Harrison-Beard-ProbSet3

July 8, 2018

```
In [982]: import numpy as np
          import scipy
          from scipy import linalg as la
          import scipy.misc
          import matplotlib.pyplot as plt
          from scipy.sparse import dok_matrix
          from numba import njit, prange
          import sympy
          import mpmath
```

1 QR Decomposition

1.1

```
In [766]: def qrd(A):
              m,n = np.shape(A)
              Q = np.copy(A)
              R = np.zeros((n,n))
              for i in range(n):
                  R[i,i] = la.norm(Q[:,i])
                  Q[:,i] = Q[:,i] / R[i,i]
                  for j in range(i+1, n):
                      R[i,j] = Q[:,j].T @ Q[:,i]
                      Q[:,j] = Q[:,j] - R[i,j] * Q[:,i]
              return Q,R
```

```
In [767]: def problem1_1():

              LB = -1e4 # lower bound for entries of A
              UB = 1e4 # upper bound for entries
              MAX_SIZE = 10 # upper bound on size of matrix

              rank_n = False
              while not rank_n:
                  m = np.random.randint(1,MAX_SIZE+1)
                  n = np.random.randint(1,MAX_SIZE+1)
                  A = (UB - LB) * np.random.rand(m,n) + LB
                  if np.linalg.matrix_rank(A) == n:
```

```
            rank_n = True
        Q1,R1 = qrd(A)
        Q2,R2 = la.qr(A, mode="economic")

        verifyR = np.allclose(np.triu(R1), R1)
        verifyQ = np.allclose(Q1.T @ Q1, np.identity(n))
        verifyQR = np.allclose(Q1 @ R1, A)

        print("  R is correct:",verifyR)
        print("  Q is correct:",verifyQ)
        print("  The QR Decomposition is correct:",verifyQR)

        print("\nMy Q:\n",Q1,"\n\nMy R:\n",R1,"\n\nScipy's Q:\n",
              Q2,"\n\nScipy's R:\n",R2)
```

```
In [768]: problem1_1()
```

```
  R is correct: True
  Q is correct: True
  The QR Decomposition is correct: True

My Q:
 [[ 0.6312998 ]
 [-0.01621734]
 [ 0.14307467]
 [-0.6794677 ]
 [ 0.34503747]]

My R:
 [[10363.22025088]]

Scipy's Q:
 [[-0.6312998 ]
 [ 0.01621734]
 [-0.14307467]
 [ 0.6794677 ]
 [-0.34503747]]

Scipy's R:
 [[-10363.22025088]]
```

1.2

```
In [771]: def absdet(A):
              return abs(np.prod(np.diag(qrd(A)[1])))
```

```
In [774]: def problem1_2():
              LB = -1 # lower bound for entries of A
```

```
    UB = 1 # upper bound for entries

    # a,b smaller to avoid roundoff error for large determinants

    MAX_SIZE = 10 # upper bound on size of A

    rank_n = False
    while not rank_n:
        n = np.random.randint(1,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(n,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    absdet1 = round(absdet(A),5)
    absdet2 = round(abs(la.det(A)),5)

    print("  The algorithm was correct:\t",(absdet1 == absdet2),
          "\n  My |det(A)|:\t\t\t",absdet1,"\n  Scipy's |det(A)|:\t\t",
           absdet2)
```

```
In [775]: problem1_2()
```

```
  The algorithm was correct:         True
  My |det(A)|:                        1.01417
  Scipy's |det(A)|:                  1.01417
```

1.3

```
In [415]: def solve_linear_system(A,b):
              Q,R = qrd(A)
              y = Q.T @ b
              n = np.shape(A)[0]
              x = np.zeros(n)

              for i in range(n-1,-1,-1):
                  x[i] = y[i]
                  for j in range(i+1,n):
                      x[i] = x[i] - R[i,j] * x[j]
                  x[i] = x[i] / R[i,i]

              return x
```

```
In [437]: def problem1_3():
              LB = -10
              UB = 10
              MAX_SIZE = 10

              rank_n = False
```

```
    while not rank_n:
        n = np.random.randint(1,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(n,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    b = (UB-LB) * np.random.rand(n) + LB

    x = solve_linear_system(A,b)

    print("x:",x)
    print("\n--> ACCURATE:",all(np.round(A @ x,5) == np.round(b,5)))
```

```
In [438]: problem1_3()
```

```
x: [-2.16276632 -0.41584287]

--> ACCURATE: True
```

1.4

```
In [776]: def householder(A):
    m,n = np.shape(A)
    R = np.copy(A)
    Q = np.eye(m)
    sign = lambda x: 1 if x>=0 else -1
    for k in range(n):
        u = np.copy(R[k:,k])
        u[0] = u[0] + sign(u[0]) * la.norm(u)
        u = u / la.norm(u)
        R[k:,k:] = R[k:,k:] - np.outer(2*u, u.T @ R[k:,k:])
        Q[k:,:] = Q[k:,:] - np.outer(2*u, u.T @ Q[k:,:])

    return Q.T, R
```

```
In [777]: def problem1_4():
    LB = -10
    UB = 10
    MAX_SIZE = 10

    rank_n = False
    while not rank_n:
        m = np.random.randint(1,MAX_SIZE+1)
        n = np.random.randint(1,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(m,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True
```

```
    QT,R = householder(A)

    print("   A shape:  ",np.shape(A),"\n",
          "  Q.T shape:",np.shape(QT),"\n",
          "  R shape:  ",np.shape(R),"\n")

    print("--> ACCURATE:",np.allclose((QT @ R),A))
```

```
In [778]: problem1_4()
```

```
   A shape:   (9, 4)
   Q.T shape: (9, 9)
   R shape:   (9, 4)

--> ACCURATE: True
```

1.5

```
In [779]: def hessenberg(A):
    m,n = np.shape(A)
    H = np.copy(A)
    Q = np.eye(m)
    sign = lambda x: 1 if x>=0 else -1
    for k in range(n-2):
        u = np.copy(H[k+1:,k])
        u[0] = u[0] + sign(u[0]) * la.norm(u)
        u = u / la.norm(u)
        H[k+1:,k:] = H[k+1:,k:] - np.outer(2*u, u.T @ H[k+1:,k:])
        H[:,k+1:] = H[:,k+1:] - 2 * np.outer(H[:,k+1:] @ u, u.T)
        Q[k+1:,:] = Q[k+1:,:] - np.outer(2*u, u.T @ Q[k+1:,:])

    return H, Q.T
```

```
In [780]: def problem1_5():
    LB = -10
    UB = 10
    MAX_SIZE = 10

    rank_n = False
    while not rank_n:
        n = np.random.randint(1,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(n,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    H1,QT1 = hessenberg(A)
    H2,QT2 = la.hessenberg(A, calc_q = True)
```

```
    print("MY FUNCTION:")
    print("  Verify H has all zeros below first subdiagonal:",
          np.allclose(np.triu(H1,-1),H1),"\n  Verify Q @ H @ Q.T = A:",
          np.allclose(QT1 @ H1 @ QT1.T, A))

    print("\nSCIPY'S FUNCTION:")
    print("  Verify H has all zeros below first subdiagonal:",
          np.allclose(np.triu(H2,-1),H2),"\n  Verify Q @ H @ Q.T = A:",
          np.allclose(QT2 @ H2 @ QT2.T, A))
```

```
In [781]: problem1_5() # mess around with QT, Q.T, and Q in problem1_5 if not getting true
```

```
MY FUNCTION:
  Verify H has all zeros below first subdiagonal: True
  Verify Q @ H @ Q.T = A: True

SCIPY'S FUNCTION:
  Verify H has all zeros below first subdiagonal: True
  Verify Q @ H @ Q.T = A: True
```

2 Least Squares and Computing Eigenvalues

2.1

```
In [579]: def solve_normal_equations(A,b):

              Q,R = qrd(A.T)

              x = la.solve_triangular(R, Q.T@b)

              return x
```

```
In [491]: def problem2_1():
              LB = -10
              UB = 10
              MAX_SIZE = 10

              rank_n = False
              while not rank_n:
                  m = np.random.randint(1,MAX_SIZE+1)
                  n = np.random.randint(1,MAX_SIZE+1)
                  A = (UB - LB) * np.random.rand(m,n) + LB
                  if np.linalg.matrix_rank(A) == n:
                      rank_n = True

              b = (UB-LB) * np.random.rand(m) + LB
```

```
    x = solve_normal_equations(A,b)

    print("x:",x)
    print("\n--> ACCURATE:",np.allclose(A @ x, b))
```

