OSM Boot Camp **Math Problem Set 4**

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16 July 2018

Exercise 6.6.

Find and identify all critical points of

$$f(x,y) = 3x^2y + 4xy^2 + xy.$$

Determine whether they are the locations of local maxima, minima, or saddle points.

Solution. Differentiating, we have

$$\begin{aligned} \mathbf{D}f(x,y) &= \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 6xy + 4y^2 + y & 2x^2 + 8xy + x \end{bmatrix}, \end{aligned}$$

which equals $\mathbf{0}$ at $(x,y) = \begin{cases} (0,0) \\ \left(0, -\frac{1}{4}\right) \\ \left(\frac{1}{3}, 0\right) \\ \left(-\frac{1}{9}, -\frac{1}{12}\right) \end{cases}$. Thus our Hessian is

$$\mathbf{D}^2 f(x,y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix},$$

which evaluates to a negative definite matrix only for $(x,y) = \left(-\frac{1}{9}, -\frac{1}{12}\right)$, so $(x,y) = \left(-\frac{1}{9}, -\frac{1}{12}\right)$ is a local maximum. For the points $(x,y) = \begin{cases} (0,0) \\ \left(0, -\frac{1}{4}\right) \\ \left(\frac{1}{3}, 0\right) \end{cases}$, the Hessian is neither positive nor negative definite, so they are saddle points. $\square$

Exercise 6.7.

An unconstrained quadratic optimization problem is an optimization problem with no constraints where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is quadratic, meaning that it can be written in the form

$$f(\mathbf{x}) = \mathbf{x}^\mathsf{T} A\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c, \tag{6.17}$$

for some square matrix $A \in \mathrm{M}_n(\mathbb{R})$ and some vector $\mathbf{b} \in \mathbb{R}^n$.

(i). Prove that for any square matrix A the matrix $Q = A^\mathsf{T} + A$ is symmetric, and $\mathbf{x}^\mathsf{T}Q\mathbf{x} = \mathbf{x}^\mathsf{T}A^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T}A\mathbf{x} = 2\mathbf{x}^\mathsf{T}A\mathbf{x}$, so (6.17) is equal to

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\mathsf{T}Q\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c \qquad (6.18).$$

Thus we may always assume that quadratic functions are of the form (6.18) with Q symmetric.

(ii). Prove that any minimizer $\mathbf{x}^*$ of f is a solution of the equation

$$Q^\mathsf{T}\mathbf{x}^* = \mathbf{b}. \tag{6.19}$$

(iii). Prove that the quadratic minimization problem (6.17) will have a solution if and only if Q is positive definite, and in that case, the minimizer is the solution of the linear system (6.19). Explain why this shows that solving the system (6.19) with positive definite Q is equivalent to solving the quadratic optimization problem (6.18).

Depending on Q, the best way to solve the linear system (6.19) is often to use optimization algorithms on the quadratic problem (6.17) instead of using linear solvers on (6.19).

Solution.

(i). If A is square, then $A^\mathsf{T} + A$ is

$$\begin{aligned}
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} &= \begin{bmatrix} 2\cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} + a_{1n} & \cdots & 2\cdot a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} 2\cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + a_{1n} & \cdots & 2\cdot a_{nn} \end{bmatrix},
\end{aligned}$$

so $(A^\mathsf{T} + A)^\mathsf{T}$ is

$$\begin{bmatrix} 2\cdot a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} + a_{1n} & \cdots & 2\cdot a_{nn} \end{bmatrix},$$

which is itself, so it is symmetric. So, we have that

$$\begin{aligned}
\mathbf{x}^\mathsf{T}Q\mathbf{x} &= \mathbf{x}^\mathsf{T}(A^\mathsf{T} + A)\mathbf{x} \\
&= \mathbf{x}^\mathsf{T}A^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T}A\mathbf{x} \\
&= \mathbf{x}^\mathsf{T}A\mathbf{x} + \mathbf{x}^\mathsf{T}A\mathbf{x} \\
&= 2\cdot \mathbf{x}^\mathsf{T}A\mathbf{x}.
\end{aligned}$$

So, we can see that

$$\begin{aligned}
\mathbf{x}^\mathsf{T} A\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c &= \frac{1}{2}\cdot\left(2\cdot\mathbf{x}^\mathsf{T} A\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \frac{1}{2}\cdot\left(\mathbf{x}^\mathsf{T} A\mathbf{x} + \mathbf{x}^\mathsf{T} A\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \frac{1}{2}\cdot\left(\mathbf{x}^\mathsf{T} A^\mathsf{T}\mathbf{x} + \mathbf{x}^\mathsf{T} A\mathbf{x}\right) - \mathbf{b}^\mathsf{T}\mathbf{x} + c \\
&= \mathbf{x}^\mathsf{T}(A^\mathsf{T} + A)\mathbf{x} \\
&= \frac{1}{2}\cdot\mathbf{x}^\mathsf{T} Q\mathbf{x} - \mathbf{b}^\mathsf{T}\mathbf{x} + c.
\end{aligned}$$

