OSM Boot Camp: **Math ProbSet1**

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Exercise 1.3.

Question. Let $X = \mathbb{R}$. Define

- $\mathcal{G}_1 = \{A \mid A \subset \mathbb{R}, A \text{ open}\}$
- $\mathcal{G}_2 = \{A \mid A \text{ is a finite union of intervals of the form } (a,b], (-\infty, b], \text{ and } (a, \infty)\}$
- $\mathcal{G}_3 = \{A \mid A \text{ is a countable union of } (a,b], (-\infty, b], \text{ and } (a, \infty)\}$

Which of these are algebras? Which are even σ-algebras?

Answer.

- $\mathcal{G}_1$ is neither an algebra nor a σ-algebra.
    - The empty set $\emptyset$ is an open subset in $\mathbb{R}$, so $\emptyset \in \mathcal{G}_1$.
    - However, $\mathcal{G}_1$ is not closed under complements. Consider $(0,1) \subset \mathbb{R}$, an element of $\mathcal{G}_1$. Note that $(0,1)^{\complement} = \mathbb{R} \smallsetminus (0,1) = (-\infty, 0] \cup [1, \infty)$, which is not open since its complement is open. Since $(0,1)^{\complement} \subset \mathbb{R}$ is not open, it is not in $\mathcal{G}_1$ by its definition. Therefore, $\mathcal{G}_1$ is not an algebra, nor a σ-algebra.

- $\mathcal{G}_2$ is an algebra, but not a σ-algebra.
    - $(*)$ We can reason that $\emptyset \in \mathcal{G}_2$, since one can construct a finite union of *zero* such intervals.
    - $(**)$ Moreover, say we have an (arbitrary) element A of $\mathcal{G}_1$,

    $$A = \left(\bigcup_{i \in \Gamma_i} (a_i^1, b_i^1] \right) \cup \left(\bigcup_{j \in \Gamma_j} (-\infty, b_i^2] \right) \cup \left(\bigcup_{k \in \Gamma_k} (a_i^3, \infty) \right).$$

    Note that we can rewrite this as

    $$A_1 = \left(\bigcup_{i \in \Gamma_i} (a_i^1, b_i^1] \right) \cup \left(-\infty, \max_{j \in \Gamma_j} \left\{ b_j^2 \right\} \right] \cup \left(\min_{k \in \Gamma_k} \left\{ a_k^3 \right\}, \infty \right),$$

    where

    $$\max_{j \in \Gamma_j} \left\{ b_j^2 \right\} \leq a_i^1 \leq b_i^1 \leq \min_{k \in \Gamma_k} \left\{ a_k^3 \right\}$$

    for all $i \in \Gamma_i$. So, we can think of A as

    $$(-\infty, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{n-2}, x_{n-1}] \cup (x_n, \infty),$$

where $x_{p+1} \geq x_p$ for all p. From here, it is clear that the complement of this set, $A^{\complement}$, then takes the form

$$(y_1, y_2] \cup \cdots \cup (y_{m-1}, y_m],$$

where $y_{p+1} \geq y_p$ for all p. This is in the form of A as outlined earlier, where $|\Gamma_i| = |\Gamma_k| = 0$. Note that the same reasoning holds for all values of $|\Gamma_i|, |\Gamma_j|$, and $|\Gamma_k|$. Thus, $\mathcal{G}_2$ is closed under complements.

- ($***$) Next, we can also see that $\mathcal{G}_2$ is also closed under finite unions, since if we consider the union of $A_1, \ldots, A_N \in \mathcal{G}_2$, where

$$\begin{aligned} A_1 &= \Big(-\infty, x_1^1\Big] \cup (x_2^1, x_3^1] \cup \cdots \cup (x_{n_1-2}^1, x_{n_1-1}^1] \cup (x_{n_1}^1, \infty) \\ &\vdots \\ A_N &= \Big(-\infty, x_1^N\Big] \cup (x_2^N, x_3^N] \cup \cdots \cup (x_{n_N-2}^N, x_{n_N-1}^N] \cup (x_{n_N}^N, \infty), \end{aligned}$$

with $x_{p+1}^i \geq x_p^i \ \forall p, \forall i$, we observe that our resultant union takes the form

$$\bigcup_{i=1}^N A_i = \underbrace{\Big(-\infty, \max_i x_1^i\Big] \cup (x_2, x_3] \cup \cdots \cup (x_{m-2}, x_{m-1}] \cup (\min_i x_{n_i}^i, \infty).}_{(\dagger)}$$

