

OSM Boot Camp: Math ProbSet3

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Exercise 4.2.

Question. Let $V = \text{span}(\{1, x, x^2\})$ be a subspace of the inner product space $L^2([0, 1]; \mathbb{R})$. Let \mathbf{D} be the derivative operator $\mathbf{D} : V \rightarrow V$ given by $\mathbf{D}[p](x) = p'(x)$. Find all the eigenvalues and eigenspaces of \mathbf{D} . What are their algebraic and geometric multiplicities?

Answer.

$\mathbf{D} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is upper-triangular, so all its eigenvalues are $\boxed{0}$, so the (alg.) mult. of $\lambda = 0$ is $\boxed{3}$.

And, $\Sigma_0(\mathbf{D}) = \boxed{\text{span}(\{1\})}$, so the geom. mult. of $\lambda = \dim \Sigma_0(\mathbf{D}) = 0$ is $\boxed{1}$.

Exercise 4.4.

Question. Recall that a matrix $A \in M_n(\mathbb{F})$ is Hermitian if $A^H = A$ and skew-Hermitian if $A^H = -A$. Using Exercise 4.3, prove that

1. a Hermitian 2×2 matrix has only real eigenvalues;
2. a skew-Hermitian 2×2 matrix has only imaginary eigenvalues.

Answer.

1. $A \in M_{2 \times 2}(\mathbb{F})$ is Hermitian so a_{11} and a_{22} are $\in \mathbb{R}$ and $a_{12} = \bar{a}_{21}$. So, $a_{12}a_{21} = \|a_{21}\|^2 \in \mathbb{R}$, and the char poly is

$$\lambda^2 - \text{tr}(A)\lambda + \det A = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - \|a_{21}\|^2,$$

so

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + \|a_{21}\|^2}}{2},$$

where $(a_{11} - a_{22})^2 + \|a_{21}\|^2 \geq 0$ so $\lambda \in \mathbb{R}$. ■

2. $A \in M_{2 \times 2}(\mathbb{F})$ is skew-Hermitian, so $a_{11} = -\bar{a}_{11}$, $a_{22} = -\bar{a}_{22}$, $a_{12} = -\bar{a}_{21}$, so $a_{11}, a_{22} \in \mathbb{C} \setminus \mathbb{R}$ and $a_{12}a_{21} = -\|a_{21}\|^2$, and $(a_{11}a_{22})/i < 0$. We have

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} + \|a_{21}\|^2,$$

so

$$\begin{aligned}\lambda &= \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} + \|a_{21}\|^2)}}{2} \\ &= \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + \|a_{21}\|^2}}{2},\end{aligned}$$

where $(a_{11} - a_{22})^2 + \|a_{21}\|^2 < 0$, so $a_{11}, a_{12}, a_{21}, a_{22}, \lambda \in \mathbb{C} \setminus \mathbb{R}$. ■

Exercise 4.6.

Question. Prove Proposition 4.1.22.

Prop 4.1.22. *The diagonal entries of an upper-triangular (or a lower-triangular) matrix are its eigenvalues.*

Answer.

- *WLOG*, A and $\lambda I - A$ upper-triangular with diags $\{\lambda - a_{i,i}\}_{i=1}^n$. For $n = 1$ we have

$$\det(\lambda I - A) = \prod_{i=1}^1 \lambda - a_{i,i}.$$

and then

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{i,j} \det(A_{i,j}),$$

where $a_{i,1} = \delta_{i,1}$, and $A_{1,1} \in M_{n-1}(\mathbb{F})$ is upper-triangular. So it follows that

$$\begin{aligned}\det(A) &= \sum_{i=1}^n (-1)^{i+1} a_{i,1} \det(A_{i,1}) \\ &= a_{1,1} \det(A_{1,1}) \\ &= (\lambda - a_{1,1}) \prod_{i=1}^{n-1} (\lambda - (a_{A_{11}, A_{11}}) \cdot i) \\ &= (\lambda - a_{1,1}) \prod_{i=2}^n (\lambda - a_{i,i}) \\ &= \prod_{i=1}^n (\lambda - a_{i,i}).\end{aligned}$$
■

Exercise 4.8.