□

(ii). This is a direct corollary of the FONC—$Q^\mathsf{T}\mathbf{x}^* = \mathbf{b}$ because $f'(\mathbf{x}) = Q^\mathsf{T}\mathbf{x} - \mathbf{b}$. □

(iii). If Q PSD, then $f''(\mathbf{x}) > 0\ \forall \mathbf{x} \in \mathbb{R}^n$. By (6.19) and the fact that Q is invertible, we know that $\mathbf{x}^* = Q^{-1}\mathbf{b}$, where $\mathbf{x}^*$ is the value where $f'(\mathbf{x}^*) = 0$. So, by SOSC, $\mathbf{x}^*$ is a *unique* minimizer of f. If $\mathbf{x}^*$ is the unique minimizer, then by SONC we know that Q is PSD and $Q^\mathsf{T}\mathbf{x}^* = \mathbf{b}$. If Q has at least one e-value equal to zero, then $\mathbf{x}^*$ is *not* unique, so Q *must* be positive definite. □

Exercise 6.11.

Consider a quadratic function $f(x) = ax^2 + bx + c$, here $a > 0$, and $b, c \in \mathbb{R}$. Show that for any initial guess $x_0 \in \mathbb{R}$, one iteration of Newton's method lands at the unique minimizer of f.

Solution. f has a minimum at $x = -\frac{b}{2a}$ where $f(x) = a\cdot\left(x + \frac{b}{2a}\right)^2 - a\cdot\left(\frac{b}{2a}\right)^2$, so the Newton's method always yields

$$\begin{aligned}
x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0} \\
&= x_0 - \frac{2ax_0 + b}{2a} \\
&= -\frac{b}{2a}.
\end{aligned}$$

□

Exercise 6.15.

Code up (in Python/NumPy) the secant method for finding a minimizer of a function. Your code should accept two initial guesses x_0 and x_1, a desired level of accuracy ε, and a callable function $f'(x)$. It should return an approximation to a minimizer of f, provided the algorithm converges for the initial conditions. For the stopping criterion, use $|x_{k+1} - x_k| < |x_k|\,\varepsilon$. Be sure your code has methods for identifying and handling cases where the sequence does not converge.

Write your code in a notebook and test it with the function $f(x) = x^4 - 14x^3 + 60x^2 + 70x$ and initial guess $x_0 = 0$. Plot the function and the points and function values at each iteration in an appropriate graph.

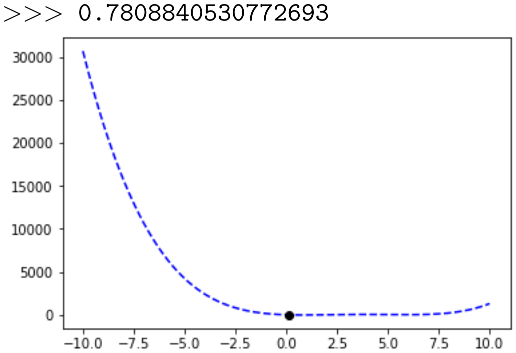
Solution.

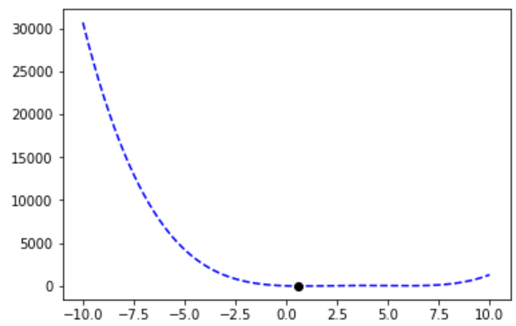
Algorithm 6.15.

```
import matplotlib.pyplot as plt
import numpy as np
from scipy import linalg as la

def f(x):
        return x**4 - 14*x**3 + 60*x**2 - 70*x

def f_pr(x):
        return 4 * x**3 - 14 * 3 * x**2 + 60 * 2 * x - 70


def problem6_15(x_0, x_1, ε, f_pr):
        MAX_IT = 1_000
        k = 0
        x_k = x_0
        x_kp1 = x_1
        X = np.linspace(-10, 10, 1_000)
        while k < MAX_IT:
                x_km1 = x_k
                x_k = x_kp1
                x_kp1 = x_k - f_pr(x_k) * (x_k - x_km1)/(f_pr(x_k) - f_pr(x_km1))
                k += 1
                plt.plot(X, f(X), "b--")
                plt.plot(x_k, f(x_k), "ko")
                plt.show()
                if la.norm(x_kp1 - x_k) < ε * la.norm(x_k):
                        break
        return x_k
```