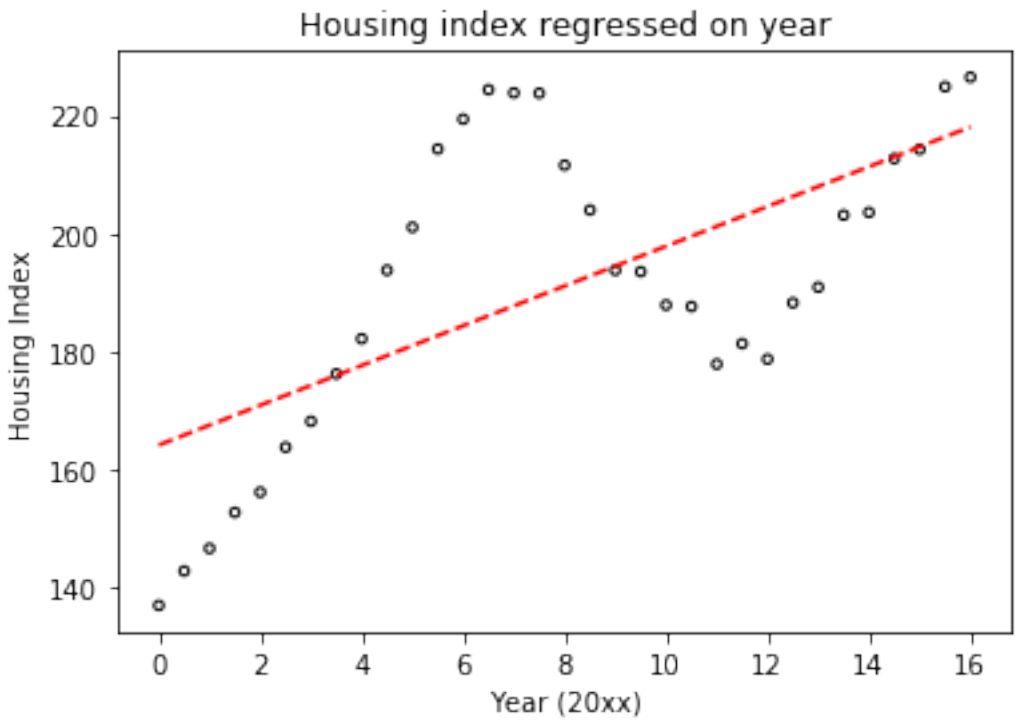
```
In [492]: problem2_1()
```

```
x: [-0.18569398 -0.38314783  0.69416316 -0.4249165   0.121046   -0.11408084
 -0.3452373 ]

--> ACCURATE: False
```

2.2

```
In [588]: def problem2_2():
    year, index = np.load("housing.npy").T
    A = year
    A = np.vstack((A, np.ones(len(year))))
    b = index

    _1, _0 = solve_normal_equations(A,b)

    plt.scatter(year, index, facecolors="none", edgecolors="k", s=12)
    plt.plot(year, year * _1 + _0, "r--")
    plt.title("Housing index regressed on year")
    plt.xlabel("Year (20xx)")
    plt.ylabel("Housing Index")

    plt.show()
```

```
In [589]: problem2_2()
```



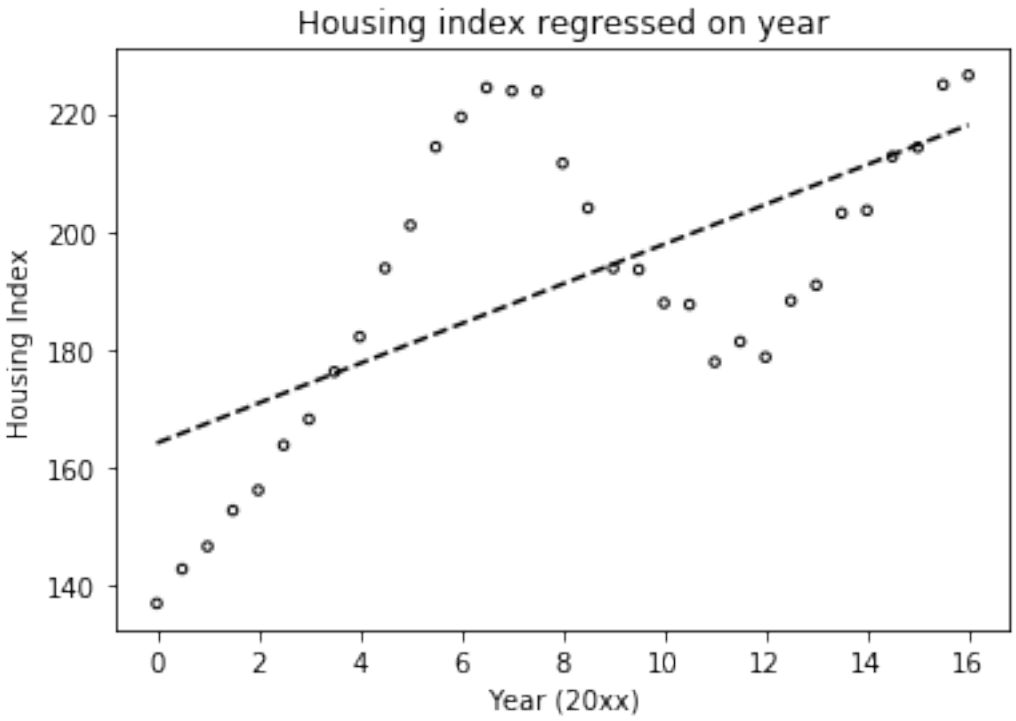
2.3

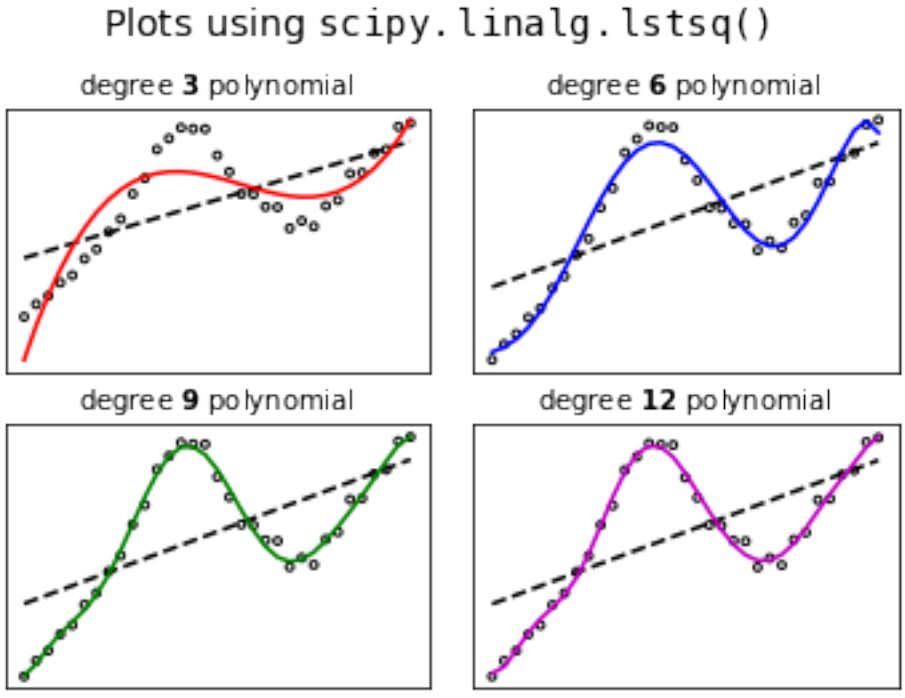
```
In [789]: def problem2_3():
              degs = [3,6,9,12]
              styles = ["r","b","g","m"]
              year, index = np.load("housing.npy").T
              A = year
              A = np.vstack((year,np.ones(len(year))))
              b = index

              _1, _0 = solve_normal_equations(A,b)


              plt.scatter(year, index, facecolors="none", edgecolors="k", s=12)
              plt.plot(year, year * _1 + _0, "k--")
              plt.title("Housing index regressed on year")
              plt.xlabel("Year (20xx)")
              plt.ylabel("Housing Index")
              plt.show()
```