Question. Let V be the span of the set $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ in the vector space $\mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$.

1. Prove that S is a basis for V .
2. Let \mathbf{D} be the derivative operator. Write the matrix representation of \mathbf{D} in the basis S .
3. Find two complementary \mathbf{D} -invariant subspaces in V .

Answer.

1. S orthonormal under inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt$ from the last homework, so all the indep. elts form a basis for the span. ■

$$2. \text{ We know that } \begin{cases} \mathbf{D} \sin(x) = \cos(x) \\ \mathbf{D} \cos(x) = -\sin(x) \\ \mathbf{D} \sin(2x) = 2 \cos(2x) \\ \mathbf{D} \cos(2x) = -2 \sin(2x) \end{cases}, \text{ so it follows straightforwardly that } \mathbf{D} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \blacksquare$$

3. $\boxed{\text{span}(\{\sin(x), \cos(x)\})}$ and $\boxed{\text{span}(\{\sin(2x), \cos(2x)\})}$.

Exercise 4.13.

Question. Let

$$A = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}.$$

Compute the transition matrix P such that $P^{-1}AP$ is diagonal.

Answer.

$$\text{Note } \det(\lambda I - A) = \lambda^2 - 1.4\lambda + 0.4, \text{ so } \lambda \in \{1, 0.4\}. \text{ Thus, } P = (\vec{v}_1 \quad \vec{v}_{0.4}) = \boxed{\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}}.$$

Exercise 4.15.

Question. Prove Theorem 4.3.12.

Thm 4.3.12. (Semisimple Spectral Mapping). If $(\lambda_i)_{i=1}^n$ are the eigenvalues of a semisimple matrix $A \in M_n(\mathbb{F})$ and $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial, then $(f(\lambda_i))_{i=1}^n$ are the eigenvalues of $f(A) = a_0I + a_1A + \cdots + a_nA^n$.

Answer.

$$\begin{aligned}
f(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\
&= a_0 P P^{-1} + a_1 P \Lambda P^{-1} + \cdots + a_n P \Lambda^n P^{-1} \\
&= P f(\Lambda) P^{-1},
\end{aligned}$$

where $f(\Lambda)$ is composed of diagonal entries, so we have $(f(\lambda_i))_{i=1}^n$. The eigenvalues are $(f(\lambda_i))_{i=1}^n$ due to similarity. ■

Exercise 4.16.

Question. Let A be the matrix in Exercise 4.13 above.

1. Compute $\lim_{n \rightarrow \infty} A^n$ with respect to the 1-norm; that is, find a matrix B such that for any $\varepsilon > 0$ there exists an $N > 0$ with $\|A^k - B\| < \varepsilon$ whenever $k < N$. *Hint:* Use Proposition 4.3.10. **Prop 4.3.10.** *If matrices $A, B \in M_n(\mathbb{F})$ are similar, with $A = P^{-1}BP$, then $A^k = P^{-1}B^kP$ for all $k \in \mathbb{N}$.*
2. Repeat part (1) for the ∞ -norm and the Frobenius norm. Does the answer depend on the choice of norm? We discuss this further in Section 5.8.
3. Find all the eigenvalues of the matrix $3I + 5A + A^3$. *Hint:* Consider using Theorem 4.3.12 (see above for the theorem).

Answer.

1. $A^n = P \Lambda^n P^{-1}$, where

$$\Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix},$$

where

$$\begin{aligned}
A^k &= \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.4^k \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\
&= \frac{1}{3} \cdot \begin{pmatrix} 2 + 0.4^k & 2 - 2 \cdot 0.4^k \\ 1 - 0.4^k & 1 + 2 \cdot 0.4^k \end{pmatrix}
\end{aligned}$$

where

$$B = \boxed{\frac{1}{3} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}},$$

and $A^k - B = \frac{1}{3} \begin{pmatrix} 0.4^k & -2 \cdot 0.4^k \\ -0.4^k & 2 \cdot 0.4^k \end{pmatrix}$ converges via the 1-norm.

