

OSM Boot Camp Econ Notes

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- **ⓘ Note.** Recall

$$\hat{\theta}_{\text{MLE}} = \theta : \max_{\theta} \log \mathcal{L}(\mathbf{x} \mid \theta)$$

$$\begin{aligned}\hat{\theta}_{\text{GMM}} &= \theta : \min_{\theta} \|m(\mathbf{x} \mid \theta) - m(\mathbf{x})\| \\ &= \arg \min_{\theta} e(\mathbf{x} \mid \theta)^{\top} \mathbf{W} e(\mathbf{x} \mid \theta).\end{aligned}$$

where we can set

$$m_1(\mathbf{x} \mid \mu, \sigma) = \begin{pmatrix} \text{E}[\mathbf{x} \mid \mu, \sigma] \\ \text{Var}[\mathbf{x} \mid \mu, \sigma] \end{pmatrix}$$

and

$$m_2(\mathbf{x} \mid \mu, \sigma)$$

- **Topic.** OLS.
 - Consider

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i,$$

where

$$\text{E}[\varepsilon_i] = 0$$

and

$$\text{E}[x_{ji}\varepsilon_i] = 0.$$

- Let

$$\hat{\theta}_{\text{GMM}} = \theta : \min_{\theta} \varepsilon^{\top} \varepsilon.$$

This is just OLS regression, in the form of GMM.

- Now consider

$$\text{E}[x_{1i}\varepsilon_i] = 0$$

and

$$\text{E}[x_{2i}\varepsilon_i] = 0.$$

- Computing a moment:

$$\varepsilon_i = y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i}.$$

Each error is a function of the data.

- Let

$$\begin{aligned}m_1(\mathbf{x} \mid \beta_0, \beta_1, \beta_2) &= \frac{1}{N} \sum_{i=1}^N \\ &= \frac{1}{N} \sum_i (y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i}) \\ &= 0.\end{aligned}$$

MLE says that choose a set of params so that that sum adds up to zero.

$$\begin{aligned} m_2(\mathbf{x} \mid \beta_0, \beta_1, \beta_2) &= \frac{1}{N} \sum_{i=1}^N x_{1i} (y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i}) \\ &= 0. \end{aligned}$$

This is another moment condition.

$$m_3(\cdot) = \cdots x_{2i} \cdots$$

and so on.

- **Topic.** Brock-Mirman model.

$$\begin{aligned} 1 &= \frac{\beta E[r_{t+1} u'(c_{t+1})]}{u'(c_t)} \\ \implies \beta E\left[\frac{r_{t+1} c_t}{c_{t+1}}\right] - 1 &= 0. \end{aligned}$$

Middle term is

$$\beta E\left[\frac{\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} c_t}{c_{t+1}}\right] - 1 = 0.$$

This is equation (9) in the notes.

- **Topic.**
 - Consider the difference between

$$\hat{\boldsymbol{\theta}}_{\text{MLE}} = \arg \max_{\boldsymbol{\theta}} \log \mathcal{L}(\mathbf{x} \mid \boldsymbol{\theta})$$

and

$$\hat{\boldsymbol{\theta}}_{\text{GMM}} = \arg \min_{\boldsymbol{\theta}} \left\| \underbrace{m(\mathbf{x} \mid \boldsymbol{\theta})}_{\text{model}} - \underbrace{m(\mathbf{x})}_{\text{data}} \right\|$$

- **Topic.**
 - Let

$$S := \# \text{ of sims } (s)$$

$$\tilde{\mathbf{x}} := \{\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_S\} \implies \tilde{\mathbf{x}}_s = \begin{pmatrix} y_{1s} x_{11s} x_{21s} \\ \vdots \\ y_{is} x_{1is} x_{2is} \\ \vdots \\ y_{Ns} x_{1Ns} x_{2Ns} \end{pmatrix}.$$

The model moments are

$$m(\tilde{\mathbf{x}} \mid \boldsymbol{\theta}) = \frac{1}{S} \sum_{s=1}^S m(\tilde{\mathbf{x}}_s \mid \boldsymbol{\theta}).$$

Note that S is usually a large number, like 10,000.

- **Topic.**
 - For SMM, we have

$$\hat{\boldsymbol{\theta}}_{\text{SMM}} = \arg \min_{\boldsymbol{\theta}} \|m(\tilde{\mathbf{x}} \mid \boldsymbol{\theta}) - m(\mathbf{x})\|.$$

In the L^2 norm way,

$$\arg \min_{\boldsymbol{\theta}} e(\tilde{\mathbf{x}} \mid \boldsymbol{\theta})^\top \mathbf{W} e(\tilde{\mathbf{x}} \mid \boldsymbol{\theta}),$$

where

$$e(\tilde{\mathbf{x}} \mid \boldsymbol{\theta}) := m(\tilde{\mathbf{x}} \mid \boldsymbol{\theta}) - m(\mathbf{x}).$$

- **Topic.**

- Taking draws from the truncated normal distribution.