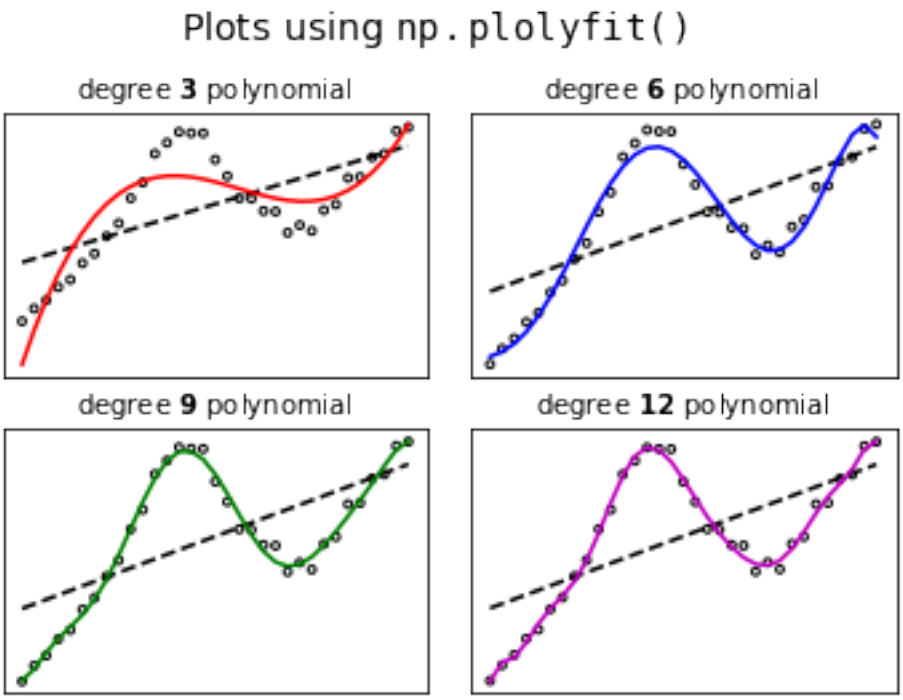
```
    for n,i in enumerate(degs):
        plt.subplot(2,2,n+1)
        plt.scatter(year, index, facecolors="none", edgecolors="k", s=8)
        plt.plot(year, year * _1 + _0, "k--")
        plt.title(r"degree $\mathbf{"+str(i)+"}$ polynomial",size=10)
        plt.xticks([])
        plt.yticks([])

        deglist = list(range(i+1))

        M = year[:, np.newaxis]**deglist
        coeffs = la.lstsq(M, b)[0]

        y = coeffs[0]
        for j in range(1,i+1):
            y += coeffs[j] * year**j
        plt.plot(year, y, styles[n])
    plt.suptitle(r"Plots using $\mathtt{scipy.linalg.lstsq()}$",size=14,y=1.02)
    plt.subplots_adjust(hspace=.2, wspace=.1)
    plt.show()
```

```
    for n,i in enumerate(degs):
        plt.subplot(2,2,n+1)
        plt.scatter(year, index, facecolors="none", edgecolors="k", s=8)
        plt.plot(year, year * _1 + _0, "k--")
        plt.title(r"degree $\mathbf{"+str(i)+"}$ polynomial",size=10)
        plt.xticks([])
        plt.yticks([])
        coeffs = np.polyfit(year, index, i)[::-1]
        y = coeffs[0]
        for j in range(1,i+1):
            y += coeffs[j] * year**j
        plt.plot(year, y, styles[n])
    plt.suptitle(r"Plots using $\mathtt{np.plolyfit()}$",size=14,y=1.02)
    plt.subplots_adjust(hspace=.2, wspace=.1)
    plt.show()
```

```
In [790]: problem2_3()
```


Housing index regressed on year
Housing Index
220
200
180
160
140
0
2
4
6
8
10
12
14
16
Year (20xx)


Plots using scipy.linalg.lstsq()
degree 3 polynomial
degree 6 polynomial
degree 9 polynomial
degree 12 polynomial

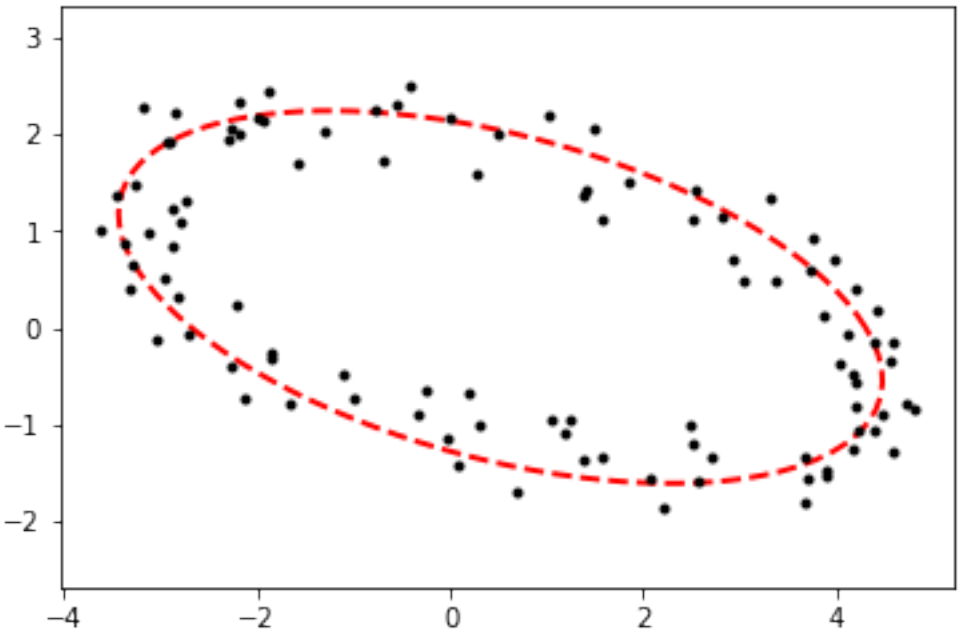


2.4

```
In [899]: def problem2_4():
              x,y = np.load("ellipse.npy").T
              A = np.column_stack((x**2, x, x*y, y, y**2))
              b = np.ones_like(x)

              a,b,c,d,e = la.lstsq(A, b)[0]

              plot_ellipse(a,b,c,d,e)

              plt.plot(x, y, 'k.')
              plt.axis("equal")
              plt.show()


In [906]: def plot_ellipse(a,b,c,d,e):
              '''
              ax^2 + bx + cxy + dy + ey^2 = 1
              '''
              theta = np.linspace(0, 2*np.pi, 200)
              cos_t, sin_t = np.cos(theta), np.sin(theta)
              A = a*(cos_t**2) + c*cos_t*sin_t + e*(sin_t**2)
              B = b*cos_t + d*sin_t
```

```
    r = (-B + np.sqrt(B**2 + 4*A)) / (2*A)
    plt.plot(r*cos_t, r*sin_t, "r--", lw=2)
    plt.gca().set_aspect("equal", "datalim")
```

In [907]:
```
problem2_4()
```



2.5

In [1020]:
```
def power_method(A, N, tol):
    m, n = A.shape
    x = np.random.random(size=n)
    x = x / la.norm(x)
    k = 0
    t = 1e8
    while (t > tol) and (k < N):
        x = A @ x
        x = x / la.norm(x)
        k += 1
    return x.T @ A @ x, x
```

In [1021]:
```
def problem2_5():
    LB = 1e-10
    UB = 10
    MAX_SIZE = 10

    rank_n = False
```

```
                while not rank_n:
                    n = np.random.randint(2,MAX_SIZE+1)
                    A = (UB - LB) * np.random.rand(n,n) + LB
                    if np.linalg.matrix_rank(A) == n:
                        rank_n = True

                evals, evecs = la.eig(A)
                loc = np.argmax(eigs)

                0, x0 = evals[loc], evecs[:,loc]

                print("       Scipy A@x == *x:",np.allclose(A @ x0, 0 * x0))

                , x = power_method(A, 100, 1e-4)

                print("power_method A@x == *x:",np.allclose(A @ x,  * x))
```

```
In [1022]: problem2_5()
```

```
       Scipy A@x == *x: True
power_method A@x == *x: True
```

2.6

```
In [1110]: def qr_algo(A, N, tol):
                n = A.shape[0]
                S = la.hessenberg(A)

                for k in range(N):
                    Q, R = qrd(S)
                    S = R @ Q
                evals = []
                i = 0
                while i < n:
                    if i == n - 1:
                        evals.append(S[i, i])
                    elif S[i + 1, i] < tol:
                        evals.append(S[i, i])
                    elif len(S[i:, i:]) == 2:
                        1 = ((S[i, i] + S[i+1, i+1]) + cmath.sqrt((S[i, i] + S[i+1, i+1]) ** 2
                                                                  - 4 * (S[i, i] * S[i+1, i+1] -
                        2 = ((S[i, i] + S[i+1, i+1]) - cmath.sqrt((S[i, i] + S[i+1, i+1]) ** 2
                                                                  - 4 * (S[i, i] * S[i+1, i+1] -
                        evals.append(1, 2)
                        i += 1
                    i += 1

                return evals
```

```
In [1111]: def problem2_6():
               LB = 1e-10
               UB = 10
               MAX_SIZE = 10

               rank_n = False
               while not rank_n:
                   n = np.random.randint(2,MAX_SIZE+1)
                   A = (UB - LB) * np.random.rand(n,n) + LB
                   if np.linalg.matrix_rank(A) == n:
                       rank_n = True

               print("Scipy's e-vals:")
               print(la.eig(A + A.T)[:-1])


               print("\nMy e-vals:")
               print(qr_algo(A + A.T, 1000, 1e-8))

In [1112]: problem2_6()

Scipy's e-vals:
(array([59.94613458+0.j, 18.82900741+0.j, -8.60436854+0.j, -5.43809062+0.j,
        1.29177465+0.j,  8.41848897+0.j]),)

My e-vals:
[59.94613458346843, 18.829007410440433, -8.60436853780165, 8.418488966739073, -5.438090622342103
```

The eigenvalues are the same!

