OSM Boot Camp **Math Problem Set 5**

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First, some imports:

```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
summer = matplotlib.cm.get_cmap('summer')
from mpl_toolkits.mplot3d import Axes3D
import scipy.optimize as opt
%matplotlib notebook
```

Exercise 8.1.

Consider the linear optimization problem

$$\begin{array}{rl} \text{maximize} & 5x - 4y \\ \text{subject to} & 2x - 3y \leq -4 \\ & x - 6y \leq 1 \\ & x + y \leq 6 \\ & x, y \geq 0. \end{array}$$

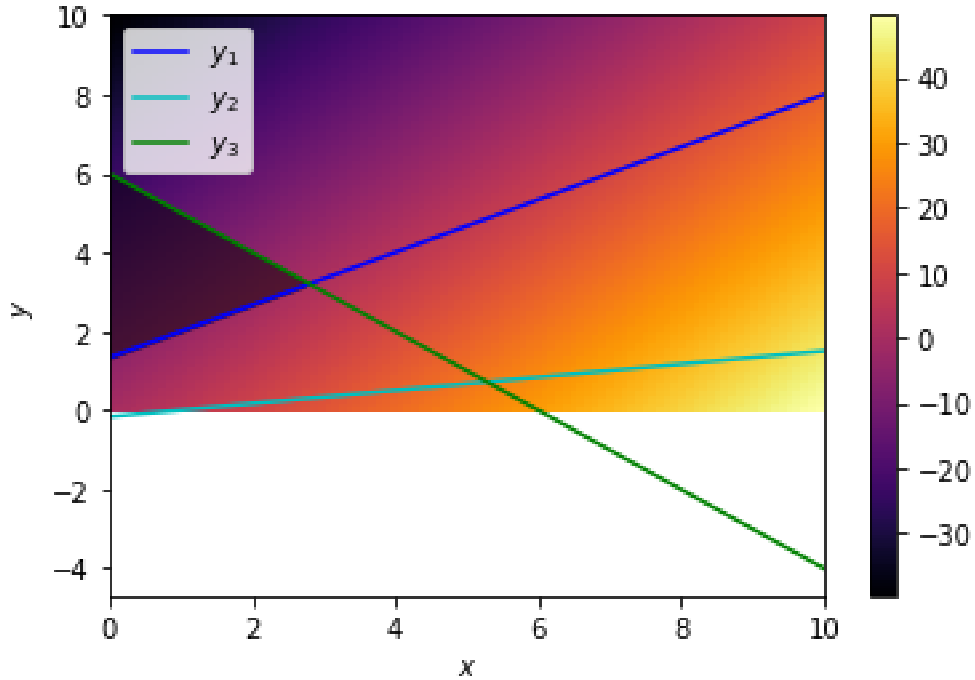
Sketch (or plot) the feasible set. Plot the objective function over the feasible set. Find an optimizer for this problem.

Solution.

```
def p8_1_1():
  x = np.linspace(0,10,100)
  y_1 = (2*x+4)/3
  y_2 = (x-1)/6
  y_3 = 6-x
  plt.plot(x,y_1,label="$y_1$",c="b")
  plt.plot(x,y_2,label="$y_2$",c="c")
  plt.plot(x,y_3,label="$y_3$",c="g")
  X,Y = np.meshgrid(x,x)
  Z = 5 * X - 4 * Y
  plt.pcolormesh(X,Y,Z,cmap="inferno")
  plt.fill_between(x,y_3,np.maximum(y_1,y_2),where=y_3>=np.maximum(y_1,y_2),color="k",alpha=.5)
  plt.ylabel("$y$")
  plt.xlabel("$x$")
```

```
    plt.colorbar()
    plt.legend()
    plt.show()

p8_1_1()
```



As we can see from the plot, the objective function is highest at the rightmost vertex, which is precisely the x and y values such that

$$\begin{aligned}
y = y_1 &= y_3 \\
\frac{2x+4}{3} &= 6 - x \\
2x + 4 &= 18 - 3x \\
5x &= 14 \\
x &= \frac{14}{5} \implies y = 6 - \frac{14}{5} = \frac{16}{5}.
\end{aligned}$$

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Exercise 8.2.

For each of the following, draw the feasible polygon, identify all the vertices, and use the Fundamental Theorem to solve the linear optimization problem (that is, check all the vertices). Give both the optimal point (optimizer) and the optimum value of the objective function.

