Partially observable systems and Predictive State Representation (PSR)

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CS 598 Statistical RL @ UIUC

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Prev. frame Next frame

video prediction

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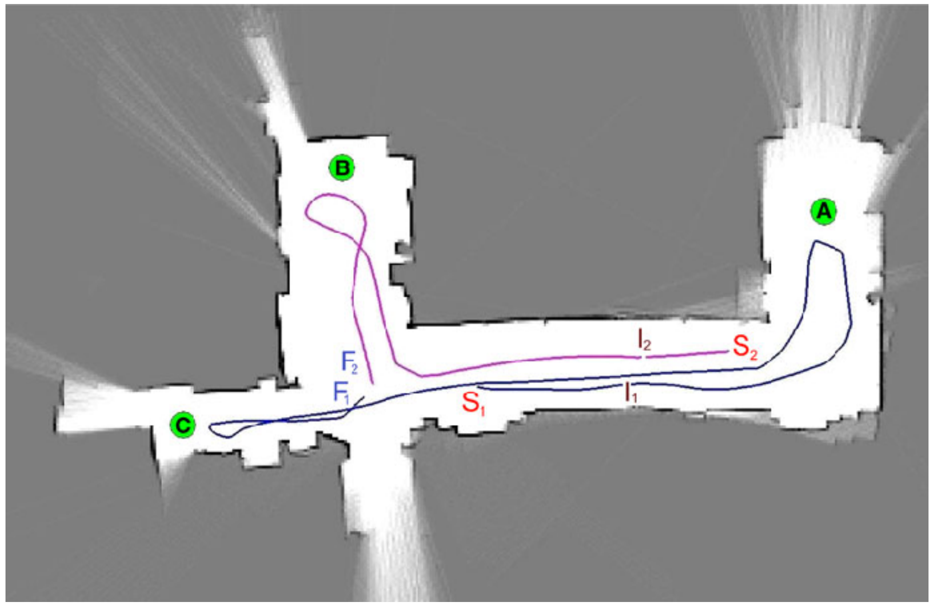

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"perceptual aliasing"

Prev. frame Next frame

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- How restrictive is Markov assumption?

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  - Need structure…

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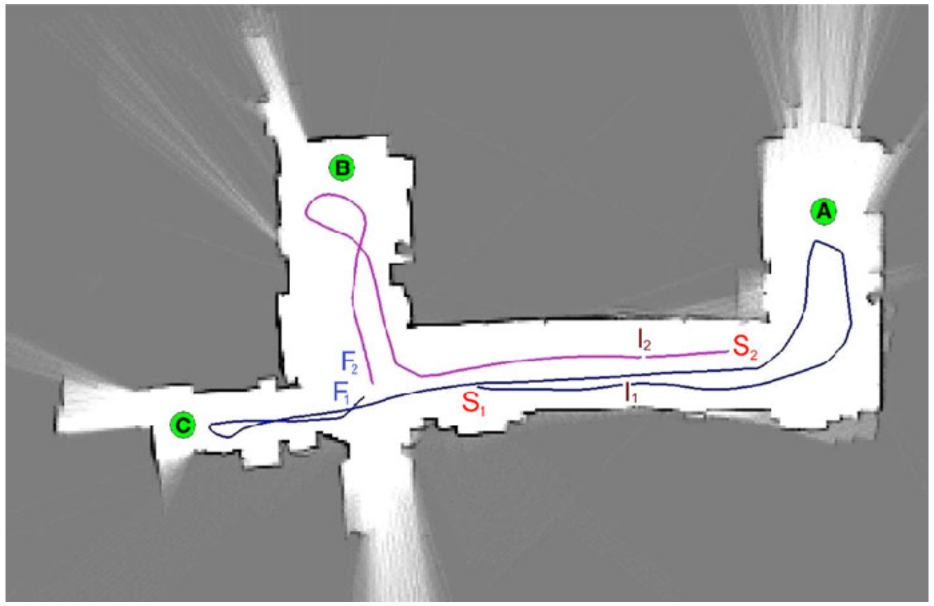