- Consider the union of $A_1, A_2, A_3, \ldots \in \mathcal{G}_2$, where

$$\begin{aligned} A_1 &= (x_1^1, x_2^1] \cup \cdots \cup (x_{n_1-1}^1, x_{n_1}^1] \\ A_2 &= (x_1^2, x_2^2] \cup \cdots \cup (x_{n_2-1}^2, x_{n_2}^2] \\ &\vdots \\ A_i &= (x_1^i, x_2^i] \cup \cdots \cup (x_{n_i-1}^i, x_{n_i}^i] \\ &\vdots \end{aligned}$$

and where each $(x_{j-1}^i, x_j^i] \ \forall i \in \mathbb{N} \ \forall j \in \{2, \ldots, n_i\}$ lies in the complement of the union of every other $(x_{j-1}^i, x_j^i]$. When this process is continued countably infinitely many times, we approach a union comprising infinitely (but countably) many sets, so it does not satisfy the criteria for being an element of $\mathcal{G}_2$; therefore, $\mathcal{G}_2$ is not closed under countable unions, so it is not a σ-algebra.

- $\mathcal{G}_3$ is both an algebra and a σ-algebra.
  - The empty set is in $\mathcal{G}_3$; see ($*$) above.
  - $\mathcal{G}_3$ is closed under complements ($**$) and finite unions ($***$). See above. The same line of reasoning used in $\mathcal{G}_2$ applies for $\mathcal{G}_3$ because finite is a stronger restriction than countable, and everything that is finite is also countable.
  - Finally, we can continue the process of unioning even more sets like above countably infinitely many times, since if $x_O \geq x_E$ in ($\dagger$) for any $O < E$ (where O is an odd number in $[1, m-1]$ and E is an even number in $[2, m-2]$), then the intervals $(\cdot, x_O]$ and $(x_E, \cdot]$ would collapse to the form $(\cdot, \cdot]$, and if $x_O < x_E$ then another interval with form $(\cdot, \cdot]$ would be included in the new union. And, if $x_O \geq \min_i x_{n_i}^i$, then the final interval in the union (when sorted in ascending order) would still remain in the form $(\cdot, \infty)$. Similarly as stated before, these results still hold for all values of $|\Gamma_i|, |\Gamma_j|$, and $|\Gamma_k|$. Since A is specified as a *countable* (not restricted to only finite) union in the problem, we don't have to worry about the counterexample mentioned for $\mathcal{G}_2$. Thus $\mathcal{G}_3$ satisfies the criteria for both algebra and σ-algebra.

Exercise 1.7.

Question. Explain why these are the "largest" and "smallest" possible σ-algebras, respectively, in the following sense: if $\mathcal{A}$ is any σ-algebra, then $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

Answer.

- In set theory, two direct corollaries of set-theoretic axioms are that every set contains the empty set as a subset, and every set contains itself as a subset. In other words, if X is a set, then $\emptyset \subseteq X$ and $X \subseteq X$. By extension, $\{\emptyset, X\} \subseteq X$. Note that $X^{\complement}$ in X is $X \setminus X = \emptyset$, and $\emptyset^{\complement} = X \setminus \emptyset = X \in \{\emptyset, X\}$, so $\{\emptyset, X\}$ is closed under complements. Also note that $\emptyset \cup X = X \in \{\emptyset, X\}$, and that $\{\emptyset, X\}$ only has just two elements, so $\{\emptyset, X\}$ must also be closed under finite and countable unions. So, $\{\emptyset, X\}$ is a σ-algebra. If we omit one of these elements, the set is no longer closed under complements, so $\{\emptyset, X\}$ must be the smallest σ-algebra for any X.
- Next, $\mathcal{P}(X)$ must always be a σ-algebra for X since for any subset $S \subseteq X$, note that $X^{\complement} = X \setminus S \in \mathcal{P}(X)$, and for any countable set of subsets $\{S_i \subseteq X\}_{i=1}^{\infty}$, we know that the union of each element $\bigcup_{i=1}^{\infty} S_i \subseteq X$ lies in X since an arbitrary union of subsets of a given set X is always itself a subset of that set X (and therefore is an element of $\mathcal{P}(X)$ by definition), since if it was not, it would contain an element that is not in X, which means that at least one element in one of the S_i's would not lie in X; but, that would make S_i no longer a subset of X, so this must not be the case. Now that it's obvious $\mathcal{P}(X)$ is a σ-algebra for X, we can reason that $\mathcal{P}(X)$ is the largest of all σ-algebras of X, since if there were one larger, say Σ, where $|\Sigma| > |\mathcal{P}(X)|$, this would mean that Σ contains as an element a set (say Z) that does not lie in $\mathcal{P}(X)$, which means that Σ would contain an element Z that is not a subset of X. This contradicts the very definition of an algebra in 1.1—any algebra $\mathscr{A}$ of X is a family of *subsets* of X.
- We can now reason that any arbitrary σ-algebra of X, say $\mathcal{A}$, must be a superset of $\{\emptyset, X\}$ and a subset of $\mathcal{P}(X)$, since $\mathcal{A}$ contains only subsets of X by definition, so every element of $\mathcal{A}$ lies in $\mathcal{P}(X)$, so that means $\mathcal{A} \subseteq \mathcal{P}(X)$. Also, since every algebra must contain $\emptyset$ by definition, we know that X must be in $\mathcal{A}$ by extension, since $\mathcal{A}$ must be closed under complements.