Let the PDF be $\phi(\mathbf{x} | \boldsymbol{\theta})$ and the CDF be $\Phi(\mathbf{x} | \boldsymbol{\theta})$. To simulate a general distribution, here are the steps:

1. Draw N values $u_i \sim \text{Unif}(0, 1)$.
2. Use $\Phi(\mathbf{x} | \boldsymbol{\theta})$ to convert u_i to x_i (the implied values from this PDF) $\implies x_i \sim \phi(\mathbf{x} | \boldsymbol{\theta})$.

⚠ **Note.** The SMM problem will be a bonus problem.

Mon, 23 Jul. 2018

- **Topic.** Lucas Tree Model.
- ① **Note.** Review of probability.
 - Start with probability space $Z := \{z_1, z_2, \dots, z_n\}$. Assume Z stays constant over time. Each z_i is mutually exclusive and exactly one must occur.
 - Take the infinite cartesian product of this set, Z^∞ . We are interested in an infinite horizon. Call this set Ω .
 - For $\omega \in \Omega$, we have $\omega = (z^1, z^2, z^3, \dots, z^t, \dots)$. Call this a *path*.
 - Random variable $X(\omega) : \Omega \rightarrow \mathbb{R}$.
- ♦ **Example.** Simple random variables.
 - Suppose $z^1 = z_3$. Then

$$\begin{aligned} X(\omega) &= \{a \mid z^1 = a\} \\ &= z_3. \end{aligned}$$

Alternatively, let

$$X'(\omega) = \{a \mid z^2 = a\}.$$

Consider two different paths:

$$(z_1, z_2, z_7, z_{14}, z_2, z_1, z_{25}, \dots)$$

and

$$(z_1, z_7, z_{14}, z_2, z_{27}, \dots).$$

These paths are isomorphic to decision/probability tree paths.

- **△ Def.** A **filtration** is a sequence of σ -algebras or partitions.
- ♦ **Example.** Consider $Z = \{1, 2\}$. Two possible paths are

$$(2, 1, 2, 2, 1, 1, 1, 2, 1, \dots)$$

and

$$(1, 2, 2, 2, 1, 1, 2, 1, \dots).$$

- We can partition this set into two possible sets:

$$\{\omega \mid z_1 = 1\}$$

and

$$\{\omega \mid z_1 = 2\}.$$

If we combined these two sets with \emptyset and Ω , we get a σ -algebra! Call this **S.A. 1**.

- Now consider **S.A. 2**. What could it be? We have

$$\{\omega \mid z_1 = 1, z_2 = 1\}, \dots, \{\omega \mid z_1 = 2, z_2 = 2\};$$

in total we have four sets. Now include \emptyset and Ω . If we include all the unions and cross-unions, we would also get a σ -algebra. In other words, this would be the σ -algebra *generated* by these sets. Note that this resultant σ -algebra is **finer** than S.A. 1. In other words, S.A. 1 is **coarser** than S.A. 2.

- We make a sequence of successively finer σ -algebras, as we learn more information. This would be called a **filtration**.
- Let $P\{1\} = \pi(1) = \frac{1}{3}$ and $\pi(2) = \frac{2}{3}$, and the outcomes IID.
- **Topic.** Consider the investor's problem,

$$\max_c \left\{ E \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \right\}.$$

Let's say we have

Asset	Price	Dividend
Asset 1	p_t	$d_t(\omega)$

where the price and the dividend are both random variables. The only things that we can consume are with the dividends that are paid out by these assets.

- Define θ_t as the household's portfolio at time t . This might be

$$\theta_t = (\theta_{1t}, \theta_{2t}, \dots, \theta_{nt}),$$

i.e., how many shares of each stock you have in your portfolio.