Test.

```
>>> problem6_15(0, 0.1, 1e-10, f_pr)
>>> 0.7808840530772693
```

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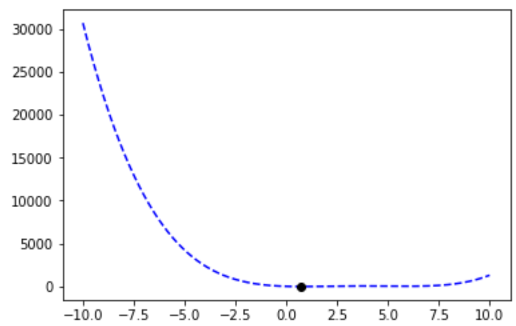
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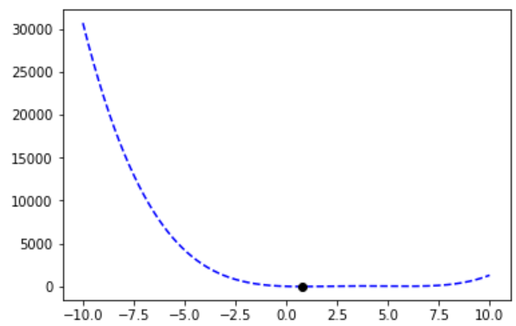
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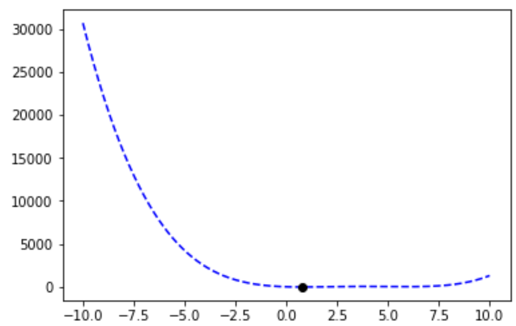
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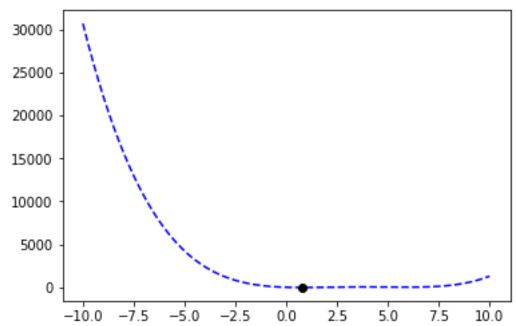
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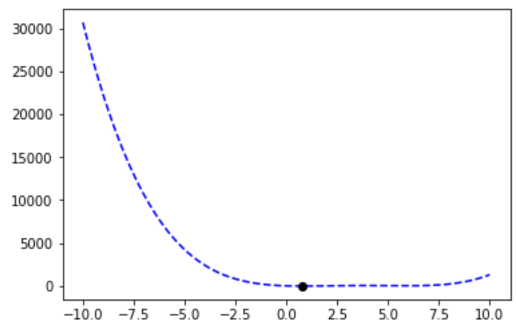

Exercise 7.1.

Prove Proposition 7.1.5.
Prop 7.1.5. *If S is a nonempty subset of V, then* $\operatorname{conv}(S)$ *is convex.*

Solution. Start by rewriting $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{conv}(S)$ as a parameterized linear combination of $\mathbf{x}_i$'s in S:

$$\mathbf{v}_1 = \sum_{j=1}^{n} \lambda_{i_j} \cdot \mathbf{x}_{i_j}, \qquad \mathbf{v}_2 = \sum_{k=1}^{m} \lambda_{i_k} \cdot \mathbf{x}_{i_k},$$

with the conditions that

$$\sum_{j=1}^{n} \lambda_{i_j} = \sum_{k=1}^{m} \lambda_{i_k} = 1.$$

It follows that

$$\begin{aligned} \lambda \cdot \mathbf{v}_1 + (1-\lambda) \cdot \mathbf{v}_2 &= \lambda \cdot \sum_{j=1}^{n} \lambda_{i_j} \cdot \mathbf{x}_{i_j} + (1-\lambda) \cdot \sum_{k=1}^{m} \lambda_{i_k} \cdot \mathbf{x}_{i_k} \\ &= \sum_{j=1}^{n} \lambda \cdot \lambda_{i_j} \cdot \mathbf{x}_{i_j} + \sum_{k=1}^{m} (1-\lambda) \cdot \lambda_{i_k} \cdot \mathbf{x}_{i_k} \\ &= \sum_{i=1}^{n+m} \big(\chi(\{i \le n\}) \cdot \lambda \cdot \lambda_{i_j} + \chi(\{i > n\}) \cdot (1-\lambda) \cdot \lambda_{i_k}\big) \cdot \mathbf{x}_i, \end{aligned}$$