3 SVD and Image Compression

3.1

```
In [782]: def compact_svd(A, tol=1e-4):
              AH = A.conj().T
              , V = la.eig(AH @ A)
               = np.sqrt()

              V = V[:,np.argsort()[::-1]]
               = np.sort()[::-1]


              r = 0
              for i in :
                  if i>tol:
                      r+=1
```

```
                _1 = [:r]
                V_1 = V[:,:r]
                U_1 = A @ V_1 / _1

                VH_1 = V_1.conj().T

                return U_1, _1, VH_1
```

In [783]:
```
def problem3_1():

    LB = -10
    UB = 10
    MAX_SIZE = 10

    rank_n = False
    while not rank_n:
        m = np.random.randint(2,MAX_SIZE+1)
        n = np.random.randint(2,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(m,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    U0, 0, VH0 = la.svd(A, full_matrices=False)

    print("--- BUILT-IN SVD ---\n")

    print(
        "   U.shape, .shape, VH.shape:\n   ",
        U0.shape, 0.shape, VH0.shape,"\n")

    print(

        "   Verify U and VH are orthonormal:",
        np.allclose(U0.T @ U0, np.identity(n)),
        np.allclose(VH0.T @ VH0,np.identity(n)),"\n\n",

        "  Verify U @  @ VH = A:",
        np.allclose(U0 @ np.diag(0) @ VH0, A),"\n\n",

        "  Verify the rank is correct:",
        np.linalg.matrix_rank(A) == len(0))


    print("\n---  COMPACT SVD  ---\n")


    U, , VH = compact_svd(A, tol = 1e-15)
```

```
            print(
                "   U.shape, .shape, VH.shape:\n   ",
                U.shape, .shape, VH.shape,"\n")

            print(

                "   Verify U and VH are orthonormal:",
                np.allclose(U.T @ U, np.identity(n)),
                np.allclose(VH.T @ VH,np.identity(n)),"\n")


            print(
                "   Verify U @  @ VH = A:",
                np.allclose(U @ np.diag() @ VH, A),"\n\n",

                "  Verify the rank is correct:",
                np.linalg.matrix_rank(A) == len())
```

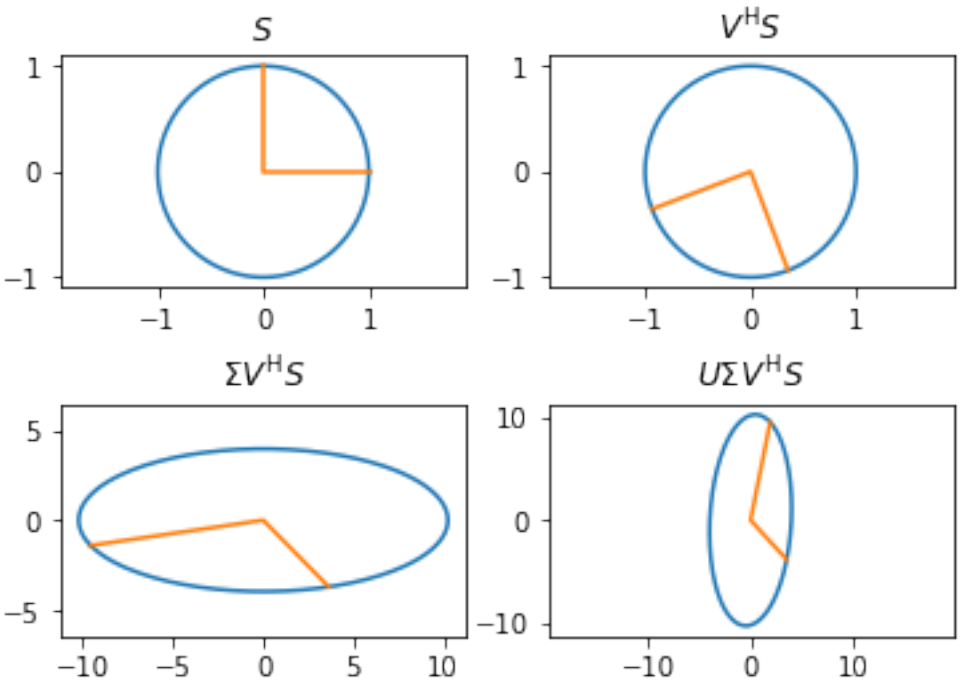
```
In [784]: problem3_1()
```

```
--- BUILT-IN SVD ---

   U.shape, .shape, VH.shape:
    (10, 4) (4,) (4, 4)

   Verify U and VH are orthonormal: True True

   Verify U @  @ VH = A: True

   Verify the rank is correct: True

---  COMPACT SVD  ---

   U.shape, .shape, VH.shape:
    (10, 4) (4,) (4, 4)

   Verify U and VH are orthonormal: True True

   Verify U @  @ VH = A: True

   Verify the rank is correct: True
```

3.2

```
In [306]: def A_to_S(A):
                = np.linspace(0, 2 * np.pi, 200)
```

```
            S = np.vstack([np.cos(), np.sin()])
            return S

In [325]: def problem3_2():
            LB = -10
            UB = 10
            A = (UB - LB) * np.random.rand(2,2) + LB

            S = A_to_S(A)

            E = np.array([[1, 0, 0],
                          [0, 0, 1]])

            U, , V = la.svd(A)
             = np.diag()

            plt.subplot(221)
            plt.plot(S[0], S[1])
            plt.plot(E[0], E[1])
            plt.title(r"$S$")
            plt.axis("equal")

            plt.subplot(222)
            plt.plot((V @ S)[0], (V @ S)[1])
            plt.plot((V @ E)[0], (V @ E)[1])
            plt.title(r"$V^{\mathsf{H}}S$")
            plt.axis("equal")

            plt.subplot(223)
            plt.plot(( @ V @ S)[0], ( @ V @ S)[1])
            plt.plot(( @ V @ E)[0], ( @ V @ E)[1])
            plt.title(r"$\Sigma V^{\mathsf{H}}S$")
            plt.axis("equal")

            plt.subplot(224)
            plt.plot((U @  @ V @ S)[0], (U @  @ V @ S)[1])
            plt.plot((U @  @ V @ E)[0], (U @  @ V @ E)[1])
            plt.title(r"$U \Sigma V^{\mathsf{H}}S$")
            plt.axis("equal")


            plt.subplots_adjust(hspace=.5, wspace=.2)
            plt.show()

In [328]: problem3_2()
```



3.3

```
In [340]: def compact_svd_2(A, s):

              m,n = np.shape(A)

              if s>np.linalg.matrix_rank(A):
                  raise ValueError("s is greater than the number of nonzero singular values of A

              vals = m*s + n*s + s

              U,,V = la.svd(A)
              U_s = U[:,:s]
              _s = [:s]
              V_s = V[:s,:]

              return U_s @ np.diag(_s) @ V_s, vals

In [345]: def problem3_3():
              LB = -10
              UB = 10
              MAX_SIZE = 10

              rank_n = False
```

```
            while not rank_n:
                m = np.random.randint(2,MAX_SIZE+1)
                n = np.random.randint(2,MAX_SIZE+1)
                A = (UB - LB) * np.random.rand(m,n) + LB
                if np.linalg.matrix_rank(A) == n:
                    rank_n = True

            s = np.random.randint(min(m,n),max(m,n)+1)

            a,b = compact_svd_2(A,s)

            print("U_s @ diag(_s) @ V_s:\n\n",a,"\n\nvals:",b)
```

In [353]:
```
problem3_3()
```

```
U_s @ diag(_s) @ V_s:

 [[ 7.72364074  8.13857641 -9.77569061  9.12699429]
 [ 6.92178331 -3.97026793 -9.19867206  5.49575643]
 [-0.81925435  1.52811771 -2.84979319 -3.78509506]
 [-4.65441984  3.70018924 -1.71173774  8.55567942]]

vals: 36
```

3.4

In [354]:
```
def compact_svd_3(A, =0.1):
            U,,V = la.svd(A)

            if  < min():
                raise ValueError(" is less than or equal to the smallest singular value of A")

            s = len([>])

            U_s = U[:,:s]
            _s = [:s]
            V_s = V[:s,:]

            return U_s @ np.diag(_s) @ V_s
```