- Households take their wealth at time t and choose to eat some of it and invest the rest of it. At time t ,

$$c_t + p_{1t}\theta_{1t} + p_{2t}\theta_{2t} + \dots + p_{nt}\theta_{nt}.$$

This represents how much the agent consumes and distributes investment across different assets at time t . Note that

$$c_t + p_{1t}\theta_{1t} + p_{2t}\theta_{2t} + \dots + p_{nt}\theta_{nt} \leq (p_{1t} + d_{1t})\theta_{1,t-1} + \dots + (p_{nt} + d_{nt})\theta_{n,t-1}.$$

In vector notation,

$$\underbrace{c_t}_{\text{not a vector}} + \mathbf{p}'_t \theta_t \leq (\mathbf{p}'_t + \mathbf{d}'_t) \theta_{t-1}.$$

Rewriting, we have

$$\begin{aligned} c_t &= \mathbf{p}'_t \theta_{t-1} + \mathbf{d}'_t \theta_{t-1} + \mathbf{p}'_t \theta_t \\ &= \mathbf{p}'_t (\theta_{t-1} - \theta_t) + \mathbf{d}'_t \theta_{t-1}. \end{aligned}$$

So our problem is

$$\max_{\theta} \left\{ \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u \left(\underbrace{\mathbf{p}'_t (\theta_{t-1} - \theta_t) + \mathbf{d}'_t \theta_{t-1}}_{\substack{\uparrow \\ \text{change in portfolio wealth}}} \right) \right] \right\}.$$

- So what is this individual going to do? Take the FOCs of the above with respect to θ_t :

$$\mathbb{E} \left[\beta^t \left(-u' \left(\underbrace{c_t}_{\substack{\uparrow \\ \mathbf{p}'_t \theta_{t-1} + \mathbf{d}'_t \theta_{t-1} + \mathbf{p}'_t \theta_t}} \right) \mathbf{p}_t + \beta u' (c_{t+1}) (\mathbf{p}_{t+1} + \mathbf{d}_{t+1}) \right) \right] = \mathbf{0}^T.$$

For row i :

$$\mathbb{E} \left[\beta^t (-u' (c_t) p_{it} + \beta u' (c_{t+1}) (p_{i,t+1} + d_{i,t+1})) \right] = 0.$$

Divide the LHS and the RHS by p_{it} :

$$\mathbb{E} \left[\beta^t \left(-u' (c_t) + \beta u' (c_{t+1}) \underbrace{\frac{p_{i,t+1} + d_{i,t+1}}{p_{it}}}_{R_{it}} \right) \right] = 0.$$

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Stochastic Discount Factor.

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$$\begin{aligned}
 E[-\beta^t u'(c_t) \mathbf{p}_t + \beta^{t+1} u'(c_{t+1}) (\mathbf{p}_{t+1} + \mathbf{d}_{t+1})] &= 0 \\
 E[-u'(c_t) \mathbf{p}_t + \beta u'(c_{t+1}) (\mathbf{p}_{t+1} + \mathbf{d}_{t+1})] &= 0 \\
 \implies -u'(c_t) + \beta E[u'(c_{t+1}) R_{it} | \Omega_t] &= 0 \\
 \implies u'(c_t) &= \beta E[u'(c_{t+1}) R_{it} | \Omega_t].
 \end{aligned}$$

$$1 = E \left[\underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{stochastic discount factor}} R_{it} \middle| \Omega_t \right].$$

Note that for small agents, R_{it} is exogenous to choices. Also,

$$\begin{aligned}
 1 &= E[m_t R_{it} | \Omega_t] \\
 1 &= \text{Cov}[m_t, R_{it} | \Omega_t] + E[m_t | \Omega_t] E[R_{it} | \Omega_t].
 \end{aligned}$$

For now, write $\text{Cov}[m_t, R_{it} | \Omega_t] =: \text{Cov}[m_t, R_{it}]$ for brevity. We have

$$\underbrace{\text{Cov}[m_t, R_{it}] + E[m_t] E[R_{it}]}_{\text{risk-return tradeoff}} = \text{Cov}[m_t, R_{jt}] + E[m_t] E[R_{jt}]$$

(for $i \neq j$ in general).

- Recall

$$m_t = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

and

$$R_{it} = \frac{\mathbf{p}_{i,t+1} + \mathbf{d}_{i,t+1}}{\mathbf{p}_{it}}.$$

① **Note.** Small investors do not have control over R ; they take it as given, and $E[R]$ is a belief based on information that the investor has. Portfolio choices θ are encapsulated in m . As an investor, we are manipulating the $\text{Cov}[m, R]$ term. In equilibrium, we all do that.

- consider the FOC

$$-u'(c_t) + \beta E \left[u'(c_{t+1}) \left(\frac{1}{\mathbf{p}_{it}} \right) \right] = 0.$$

Rearranging, we have

$$\mathbf{p}_{it} = \beta E \left[\frac{u'(c_{t+1})}{u'(c_t)} \right].$$

This is the **expected discount factor**.