where

$$\sum_{i=1}^{n+m} \big(\chi(\{i \le n\}) \cdot \lambda \cdot \lambda_{i_j} + \chi(\{i > n\}) \cdot (1-\lambda) \cdot \lambda_{i_k}\big) = \sum_{j=1}^{n} \lambda \cdot \lambda_{i_j} + \sum_{i=1}^{m} (1-\lambda) \cdot \lambda_{i_k},$$

so

$$\lambda \cdot \sum_{j=1}^{n} \lambda_{i_j} + (1-\lambda) \cdot \sum_{k=1}^{m} \lambda_{i_k} = 1,$$

so $\lambda \cdot \mathbf{v}_1 + (1-\lambda) \cdot \mathbf{v}_2 \in \operatorname{conv}(S)$ for any $\lambda \in (0,1)$. $\square$

Exercise 7.2.

Prove that:

(i). A hyperplane is convex.

(ii). A half space is convex.

Solution.

(i). For hyperplane $H = \{\mathbf{x} \in V | \langle \mathbf{a}, \mathbf{x} \rangle = b\}$, for $\mathbf{a} \neq \mathbf{0} \in V$ and $b \in \mathbb{R}$, consider that for arbitrary $\mathbf{v}, \mathbf{u} \in H$, we have

$$\begin{aligned}\langle \mathbf{a}, \lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \rangle &= \langle \mathbf{a}, \lambda \cdot \mathbf{v} \rangle + \langle \mathbf{a}, (1-\lambda) \cdot \mathbf{u} \rangle \\ &= \lambda \cdot \langle \mathbf{a}, \mathbf{v} \rangle + (1-\lambda) \cdot \langle \mathbf{a}, \mathbf{u} \rangle \\ &= \lambda \cdot b + (1-\lambda) \cdot b \\ &= b,\end{aligned}$$

so $\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \in H$. □

(ii). For halfspace $H = \{\mathbf{x} \in V | \langle \mathbf{a}, \mathbf{x} \rangle \leq b\}$, for $\mathbf{a} \neq \mathbf{0} \in V$ and $b \in \mathbb{R}$, consider that for arbitrary $\mathbf{v}, \mathbf{u} \in H$, we have

$$\begin{aligned}\langle \mathbf{a}, \lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \rangle &= \langle \mathbf{a}, \lambda \cdot \mathbf{v} \rangle + \langle \mathbf{a}, (1-\lambda) \cdot \mathbf{u} \rangle \\ &= \lambda \cdot \langle \mathbf{a}, \mathbf{v} \rangle + (1-\lambda) \cdot \langle \mathbf{a}, \mathbf{u} \rangle \\ &\leq \lambda \cdot b + (1-\lambda) \cdot b \\ &= b,\end{aligned}$$

so $\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u} \in H$. □

Exercise 7.4.

Prove the following theorem: Let $C \subset \mathbb{R}^n$ be nonempty, closed, and convex. A point $\mathbf{p} \in C$ is the projection of $\mathbf{x}$ onto C if and only if

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0, \qquad \forall \mathbf{y} \in C. \tag{7.14}$$

Prove the statements below and then write a complete proof of the theorem:

(i). $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2 + \|\mathbf{p} - \mathbf{y}\|^2 + \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle$.

(ii). If (7.14) holds, then $\|\mathbf{x} - \mathbf{y}\| > \|\mathbf{x} - \mathbf{p}\|$ for all $\mathbf{y} \in C$, $\mathbf{y} \neq \mathbf{p}$. *Hint*: Use the identity in (i).

(iii). If $\mathbf{z} = \lambda \mathbf{y} + (1-\lambda)\mathbf{p}$, where $0 \leq \lambda \leq 1$, then

$$\|\mathbf{x} - \mathbf{z}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2 + 2\lambda \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle + \lambda^2 \|\mathbf{y} - \mathbf{p}\|^2. \tag{7.15}$$

(iv). If $\mathbf{p}$ is a projection of $\mathbf{x}$ onto the convex set C, then $\langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle \geq 0$ for all $\mathbf{y} \in C$. *Hint*: Use (7.15) to show that

$$0 \leq 2 \langle \mathbf{x} - \mathbf{p}, \mathbf{p} - \mathbf{y} \rangle + \lambda \|\mathbf{y} - \mathbf{p}\|^2, \qquad \forall \mathbf{y} \in C, \ \lambda \in [0,1].$$

Solution.