2. The answer does *not* depend on choice of norm: The ∞ -norm goes to zero, as above; the Frobenius norm is

$$\begin{aligned}
\|A^k - B\| &= \frac{1}{3} \sqrt{\text{tr} \left(\begin{pmatrix} 0.4^k & -0.4^k \\ -2 \cdot 0.4^k & 2 \cdot 0.4^k \end{pmatrix} \begin{pmatrix} 0.4^k & -2 \cdot 0.4^k \\ -0.4^k & 2 \cdot 0.4^k \end{pmatrix} \right)} \\
&= \frac{1}{3} \sqrt{\text{tr} \left(\begin{pmatrix} 2 \cdot 0.4^{2k} & -4 \cdot 0.4^{2k} \\ -4 \cdot 0.4^{2k} & 8 \cdot 0.4^{2k} \end{pmatrix} \right)} \\
&= \sqrt{10 \cdot 0.4^{2k}},
\end{aligned}$$

which also converges to zero!

3. $f(1) = 9$ and $f(0.4) = 5.064$ since the e-vals were 1 and 0.4 originally, and $f(x) = 3 + 5x + x^3$, the e-vals of A .

Exercise 4.18.

Question. Prove: If λ is an eigenvalue of the $A \in M_n(\mathbb{F})$, then there exists a nonzero row vector \vec{x}^T such that $\vec{x}^T A = \lambda \vec{x}^T$.

Answer.

$\det(\lambda I - A) = \det(\lambda I - A^T)$, so e-vals of A are also e-vals of A^T . So $\exists \vec{v} \in \mathbb{F}^n$ s.t. $A^T \vec{v} = \lambda \vec{v}$, so it follows that $\vec{v}^T A = \lambda \vec{v}^T$. ■

Exercise 4.20.

Question. Prove Lemma 4.4.2.

Lemma 4.4.2. *If A is Hermitian and orthonormally similar to B , then B is also Hermitian.*

Answer.

From the given we know that $B = PAP^{-1}$ for some orthonormal P . A Hermitian so

$$\begin{aligned} B^H &= (PAP^{-1})^H \\ &= (PA P^H)^H \\ &= (P^H)^H A^H P^H \\ &= P A P^H \\ &= B. \end{aligned}$$

■

Exercise 4.24.

Question. Given $A \in M_n(\mathbb{C})$, define the *Rayleigh quotient* as

$$\rho(\vec{x}) = \frac{\langle \vec{x}, A\vec{x} \rangle}{\|\vec{x}\|^2},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{F}^n . Show that the Rayleigh quotient can only take on real values for Hermitian matrices and only imaginary values for skew-Hermitian matrices.

Answer.

$\|\vec{x}\|^2 \in \mathbb{R}$, so it suffices to show that $\langle \vec{x}, A\vec{x} \rangle = \vec{x}^H A \vec{x}$:

$$\begin{aligned} \overline{\vec{x}^H A \vec{x}} &= (\vec{x}^H A \vec{x})^H \\ &= \vec{x}^H A^H \vec{x} \\ &= \begin{cases} \vec{x}^H A \vec{x} & (\langle \vec{x}, A\vec{x} \rangle \in \mathbb{R}) \\ -\vec{x}^H A \vec{x} & (\langle \vec{x}, A\vec{x} \rangle \in \mathbb{C} \setminus \mathbb{R}). \end{cases} \end{aligned}$$

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Exercise 4.25.

Question. Let $A \in M_n(\mathbb{C})$ be a normal matrix with eigenvalues $(\lambda_1, \dots, \lambda_n)$ and corresponding orthonormal eigenvectors $(\vec{x}_1, \dots, \vec{x}_n)$.

1. Show that the identity matrix can be written $I = \vec{x}_1 \vec{x}_1^H + \dots + \vec{x}_n \vec{x}_n^H$. *Hint:* What is $(\vec{x}_1 \vec{x}_1^H + \dots + \vec{x}_n \vec{x}_n^H) \vec{x}_j$?
2. Show that A can be written $A = \lambda_1 \vec{x}_1 \vec{x}_1^H + \dots + \lambda_n \vec{x}_n \vec{x}_n^H$. This is called an *outer product expansion*.