- **Gross return:**

$$\begin{aligned}
 R_{it} &= \underbrace{1 + r_{it}}_{(*)} \\
 &= \frac{1}{p_{it}} \\
 &= \frac{1}{\beta E \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]}.
 \end{aligned}$$

This is the implied rate of return by this model. This can be used to estimate the parameters of the SDF through GMM. This is a **moment condition**. Note that GMM was developed by Lars Hansen specifically to estimate these particular moment conditions. Also note that $(*)$ is something that we can easily observe. We can use all the expected marginal utilities information to estimate what people's β is.

- We would assume that

$$u(c_{t+1}) = \frac{c_{t+1}^{1-\gamma}}{1-\gamma}.$$

- If we take the FOC

$$\begin{aligned}
 g(\gamma, \beta, R) &= \beta E \left[\frac{u'(c_{t+1})}{u'(c_t)} R_{it} \right] - 1 \\
 &= 0
 \end{aligned}$$

as a *moment condition*. We will do this on the homework. Every period, calculate

$$E \left[\frac{u'(c_{t+1})}{u'(c_t)} R_{it} - 1 \right].$$

Figure out what the average is, adjust the γ , and figure out what γ would make this condition equal to zero. (This is Questions 1 & 2.)

Kyle (1985) Model

- Agents:
 - **Market Makers:** The ones submitting limit orders. We want to understand this party. To these guys, V is a random variable, but they know the distribution of V . They don't know which traders are informed or uninformed. They *observe* $Y = X + U$ (the *sum* of informed and uninformed demand).
 - **Informed Traders:** The ones who actually *know* what V is exactly. They also know the distribution of V that market makers know. Their optimal demand is $X(V)$.
 - **Noise Traders** or **Uninformed/Liquidity Traders:** Individuals that are trading in ways that are uncorrelated with the future value of the asset. *There is no information in this trade.* No "market timing" effects involved. They know nothing. They could be bad traders, or they could just be laypeople selling stock in order to gain idiosyncratic personal financial liquidity. We assume $U \sim \mathcal{N}(0, \sigma_u^2)$. **Assume that U, V are uncorrelated.**
- Let future value of asset is V . By "future" for this model, we refer to a small time horizon, i.e., less than a week or so. Assume $V \sim \mathcal{N}(p_0, \Sigma_0)$. The market makers know this distribution.

Price Function.

△ **Def.** In **equilibrium**, a market maker sets a **price function** $P(Y)$, where $Y = X + U$. The informed traders also have a **demand** such that $E \left[(V - P(Y)) \underbrace{(-Y)}_{\text{position}} \middle| Y \right] = 0$ (competitive risk-neutral market makers), and where $X(U)$ maximizes $E[(V - P(Y))X(U) | V]$. *In general, this is a very hard system of equations to solve.*

- To solve for the equilibrium, we use *guess and check*. Assume that the equilibrium is $P(Y) = \mu + \lambda Y$. Assume linear. *Note that this is the place where you would expect to see linearity*—risk-neutral and linear-preference assumptions. Take this as given, and move directly to

$$E[(V - P(Y))X(U)|V].$$

They want to

$$\max_X E[(V - \mu - \lambda(X + U))X|V].$$

Note that for informed traders, U is the only thing that is unknown. Now some simplification:

$$\begin{aligned} &= E[VX - \mu X - \lambda X^2 - \lambda UX|V] \\ &= VX - \mu X - \lambda X^2 - \lambda X \underbrace{E[U|V]}_{=0} \\ &= VX - \mu X - \lambda X^2. \end{aligned}$$

Calculating the FOC and solving for optimal X ,

$$V - \mu - 2\lambda X = 0$$

so

$$\begin{aligned} X &= \frac{V}{2\lambda} - \frac{\mu}{2\lambda} \\ &= -\frac{\mu}{2\lambda} + \frac{1}{2\lambda}V. \end{aligned}$$

Note that the second derivative is negative if $\lambda > 0$. This is an important consideration.

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Review

- NT: $U \sim \mathcal{N}(0, \sigma_U^2)$, $V \sim \mathcal{N}(p_0, \Sigma_0)$, $Y = X + U$.
- MM: $E[(V - P)(-Y) | Y]$, $P(Y) = \mu + \lambda Y$
- IT:

$$\max_X E[(V - P(X + U))X | V]$$

$$X = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda}V.$$

Market Makers want to know what V is based on all the information they have. They will never know X or U to any amount of certainty.