(i).

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}+\mathbf{p}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{x}-\mathbf{p}\rangle + \langle \mathbf{p}-\mathbf{y}, \mathbf{p}-\mathbf{y}\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle .\end{aligned}$$

□

(ii). $\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \geq 0 \; \forall \mathbf{y}\in C$ and $\langle \mathbf{x},\mathbf{x}\rangle \geq 0$ and $\langle \mathbf{x},\mathbf{x}\rangle = 0 \iff \mathbf{x}=\mathbf{0}$ and $\mathbf{y}\neq\mathbf{p}$ so $\|\mathbf{y}-\mathbf{p}\|^2 > 0$, and thus

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{y}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle \\ &> \|\mathbf{x}-\mathbf{p}\|^2 ,\end{aligned}$$

where $\langle \cdot,\cdot\rangle \geq 0$ by definition, so

$$\|\mathbf{x}-\mathbf{y}\|^2 > \|\mathbf{x}-\mathbf{p}\|^2 .$$

□

(iii).

$$\begin{aligned}\mathbf{p}-\mathbf{z} &= \mathbf{p}-\lambda\cdot\mathbf{y}-(1-\lambda)\cdot\mathbf{p} \\ &= \lambda\cdot(\mathbf{p}-\mathbf{y}),\end{aligned}$$

so by (i),

$$\begin{aligned}\|\mathbf{x}-\mathbf{z}\|^2 &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\mathbf{p}-\mathbf{z}\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{z}\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + \|\lambda(\mathbf{p}-\mathbf{y})\|^2 + 2\cdot\langle \mathbf{x}-\mathbf{p}, \lambda\cdot(\mathbf{p}-\mathbf{y})\rangle \\ &= \|\mathbf{x}-\mathbf{p}\|^2 + 2\cdot\lambda\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \lambda^2\cdot\|\mathbf{p}-\mathbf{y}\|^2 .\end{aligned}$$

□

(iv). For $\mathbf{y}\in C$, note that $\lambda\cdot\mathbf{y}+(1-\lambda)\cdot\mathbf{p}$ is also in C for $\lambda\in[0,1]$. From (7.15), if $\mathbf{p}$ is a projection of $\mathbf{x}$ onto C, then $\|\mathbf{x}-\mathbf{p}\| < \|\mathbf{x}-\mathbf{v}\|$ for $\mathbf{v}\neq\mathbf{p}$, $\mathbf{v}\in C$. From (iii), we have

$$\begin{aligned}0 &\leq \|\mathbf{x}-\mathbf{z}\|^2 - \|\mathbf{x}-\mathbf{p}\|^2 \\ &= 2\cdot\lambda\cdot\langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \lambda^2\cdot\|\mathbf{y}-\mathbf{p}\|^2 ,\end{aligned}$$

so

$$0 \leq \langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle + \frac{1}{2}\cdot\lambda\cdot\|\mathbf{y}-\mathbf{p}\|^2 ,$$

and for $\lambda=0$ we have

$$0 \leq \langle \mathbf{x}-\mathbf{p}, \mathbf{p}-\mathbf{y}\rangle .$$

□

For the proof: From (ii), $\|\mathbf{x}-\mathbf{y}\| > \|\mathbf{x}-\mathbf{p}\|$ for all $\mathbf{y}\in C$ where $\mathbf{y}\neq\mathbf{p}$, which is simply the definition of projection. From (iv), we can see the converse is true. □

Exercise 7.8.

Prove that if $f:\mathbb{R}^m\to\mathbb{R}$ is convex, if $A\in \mathrm{M}_{m\times n}(\mathbb{R})$, and if $\mathbf{b}\in\mathbb{R}^m$, then the function $g:\mathbb{R}^m\to\mathbb{R}$ given by $g(\mathbf{x}) = f(A\mathbf{x}+\mathbf{b})$ is convex.

Solution. For $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have that

$$\begin{aligned} g(\lambda \cdot \mathbf{x}_1 + (1-\lambda)\cdot \mathbf{x}_2) &= f(\lambda \cdot A\mathbf{x}_1 + (1-\lambda)\cdot A\mathbf{x}_2 + \mathbf{b}) \\ &= f(\lambda \cdot (A\mathbf{x}_1 + \mathbf{b}) + (1-\lambda)\cdot (A\mathbf{x}_2 + \mathbf{b})) \\ &\leq \lambda \cdot f(A\mathbf{x}_1 + \mathbf{b}) + (1-\lambda)\cdot f(A\mathbf{x}_2 + \mathbf{b}) \\ &= \lambda \cdot g(\mathbf{x}_1) + (1-\lambda)\cdot g(\mathbf{x}_2). \end{aligned}$$

□

Exercise 7.12.