(*i*) .

$$\begin{array}{rl} \text{maximize} & 3x_1 + x_2 \\ \text{subject to} & x_2 + 3x_2 \leq 15 \\ & 2x_1 + 3x_2 \leq 18 \\ & x_1 - x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{array}$$

(*ii*) .

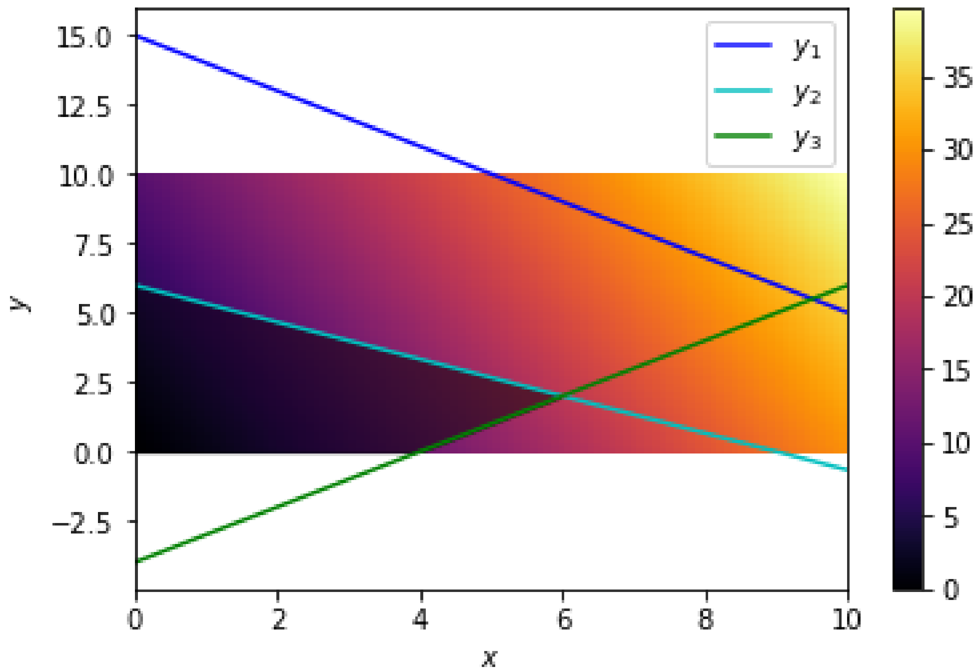
$$\begin{array}{rl} \text{maximize} & 4x + 6y \\ \text{subject to} & -x + y \leq 11 \\ & x + y \leq 27 \\ & 2x + 5y \leq 90 \\ & x, y \geq 0. \end{array}$$

Solution.

(*i*) .

```
def p8_2_1():
    x = np.linspace(0,10,100)
    y_1 = 15-x
    y_2 = (18-2*x)/3
    y_3 = x-4
    plt.plot(x,y_1,label="$y_1$",c="b")
    plt.plot(x,y_2,label="$y_2$",c="c")
    plt.plot(x,y_3,label="$y_3$",c="g")
    X,Y = np.meshgrid(x,x)
    Z = 3*X + Y
    plt.pcolormesh(X,Y,Z,cmap="inferno")
    y_top = np.minimum(y_1,y_2)
    y_bot = np.maximum(y_3,0)
    plt.fill_between(x,y_top,y_bot,where=y_top>=y_bot,color="k",alpha=.5)
    plt.ylabel("$y$")
    plt.xlabel("$x$")
    plt.colorbar()
    plt.legend()
    plt.show()

p8_2_1()
```



As we can see, the objective function increases in the northeast direction, so the vertex at the intersection of y_2 and y_3 is where the optimum is. This is precisely