Exercise 1.10.

Question. Prove the following Proposition:
Let $\{\mathcal{S}_\alpha\}$ be a family of σ-algebras on X. Then $\bigcap_\alpha \mathcal{S}_\alpha$ is also a σ-algebra.

Answer.

- $\emptyset \in \bigcap_\alpha \mathcal{S}_\alpha$ because if $\mathcal{S}_\alpha$ is a σ-algebra on X, then $\emptyset \in \mathcal{S}_\alpha$ by definition, for all α.
- $\bigcap_\alpha \mathcal{S}_\alpha$ is closed under complements because if the set S is an element of $\bigcap_\alpha \mathcal{S}_\alpha$, then that means the elements of S are themselves elements of all of the $\mathcal{S}_\alpha$'s. Since each $\mathcal{S}_\alpha$ is a σ-algebra, we know the complement of each element of S is also in each $\mathcal{S}_\alpha$. This means that the complement of S must be in $\bigcap_\alpha \mathcal{S}_\alpha$.
- $\bigcap_\alpha \mathcal{S}_\alpha$ is closed under finite unions: Consider $S_1, \ldots, S_N \in \bigcap_\alpha \mathcal{S}_\alpha$. Since each S_i (for $i \in \mathbb{N} \cap [1, N]$) is in $\bigcap_\alpha \mathcal{S}_\alpha$, this means that each S_i is in each $\mathcal{S}_\alpha$ for all α. And since each $\mathcal{S}_\alpha$ is a σ-algebra, this means that $\bigcup_i S_i$ is also in each $\mathcal{S}_\alpha$. This means that $\bigcup_i S_i$ must be in the intersection $\bigcap_\alpha \mathcal{S}_\alpha$.

- $\bigcap_\alpha \mathcal{S}_\alpha$ is closed under countable unions: Consider $S_1, S_2, \ldots \in \bigcap_\alpha \mathcal{S}_\alpha$. Since each S_i (for $i \in \mathbb{N}_{>0}$) is in $\bigcap_\alpha \mathcal{S}_\alpha$, this means that each S_i is in each $\mathcal{S}_\alpha$ for all α. And since each $\mathcal{S}_\alpha$ is a σ-algebra, this means that $\bigcup_i S_i$ is also in each $\mathcal{S}_\alpha$. This means that $\bigcup_i S_i$ must be in the intersection $\bigcap_\alpha \mathcal{S}_\alpha$. ■

Exercise 1.17.

Question. Let $(X, \mathcal{S}, \mu)$ be a measure space. Prove the following:

- μ is **monotone**: If $A, B \in \mathcal{S}$, $A \subset B$, then $\mu(A) \leq \mu(B)$.
- μ is **countably sub-additive**: If $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$, then $\mu\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mu(A_i)$.

Answer.

- **Monotonicity**: Let If $A, B \in \mathcal{S}$, $A \subset B$ (strictly), then $B \setminus A \neq \emptyset$. This means that there are some elements in B that are also in $\mathcal{S}$ that are not in A. Call the set of all such elements $(B \setminus A)$. This means that $(B \setminus A) \cap A = \emptyset$ and so by the second criterion of measure, we know by definition that $\mu(A \cup (B \setminus A)) = \mu(A) + \mu((B \setminus A))$. Since $(B \setminus A)$ contains at least one element, we can say $(B \setminus A) \neq \emptyset$ so $\mu((B \setminus A)) \geq 0$. But, note that $A \cup (B \setminus A)$ is simply B, so $\mu(A \cup (B \setminus A)) = \mu(B) = \mu(A) + \mu((B \setminus A))$. Since $\mu((B \setminus A)) \geq 0$, this means that $\mu(B) \geq \mu(A)$.
- **Countable sub-additivity**: *Since $\mathcal{A}$ was never defined in this exercise, I will assume the author means to refer to $\mathcal{S}$, the σ-algebra on X defined under the given measure space.* If all the A_i's are disjoint, then by the definition of measure, $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$. If at least one of the A_i's overlaps nontrivially with another A_j where $i \neq j$, then there is an element in $A_i \cap A_j$ contained strictly within $\bigcup_{i=1}^\infty A_i$. (If the sets were equal, then $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \mu(A_i)$ for any i, which is obviously less than or equal to $\sum_{i=1}^\infty \mu(A_i)$, since the image of μ is nonnegative.) Then, we know that $\bigcup_{i=1}^\infty A_i$ must strictly a subset of what we will write as $\bigcup_{i=1}^\infty B_i$, where each B_i has the same measure as each A_i, but there are no nonempty intersections between the B_i's. Since the B_i's are disjoint, then by the definition of measure, $\mu\left(\bigcup B_i\right) = \sum \mu(B_i)$. According to monotonicity, we know that $\mu\left(\bigcup A_i\right) \leq \mu\left(\bigcup B_i\right) = \sum \mu(B_i)$. But remember that each B_i has the same measure as each A_i, so we arrive at $\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$. ■