Prove the following:

(i). The set $\mathrm{PD}_n(\mathbb{R})$ of positive-definite matrices in $\mathrm{M}_n(\mathbb{R})$ is convex.

(ii). The function $f(X) = \log(\det(X))$ is convex on $\mathrm{PD}_n(\mathbb{R})$. To prove this, show the following:

(a). The function f is convex if for every $A, B \in \mathrm{PD}_n(\mathbb{R})$ the function $g(t) : [0,1] \to \mathbb{R}$ given by $g(t) = f(tA + (1-t)B)$ is convex.

(b). Use the fact that positive definite matrices are normal to show that there is an S such that $S^{\mathsf{H}}S = A$ and

$$\begin{aligned} g(t) &= -\log(\det(S^{\mathsf{H}}(tI + (1-t)(S^{\mathsf{H}})^{-1}BS^{-1})S)) \\ &= -\log(\det(A)) - \log(\det(tI + (1-t)(S^{\mathsf{H}})^{-1}BS^{-1})). \end{aligned}$$

(c). Show that

$$g(t) = -\sum_{i=1}^{n} \log(t + (1-t)\lambda_i) - \log(\det(A)),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $(S^{\mathsf{H}})^{-1}BS^{-1}$.

(d). Prove that $g''(t) \geq 0$ for all $t \in [0,1]$.

Solution.

(i). For $A_1, A_2 \in \mathrm{PD}_n(\mathbb{R})$, $\lambda \in [0,1]$, and $\mathbf{x} \in \mathbb{R}^n$, note that by positive definiteness it follows that

$$\begin{aligned} \mathbf{x}^{\mathsf{T}}(\lambda \cdot A_1 + (1-\lambda)\cdot A_2)\mathbf{x} &= \lambda \cdot \left(\mathbf{x}^{\mathsf{T}} A_1 \mathbf{x}\right) + (1-\lambda)\cdot \left(\mathbf{x}^{\mathsf{T}} A_2 \mathbf{x}\right) \\ &> 0. \end{aligned}$$

□

(ii).

(a). Fix $t_A, t_B \in \mathbb{R}$, $A, B \in \mathrm{PD}_n(\mathbb{R})$, $\lambda \in [0,1]$ arbitrarily. Then

$$\lambda \cdot g(t_A) + (1-\lambda)\cdot g(t_B) = \lambda \cdot f(t_A \cdot A + (1-t_A)\cdot B) + (1-\lambda)\cdot f(t_B \cdot A + (1 - t_B \cdot B)$$

and

$$\begin{aligned} g(\lambda \cdot t_A + (1-\lambda)\cdot t_B &= f((\lambda \cdot t_A + (1-\lambda)\cdot t_B)\cdot A + (1 - \lambda \cdot t_A + (1-\lambda)\cdot t_B)\cdot B) \\ &= f(\lambda \cdot (t_A \cdot A + (1-t_A)\cdot B) + (1-\lambda)\cdot (t_B \cdot A + (1-t_B)\cdot B)) \end{aligned}$$

so

$$\begin{aligned} & f\left(\lambda \cdot\left(t_A \cdot A+\left(1-t_A\right) \cdot B\right)+(1-\lambda) \cdot\left(t_B \cdot A+\left(1-t_B\right) \cdot B\right)\right) \\ \leq\ & \lambda \cdot f\left(t_A \cdot A+\left(1-t_A\right) \cdot B\right)+(1-\lambda) \cdot f\left(t_B \cdot A+\left(1-t_B\right) \cdot B\right). \end{aligned}$$

□

(b). $A \in \mathrm{PD}_n(\mathbb{R})$ so $\exists S$, nonsingular, such that $A = S^{\mathsf{H}} S$:

$$t \cdot A+(1-t) \cdot B=S^{\mathsf{H}} \cdot\left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right) S,$$

so

$$\begin{aligned} g(t) &= -\log (\det (t \cdot A+(1-t) \cdot B)) \\ &= -\log \left(\det \left(S^{\mathsf{H}}\left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right) S\right)\right) \\ &= -\log \left(\det \left(S^{\mathsf{H}}\right)\right)-\log \left(\det \left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right)\right) \\ &\quad -\log (\det (S)) \\ &= -\log \left(\det \left(S^{\mathsf{H}}\right) \cdot \det (S)\right)-\log \left(\det \left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right)\right) \\ &= -\log (\det (A))-\log \left(\det \left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right)\right). \end{aligned}$$