Answer.

1. $\vec{x}_j^H \vec{x}_i = 0$ if $i \neq j$ by orthonormality, so $(\sum_{i=1}^n \vec{x}_i \vec{x}_i^H) \vec{y} = \vec{y}$ for some \vec{y} gives us

$$\begin{aligned} \left(\sum_{i=1}^n \vec{x}_i \vec{x}_i^H \right) \left(\sum_{i=1}^n \alpha_i \vec{x}_i \right) &= \sum_{i=1}^n \vec{x}_i \vec{x}_i^H \vec{x}_i \alpha_i \\ &= \sum_{i=1}^n \vec{x}_i \alpha_i \\ &= \underbrace{\sum_{i=1}^n \alpha_i \vec{x}_i}_{\vec{y}} \end{aligned}$$

■

- 2.

$$\begin{aligned} A\vec{y} &= A \left(\sum_{i=1}^n \alpha_i \vec{x}_i \right) \\ &= \sum_{i=1}^n \alpha_i A \vec{x}_i \\ &= \sum_{i=1}^n \alpha_i \lambda_i \vec{x}_i \end{aligned}$$

and

$$\begin{aligned}
 \underbrace{\left(\sum_{i=1}^n \lambda_i \vec{x}_i \vec{x}_i^H \right)}_A \vec{y} &= \left(\sum_{i=1}^n \lambda_i \vec{x}_i \vec{x}_i^H \right) \left(\sum_{i=1}^n \alpha_i \vec{x}_i \right) \\
 &= \sum_{i=1}^n \lambda_i \vec{x}_i \vec{x}_i^H \vec{x}_i \alpha_i \\
 &= \sum_{i=1}^n \alpha_i \lambda_i \vec{x}_i
 \end{aligned}$$

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Exercise 4.27.

Question. Assume $A \in M_n(\mathbb{F})$ is positive definite. Prove that all its diagonal entries are real and positive.

Answer.

A PD so $\underbrace{\vec{e}_i^H A \vec{e}_i}_{a_{ii} \in \mathbb{R}} > 0$ for $\vec{e}_i \in \text{basis}_{\perp}(A)$.

■

Exercise 4.28.

Question. Assume $A, B \in M_n(\mathbb{F})$ are positive semidefinite. Prove that

$$0 \leq \text{tr}(AB) \leq \text{tr}(A) \text{tr}(B),$$

and use this result to prove that $\|\cdot\|_{\text{F}}$ is a matrix norm.

Answer.

A, B PSD so $\exists S_A, S_A^H, S_B, S_B^H$ such that

$$S_A^H S_A = A$$

and

$$S_B^H S_B = B,$$

so

$$\begin{aligned}
 \text{tr}(AB) &= \text{tr}(S_A^H S_A S_B^H S_B) \\
 &= \text{tr}(S_B S_A^H S_A S_B^H) \\
 &= \text{tr}\left((S_A S_B^H)^H S_A S_B^H\right) \\
 &\geq 0.
 \end{aligned}$$

Diagonalizing,

$$\begin{aligned}
 \operatorname{tr}(A) &= \operatorname{tr}(P_A D_A P_A^{-1}) \\
 &= \operatorname{tr}(P_A^{-1} P_A D_A) \\
 &= \operatorname{tr}(D_A) \\
 &= \sum_i \lambda_{A_i}.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \operatorname{tr}(AB) &= \operatorname{tr}(P_A D_A P_A^{-1} P_B D_B P_B^{-1}) \\
 &= \operatorname{tr}(P_A P_A^{-1} P_B D_A D_B P_B^{-1}) \\
 &= \operatorname{tr}(P_B^{-1} P_B D_A D_B) \\
 &= \operatorname{tr}(D_A D_B) \\
 &= \sum_i \lambda_{A_i} \lambda_{B_i} \\
 &\leq \left(\sum_i \lambda_{A_i} \right) \left(\sum_i \lambda_{B_i} \right) \\
 &= \operatorname{tr}(A) \operatorname{tr}(B).
 \end{aligned}$$