Exercise 1.18.

Question. Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda : \mathcal{S} \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure on $(X, \mathcal{S})$.

Answer.

- The intersection of the empty set with any other set is the empty set, so $\lambda(\emptyset) = \mu(\emptyset) = 0$.
- The measure of union of disjoint sets $A_1, A_2, \ldots$ is $\lambda\left(\bigcup A_i\right) = \mu\left(\left(\bigcup A_i\right) \cap B\right) = \mu\left((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots\right)$, with each $A_i \cap B$ disjoint, so that equals $\sum_i \mu(A_i \cap B)$ by definition of the measure μ, which is simply $\sum_i \lambda(A_i)$. Therefore, λ is also a measure. ■

Exercise 1.20.

Question. If μ is a measure on $(X, \mathcal{S})$, then prove it is continuous from below in the sense that:

$$(A_1 \supset A_2 \supset \cdots, A_i \in \mathcal{S}, \mu(A_1) < \infty) \implies \left(\lim_{n\to\infty} \mu(A_n) = \mu \left(\bigcap_{i=1}^{\infty} A_i \right) \right).$$

Answer.

- Define $B_m = A_m \smallsetminus A_{m+1}$ for $m \in \mathbb{N}_{>0}$. Then, $A_i = (\bigcap_{i=1}^{\infty} A_i) \cup (\bigcup_{m=n}^{\infty} B_m)$, and the sets in the resultant union are all disjoint, so

$$\mu(A_n) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \sum_{m=n}^{\infty} \mu(B_m).$$

So, if $\mu(A_N) < \infty$ for some $N \in \mathbb{N}_{>0}$, then $\mu(\bigcap_{i=1}^{\infty} A_i) < \infty$ and $\sum_{m=N}^{\infty} \mu(B_m) < \infty$, and so:

$$\begin{aligned} \lim_{n\to\infty} \mu(A_n) &= \lim_{n\to\infty} \left(\mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \sum_{m=n}^{\infty} \mu(B_m) \right) \\ &= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(B_m) \\ &= \mu\left(\bigcap_{i=1}^{\infty} A_i\right). \end{aligned}$$

■

Exercise 2.10.

Question. Consider the Carathéodory Construction: Let μ^* be an outer measure on X, and consider the collection $\mathcal{M}$ of subsets $E \subset X$ such that for every $B \subset X$,

$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$$

Then $\mathcal{M}$ is a σ-algebra, and $\bar{\mu} : \mathcal{M} \to [0, \infty]$ defined by $\bar{\mu} : \mu^*|_{\mathcal{M}}(\bar{\mu}(E) = \mu^*(E)$ for $E \in \mathcal{M})$ is a measure on $\mathcal{M}$.
Explain why

$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$$

could be replaced by

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$$

Answer.