□

(c). $A, B \in \mathrm{PD}_n(\mathbb{R})$ so $B^{-1} \in \mathrm{PD}_n(\mathbb{R})$ and

$$\left(\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right)^{-1}=S B^{-1} S^{\mathsf{H}} \in \mathrm{PD}_n(\mathbb{R}),$$

so

$$\left(S^{\mathsf{H}}\right)^{-1} B S^{-1} \in \mathrm{PD}_n(\mathbb{R}),$$

so for the e-vals λ_i and e-vecs $\mathbf{v}_i$ of $\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}$, we know

$$\begin{aligned} \left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right) \mathbf{v}_i &= t \cdot \mathbf{v}_i+(1-t) \cdot \lambda_i \cdot \mathbf{v}_i \\ &= \left(t+(1-t) \cdot \lambda_i\right) \cdot \mathbf{v}_i, \end{aligned}$$

so the e-vals of $t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}$ are of the form $t+(1-t) \cdot \lambda_i$, so

$$\begin{aligned} & -\log (\det (A))-\log \left(\det \left(t \cdot I+(1-t) \cdot\left(S^{\mathsf{H}}\right)^{-1} B S^{-1}\right)\right) \\ =\ & -\log (\det (A))-\log \left(\prod_{i=1}^n\left(t+(1-t) \cdot \lambda_i\right)\right) \\ =\ & -\log (\det (A))-\sum_{i=1}^n \log \left(\left(t+(1-t) \cdot \lambda_i\right)\right). \end{aligned}$$

□

(d). From above, it follows that

$$g^{\prime}(t)=\sum_{i=1}^n \frac{1-\lambda_i}{t+(1-t) \cdot \lambda_i},$$

so

$$\begin{aligned} g^{\prime \prime}(t) &= \sum_{i=1}^n \frac{\left(1-\lambda_i\right)^2}{\left(t+(1-t) \cdot \lambda_i\right)^2} \\ &\geq 0. \end{aligned}$$

□

Exercise 7.13.

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above, prove that f is constant.

Solution. Proof by contradiction: Suppose f is *not* constant. Then, there must exist some vectors $\mathbf{v} > \mathbf{u}$ such that $f(\mathbf{v}) > f(\mathbf{u})$. Then, for all $\mathbf{x} > \mathbf{v}$,

$$0 < \frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}} \leq 1$$

and

$$0 \leq 1 - \frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}} = \frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}} < 1.$$

Note that

$$\begin{aligned}
\mathbf{v} &= \left(\frac{\mathbf{x}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{v} \\
&= \frac{\mathbf{xv}-\mathbf{xu}+\mathbf{xu}-\mathbf{vu}}{\mathbf{x}-\mathbf{u}} \\
&= \left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{x} \\
&= \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u} \\
&= \left(1-\left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\right)\mathbf{x} + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u},
\end{aligned}$$

where

$$\begin{aligned}
f(\mathbf{v}) &= f\left(\left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{x} + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right)\mathbf{u}\right) \\
&\leq \left(\frac{\mathbf{v}-\mathbf{u}}{\mathbf{x}-\mathbf{u}}\right) f(\mathbf{x}) + \left(\frac{\mathbf{x}-\mathbf{v}}{\mathbf{x}-\mathbf{u}}\right) f(\mathbf{u}),
\end{aligned}$$

so

$$\begin{aligned}
f(\mathbf{x}) &\geq \frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}} f(\mathbf{v}) - \frac{\mathbf{x}-\mathbf{v}}{\mathbf{v}-\mathbf{u}} f(\mathbf{u}) \\
&= \frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}} f(\mathbf{v}) - \frac{\mathbf{x}-\mathbf{u}+\mathbf{u}-\mathbf{v}}{\mathbf{v}-\mathbf{u}} f(\mathbf{u}) \\
&= \underbrace{\frac{\mathbf{x}-\mathbf{u}}{\mathbf{v}-\mathbf{u}}}_{>0}\underbrace{(f(\mathbf{v})-f(\mathbf{u}))}_{>0} + f(\mathbf{u}) \\
&> f(\mathbf{u}),
\end{aligned}$$

so for any arbitrary value $U \in \mathbb{R}$ there is an $\mathbf{x}$ where $f(\mathbf{x}) > U$, so f is unbounded, which contradicts our assumptions. Therefore f is constant. □

Exercise 7.20.