$\|\cdot\|_F$ is a mx norm because $\|A\| = \sqrt{\operatorname{tr}(A^H A)} \geq 0$, and “ $=$ ” $\iff A = \vec{0}$, and

$$\begin{aligned}
 \|cA\| &= \sqrt{\operatorname{tr}(c^H A^H A c)} \\
 &= c \sqrt{\operatorname{tr}(A^H A)} \\
 &= c \|A\|,
 \end{aligned}$$

and

$$\begin{aligned}
 \|A + B\|_F^2 &= \operatorname{tr}((A + B)^H (A + B)) \\
 &= \operatorname{tr}(A^H A + B^H B + A^H B + A^H B) \\
 &= \operatorname{tr}(A^H A) + \operatorname{tr}(B^H B) + \operatorname{tr}(A^H B + A^H B) \\
 &\leq \operatorname{tr}(A^H A) + \operatorname{tr}(B^H B) + 2 \|A\|_F \|B\|_F \\
 &= \|A\|_F^2 + \|B\|_F^2 + 2 \|A\|_F \|B\|_F \\
 &= (\|A\|_F + \|B\|_F)^2.
 \end{aligned}$$

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Exercise 4.31.

Question. Assume $A \in M_{m \times n}(\mathbb{F})$ and A is not identically zero. Prove that

1. $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A ;
2. If A is invertible, then $\|A^{-1}\|_2 = \sigma_n^{-1}$;
3. $\|A^H\|_2^2 = \|A^T\|_2^2 = \|A^H A\|_2 = \|A\|_2^2$;
4. if $U \in M_m(\mathbb{F})$ and $V \in M_n(\mathbb{F})$ are orthonormal, then $\|UAV\|_2 = \|A\|_2$.

Answer.

1. $A^H A$ normal, so it has *orthonormal* e-vecs $\{\vec{v}_i\}_{i=1}^n$ with respective e-vals $\{\sigma_i^2\}_{i=1}^n$,

$$\begin{aligned}
 \|A\|_2 &= \sup_{\|\vec{u}\|=1} \|A\vec{u}\| \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\langle A\vec{u}, A\vec{u} \rangle} \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\langle \vec{u}, A^H A \vec{u} \rangle} \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\left\langle \sum_{i=1}^n \alpha_i \vec{v}_i, \sum_{i=1}^n \alpha_i \sigma_i^2 \vec{v}_i \right\rangle} \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\left\langle \sum_{i=1}^n \alpha_i \vec{v}_i, \sum_{i=1}^n \alpha_i \sigma_i^2 \vec{v}_i \right\rangle} \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\sum_{i=1}^n |\alpha_i|^2 \sigma_i^2} \\
 &= \sqrt{\sigma_i^2} \\
 &= \sigma_1,
 \end{aligned}$$

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2. $A\vec{v} = \lambda\vec{v}$, so $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$:

$$\begin{aligned}
 \|A^{-1}\|_2 &= \sup_{\|\vec{u}\|=1} \|A^{-1}\vec{u}\| \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\langle A^{-1}\vec{u}, A^{-1}\vec{u} \rangle} \\
 &= \sup_{\|\vec{u}\|=1} \sqrt{\langle \vec{u}, (A^H A)^{-1} \vec{u} \rangle},
 \end{aligned}$$

so

$$\sup_{\|\vec{u}\|=1} \sqrt{\sum_{i=1}^n |c_i|^2 \frac{1}{\sigma_i^2}} = \sigma_n^{-1}.$$

■

3. $A = U\Sigma V^H$, for U, V o.n.:

$$\begin{aligned}
 A^H &= (U\Sigma V^H)^H \\
 &= V\Sigma^H U^H \\
 &= V\Sigma U^H,
 \end{aligned}$$

$$\begin{aligned}
 A^T &= (U\Sigma V^H)^T \\
 &= \bar{V}\Sigma^T U^T \\
 &= \bar{V}\Sigma U^T,
 \end{aligned}$$

for \bar{V}, U^\top o.n.. Moreover,

$$\begin{aligned} A^H A &= (U \Sigma V^H)^H U \Sigma V^H \\ &= V \Sigma^H U^H U \Sigma V^H \\ &= V \Sigma^H U^H U \Sigma V^H \\ &= V \tilde{\Sigma}^2 V^H, \end{aligned}$$