In [389]:
```
def problem3_4():

             = 0.4

            LB = -10
            UB = 10
            MAX_SIZE = 10
```

```
    rank_n = False
    while not rank_n:
        m = np.random.randint(2,MAX_SIZE+1)
        n = np.random.randint(2,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(m,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    print("A:\n",A,"\n\n")
    print("proper singular vals of A:\n",la.svd(A)[1],"\n\n")
    print("my singular vals of A:\n",compact_svd_3(A, =),"\n\n")
    print("normed differece:\n",la.norm(A - compact_svd_3(A,=)))
```

```
In [395]: problem3_4()
```

```
A:
 [[-8.14008872  8.8974224  -0.20543243  9.01534961 -7.55091274  9.204904  ]
 [ 1.74478595  4.77436014  6.87386709 -0.82403186  1.09794414  6.61596462]
 [ 2.03170595 -6.78766454  8.39982062 -8.90647696  3.87701378  8.66322419]
 [-4.28915763 -4.94207401 -5.93352352 -5.24978438 -7.58635383  1.97710859]
 [ 1.16450135  7.80141351 -4.44290169  8.03013323 -9.44867885  8.05407065]
 [-9.53613868  1.68859368  1.98162728  2.04886406  6.98879612  8.2807711 ]
 [-7.3645293  -6.74281246 -6.12340625 -6.3796139  -6.88362993  7.96240487]]


proper singular vals of A:
 [26.7263637  21.83089036 19.63344719 12.35442399  4.31631604  2.32914168]


my singular vals of A:
 [[0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]]


normed differece:
 41.86938996792117
```

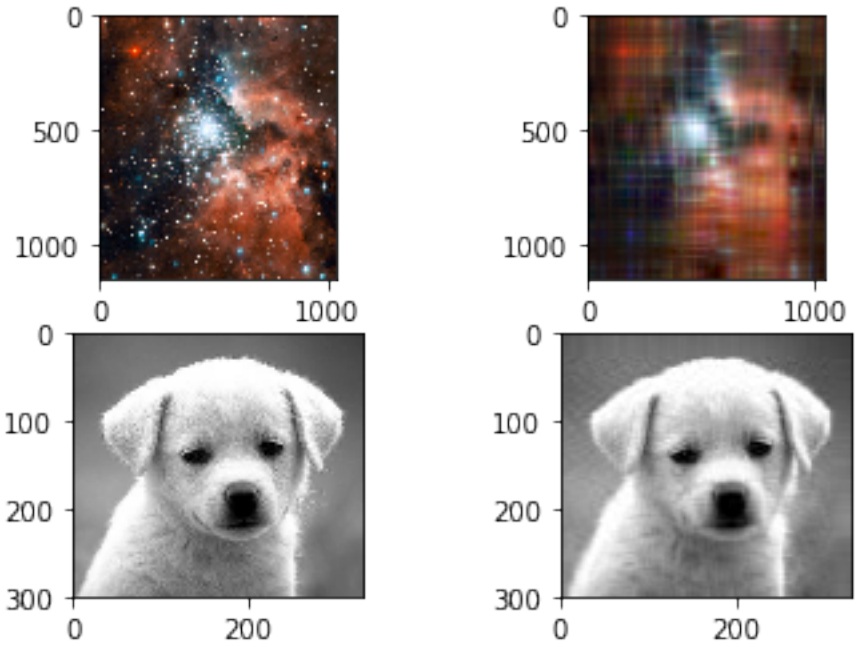
3.5

```
In [412]: def compress(file, s):

              im = plt.imread(file)/255
```

```
            if im.ndim == 3:
                C=[]
                V=0
                for i in range(3):
                    col = im[:,:,i]
                    col_s,v = compact_svd(col, s)
                    V+=v
                    col_s[col_s<0] = 0
                    col_s[col_s>1] = 1
                    C.append(col_s)
                return np.dstack(C),V

            elif im.ndim == 2:
                im_s,v = compact_svd(im, s)
                im_s[im_s<0] = 0
                im_s[im_s>1] = 1
                return im_s, v
```

In [419]:
```
def problem3_5():

    im1 = plt.imread("hubble.jpg")/255
    im1_c, v1_c = compress("hubble.jpg",5)

    plt.subplot(221).imshow(im1)
    plt.subplot(222).imshow(im1_c)

    im2 = plt.imread("puppy.jpg")/255
    im2_c, v2_c = compress("puppy.jpg",25)

    plt.subplot(223).imshow(im2, cmap="gray")
    plt.subplot(224).imshow(im2_c, cmap="gray")
    plt.show()
```

In [420]:
```
problem3_5()
```



4 The Drazin Inverse

4.1

```
In [785]: def check_drazin(A, k, A_D):
              return (
              np.allclose(A @ A_D, A_D @ A) and
              np.allclose(np.linalg.matrix_power(A, k+1)
                  @ A_D, np.linalg.matrix_power(A, k)) and
              np.allclose(A_D @ A @ A_D, A_D)
              )

In [786]: def problem4_1():

              A = np.array([[1, 3, 0, 0],
                            [0, 1, 3, 0],
                            [0, 0, 1, 3],
                            [0, 0, 0, 0]])
              k_A = 1
              A_D = np.array([[1, -3, 9, 81],
                              [0, 1, -3, -18],
                              [0, 0, 1, 3],
                              [0, 0, 0, 0]])
              print("  A^D is the Drazin inverse of A for index "+
                    str(k_A)+":",check_drazin(A, k_A, A_D),"\n")
```

```
            B = np.array([[1, 1, 3],
                          [5, 2, 6],
                          [-2, -1, -3]])
            k_B = 3
            B_D = np.zeros((3, 3))
            print("  B^D is the Drazin inverse of B for index "+
                  str(k_B)+":",check_drazin(B, k_B, B_D))
```

```
In [787]: problem4_1()
```

```
  A^D is the Drazin inverse of A for index 1: True

  B^D is the Drazin inverse of B for index 3: True
```

4.2

```
In [788]: def compute_drazin(A, k, tol=1e-3):

            n=np.shape(A)[0]

            gter = lambda x: (
                abs(x) > tol) # "greater than" function

            leq = lambda x: (
                abs(x) <= tol) # "less than or equal to" function

            Q_1,S,k_1 = la.schur(A, sort=gter) # using the lambda function
            Q_2,T,k_2 = la.schur(A, sort=leq) # using the lambda function

            U = np.hstack([S[:,:k_1], T[:,:(n-k_1)]])
            U_inv = np.linalg.inv(U)
            V = U_inv @ A @ U
            Z = np.zeros((n,n))

            if k!=0:
                M_inv = np.linalg.inv(V[:k_1,:k_1])
                Z[:k_1,:k_1] = M_inv

            return U @ Z @ U_inv
```

```
In [789]: def problem4_2():

            LB = -10
            UB = 10
            MAX_SIZE = 10
```

```
    rank_n = False
    while not rank_n:
        n = np.random.randint(2,MAX_SIZE+1)
        A = (UB - LB) * np.random.rand(n,n) + LB
        if np.linalg.matrix_rank(A) == n:
            rank_n = True

    k_A = 5
    A_D = compute_drazin(A, k_A, tol=1e-3)

    print("  A^D @ A:\n",np.round(np.abs( A_D @ A )),"\n")

    print("  A^D is the Drazin inverse of A for index "+
          str(k_A)+":",check_drazin(A, k_A, A_D))
```

In [790]:
```
problem4_2()
```

```
  A^D @ A:
 [[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]

  A^D is the Drazin inverse of A for index 5: True
```

4.3

In [791]:
```
def resistance(A):

    n = A.shape[0]

    D = np.diag(A.sum(axis=1))

    L = D-A

    R = np.zeros((n, n))

    for i in range(n):
        for j in range(n):
            L_tilde = L.copy()
            L_tilde[j,:] = np.eye(n)[j]
            L_tilde = compute_drazin(L_tilde, k=5, tol=1e-6)
            if i!=j: R[i,j] = L_tilde[i,i]

    return R
```

In [792]:
```
def problem4_3():
```

```
    A_1 = np.array([[0, 1, 0, 0],
                    [1, 0, 1, 0],
                    [0, 1, 0, 1],
                    [0, 0, 1, 0]])

    A_2 = np.array([[0, 1],
                    [1, 0]])

    print(
        "  Resistence matrix for A_1 =\n",A_1,":\n\n",
        np.round(resistance(A_1)),"\n\n",
        "  Resistence matrix for A_2 =\n",A_2,":\n\n",
        np.round(resistance(A_2))
    )
```

```
In [793]: problem4_3()
```

```
  Resistence matrix for A_1 =
 [[0 1 0 0]
 [1 0 1 0]
 [0 1 0 1]
 [0 0 1 0]] :

 [[0. 1. 2. 3.]
 [1. 0. 1. 2.]
 [2. 1. 0. 1.]
 [3. 2. 1. 0.]]

   Resistence matrix for A_2 =
 [[0 1]
 [1 0]] :

 [[0. 1.]
 [1. 0.]]
```