- The proof of the theorem of the Carathéodory Construction is already given, so I will explain simply why one can replace
$$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$$
with
$$\mu^*(B) = \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right),$$
based primarily on the proof given. This replacement is possible because:
    - If $E = \emptyset$, then $\mu^*(B) = \mu^*(B \cap \emptyset) + \mu^*(B \cap X) = \mu^*(\emptyset) + \mu^*(B)$, which is simply $0 + \mu^*(B)$.
    - $\mathcal{M}$ is still closed under complements because the expression $\mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$ is still symmetric by its construction.
    - $\mathcal{M}$ is closed under countable unions.
        - $\mathcal{M}$ is closed under finite unions. Let $E, F \in \mathcal{M}$. Want to show $E \cup F \in \mathcal{M}$. We observe that
$$B \cap (E \cup F) = \left(B \cap E \cap F^{\complement}\right) \cup \left(B \cap E^{\complement} \cap F\right) \cup (B \cap E \cap F)$$
and because $F \in \mathcal{M}$,we have both
$$\mu^*\left(B \cap E \cap F^{\complement}\right) + \mu^*(B \cap E \cap F) = \mu^*(B \cap E)$$
and
$$\mu^*\left(B \cap E^{\complement} \cap F^{\complement}\right) + \mu^*\left(B \cap E^{\complement} \cap F\right) = \mu^*\left(B \cap E^{\complement}.\right)$$
Now using ***the definition of measure*** (because the sets $B \cap E^{\complement} \cap F^{\complement}$, $B \cap E \cap F$, $B \cap E \cap F^{\complement}$, and $B \cap E^{\complement}, F$ are all pairwise disjoint, so we can add their measures to get the measure of the disjoint union), the specified above equalities, and the fact that $E \in \mathcal{M}$, we can deduce that
$$\begin{aligned}\mu^*(B \cap (E \cup F)) + \mu^*\left(B \cap (E \cup F)^{\complement}\right) &= \mu^*\left(B \cap E \cap F^{\complement}\right) + \mu^*(B \cap E \cap F) \\ &\quad + \mu^*\left(B \cap E^{\complement} \cap F^{\complement}\right) + \mu^*\left(B \cap E^{\complement} \cap F\right) \\ &= \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right) \\ &= \mu^*(B).\end{aligned}$$
This shows closedness under finite union.
    - Now take $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$. Without loss of generality, we take the E_n to be pairwise disjoint. Define $E = \bigcup_{n \in \mathbb{N}} E_n$. By closedness under finite union, we have
$$\begin{aligned}\mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) &= \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right) \cap E_N\right) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right) \cap E_N^{\complement}\right) \\ &= \mu^*(B \cap E_N) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{N-1} E_n\right)\right) \\ &\vdots \\ &= \sum_{n=1}^{N} \mu^*(B \cap E_n).\end{aligned}$$

Next, note that by our specified new criterion $\mu^*(B) = \mu^*(B \cap E) + \mu^*\left(B \cap E^{\complement}\right)$ and the fact that the E_n's are disjoint, we have that for all N,

$$\begin{aligned} \mu^*(B) &= \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right)^{\complement}\right) \\ &= \sum_{n=1}^{N} \mu^*(B \cap E_n) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right)^{\complement}\right), \end{aligned}$$

where $\mu^*(B)$ does not depend on N, so this holds for the limit. Thus,

$$\begin{aligned} \mu^*(B) &= \sum_{n=1}^{\infty} \mu^*(B \cap E_n) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{\infty} E_n\right)^{\complement}\right) \\ &= \mu^*\left(B \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) + \mu^*\left(B \cap \left(\bigcup_{n=1}^{\infty} E_n\right)^{\complement}\right), \end{aligned}$$

with the equality still holding because as the number of pairwise disjoint sets in E_n approaches ∞, either the size of each set successive approaches zero, or the union approaches the entire space. Therefore we know that $\mathcal{M}$ is a σ-algebra.

- $\bar{\mu}$ is a measure because if we replace the first line from above and replace B by $\bigcup_{n=1}^{\infty} E_n$, we get $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n)$.

Exercise 2.14.

Question. Why is it true that the Borel-algebra $\mathcal{B}(\mathbb{R})$ is a subset of $\mathcal{M}$? Hint: Carathéodory does most of the work—you only need to show that $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$.

Additional information:

- **Premeasure:** $\nu : \mathcal{A} \to [0, \infty]$ is called a premeasure if it satisfies:
  - $\nu(\emptyset) = 0$.
  - $\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n)$ for $\{A_n\} \subset \mathcal{A}$, pairwise disjoint.
- **Outer Measure:** Given X, we call $\mu^* : \mathcal{P}(X) \to [0, \infty]$ an outer measure if:
  - $\mu^*(\emptyset) = 0$.
  - monotone: $A \subset B \implies \mu^*(A) \le \mu^*(B)$.
  - countably sub-additive: $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)\ \forall A_i \in \{A_i\}_{i=1}^{\infty}$.
- **Carathéodory Construction:** Let μ^* be an outer measure on X, and consider the collection $\mathcal{M}$ of subsets $E \subset X$ such that for every $B \subset X$,

$$\mu^*(B) \ge \mu^*(B \cap E) + \mu^*(B \cap E^{\complement}).$$

  Then $\mathcal{M}$ is a σ-algebra and $\bar{\mu} : \mathcal{M} \to [0, \infty]$ defined by $\bar{\mu} = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.

