A} + (1-t) \cdot \mathbf{B} = \mathbf{S}^{\mathsf{H}} \cdot \left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right) \mathbf{S},$$

so

$$\begin{aligned} g(t) &= -\log\left(\det\left(t \cdot \mathbf{A} + (1-t) \cdot \mathbf{B}\right)\right) \\ &= -\log\left(\det\left(\mathbf{S}^{\mathsf{H}}\left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right) \mathbf{S}\right)\right) \\ &= -\log\left(\det\left(\mathbf{S}^{\mathsf{H}}\right)\right) - \log\left(\det\left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right)\right) \\ &\quad -\log\left(\det\left(\mathbf{S}\right)\right) \\ &= -\log\left(\det\left(\mathbf{S}^{\mathsf{H}}\right) \cdot \det\left(\mathbf{S}\right)\right) - \log\left(\det\left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right)\right) \\ &= -\log\left(\det\left(\mathbf{A}\right)\right) - \log\left(\det\left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right)\right). \end{aligned}$$

□

(c). Note that $\mathbf{A}, \mathbf{B} \in \mathrm{PD}_n(\mathbb{R})$ so $\mathbf{B}^{-1} \in \mathrm{PD}_n(\mathbb{R})$ and

$$\left(\left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right)^{-1} = \mathbf{S}\mathbf{B}^{-1}\mathbf{S}^{\mathsf{H}} \in \mathrm{PD}_n(\mathbb{R}),$$

so

$$\left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1} \in \mathrm{PD}_n(\mathbb{R}),$$

so we know that for the e-vals λ_i and e-vecs $\mathbf{v}_i$ of $\left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}$,

$$\begin{aligned} \left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right) \mathbf{v}_i &= t \cdot \mathbf{v}_i + (1-t) \cdot \lambda_i \cdot \mathbf{v}_i \\ &= \left(t + (1-t) \cdot \lambda_i\right) \cdot \mathbf{v}_i, \end{aligned}$$

so the e-vals of the resultant matrix of $t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}$ are of the form $t + (1-t) \cdot \lambda_i$, so

$$\begin{aligned} & -\log\left(\det\left(\mathbf{A}\right)\right) - \log\left(\det\left(t \cdot \mathbf{I} + (1-t) \cdot \left(\mathbf{S}^{\mathsf{H}}\right)^{-1} \mathbf{B}\mathbf{S}^{-1}\right)\right) \\ = \; & -\log\left(\det\left(\mathbf{A}\right)\right) - \log\left(\prod_{i=1}^{n} \left(t + (1-t) \cdot \lambda_i\right)\right) \\ = \; & -\log\left(\det\left(\mathbf{A}\right)\right) - \sum_{i=1}^{n} \log\left(\left(t + (1-t) \cdot \lambda_i\right)\right). \end{aligned}$$

□

(d). From above, it follows that

$$g'(t) = \sum_{i=1}^{n} \frac{1-\lambda_i}{t + (1-t) \cdot \lambda_i},$$

so

$$\begin{aligned} g''(t) &= \sum_{i=1}^{n} \frac{\left(1-\lambda_i\right)^2}{\left(t + (1-t) \cdot \lambda_i\right)^2} \\ &\geq 0. \end{aligned}$$

□

Exercise 7.13.

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above, prove that f is constant.

Solution. Proof by contradiction: Suppose f is *not* constant. Then, there must exist some vectors $\mathbf{v} > \mathbf{u}$ such that $f(\mathbf{v}) > f(\mathbf{u})$. Then, for all $\mathbf{x} > \mathbf{v}$,

$$0 < \frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}} \leq 1$$

and

$$0 \leq 1 - \frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}} \quad = \quad \frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}} < 1.$$

Note that

$$\begin{aligned}
\mathbf{v} &= \left(\frac{\mathbf{x}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{v} \\
&= \frac{\mathbf{xv}-\mathbf{xu}+\mathbf{xu}-\mathbf{vu}}{\mathbf{x}-\mathbf{u}} \\
&= \left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{x} \\
&= \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u} \\
&= \left(1-\left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\right)\mathbf{x} + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u},
\end{aligned}$$

where

$$\begin{aligned}
f(\mathbf{v}) &= f\left(\left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{x} + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u}\right) \\
&\leq \left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)f(\mathbf{x}) + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)f(\mathbf{u}),
\end{aligned}$$

so

$$\begin{aligned}
f(\mathbf{x}) &\geq \frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}}f(\mathbf{v}) - \frac{\mathbf{x}-\mathbf{v}}{\mathbf{v}-\mathbf{u}}f(\mathbf{u}) \\
&= \frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}}f(\mathbf{v}) - \frac{\mathbf{x}-\mathbf{u}+\mathbf{u}-\mathbf{v}}{\mathbf{v}-\mathbf{u}}f(\mathbf{u}) \\
&= \underbrace{\frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}}}_{>0}\underbrace{(f(\mathbf{v})-f(\mathbf{u}))}_{>0} + f(\mathbf{u}) \\
&> f(\mathbf{u}),
\end{aligned}$$

so for any arbitrary value $U \in \mathbb{R}$ there is an $\mathbf{x}$ where $f(\mathbf{x}) > U$, so f is unbounded, which contradicts our assumptions. Therefore f is constant. $\square$

Exercise 7.20.