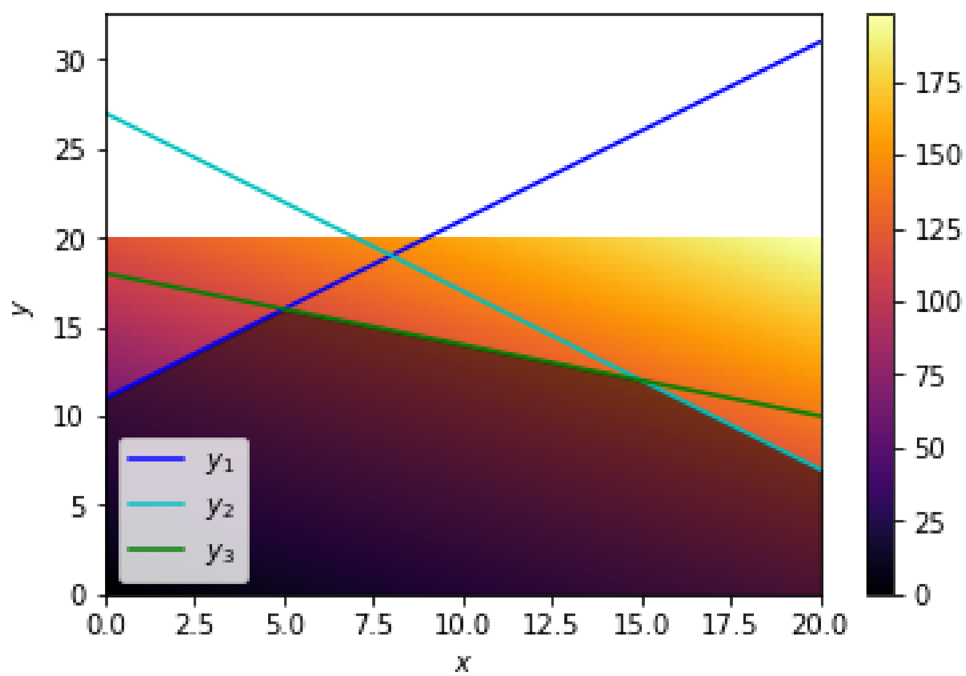
$$\begin{aligned} y = y_2 &= y_3 \\ \frac{18-2x}{3} &= x-4 \\ 18-2x &= 3x-12 \\ 30 &= 5x \\ 6 &= x \implies y = 6-4 = 2. \end{aligned}$$

■

(ii) .

```
def p8_2_2():
   x = np.linspace(0,20,100)
   y_1 = 11+x
   y_2 = 27-x
   y_3 = (90-2*x)/5
   plt.plot(x,y_1,label="$y_1$",c="b")
   plt.plot(x,y_2,label="$y_2$",c="c")
   plt.plot(x,y_3,label="$y_3$",c="g")
   X,Y = np.meshgrid(x,x)
   Z = 4*X + 6*Y
   plt.pcolormesh(X,Y,Z,cmap="inferno")
   y_top =  np.amin(np.array([y_1,y_2,y_3]),axis=0)
   y_bot = 0
   plt.fill_between(x,y_top,y_bot,where=y_top>=y_bot,color="k",alpha=.5)
   plt.ylabel("$y$")
   plt.xlabel("$x$")
   plt.colorbar()
   plt.legend()
   plt.show()
```

```
p8_2_2()
```



As we can see above, the optimum is at the vertex of the intersection between y_2 and y_3, which is precisely

$$\begin{aligned} y = y_2 &= y_3 \\ 27 - x &= \frac{90 - 2x}{5} \\ 135 - 5x &= 90 - 2x \\ 45 &= 3x \\ 15 &= x \implies y = 27 - 15 = 12. \end{aligned}$$

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Exercise 8.3.

Kenny's Toy Co. manufactures two types of toys: a GI Barb soldier and a Joey doll. A GI Barb soldier sells for $\$12$ and uses $\$5$ worth of raw materials. Each soldier that is manufactured increases Kenny's general overhead costs by $\$3$. A Joey doll sells for $\$10$ and uses $\$3$ worth of raw material. Each Joey doll built increases Kenny's overhead costs by $\$4$. The manufacture of soldiers and dolls require two types of labor: modeling and finishing. A soldier requires 15 minutes of finishing labor and 2 minutes of molding labor. A Joey doll requires 10 minutes of finishing and 2 minutes of modeling labor. Each week, Kenny can obtain all of the needed raw material but only 30 finishing hours and 5 molding hours of labor. Demand for GI Barb is unlimited but at most 200 Joey dolls are bought each week. Formulate a linear optimization problem in standard form whose solution would maximize Kenny's profit on these toys.

Solution. For quantity vector $\mathbf{x} = \begin{pmatrix} x_{\text{GI Barb}} \\ x_{\text{Joey}} \end{pmatrix}$, we have the problem

$$\begin{array}{rl} \text{maximize} & 4x_{\text{GI Barb}} + 3x_{\text{Joey}} \\ \text{subject to} & 15x_{\text{GI Barb}} + 10x_{\text{Joey}} \leq 1800 \\ & 2x_{\text{GI Barb}} + 2x_{\text{Joey}} \leq 300 \\ & x_{\text{Joey}} \leq 200 \\ & \mathbf{x} \succeq \mathbf{0}. \end{array}$$

■

Exercise 8.4.