where $\tilde{\Sigma}$ has its sing. vals. along the main diag. By (4), then $\|A^H\|_2^2 = \|A^\top\|_2^2 = \|A\|_2^2$, and

$$\begin{aligned} \|A^H A\|_2 &= \|V \tilde{\Sigma}^2 V^H\| \\ &= \sup_{\|\vec{v}\|=1} \|V \tilde{\Sigma}^2 V^H \vec{v}\| \\ &= \sup_{\|V^H \vec{v}\|=1} \|V^H V \tilde{\Sigma}^2 V^H \vec{v}\| \\ &= \sup_{\|\tilde{\vec{v}}\|=1} \|\Sigma^2 \tilde{\vec{v}}\| \\ &= \sup_{\|\tilde{\vec{v}}\|=1} \sqrt{\sum_{i=1}^n |c_i|^2 \sigma_i^4} \\ &= \|A\|_2^2. \end{aligned}$$

■

4. W o.n. and full rank and bijective so $W^{-1} = W^H$:

$$\begin{aligned} \|UAV\|_2 &= \sup_{\|\vec{v}\|=1} \sqrt{\langle UAV \vec{v}, UAV \vec{v} \rangle} \\ &= \sup_{\|\vec{v}\|=1} \sqrt{\langle AV \vec{v}, U^H U AV \vec{v} \rangle} \\ &= \sup_{\|\vec{v}\|=1} \sqrt{\langle AV \vec{v}, AV \vec{v} \rangle} \\ &= \sup_{\|\tilde{\vec{v}}\|=1} \sqrt{\langle A \tilde{\vec{v}}, A \tilde{\vec{v}} \rangle} \\ &= \|A\|_2. \end{aligned}$$

■

Exercise 4.32.

Question. Assume $A \in M_{m \times n}(\mathbb{F})$ is of rank r . Prove that

1. if $U \in M_m(\mathbb{F})$ and $V \in M_n(\mathbb{F})$ are orthonormal, then $\|UAV\|_F = \|A\|_F$;
2. $\|A\|_F = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2)^{1/2}$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values of A .

Answer.

1.

$$\begin{aligned}
\|UAV\|_F &= \sqrt{\text{tr}(V^H A^H U^H U A V)} \\
&= \sqrt{\text{tr}(V^H A^H A V)} \\
&= \sqrt{\text{tr}(A^H A V V^H)} \\
&= \sqrt{\text{tr}(A^H A)} \\
&= \|A\|_F.
\end{aligned}$$

■

2.

$$\begin{aligned}
\|A\|_F &= \|U\Sigma V^H\|_F \\
&= \|\Sigma\|_F \\
&= \sqrt{\text{tr}(\Sigma^H \Sigma)} \\
&= \sqrt{\sum_{i=1}^n \sigma_i^2}.
\end{aligned}$$

■

Exercise 4.33.

Question. Assume $A \in M_n(\mathbb{F})$. Prove that

$$\|A\|_2 = \sup_{\substack{\|\vec{x}\|_2 = 1 \\ \|\vec{y}\|_2 = 1}} |\vec{y}^H A \vec{x}|.$$

Hint: Use Exercise 4.31 (above).

Answer.

$$\begin{aligned}
\sup_{\substack{\|\vec{x}\|_2 = 1 \\ \|\vec{y}\|_2 = 1}} |\vec{y}^H A \vec{x}| &= \sup_{\substack{\|\vec{x}\|_2 = 1 \\ \|\vec{y}\|_2 = 1}} |\vec{y}^H U \Sigma V^H \vec{x}| \\
&= \sup_{\substack{\|\tilde{\vec{x}}\|_2 = 1 \\ \|\tilde{\vec{y}}\|_2 = 1}} |\tilde{\vec{y}}^H \Sigma \tilde{\vec{x}}| \\
&= \|A\|_2.
\end{aligned}$$

■

Exercise 4.36.

Question. Give an example of a 2×2 matrix whose determinant is nonzero and whose singular values are not equal to any of its eigenvalues.