4.4

```
In [794]: class LinkPredictor:

    def __init__(self, file):

        self.file = file

        nodes = []
        with open("social_network.csv","r") as f:
            for i in f:
                nodes.append(i.strip().split(","))
```

```
            nodes = np.array(nodes)
            names = np.unique(nodes)

            names_d = dict(zip(names, range(len(names))))
            nodes_d = np.array( [ [ names_d[i], names_d[j]]
                                        for i,j in nodes[:,]     ] )
            A = np.zeros((len(names),len(names)))

            for e in nodes_d:
                i,j = e
                A[i,j] = 1
                A[j,i] = 1

            self.A, self.names, self.R = A, names, resistance(A)

        def predict_link(self, node=None):

            R,A,names = self.R,self.A,self.names
            R[A==1] = 0

            if node == None:
                minarg = np.argwhere(R == np.min(R[R>0])).flatten()
                return names[minarg[0]],names[minarg[1]]

            elif node.isalpha():
                if node not in names: raise ValueError("Not in the network")

                R_b = R[:,np.argwhere(names == node)]
                minarg = np.argwhere(R_b == np.min(R_b[R_b>0])).flatten()
                return node,names[minarg[0]]

        def add_link(self, node_1, node_2):

            names = self.names

            if ((node_1 not in names) or
                (node_2 not in names)):
                raise ValueError("Not in the network")

            self.A[np.argwhere(names==node_1),np.argwhere(names==node_2)],\
            self.A[np.argwhere(names==node_2),np.argwhere(names==node_1)],\
            self.R = 1,1,resistance(self.A)

In [795]: def problem4_4():

        net = LinkPredictor("social_network.csv")

        print("  Should be Oliver, Emily:",
```

```
                  str(net.predict_link()[0])+",",
                  net.predict_link()[1],"\n")

            print("  Should be Carol:",
                net.predict_link("Melanie")[1],"\n")

            print("  Should be Sonia:",
                net.predict_link("Alan")[1],"\n")

            net.add_link("Sonia", "Alan")
            print("  Should be Piers:",
                net.predict_link("Alan")[1],"\n")

            net.add_link("Piers", "Alan")
            print("  Should be Abigail:",
                net.predict_link("Alan")[1])
```

```
In [796]: problem4_4()

  Should be Oliver, Emily: Oliver, Emily

  Should be Carol: Carol

  Should be Sonia: Sonia

  Should be Piers: Piers

  Should be Abigail: Abigail
```

4.5

See Problem 4 above for the implemented methods and tests.

5 The PageRank Algorithm

5.1

```
In [582]: def adj(file, N):
            A = np.zeros((N, N))

            nodes = []
            with open(file,"r") as f:
                for v in f:
                    try: nodes.append(list(map(int,v.strip().split())))
                    except: pass
            for i in range(N):
                for j in range(N):
```

```
                    if [i,j] in nodes:
                        A[i,j]=1
            return dok_matrix(A).toarray()
```

```
In [583]: def problem5_1():
            print(adj("matrix.txt",8))
```

```
In [584]: problem5_1()
```

```
[[0. 0. 0. 0. 0. 0. 0. 1.]
 [1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0.]
 [1. 0. 1. 0. 0. 0. 1. 0.]
 [1. 0. 0. 0. 0. 1. 1. 0.]
 [1. 0. 0. 0. 0. 0. 1. 0.]
 [1. 0. 0. 0. 0. 0. 0. 0.]
 [1. 0. 0. 0. 0. 0. 0. 0.]]
```

5.2

```
In [602]: def K(A,N):
            A[A.sum(axis=1)==0,:] = np.ones(N)
            return A.T / A.sum(axis=1)
```

```
In [603]: def problem5_2():
            print(K(adj("matrix.txt",8),8))
```

```
In [604]: problem5_2()
```

```
[[0.         1.         0.125      0.33333333 0.33333333 0.5
  1.         1.        ]
 [0.         0.         0.125      0.         0.         0.
  0.         0.        ]
 [0.         0.         0.125      0.33333333 0.         0.
  0.         0.        ]
 [0.         0.         0.125      0.         0.         0.
  0.         0.        ]
 [0.         0.         0.125      0.         0.         0.
  0.         0.        ]
 [0.         0.         0.125      0.         0.33333333 0.
  0.         0.        ]
 [0.         0.         0.125      0.33333333 0.33333333 0.5
  0.         0.        ]
 [1.         0.         0.125      0.         0.         0.
  0.         0.        ]]
```

5.3

```
In [700]: def page_rank_ss(A, N=None, d=0.85, tol=1e-5):

              if N == None: N = A.shape[0]

              A = A[:N+1,:N+1]

              pr_ss = np.ones(N)/np.ones(N).sum()
              norm=100
              while norm>tol:
                  pr_ss_pr = d * K(A,N)@pr_ss + ((1-d)/N) * np.ones(N)
                  norm = la.norm(pr_ss - pr_ss_pr)
                  pr_ss = pr_ss_pr

              return pr_ss

In [701]: def problem5_3():
              print(page_rank_ss(adj("matrix.txt",8)))

In [702]: problem5_3()

[0.43868966 0.02171029 0.02786154 0.02171029 0.02171029 0.02786154
 0.04585394 0.39460246]
```

5.4

```
In [637]: def page_rank_ss_2(A, N=None, d=0.85, tol=1e-5):

              if N == None: N = A.shape[0]

              A = A[:N+1,:N+1]

              evals,evecs = la.eig(d * K(A,N) + ((1-d)/N) * np.ones((N,N)))

              return evecs[:,evals.argmax()]

In [638]: def problem5_4():
              print(page_rank_ss_2(adj("matrix.txt",8)))

In [639]: problem5_4()

[-0.73812911 -0.03652896 -0.04687883 -0.03652896 -0.03652896 -0.04687883
 -0.07715221 -0.66393871]
```

5.5

```
In [729]: def ncaa():

              win_loss = []
              with open("ncaa2013.csv","r") as f:
                  f.readline()
                  for e in f: win_loss.append(e.strip().split(","))
              win_loss = np.array(win_loss)


              teams = np.unique(win_loss.flatten())

              teams_d = dict(zip(teams,range(len(teams))))


              wins_losses = np.array([[teams_d[win], teams_d[loss]] for win, loss in win_loss[:,

              return (teams, wins_losses)

In [734]: def problem5_5():

              mx = np.zeros((len(ncaa()[0]),len(ncaa()[0])))

              for i in ncaa()[1]:
                  mx[i[0], i[1]] = 1

              print("I could not get this to work :( \n",
                  ncaa()[0][page_rank_ss(mx,d=.3).argsort()[-5:][::-1]])

In [735]: problem5_5()

I could not get this to work :(
 ['Coppin St' 'Longwood' 'Grambling' 'MD E Shore' 'Maine']
```

6 Conditioning and Stability

6.1

```
In [757]: def condition_number(A):
              return (np.inf if min(la.svdvals(A))==0 else max(la.svdvals(A))/min(la.svdvals(A))

In [762]: def problem6_1():

              LB = -10
              UB = 10
              MAX_SIZE = 10
```

```
                rank_n = False
                while not rank_n:
                    n = np.random.randint(2,MAX_SIZE+1)
                    A = (UB - LB) * np.random.rand(n,n) + LB
                    if np.linalg.matrix_rank(A) == n:
                        rank_n = True

                print("  Condition number:",condition_number(A),"\n")
                print("  According to Numpy:",np.linalg.cond(A),"\n")
                print("  For an identity matrix:",condition_number(np.eye(n)))
```

```
In [763]: problem6_1()
```

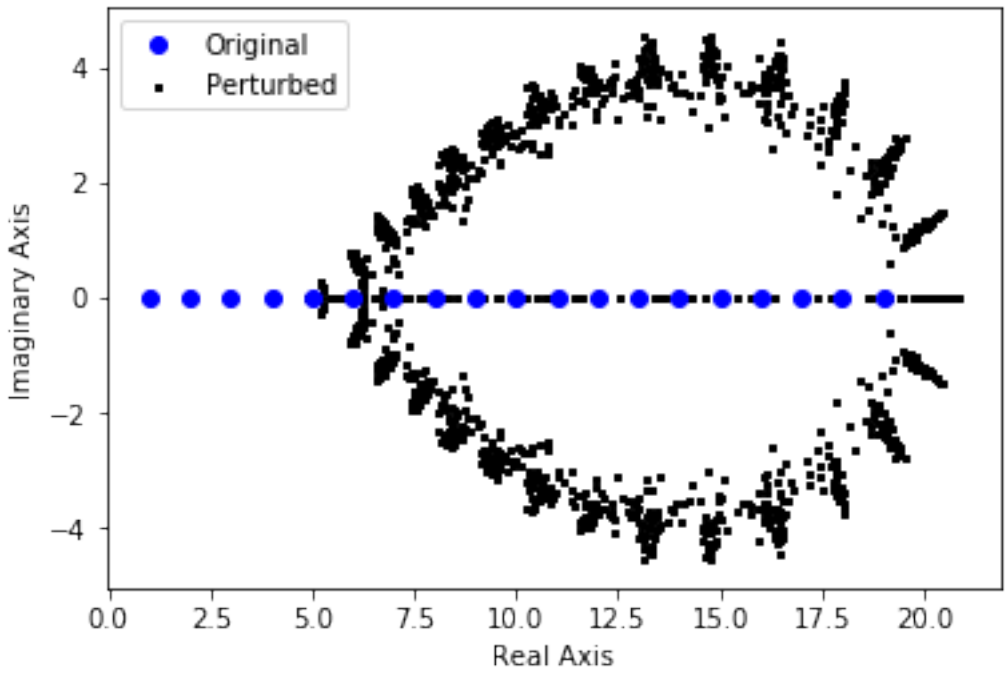
```
  Condition number: 63.53482183755394

  According to Numpy: 63.53482183755394

  For an identity matrix: 1.0
```