Consider the following network, where the weights of each edge represent the carrying cost per unit of that edge.

$$\begin{array}{ccccc} Ⓐ & \overset{2}{\rightarrow} & Ⓑ & \overset{5}{\rightarrow} & Ⓒ \\ \downarrow^{5} & {}^{2}\swarrow & {}^{7}\downarrow & \searrow^{9} & {}^{2}\downarrow \\ Ⓓ & \underset{4}{\rightarrow} & Ⓔ & \underset{3}{\rightarrow} & Ⓕ \end{array}$$

Assume that the supply or demand at the nodes is $b_A = 10$, $b_B = 1$, $b_C = -2$, $b_D = -3$, $b_E = 4$, $b_F = -10$, and that the capacity of each edge is bounded by 6. Write a linear optimization problem in standard form whose solution ives the optimal (cheapest) flow in this network with these constraints.

Solution. For flow vector $\mathbf{x} = \begin{pmatrix} x_{\text{AB}} \\ x_{\text{BC}} \\ \vdots \\ x_{\text{EF}} \end{pmatrix}$, our problem is

$$\begin{array}{rl} \text{maximize} & 2x_{\text{AB}} + 5x_{\text{AD}} + 5x_{\text{BC}} + 2x_{\text{BD}} + 7x_{\text{BE}} + 9x_{\text{BF}} + 2x_{\text{CF}} + 4x_{\text{DE}} + 3x_{\text{EF}} \\ \text{subject to} & x_{\text{AB}} + x_{\text{AD}} = 10 \\ & x_{\text{BC}} + x_{\text{BD}} + x_{\text{BE}} + x_{\text{BF}} - x_{\text{AB}} = 1 \\ & x_{\text{CF}} - x_{\text{BC}} = -2 \\ & x_{\text{DE}} - x_{\text{AD}} - x_{\text{BD}} = -3 \\ & x_{\text{EF}} - x_{\text{BE}} - x_{\text{DE}} = 4 \\ & -x_{\text{BF}} - x_{\text{CF}} - x_{\text{EF}} = -10 \\ & \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{6}. \end{array}$$

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Exercise 8.5.

For each of the linear problems in Exercise 8.2, solve the linear problem using the simplex algorithm. Show the dictionary after each pivot. Give both the optimal point (optimizer) and the optimum value of the objective function. Verify that your answers agree with those you got in Exercise 8.2.

Solution.

(i) .

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{subject to} & x_1 + 3x_2 + w_1 = 15 \\ & 2x_1 + 3x_2 + w_2 = 18 \\ & x_1 - x_2 + w_3 = 4 \\ & x_1, x_2, w_1, w_2, w_3 \geq 0. \end{array}$$

$$\begin{array}{ccccccc} \zeta & = & & & 3x_1 & + & x_2 \\ \hline w_1 & = & 15 & - & x_1 & - & 3x_2 \\ w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\ w_3 & = & 4 & - & x_1 & + & x_2 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 12 & + & 4x_2 & - & 3w_3 \\ \hline w_1 & = & 11 & - & 4x_2 & + & w_3 \\ w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\ x_1 & = & 4 & + & x_2 & - & w_3 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\ \hline w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\ x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\ x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3. \end{array}$$

This problem has optimizer $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ with optimum 20.

(ii) .

$$\begin{array}{ll} \text{maximize} & 4x + 6y \\ \text{subject to} & -x + 3y + w_1 = 11 \\ & x + y + w_2 = 27 \\ & 2x + 5y + w_3 = 90 \\ & x, y, w_1, w_2, w_3 \geq 0. \end{array}$$

$$\begin{array}{ccccccc} \zeta & = & & & 4x & + & 6y \\ \hline w_1 & = & 11 & + & x & - & y \\ w_2 & = & 27 & - & x & - & y \\ w_3 & = & 90 & - & 2x & - & 5y \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 66 & + & 10x & - & 6w_1 \\ \hline y & = & 11 & + & x & - & w_1 \\ w_2 & = & 16 & - & 2x & + & w_1 \\ w_3 & = & 35 & - & 7x & + & 5w_1 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 116 & + & \frac{8}{7}w_1 & - & \frac{10}{7}w_3 \\ \hline y & = & 16 & - & \frac{2}{7}w_1 & - & \frac{1}{7}w_3 \\ w_2 & = & 6 & - & \frac{3}{7}w_1 & + & \frac{2}{7}w_3 \\ x & = & 5 & + & \frac{5}{7}w_1 & - & \frac{1}{7}w_3 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 132 & - & \frac{8}{3}w_2 & - & \frac{2}{7}w_3 \\ \hline y & = & 12 & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \\ w_1 & = & 14 & - & \frac{7}{3}w_2 & + & \frac{2}{3}w_3 \\ x & = & 15 & - & \frac{5}{3}w_2 & + & \frac{1}{3}w_3. \end{array}$$

This problem has optimizer $\begin{pmatrix} 15 \\ 12 \end{pmatrix}$ with optimum 132.