Prove Proposition 7.4.3.
Prop 7.4.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $-f$ is also convex, then f is affine.*

Solution. For any $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) \leq \lambda \cdot f(\mathbf{v}) + (1-\lambda) \cdot f(\mathbf{u})$$

and

$$-f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) \leq -\lambda \cdot f(\mathbf{v}) - (1-\lambda) \cdot f(\mathbf{u})$$

so

$$f(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) = \lambda \cdot f(\mathbf{v}) + (1-\lambda) \cdot f(\mathbf{u}).$$

Now define

$$\tilde{f}(\mathbf{x}) \coloneqq f(\mathbf{x}) - f(\mathbf{0}) = 0,$$

so we have

$$\tilde{f}(\lambda \cdot \mathbf{v} + (1-\lambda) \cdot \mathbf{u}) = \lambda \cdot \tilde{f}(\mathbf{v}) + (1-\lambda) \cdot \tilde{f}(\mathbf{u}),$$

and choose $\mathbf{u} = \mathbf{0}$. Then $\tilde{f}(\lambda \cdot \mathbf{v}) = \lambda \cdot \tilde{f}(\mathbf{u})$ for $\lambda \in [0,1]$. So $\tilde{f}$ is linear and the mappings taking $\mathbf{x}$ to $\mathbf{x}/\lambda$ and $-\mathbf{x}$ are bijective, and for all $\mathbf{v} = \frac{\mathbf{x}}{\lambda} \in \mathbb{R}^n$, we have that $f(\mathbf{x}/\lambda) = \frac{1}{\lambda} \cdot f(\mathbf{x})$, so $f(\lambda \cdot \mathbf{v}) = \lambda \cdot f(\mathbf{v})$. So we can prove linearity of $\tilde{f}$ (and thus affinity of f) by observing that for $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\tilde{f}(\alpha \cdot \mathbf{v} + \beta \cdot \mathbf{u}) = \alpha \cdot \tilde{f}(\mathbf{v}) + \beta \cdot \tilde{f}(\mathbf{u}).$$

□

Exercise 7.21.

Prove Proposition 7.4.11.
Prop 7.4.11. *If $D \subset \mathbb{R}$ with $f : \mathbb{R}^n \to D$, and if $\phi : D \to \mathbb{R}$ is a strictly increasing function, then $\mathbf{x}^*$ is a local minimizer for the problem*

$$\begin{aligned} \text{minimize} \quad & \phi \circ f(\mathbf{x}) \\ \text{subject to} \quad & G(\mathbf{x}) \preceq \mathbf{0} \\ & H(\mathbf{x}) = \mathbf{0} \end{aligned}$$

if and only if $\mathbf{x}^*$ is a local minimizer for the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & G(\mathbf{x}) \preceq \mathbf{0} \\ & H(\mathbf{x}) = \mathbf{0}. \end{aligned}$$

Solution.

- If $\mathbf{x}^*$ is a minimizer of the second problem, then, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for $\mathbf{x} \in U$, some open neighborhood. ϕ is *strictly increasing* so $\phi(\mathbf{v}) > \phi(\mathbf{u})$ iff $\mathbf{v} > \mathbf{u}$. So for $\mathbf{x} \in U$, $\phi \circ f(\mathbf{x}) \geq \phi \circ f(\mathbf{x}^*)$. So, $\mathbf{x}^*$ is a minimizer of U under $\phi \circ f(\mathbf{x})$.
- Next, if $\mathbf{x}^*$ is a minimizer for the first problem, observe $\phi \circ f(\mathbf{x}^*) \leq \phi \circ f(\mathbf{x})$ for $\mathbf{x} \in U$, some open neighborhood. There cannot be no minimizers in an open subset U' of U for the second problem because then there would be some other vectors $\mathbf{v}$ and $\mathbf{u}$ where $f(\mathbf{v}) < f(\mathbf{x}) < f(\mathbf{u})$. If $\mathbf{x}^* \in U'$, $\phi \circ f(\mathbf{v}) < \phi \circ f(\mathbf{x}^*) < \phi \circ f(\mathbf{u})$, contradicting earlier assumptions.
- If there are *infinitely many* minimizers (where none of which are $\mathbf{x}^*$) then there would be a sequence of minimizers converging to $\mathbf{x}^*$, in which case for any U', there exists a minimizer $\hat{\mathbf{x}}^*$ such that $f(\hat{\mathbf{x}}^*) \leq f(\mathbf{x})$ for $\mathbf{x} \in U'$. So $\phi \circ f(\hat{\mathbf{x}}^*) \leq \phi \circ f(\mathbf{x}^*)$ and $\phi \circ (f(\hat{\mathbf{x}})) = \phi \circ f(\mathbf{x}^*)$ because $\mathbf{x}^*$ is a minimizer for the first problem.

- ϕ is *strictly* increasing, so $\phi(\mathbf{v}) = \phi(\mathbf{u})$ iff $\mathbf{v} = \mathbf{u}$, so $f(\mathbf{x}) \geq f(\hat{\mathbf{x}}^*) = f(\mathbf{x}^*)$ for $\mathbf{x} \in U'$ and $\mathbf{x}^*$ is a minimizer of the second problem. $\square$