- **Generation of outer measure:** Let $\mathcal{A}$ be a collection of subsets of X, containing $\emptyset$ and X. Let ν satisfy $\nu(\emptyset) = 0$. Then, $\mu^* : \mathcal{P}(X) \to [0, \infty]$ defined as

$$\mu^*(B) := \inf \left\{ \sum_{n \in \mathbb{N}} \nu(A_n) \,\middle|\, \{A_n\} \subset \mathcal{A}, B \subset \bigcup_{n \in \mathbb{N}} A_n \right\}$$

constitutes an outer measure on X. We call it the outer measure generated by ν.

- **Carathéodory Extension Theorem:** Let $(\mathcal{A}, \nu)$ be an algebra-premeasure pair on X. Let μ^* denote the outer measure generated by ν, and let $\mathcal{M}$ denote the σ-algebra from the Carathéodory construction. Let $\sigma(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$ and $\mu := \mu^*|_{\sigma(\mathcal{A})}$. Then, $\sigma(\mathcal{A}) \subset \mathcal{M}$, and $\mu|_{\mathcal{A}} = \mu^*|_{\mathcal{A}} = \nu$.

- **Borel σ-algebra $\mathcal{B}(\mathbb{R})$:** Let X be a metric space, and let $\mathcal{O}$ denote the collection of open sets of X. $\sigma(\mathcal{O})$ is thus the smallest σ-algebra containing all open sets of X.

Answer.

- The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all the open sets of $\mathbb{R}$, collectively called $\mathcal{O}$. In the Carathéodory Extension Theorem, we see that for an algebra-premeasure pair $(\mathcal{A}, \nu)$ with set X and accompanying σ-algebra $\mathcal{M}$, the σ-algebra generated by $\mathcal{A}$ is a subset of $\mathcal{M}$. Thus, in order to show that $\mathcal{B}(\mathbb{R})$ is a subset of $\mathcal{M}$, it suffices to show that $\sigma(\mathcal{O}) = \sigma(\mathcal{A})$. Note that if the underlying set is $X = \mathbb{R}$, then we know that every subset of $\mathbb{R}$ is contained within an open subset of $\mathbb{R}$, so the σ-algebras of all the open sets of $\mathbb{R}$ (the Borel σ-algebra on $\mathbb{R}$) is equivalent to the σ-algebra $\sigma(\mathcal{A})$ generated by any collection of subsets of $\mathbb{R}$.

Exercise 3.1.

Question. Prove that every countable subset of the real line has Lebesgue measure 0.

Answer.

- Consider the case for the Lebesgue outer measure. Let $x \in \mathbb{R}$. Then $\{x\} \subset [x - \varepsilon, x + \varepsilon]$ for all $\varepsilon > 0$ and so $\lambda^*(\{x\}) = 0$ for all $x \in \mathbb{R}$. If $X = \{x_1, x_2, \ldots\} = \bigcup_{n=1}^{\infty} \{x_n\}$ is countable, then $\lambda^*(X) \leq \sum_{n=1}^{\infty} \lambda^*(\{x_n\}) = 0$, so $\lambda^*(X) = 0$. Since every countable subset of the real line has Lebesgue outer measure 0, then the outer measure restricted to $\mathcal{M}$, namely the Lebesgue measure, must also have measure 0 for such sets. ■

Exercise 3.4.

Question. Explain why the set $\{x \in X \mid f(x) < a\}$ could be replaced by any of the following:

$$\{x \in X \mid f(x) \leq a\}$$

$$\{x \in X \mid f(x) > a\}$$

$$\{x \in X \mid f(x) \geq a\}.$$

Answer.

- If $\{x \in X \mid f(x) < a\} \in \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra, then the complement $\{x \in X \mid f(x) < a\}^{\complement} = X \setminus \{x \in X \mid f(x) < a\} = \{x \in X \mid f(x) \geq a\}$ is in $\mathcal{M}$. Since $\{x \in X \mid f(x) = a\} \subseteq \{x \in X \mid f(x) \geq a\} \in \mathcal{M}$ and $\mathcal{M}$ is closed under countable unions, we know $\{x \in X \mid f(x) = a\} \cup \{x \in X \mid f(x) < a\} = \{x \in X \mid f(x) \leq a\} \in \mathcal{M}$, and so $\{x \in X \mid f(x) \leq a\}^{\complement} = \{x \in X \mid f(x) > a\} \in \mathcal{M}$.

Exercise 3.7.