■

Exercise 8.6.

Solve the Kenny's Toys linear problem of Exercise 8.3 using the simplex algorithm. Show the dictionary after each pivot. Give both the optimal choice of how much of each toy to manufacture and the maximal profit.

Solution.

$$\begin{array}{rl} \text{maximize} & x_{\text{GI Barb}} + 3x_{\text{Joey}} \\ \text{subject to} & 15x_{\text{GI Barb}} + 10x_{\text{Joey}} + w_1 = 1800 \\ & 2x_{\text{GI Barb}} + 2x_{\text{Joey}} + w_2 = 300 \\ & x_{\text{Joey}} + w_3 = 200 \\ & \mathbf{x}, \mathbf{w} \succeq \mathbf{0}. \end{array}$$

$$\begin{array}{ccccccc} \zeta & = & & & 4x_{\text{GI Barb}} & + & 3x_{\text{Joey}} \\ \hline w_1 & = & 1800 & - & 15x_{\text{GI Barb}} & - & 10x_{\text{Joey}} \\ w_2 & = & 300 & - & 2x_{\text{GI Barb}} & - & 2x_{\text{Joey}} \\ w_3 & = & 200 & - & x_{\text{Joey}} & & \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 450 & + & x_{\text{GI Barb}} & - & \frac{3}{2}w_2 \\ \hline w_1 & = & 300 & - & 5x_{\text{GI Barb}} & + & 5w_2 \\ x_{\text{Joey}} & = & 150 & - & x_{\text{GI Barb}} & - & \frac{1}{2}w_2 \\ w_3 & = & 50 & + & x_{\text{GI Barb}} & + & \frac{1}{2}w_2 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & 510 & - & \frac{1}{5}w_1 & - & \frac{1}{2}w_2 \\ \hline x_{\text{GI Barb}} & = & 60 & - & \frac{1}{5}w_1 & + & w_2 \\ x_{\text{Joey}} & = & 90 & + & \frac{1}{5}w_1 & - & \frac{3}{2}w_2 \\ w_3 & = & 110 & - & \frac{1}{5}w_1 & + & \frac{3}{2}w_2. \end{array}$$

This problem has optimizer $\mathbf{x} = \begin{pmatrix} x_{\text{GI Barb}} \\ x_{\text{Joey}} \end{pmatrix} = \begin{pmatrix} 60 \\ 90 \end{pmatrix}$ with optimum $\$150$. ■

Exercise 8.7.

Solve the following linear problems using the simplex algorithm. Show the dictionary after each pivot. If there is a solution, give an optimal point and the optimal value. If there isn't a solution, tell whether the problem is unbounded or infeasible.

(i).

$$\begin{array}{rl} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -4x_1 - 2x_2 \leq -8 \\ & -2x_1 + 3x_2 \leq 6 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

(ii).

$$\begin{array}{rl} \text{maximize} & 5x_1 + 2x_2 \\ \text{subject to} & 5x_1 + 3x_2 \leq 15 \\ & 3x_1 + 5x_2 \leq 15 \\ & 4x_1 - 3x_2 \leq -12 \\ & x_1, x_2 \geq 0. \end{array}$$

(iii).

$$\begin{array}{rl} \text{maximize} & -3x_1 + x_2 \\ \text{subject to} & x_2 \leq 4 \\ & -2x_1 + 3x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{array}$$

Solution.

(i). We have

$$\begin{array}{rl} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -4x_1 - 2x_2 + x_3 - 8 \\ & -2x_1 + 3x_2 + x_4 = 6 \\ & x_1 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0, \end{array}$$

with auxiliary problem

$$\begin{array}{rl} \text{maximize} & -x_0 \\ \text{subject to} & -4x_1 - 2x_2 + x_3 - x_0 = -8 \\ & -2x_1 + 3x_2 + x_4 - x_0 = 6 \\ & x_1 + x_5 - x_0 = 3 \\ & x_0, x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Solving using simplex, we have:

$$\begin{array}{ccccccccc} \zeta & = & & & & & & - & x_0 \\ \hline x_3 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\ x_4 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\ x_5 & = & 3 & - & x_1 & & & + & x_0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc} \zeta & = & -8 & + & 4x_1 & + & 2x_3 & - & x_3 \\ \hline x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & x_3 \\ x_4 & = & 14 & - & 2x_1 & - & 5x_2 & + & x_3 \\ x_5 & = & 11 & - & 5x_1 & - & 2x_2 & + & x_3 \end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & - & x_0 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & - & \frac{1}{4}x_0 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & + & \frac{1}{2}x_0 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & + & \frac{5}{4}x_0
\end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr}
\zeta & = & 2 & + & \frac{3}{2}x_2 & + & \frac{1}{4}x_3 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3
\end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr}
\zeta & = & 3 & + & 2x_2 & - & x_5 \\
\hline
x_1 & = & 3 & & & - & x_5 \\
x_4 & = & 12 & - & 3x_2 & - & 2x_5 \\
x_3 & = & 4 & + & 2x_2 & - & 4x_5
\end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcr}
\zeta & = & 11 & - & \frac{2}{3}x_4 & - & \frac{7}{3}x_5 \\
\hline
x_1 & = & 3 & & & - & x_5 \\
x_2 & = & 4 & - & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 \\
x_3 & = & 4 & - & \frac{2}{3}x_4 & - & \frac{16}{3}x_5.
\end{array}$$

This problem has optimizer $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and optimum 11.

(ii). We have

$$\begin{array}{rl}
\text{maximize} & 5x_1 + 2x_2 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 = 15 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& 4x_1 - 3x_2 + x_4 = -12 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0,
\end{array}$$

with auxiliary problem

$$\begin{array}{rl}
\text{maximize} & -x_0 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 - x_0 = 15 \\
& 3x_1 + 5x_2 + x_4 - x_0 = 15 \\
& 4x_1 - 3x_2 + x_5 - x_0 = -12 \\
& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{array}$$

Using simplex, we have:

$$\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
x_4 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
x_5 & = & -12 & - & 4x_1 & + & 3x_2 & + & x_0
\end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcrcr}
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & x_5 \\
\hline
x_3 & = & 27 & - & x_1 & - & 6x_2 & + & x_5 \\
x_4 & = & 27 & + & x_1 & - & 8x_2 & + & x_5 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & x_5
\end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc}
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}x_4 & - & \frac{5}{8}x_5 \\ \hline
x_3 & = & \frac{27}{4} & - & \frac{7}{4}x & + & \frac{3}{4}x_4 & + & \frac{1}{4}x_5 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}x_4 & + & \frac{1}{8}x_5 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}x_4 & + & \frac{5}{8}x_5.
\end{array}$$

Unfortunately, there are no feasible solutions.

(iii). We have

$$\begin{array}{rl}
\text{maximize} & -3x_1 + x_2 \\
\text{subject to} & x_2 + x_3 = 4 \\
& -2x_1 + 3x_2 + x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{array}$$

Solving using simplex, we have:

$$\begin{array}{ccccccc}
\zeta & = & & - & 3x_1 & + & x_2 \\ \hline
x_3 & = & 4 & & & - & x_2 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2
\end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc}
\zeta & = & 2 & - & \frac{7}{3}x_1 & - & \frac{1}{3}x_4 \\ \hline
x_3 & = & 2 & - & \frac{2}{3}x_1 & + & \frac{1}{3}x_4 \\
x_2 & = & 2 & + & \frac{2}{3}x_1 & - & \frac{1}{3}x_4.
\end{array}$$

This problem has optimizer $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ with optimum 2.

■

Exercise 8.8.

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded, but where the objective function still has a unique feasible maximizer.

Solution.

$$\begin{array}{rl}
\text{maximize} & -x_1 - x_2 - x_3 \\
\text{subject to} & -x_1 + x_2 \leq 1 \\
& x_1 - x_2 \leq 1 \\
& x_1 - x_3 \leq 1 \\
& -x_1 + x_3 \leq 1 \\
& \mathbf{x} \succeq \mathbf{0}.
\end{array}$$

■

Exercise 8.9.

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded and where the objective function has no maximizer.

Solution.

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + x_3 \\ \text{subject to} & -x_1 + x_2 \leq 1 \\ & x_1 - x_2 \leq 1 \\ & x_1 - x_3 \leq 1 \\ & -x_1 + x_3 \leq 1 \\ & \mathbf{x} \succeq \mathbf{0}. \end{array}$$

■

Exercise 8.10.

Give an example of a three-dimensional linear problem where the feasible region is empty.