Answer.

$$A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad a > b > 0.$$

■

Exercise 4.38.

Question. Prove Proposition 4.6.2.

Prop 4.6.2. If $A \in M_{m \times n}(\mathbb{F})$, then the Moore-Penrose pseudoinverse of A satisfies the following:

1. $AA^\dagger A = A$.
2. $A^\dagger AA^\dagger = A^\dagger$.
3. $(AA^\dagger)^\mathsf{H} = AA^\dagger$.
4. $(A^\dagger A)^\mathsf{H} = A^\dagger A$.
5. $AA^\dagger = \text{proj}_{\mathcal{R}(A)}(\cdot)$ is the orthogonal projection onto $\mathcal{R}(A)$.
6. $A^\dagger A = \text{proj}_{\mathcal{R}(A^\mathsf{H})}(\cdot)$ is the orthogonal projection onto $\mathcal{R}(A^\mathsf{H})$.

Answer.

1. U, V full rank and $VV^\mathsf{H} \neq I$ so

$$\begin{aligned} U^\mathsf{H}U &= I \\ &= V^\mathsf{H}V, \end{aligned}$$

where

$$\begin{aligned} A^\dagger A &= V_1 \Sigma_1^{-1} U_1^\mathsf{H} U_1 \Sigma_1 V_1^\mathsf{H} \\ &= V_1 \Sigma_1^{-1} \Sigma_1 V_1^\mathsf{H} \\ &= V_1 V_1^\mathsf{H}, \end{aligned}$$

so

$$\begin{aligned} AA^\dagger A &= A V_1 V_1^\mathsf{H} \\ &= U_1 \Sigma_1 V_1^\mathsf{H} V_1 V_1^\mathsf{H} \\ &= U_1 \Sigma_1 V_1^\mathsf{H} \\ &= A. \end{aligned}$$

■

2.

$$\begin{aligned}
A^\dagger A A^\dagger &= V_1 V_1^H V_1 \Sigma_1^{-1} U_1^H \\
&= V_1 \Sigma_1^{-1} U_1^H \\
&= A^\dagger.
\end{aligned}$$

■

3.

$$\begin{aligned}
(AA^\dagger)^H &= (U_1 U_1^H)^H \\
&= (U_1^H)^H U_1^H \\
&= U_1 U_1^H \\
&= AA^\dagger.
\end{aligned}$$

■

4.

$$\begin{aligned}
(A^\dagger A)^H &= (V_1 V_1^H)^H \\
&= (V_1^H)^H V_1^H \\
&= V_1 V_1^H \\
&= A^\dagger A.
\end{aligned}$$

■

5. Consider

$$\begin{aligned}
\langle A\vec{v}, \vec{v} - AA^\dagger \vec{v} \rangle &= \langle \vec{v}, (A^H - A^H AA^\dagger) \vec{v} \rangle \\
&= \langle \vec{v}, (A^H - V_1 \Sigma_1 U_1^H U_1 U_1^H) \vec{v} \rangle \\
&= \langle \vec{v}, (A^H - V_1 \Sigma_1 U_1^H) \vec{v} \rangle \\
&= \langle \vec{v}, (A^H - A^H) \vec{v} \rangle \\
&= \langle \vec{v}, 0 \rangle \\
&= 0,
\end{aligned}$$

for fixed $\vec{v} \in \mathbb{F}^n$. $\vec{v} \in \mathcal{R}(A)$ surj, and $\dim(\mathcal{R}(A)) = \text{rank}(\underbrace{U_1 U_1^H}_{AA^\dagger})$. AA^\dagger proj bc $(AA^\dagger)^2 = AA^\dagger$ and (1),

above.

■

6.

$$\begin{aligned}
A^H (A^H)^\dagger &= V_1 \Sigma_1 U_1^H U_1 (\Sigma_1^H)^{-1} V_1^H \\
&= V_1 \Sigma_1 \Sigma_1^{-1} V_1^H \\
&= V_1 \Sigma_1 \Sigma_1^{-1} V_1^H = V_1 V_1^H \\
&= A^\dagger A.
\end{aligned}$$

■