Question.
Theorem: Let $f, g, \{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}$ be measurable functions on $(X, \mathcal{M})$, and let $F : (\operatorname{im}(f), \operatorname{im}(g)) \to \mathbb{R}$ be continuous. Then the following are measurable:

1. $f + g$, $f \cdot g$, $\max\{f, g\}$, $\min\{f, g\}$, $|f|$.
2. $\sup_{n\in\mathbb{N}} f_n(x)$, $\inf_{n\in\mathbb{N}} f_n(x)$.
3. $\limsup_{n\to\infty} f_n(x)$, $\liminf_{n\to\infty} f_n(x)$.
4. $F(f(x), g(x))$.

Explain why $(2) \wedge (4) \implies (1)$.

Answer.

- note that $\sup_{n\in\mathbb{N}} f_n(x) \geq \max_{n\in\mathbb{N}} f_n(x)$ and $\inf_{n\in\mathbb{N}} f_n(x) \leq \min_{n\in\mathbb{N}} f_n(x)$, so if

$$\left\{x \in X \mid \sup_{n\in\mathbb{N}} f_n(x) < a\right\} \in \mathcal{M}$$

then

$$\left\{x \in X \mid \max_{n\in\mathbb{N}} f_n(x) < a\right\} \in \mathcal{M};$$

likewise, if

$$\left\{x \in X \mid \inf_{n\in\mathbb{N}} f_n(x) > a\right\} \in \mathcal{M}$$

then

$$\left\{x \in X \mid \min_{n\in\mathbb{N}} f_n(x) > a\right\} \in \mathcal{M}.$$

Next, note that if there is a continuous mapping that takes the images of f and g to $\mathbb{R}$ that is also measurable, then $f + g$ and fg, the sum and product of the images, are also continuous and so they are also measurable. Finally, the absolute value $|f|$ can be defined as $\max\{f, 0\} - \min\{f, 0\}$, which is a sum of measurable functions and so is measurable itself, as shown in the first part of (1).

Exercise 3.14.

Question.
Theorem: For $f : X \to \mathbb{R}$,

1. $\exists \{s_n\}$, such that $s_n \to f$ pointwise, i.e., for any $x \in X$, $\lim_{n\to\infty} s_n(x) = f(x)$,
2. if f is measurable, $\{s_n\}$ may be taken to be measurable,
3. if $f \geq 0$, $\{s_n\}$ may be taken as an increasing sequence (i.e., $s_n \leq s_{n+1}$), and
4. if f is bounded, the convergence in (1) is uniform.

Prove (4).

Answer.

- This is an applied case of the Simple Function Approximation Lemma. Consider the approximation from below,
$$\phi_n \coloneqq \sum_{i=1}^{n\cdot 2^n} \frac{i}{2^n}\chi_{E_i^n} + n\chi_{E_\infty^n},$$
where
$$E_i^n \coloneqq f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)$$
is defined for $n \in \mathbb{N}_{>0}$ and $i \in \mathbb{N} \cap [1, 2^n]$ and
$$E_\infty^n \coloneqq f^{-1}([n,\infty))$$
for $n \in \mathbb{N}_{>0}$. Note that if f is bounded, then E_∞^n is empty for large enough n. Since $f - \phi_n < \frac{1}{2^n}$ on $(E_\infty^n)^\complement$—which approaches the whole space for large n—it follows that the convergence is uniform. Note that analogous reasoning holds for the approximation from above. ■

Exercise 4.13.

Question. Prove: If f is measurable, $\|f\| < M$ on $E \in \mathcal{M}$ and $\mu(E) < \infty$, then $f \in \mathscr{L}^1(\mu, E)$.

Answer.

- We know that f is measurable, and $\|f\| = f^+ + f^-$ (by Remark 4.10) is bounded, namely $\|f\| < M$ on $E \in \mathcal{M}$. By Definition 4.9, if both $\int_E f^+ \, \mathrm{d}\mu$ and $\int_E f^- \, \mathrm{d}\mu$ are finite, then f is integrable with respect to μ; i.e., we can say that $f \in \mathscr{L}^1$. ■

Exercise 4.14.

Question. Prove: If $f \in \mathscr{L}^1(\mu, E)$, then f is finite almost everywhere on E.

Answer.