Solution.

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + x_3 \\ \text{subject to} & -x_1 + x_2 \leq -1 \\ & x_1 - x_2 \leq -1 \\ & x_1 - x_3 \leq -1 \\ & -x_1 + x_3 \leq -1 \\ & \mathbf{x} \succeq \mathbf{0}. \end{array}$$

■

Exercise 8.11.

Give an example of a three-dimensional linear problem where the feasible region is nonempty, closed, and bounded, but $\mathbf{0}$ is not feasible. Write an auxiliary problem (for which $\mathbf{0}$ is feasible) whose solution gives a feasible vertex for starting the original problem.

Solution.

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + x_3 \\ \text{subject to} & -x_1 + x_2 \leq 1 \\ & x - x_2 \leq 1 \\ & x_1 - x_3 \leq 1 \\ & -x_1 + x_3 \leq 1 \\ & -x_1 - x_2 - x_3 \leq -1 \\ & x_1 + x_2 + x_3 \leq 5 \\ & \mathbf{x} \succeq \mathbf{0}, \end{array}$$

with corresponding auxiliary problem

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & -x_1 + x_2 + x_0 \leq 1 \\ & x_1 - x_2 + x_0 \leq 1 \\ & x_1 - x_3 + x_0 \leq 1 \\ & -x + x_3 + x_0 \leq 1 \\ & -x_1 - x_2 - x_3 + x_0 \leq -1 \\ & x_1 + x_2 + x_3 + x_0 \leq 5 \\ & \mathbf{x} \succeq \mathbf{0}. \end{array}$$

■

Exercise 8.12.

Solve the following linear problem using Bland's rule to resolve degeneracy. Show the dictionary after each pivot. Give the optimal point and the optimum value of the objective function.

$$\begin{array}{rl} \text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0 \\ & x_1 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

Solution. We have

$$\begin{array}{rl} \text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\ & x_1 + x_7 = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0. \end{array}$$

Solving using simplex, we have

$$\begin{array}{rcrcrcrcrcr} \zeta & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\ \hline x_5 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\ x_6 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\ x_7 & = & 1 & - & x_1 & & & & & & \end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcrcrcr} \zeta & = & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_5 \\ \hline x_1 & = & & & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_5 \\ x_6 & = & & & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_5 \\ x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_5 \end{array}$$

$$\Downarrow$$

$$\begin{array}{rcrcrcrcrcr} \zeta & = & 1 & - & 30x_2 & - & 42x_4 & - & 18x_5 & - & x_7 \\ \hline x_1 & = & 1 & & & & & & & - & x_7 \\ x_6 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_5 & - & 2x_7 \\ x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_5 & - & x_7. \end{array}$$

Our optimizer is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ with optimum 1. ■

Exercise 8.15.

Prove the Weak Duality Theorem: If $\mathbf{x}$ is a primal feasible point and $\mathbf{y}$ is any feasible point of the dual then $\mathbf{c}^{\mathsf{T}}\mathbf{x} \leq \mathbf{b}^{\mathsf{T}}\mathbf{y}$.

Solution. Let $\mathbf{x} \in \mathbb{R}^n$ be feasible for the primal and $\mathbf{y} \in \mathbb{R}^m$ be feasible for the dual. It follows that $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$, for which it then follows

$$\begin{array}{rcl}
\mathbf{b} & \geq & \mathbf{A}\mathbf{x} \\
& \Downarrow & \\
\mathbf{b}^\mathsf{T} & \geq & \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T} \\
& \Downarrow & \\
\mathbf{b}^\mathsf{T}\mathbf{y} & \geq & \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{y} \\
& \geq & \mathbf{x}^\mathsf{T}\mathbf{c} \\
& \geq & \mathbf{c}^\mathsf{T}\mathbf{x}.
\end{array}$$

■

Exercise 8.17.

Prove that the dual of the dual of a linear problem is the primal problem.

Solution. For the primal problem

$$\begin{array}{rl}
\text{maximize} & \mathbf{c}^\mathsf{T}\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \\
& \mathbf{x} \succeq \mathbf{0}
\end{array}$$

with dual

$$\begin{array}{rl}
\text{maximize} & \mathbf{b}^\mathsf{T}\mathbf{y} \\
\text{subject to} & \mathbf{A}^\mathsf{T}\mathbf{y} \succeq \mathbf{c} \\
& \mathbf{y} \succeq \mathbf{0},
\end{array}$$

yielding double dual

$$\begin{array}{rl}
\text{maximize} & \mathbf{c}^\mathsf{T}\mathbf{z} \\
\text{subject to} & \left(\mathbf{A}^\mathsf{T}\right)^\mathsf{T}\mathbf{z} \succeq \mathbf{b} \\
& \mathbf{z} \succeq \mathbf{0},
\end{array}$$

which is simply

$$\begin{array}{rl}
\text{maximize} & \mathbf{c}^\mathsf{T}\mathbf{z} \\
\text{subject to} & \mathbf{A}\mathbf{z} \succeq \mathbf{b} \\
& \mathbf{z} \succeq \mathbf{0}.
\end{array}$$

■

Exercise 8.18.