6.2

```
In [896]: def perturb(N=20):

                roots = np.arange(1, N)

                x,i = sympy.symbols("x"),sympy.symbols("i")
                coeffs = np.array(sympy.poly_from_expr(sympy.product(x-i, (i,1,N-1)))[0].all_coeff

                for i in range(100):
                    r = np.random.normal(loc=1,scale=1e-10,size=N)

                    coeffs_pr = coeffs*r
                    roots_pr = np.roots(np.poly1d(coeffs_pr))
                    roots,roots_pr = np.sort(roots),np.sort(roots_pr)
                    plt.scatter(roots_pr.real,roots_pr.imag,marker=",",color="k",s=2,label="Pertur

                plt.plot(roots,np.zeros(N-1),"bo",label="Original")

                plt.xlabel("Real Axis")
                plt.ylabel("Imaginary Axis")
                plt.legend(["Original", "Perturbed"])
                plt.show()

                print("  Absolute:",la.norm(roots_pr-roots,np.inf)/la.norm(r))
                print("  Relative:",la.norm(roots, np.inf)/la.norm(roots_pr,np.inf))
```

```
In [897]: def problem6_2():
                perturb()
```

```
In [898]: problem6_2()
```




```
Absolute: 0.8999654602202808
Relative: 0.951197269595549
```

6.3

```
In [915]: def condition_number_2(A):
              re,im = np.random.normal(0, 1e-10, A.shape),np.random.normal(0, 1e-10, A.shape)

              evals,evals_tilde = la.eigvals(A), la.eigvals(A + re + 1j*im)

              return (la.norm(evals-evals_tilde)/la.norm(re + 1j*im),
                      (la.norm(A)/la.norm(evals)) * (la.norm(evals-evals_tilde)/la.norm(re + 1j*

In [916]: def problem6_3():

              LB = -10
              UB = 10
              MAX_SIZE = 10

              rank_n = False
              while not rank_n:
                  n = np.random.randint(2,MAX_SIZE+1)
```

```
            A = (UB - LB) * np.random.rand(n,n) + LB
            if np.linalg.matrix_rank(A) == n:
                rank_n = True

        print("Absolute, Relative:",condition_number_2(A))
```

```
In [917]: problem6_3()
```

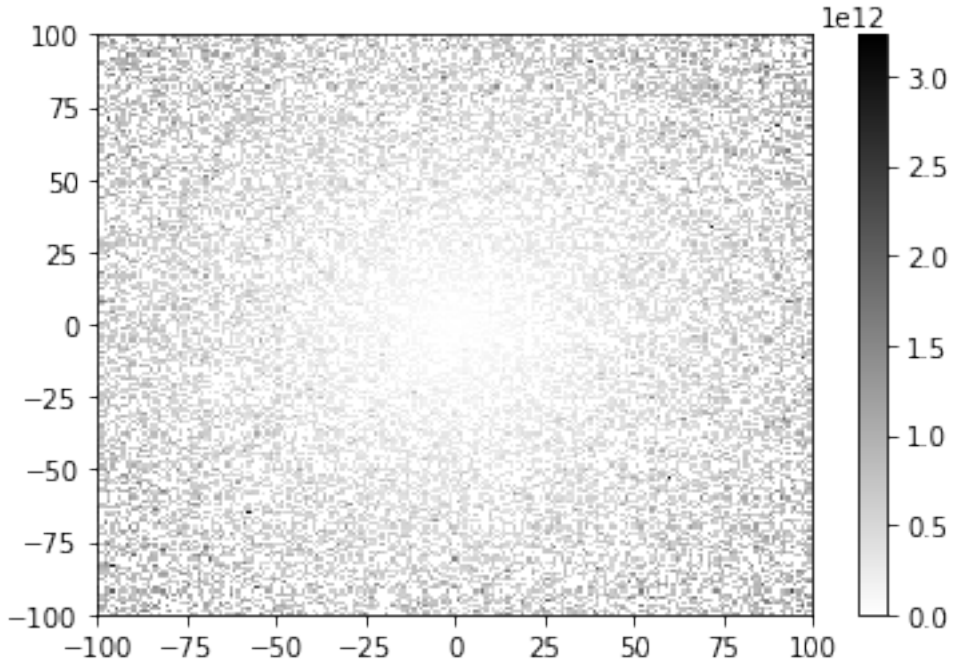
```
Absolute, Relative: (24192357342.692562, 41377870928.61217)
```

6.4

```
In [932]: def relative_condition_number_plot(bounds=list, res=int):

            X,Y = np.meshgrid(np.linspace(bounds[0],bounds[1],res),
                              np.linspace(bounds[2],bounds[3],res))

            nums = np.empty((res, res))
            for i in range(len(np.linspace(bounds[0],bounds[1],res))):
                for j in range(len(np.linspace(bounds[2],bounds[3],res))):
                    A=np.array([[1,np.linspace(bounds[0],bounds[1],res)[i]],
                                [np.linspace(bounds[0],bounds[1],res)[j],1]])
                    nums[i,j]=condition_number_2(A)[1]

            plt.pcolormesh(X,Y,nums,cmap="gray_r")
            plt.colorbar()
            plt.show()
```

```
In [933]: def problem6_4():
            relative_condition_number_plot([-100,100,-100,100],200)
```