- Let $E_n \coloneqq f^{-1}((n,\infty))$. Then we know that $n\mu(E_n) \leq \int_{E_n} f \,\mathrm{d}\mu$. Then:
$$\begin{aligned} \mu(E_n) &\leq \frac{1}{n}\int_{E_n} f \,\mathrm{d}\mu \\ &\leq \frac{1}{n}\int_{E} f \,\mathrm{d}\mu. \end{aligned}$$
Also, $\lim_{n\to\infty} \mu(E_n) \to 0$ because $\int_E f \,\mathrm{d}\mu < \infty$. Next, since
$$f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} E_n,$$
where $E_{n+1} \subset E_n$ for all n, and that there is an N such that $\mu(E_N) < \infty$, then by *continuity from above* we have that
$$\mu(f^{-1}(\{\infty\})) = \lim_{n\to\infty} \mu(E_n) = 0.$$
Thus, it follows that f must be finite everywhere, except on a set of measure zero; i.e., f is finite almost everywhere. ■

Exercise 4.15.

Question. Prove: $f, g \in \mathscr{L}^1(\mu, E)$, $f \leq g$ on $E \implies \int_E f \,\mathrm{d}\mu \leq \int_E g \,\mathrm{d}\mu$.

Answer.

- We know $f(x) \leq g(x)$ for all $x \in E$ and we know $0 \leq g(x) - f(x)$ for all $x \in E$. Note $\int_E f \,\mathrm{d}\mu \geq 0$ for $f(x) \geq 0 \;\forall x$, so it follows that
$$\int_E (g - f) \,\mathrm{d}\mu \geq 0.$$
Then, by linearity of the Lebesgue integral for (Lebesgue) measurable functions, we have
$$\int_E f \,\mathrm{d}\mu \leq \int_E g \,\mathrm{d}\mu.$$
■

Exercise 4.16.

Question. Prove: If $f \in \mathscr{L}^1(\mu, E)$, $A \in \mathcal{M}$, $A \subset E \implies f \in \mathscr{L}^1(\mu, A)$.

Answer.

- This follows straightforwardly from the definitions. If $f \in \mathscr{L}^1(\mu, E)$ for $E \in \mathcal{M}$, that is, if f is integrable with respect to μ, then $\int_E f^+ \,\mathrm{d}\mu$ and $\int_E f^- \,\mathrm{d}\mu$ are *by definition* finite. Since $A \subset E$, we can also say that $\int_E f^+|_A \,\mathrm{d}\mu = \int_A f^+ \,\mathrm{d}\mu$ and $\int_E f^-|_A \,\mathrm{d}\mu = \int_A f^- \,\mathrm{d}\mu$ are finite. Then, by definition of integrability, it follows that $f \in \mathscr{L}^1(\mu, A)$, given $A \in \mathcal{M}$ of course. ■

Exercise 4.21.

Question. Prove that if $A, B \in \mathcal{M}$, $B \subset A$, and $\mu(A \setminus B) = 0$, then if $f \in \mathscr{L}^1$,

$$\int_A f \, \mathrm{d}\mu \leq \int_B f \, \mathrm{d}\mu.$$

Answer.

- Consider the disjoint sets B and $A \setminus B$. Since $B \subset A$, then $A = B \cup (A \setminus B)$. From Remark 4.18 and Theorem 4.19, we know the Lebesgue integral is countably sub-additive for simple functions, and Lebesgue integrals can be approximated arbitrarily precisely by simple functions. The sub-additivity property gives us that $\int_A f \, \mathrm{d}\mu = \int_B f \, \mathrm{d}\mu + \int_{A \setminus B} f \, \mathrm{d}\mu$, but $\mu(A \setminus B) = 0$ so we have that $\int_B f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu$ for the lower bound on $\int_B f \, \mathrm{d}\mu$. ■

Exercise 4.28.

Question. If $f \in \mathscr{L}^1(\mu, E)$, then f is finite almost everywhere on E.

Answer.

- See my answer for Exercise 4.14:
- Let $E_n \coloneqq f^{-1}((n, \infty))$. Then we know that $n\mu(E_n) \leq \int_{E_n} f \, \mathrm{d}\mu$. Then:

$$\begin{aligned} \mu(E_n) &\leq \frac{1}{n} \int_{E_n} f \, \mathrm{d}\mu \\ &\leq \frac{1}{n} \int_E f \, \mathrm{d}\mu. \end{aligned}$$

  Also, $\lim_{n \to \infty} \mu(E_n) \to 0$ because $\int_E f \, \mathrm{d}\mu < \infty$. Next, since

$$f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} E_n,$$

  where $E_{n+1} \subset E_n$ for all n, and that there is an N such that $\mu(E_N) < \infty$, then by *continuity from above* we have that

$$\mu(f^{-1}(\{\infty\})) = \lim_{n \to \infty} \mu(E_n) = 0.$$

  Thus, it follows that f must be finite everywhere, except on a set of measure zero; i.e., f is finite almost everywhere. ■

Exercise 4.30.

(This is listed in the notes, but no question is given.)