Give the dual of the linear problem

$$\begin{array}{rl}
\text{maximize} & x_1 + x_2 \\
\text{subject to} & 2x_1 + x_2 \leq 3 \\
& x_1 + 3x_2 \leq 5 \\
& 2x_1 + 3x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{array}$$

Solution. Our primal problem is

$$\begin{array}{rl} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 + w_1 = 3 \\ & x_1 + 3x_2 + w_2 = 5 \\ & 2x_1 + 3x_2 + w_3 = 4 \\ & x_1, x_2, w_1, w_2, w_3 \geq 0. \end{array}$$

Solving with simplex, we have

$$\begin{array}{ccccccc} \zeta & = & & & x_1 & + & x_2 \\ \hline w_1 & = & 3 & - & 2x_1 & - & x_2 \\ x_2 & = & 5 & - & x_1 & - & 3x_2 \\ x_5 & = & 4 & - & 2x_1 & - & 3x_2 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & \frac{3}{2} & + & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\ \hline x_1 & = & \frac{3}{2} & - & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\ w_2 & = & \frac{7}{2} & - & \frac{5}{2}x_2 & + & \frac{1}{2}w_1 \\ w_3 & = & 1 & - & 2x_2 & + & w_1 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccc} \zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_1 \\ \hline x_1 & = & \frac{5}{4} & - & \frac{3}{4}w_1 & + & \frac{1}{4}w_3 \\ w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\ x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3. \end{array}$$

with optimizer $\begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix}$ and optimum $\frac{7}{4}$. The dual problem is

$$\begin{array}{rl} \text{maximize} & -3y_1 - 5y_2 - 4y_3 \\ \text{subject to} & -2y_1 - y_2 - 2y_3 + v_1 - v_0 = -1 \\ & -y_1 - 3y_2 - 3y_3 + v_2 - v_0 = -1 \\ & y_1, y_2, y_3, v_1, v_2 \geq 0. \end{array}$$

Solving through simplex, we have

$$\begin{array}{ccccccccccc} \zeta & = & & & & & & & & - & v_0 \\ \hline v_1 & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & + & v_0 \\ v_2 & = & -1 & + & y_1 & + & 3y_2 & + & 3y_3 & + & v_0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccccc} \zeta & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & - & v_1 \\ \hline v_0 & = & 1 & - & 2y_1 & - & y_2 & - & 2y_3 & + & v_1 \\ v_2 & = & & - & y_1 & + & 2y_2 & + & y_3 & + & v_1 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccccc} \zeta & = & & & & & & & & - & v_0 \\ \hline y_2 & = & 1 & - & 2y_1 & - & 2y_3 & + & v_1 & - & v_0 \\ v_2 & = & 2 & - & 5y_1 & - & 3y_3 & + & 3v_1 & - & 2v_0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc} \zeta & = & -2 & + & y_1 & - & 3y_2 & - & 2v_1 \\ \hline y_3 & = & \frac{1}{2} & - & y_1 & - & \frac{1}{2}y_2 & + & \frac{1}{2}v_1 \\ v_2 & = & \frac{1}{2} & - & 2y_1 & + & \frac{3}{2}y_2 & + & \frac{3}{2}v_1 \end{array}$$

$$\Downarrow$$

$$\begin{array}{ccccccccc}
\zeta & = & -\frac{7}{4} & - & \frac{3}{2}y_2 & - & \frac{5}{4}v_1 & - & \frac{1}{2}v_2 \\
\hline
y_3 & = & \frac{1}{4} & - & \frac{3}{2}y_2 & - & \frac{1}{4}v_1 & + & \frac{1}{2}v_2 \\
y_1 & = & \frac{1}{4} & + & \frac{3}{2}y_2 & + & \frac{3}{4}v_1 & - & \frac{1}{2}v_2.
\end{array}$$

with optimizer $\begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}$ and optimum $\frac{7}{4}$. ■