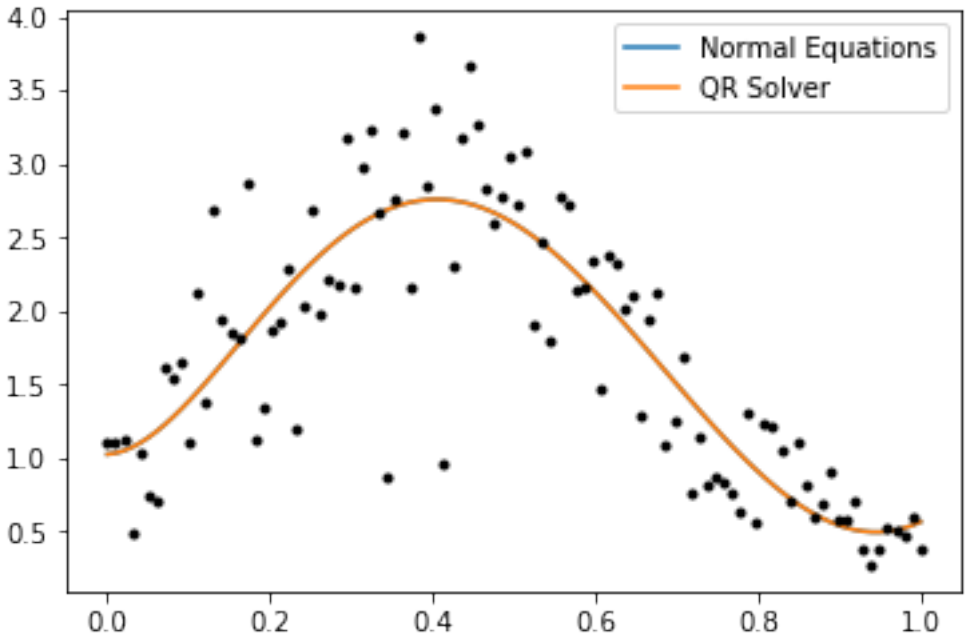
```
In [934]: problem6_4()
```



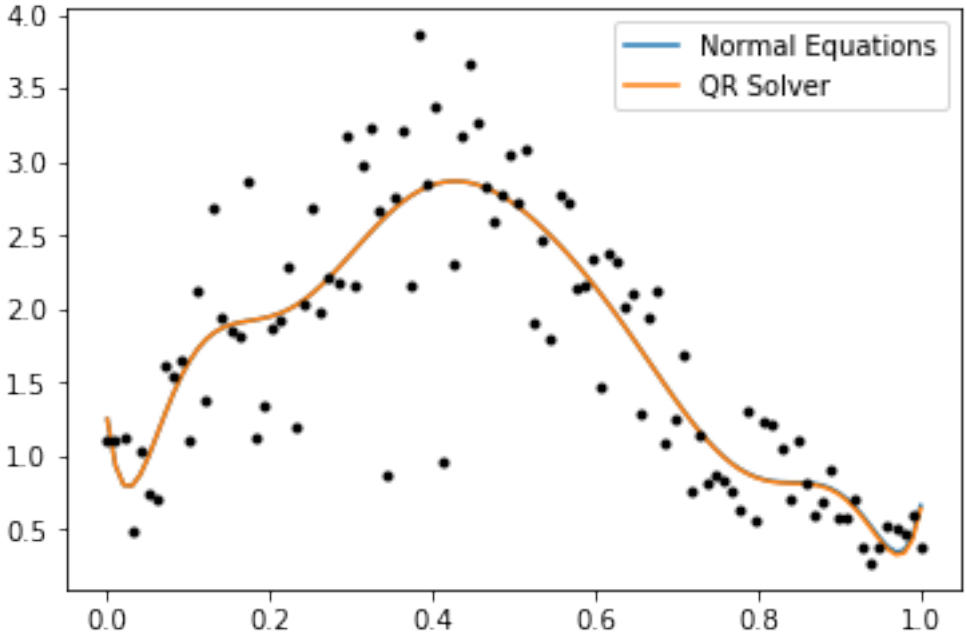
6.5

```
In [965]: def poly(n):

              x,y = np.load("stability_data.npy").T
              A = np.vander(x,n+1)

              x_inv = la.inv(A.T@A)@A.T@y
              error_inv = la.norm(A@x_inv-y)
              print("Inverse Forward Error:",error_inv)

              Q,R = la.qr(A, mode="economic")
              x_qr = la.solve_triangular(R,Q.T@y)
              error_qr = la.norm(A@x_qr-y)
              print("QR Forward Error:",error_qr)

              plt.plot(x,np.polyval(x_inv,x),label="Normal Equations")
              plt.plot(x,np.polyval(x_qr,x),label="QR Solver")
              plt.plot(x,y,"k.")
              plt.legend()
              plt.show()

In [966]: def problem6_5():
              for i in range(5,25,5):
                  poly(i)
```

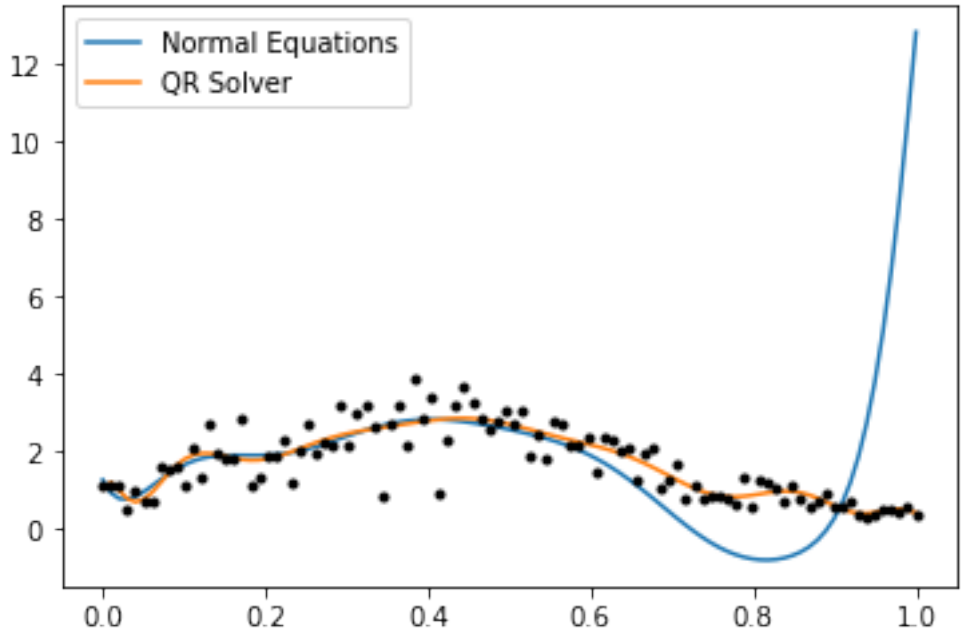
```
In [967]: problem6_5()

Inverse Forward Error: 4.854970503107442
QR Forward Error: 4.85497050310744
```



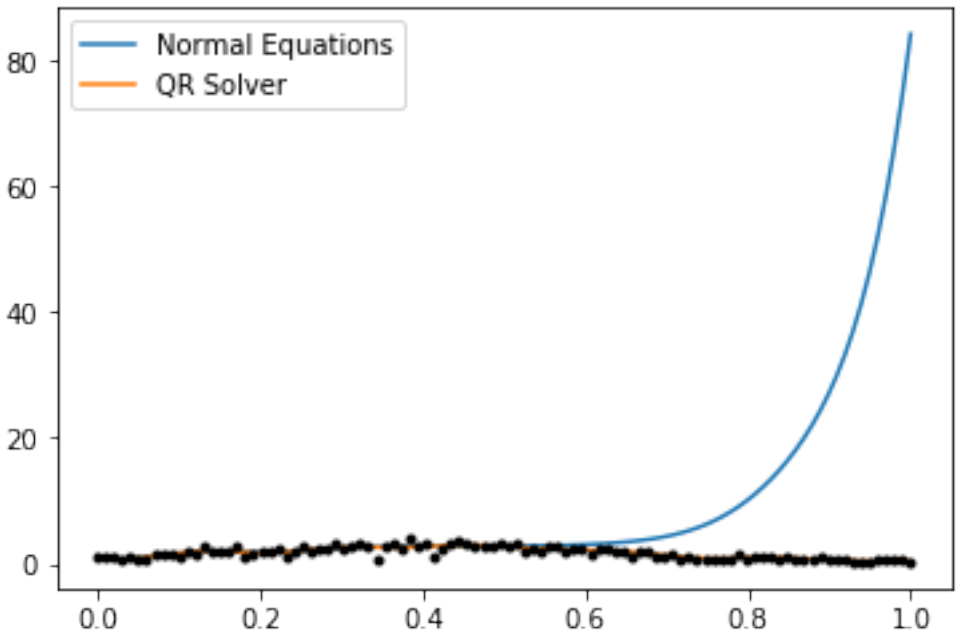

```
Inverse Forward Error: 4.6446869902282035
QR Forward Error: 4.644309587544681
```




```
Inverse Forward Error: 21.495620902404788
QR Forward Error: 4.527556122356962
```

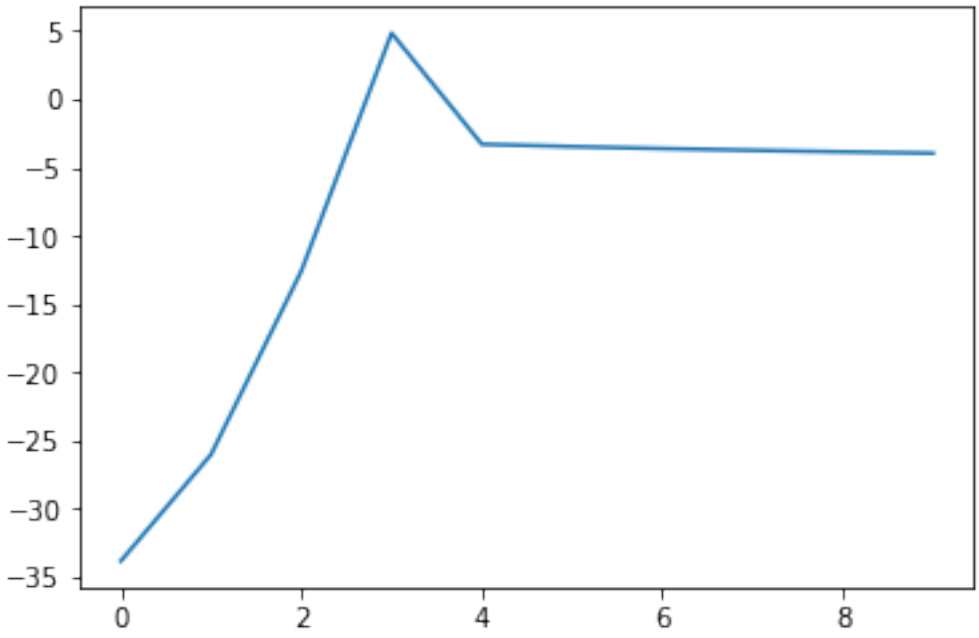


```
Inverse Forward Error: 183.0698894317729
QR Forward Error: 4.46208697902859
```



6.6

```
In [986]: def relative_forward_error():


              n_list = range(5, 55, 5)
              error_list = np.empty(10)
              x = sympy.symbols("x")
              for j in range(len(n_list)):
                  i = sympy.N(sympy.integrate(x**n_list[j]*sympy.exp(x-1), (x,0,1)))
                  error_list[j] = np.abs(i - ((-1)**n_list[j] *
                      sympy.subfactorial(n_list[j]) + (-1)**(n_list[j]+1) *
                      sympy.factorial(n_list[j]) / mpmath.e))

              plt.plot(np.log(error_list))
              plt.show()

In [987]: def problem6_6():
              relative_forward_error()

In [988]: problem6_6()
```


5
0
−5
−10
−15
−20
−25
−30
−35
0
2
4
6
8