- Market makers try to find

$$E[V | X + U = Y].$$

Note that

$$\begin{aligned} Y &= X + U \\ &= \underbrace{-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}V}_{\downarrow} + U. \\ &\mathcal{N}\left(-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}p_0, \frac{1}{4\lambda^2}\Sigma_0 + \sigma_U^2\right) \end{aligned}$$

- What is the relationship between Y and V ?
 - To solve, find the covariance:

$$\begin{aligned} \text{Cov}[Y, V] &= E[(Y - E[Y])(V - E[V])] \\ &= E\left[\left(-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}V + U - \left(-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}p_0\right)\right)(V - p_0)\right] \\ &= E\left[\left(\frac{1}{2\lambda}(V - p_0) + U\right)(V - p_0)\right] \\ &= E\left[\frac{1}{2\lambda}(V - p_0)^2 + \frac{1}{2\lambda}U(V - p_0)\right] \\ &= \frac{1}{2\lambda}E[(V - p_0)^2] + \frac{1}{2\lambda}E[U(V - p_0)] \xrightarrow{0} \\ &= \frac{1}{2\lambda}\Sigma_0. \end{aligned}$$

- The question we have in general is that, what is

$$E[V | Y]?$$

Suppose we have RVs A and B where $A, B \sim \mathcal{N}((E[A], E[B]), (\text{Cov}[A, B], \sigma_A^2, \sigma_B^2))$. Let

$$E[A | B] = \underbrace{E[A]}_{\text{initial belief}} + \underbrace{\frac{\text{Cov}[A, B]}{\sigma_B^2}(B - E[B])}_{\text{update belief}}.$$

$$E[V | Y] = p_0 + \frac{\frac{1}{2\lambda}\Sigma_0}{\left(\frac{1}{2\lambda}\right)^2\Sigma_0 + \sigma_U^2} \left(Y - \left(-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}p_0\right)\right).$$

In equilibrium, we have

$$\begin{aligned} -Y E[V - p(Y) | Y] &= 0 \\ -Y (E[V | Y] - p(Y)) &= 0 \\ \Downarrow \\ E[V | Y] &= p(Y). \end{aligned}$$

Note that

$$P(Y) = \mu + \lambda Y$$

and

$$P(Y) = p_0 + \underbrace{\frac{\frac{1}{2\lambda}\Sigma_0}{\frac{1}{4\lambda^2}\Sigma_0 + \sigma_U^2}}_{(\dagger)} \left(Y - \left(-\frac{\mu}{2\lambda} + \frac{1}{2\lambda}p_0 \right) \right).$$

Solving for λ involves solving

$$\lambda = \frac{\frac{1}{2\lambda}\Sigma_0}{\frac{1}{4\lambda^2}\Sigma_0 + \sigma_U^2}.$$

Solving for λ itself, we have

$$\begin{aligned} \lambda &= \frac{\frac{1}{2\lambda}\Sigma_0}{\frac{1}{4\lambda^2}\Sigma_0 + \sigma_U^2} \\ \left(\frac{1}{4\lambda^2}\Sigma_0 + \sigma_U^2 \right) \lambda &= \frac{1}{2\lambda}\Sigma_0 \\ \frac{1}{4\lambda}\Sigma_0 + \lambda\sigma_U^2 &= \frac{1}{2\lambda}\Sigma_0 \\ \lambda\sigma_U^2 &= -\frac{1}{2\lambda}\Sigma_0 \\ \lambda &= \sqrt{-\frac{\Sigma_0}{2\sigma_U^2}} \\ &\vdots \\ &= \boxed{\frac{\sqrt{\Sigma_0}}{2\sigma_U}}. \end{aligned}$$

Note that $\mu = p_0$ is a solution to (\dagger) . Solution:

$$P(Y) = p_0 + \frac{\sqrt{\Sigma_0}}{2\sigma_U} Y.$$

So,

$$\begin{aligned} X(U) &= -\frac{\mu}{2\lambda} + \frac{1}{2\lambda} V \\ &= \frac{-p_0}{\left(\frac{\sqrt{\Sigma_0}}{\sigma_U} \right)} + \underbrace{\frac{\sigma_U}{\sqrt{\Sigma_0}}}_{\text{ratio of two SDs}} V. \end{aligned}$$

In other words, *If there's a bunch of dumb people out there, then a smart trader doesn't have to carefully hide the aggression of his or her trading strategies / demand of trades.*

- Note that p_0 is the BBO and represents the intersection of demand and supply in the capital market's limit order book.
 - *Note:* a flat LOB means that σ_U is small compared to Σ_0 , which tells us that there is not very much noise compared to meaningful/informed trades.