Prove Proposition 7.4.3.
Prop 7.4.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $-f$ is also convex, then f is affine.*

Solution. For any $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) \leq \lambda \cdot f(\mathbf{v}) + (1-\lambda) \cdot f(\mathbf{u})$$

and

$$-f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) \leq -\lambda \cdot f(\mathbf{v}) - (1-\lambda) \cdot f(\mathbf{u})$$

so

$$f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) = \lambda \cdot f(\mathbf{v}) + (1-\lambda) \cdot f(\mathbf{u}).$$

Now define

$$\tilde{f}(\mathbf{x}) \coloneqq f(\mathbf{x}) - f(\mathbf{0}) = 0,$$

so we have

$$\tilde{f}(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) = \lambda \cdot \tilde{f}(\mathbf{v}) + (1-\lambda) \cdot \tilde{f}(\mathbf{u}),$$

and choose $\mathbf{u} = \mathbf{0}$. Then $\tilde{f}(\lambda \cdot \mathbf{v}) = \lambda \cdot \tilde{f}(\mathbf{u})$ for $\lambda \in [0,1]$. So $\tilde{f}$ is linear and the mappings taking $\mathbf{x}$ to $\mathbf{x}/\lambda$ and $-\mathbf{x}$ are bijective, and for all $\mathbf{v} = \frac{\mathbf{x}}{\lambda} \in \mathbb{R}^n$, we have that $f(\mathbf{x}/\lambda) = \frac{1}{\lambda} \cdot f(\mathbf{x})$, so $f(\lambda \cdot \mathbf{v}) = \lambda \cdot f(\mathbf{v})$. So we can prove linearity of $\tilde{f}$ (and thus affinity of f) by observing that for $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\tilde{f}(\alpha \cdot \mathbf{v} + \beta \cdot \mathbf{u}) = \alpha \cdot \tilde{f}(\mathbf{v}) + \beta \cdot \tilde{f}(\mathbf{u}).$$

□

Exercise 7.21.

Prove Proposition 7.4.11.

Prop 7.4.11. *If $D \subset \mathbb{R}$ with $f : \mathbb{R}^n \to D$, and if $\phi : D \to \mathbb{R}$ is a strictly increasing function, then $\mathbf{x}^*$ is a local minimizer for the problem*

$$\begin{array}{rl} \text{minimize} & \phi \circ f(\mathbf{x}) \\ \text{subject to} & G(\mathbf{x}) \preceq \mathbf{0} \\ & H(\mathbf{x}) = \mathbf{0} \end{array}$$

if and only if $\mathbf{x}^*$ is a local minimizer for the problem

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & G(\mathbf{x}) \preceq \mathbf{0} \\ & H(\mathbf{x}) = \mathbf{0}. \end{array}$$

Solution.

- If $\mathbf{x}^*$ is a minimizer of the second problem, then, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for $\mathbf{x} \in U$, some open neighborhood. ϕ is *strictly increasing* so $\phi(\mathbf{v}) > \phi(\mathbf{u})$ iff $\mathbf{v} > \mathbf{u}$. So for $\mathbf{x} \in U$, $\phi \circ f(\mathbf{x}) \geq \phi \circ f(\mathbf{x}^*)$. So, $\mathbf{x}^*$ is a minimizer of U under $\phi \circ f(\mathbf{x})$.
- Next, if $\mathbf{x}^*$ is a minimizer for the first problem, observe $\phi \circ f(\mathbf{x}^*) \leq \phi \circ f(\mathbf{x})$ for $\mathbf{x} \in U$, some open neighborhood. There cannot be no minimizers in an open subset U' of U for the second problem because then there would be some other vectors $\mathbf{v}$ and $\mathbf{u}$ where $f(\mathbf{v}) < f(\mathbf{x}) < f(\mathbf{u})$. If $\mathbf{x}^* \in U'$, $\phi \circ f(\mathbf{v}) < \phi \circ f(\mathbf{x}^*) < \phi \circ f(\mathbf{u})$, contradicting earlier assumptions.
- If there are *infinitely many* minimizers (where none of which are $\mathbf{x}^*$) then there would be a sequence of minimizers converging to $\mathbf{x}^*$, in which case for any U', there exists a minimizer $\hat{\mathbf{x}}^*$ such that $f(\hat{\mathbf{x}}^*) \leq f(\mathbf{x})$ for $\mathbf{x} \in U'$. So $\phi \circ f(\hat{\mathbf{x}}^*) \leq \phi \circ f(\mathbf{x}^*)$ and $\phi \circ (f(\hat{\mathbf{x}})) = \phi \circ f(\mathbf{x}^*)$ because $\mathbf{x}^*$ is a minimizer for the first problem.

- ϕ is *strictly* increasing, so $\phi(\mathbf{v}) = \phi(\mathbf{u})$ iff $\mathbf{v} = \mathbf{u}$, so $f(\mathbf{x}) \geq f(\hat{\mathbf{x}}^*) = f(\mathbf{x}^*)$ for $\mathbf{x} \in U'$ and $\mathbf{x}^*$ is a minimizer of the second problem. $\square$