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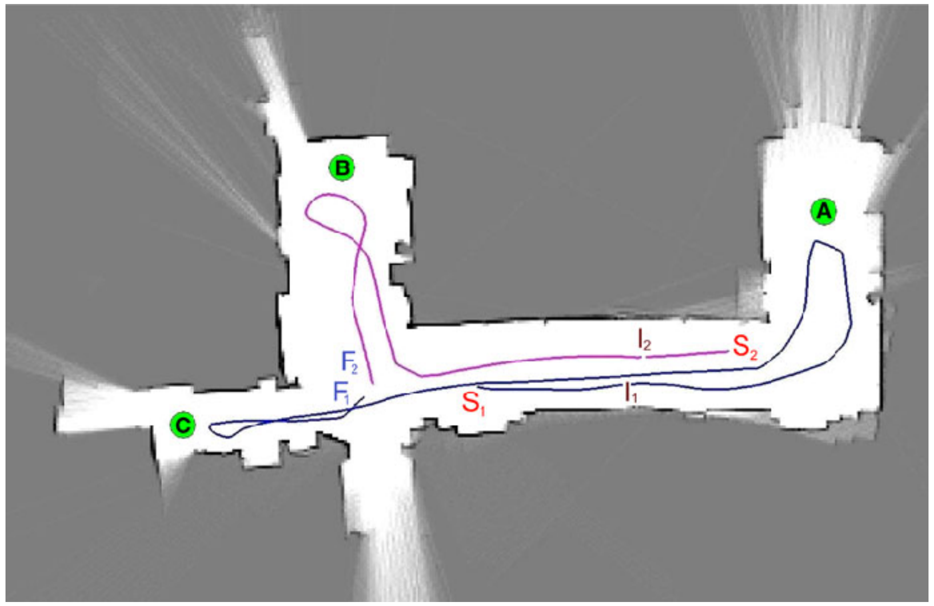

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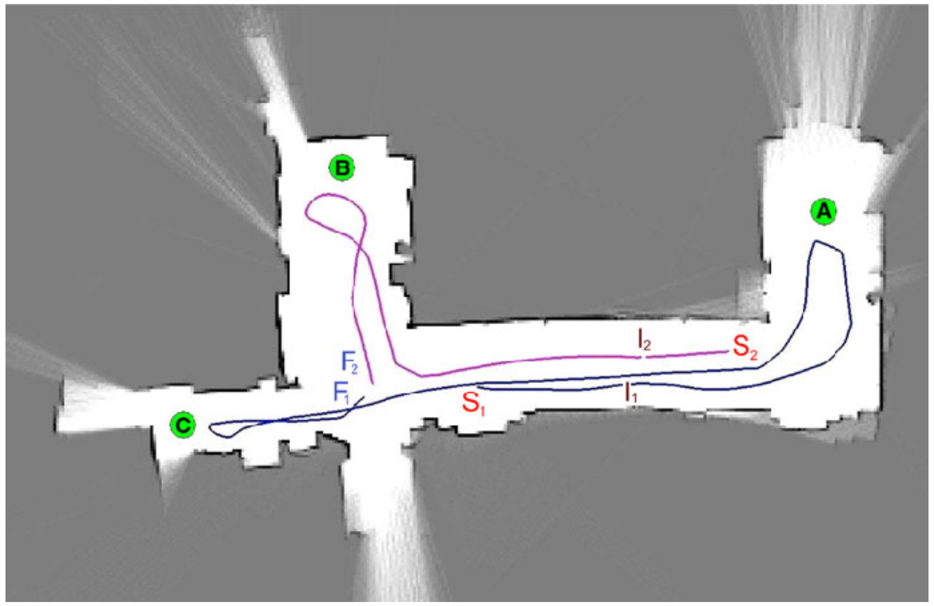
    - General PO system: you always visit a new location
    - With structural assumptions: the building only has this many different rooms. You will return to one or another.



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- Markov chain is special case: identity emission

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- Wrong: If emission discards all information, the process becomes Markov!

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- Again, one most generic way to specify a PO system is just $\Pr[o' \mid o_{1:\tau}]$, or $\Pr[o' \mid h]$ for short (h for history)

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- What does state mean in the PO setting?

Definition: **State** is a **function of history**, ϕ, that is a **sufficient statistics** for **predicting future**. That is, for all $t{:=}o_{\tau+1:\tau+k}$ and $h{:=}o_{1:\tau}$,

$$\Pr[t \mid h] = \Pr[t \mid \phi(h)]$$

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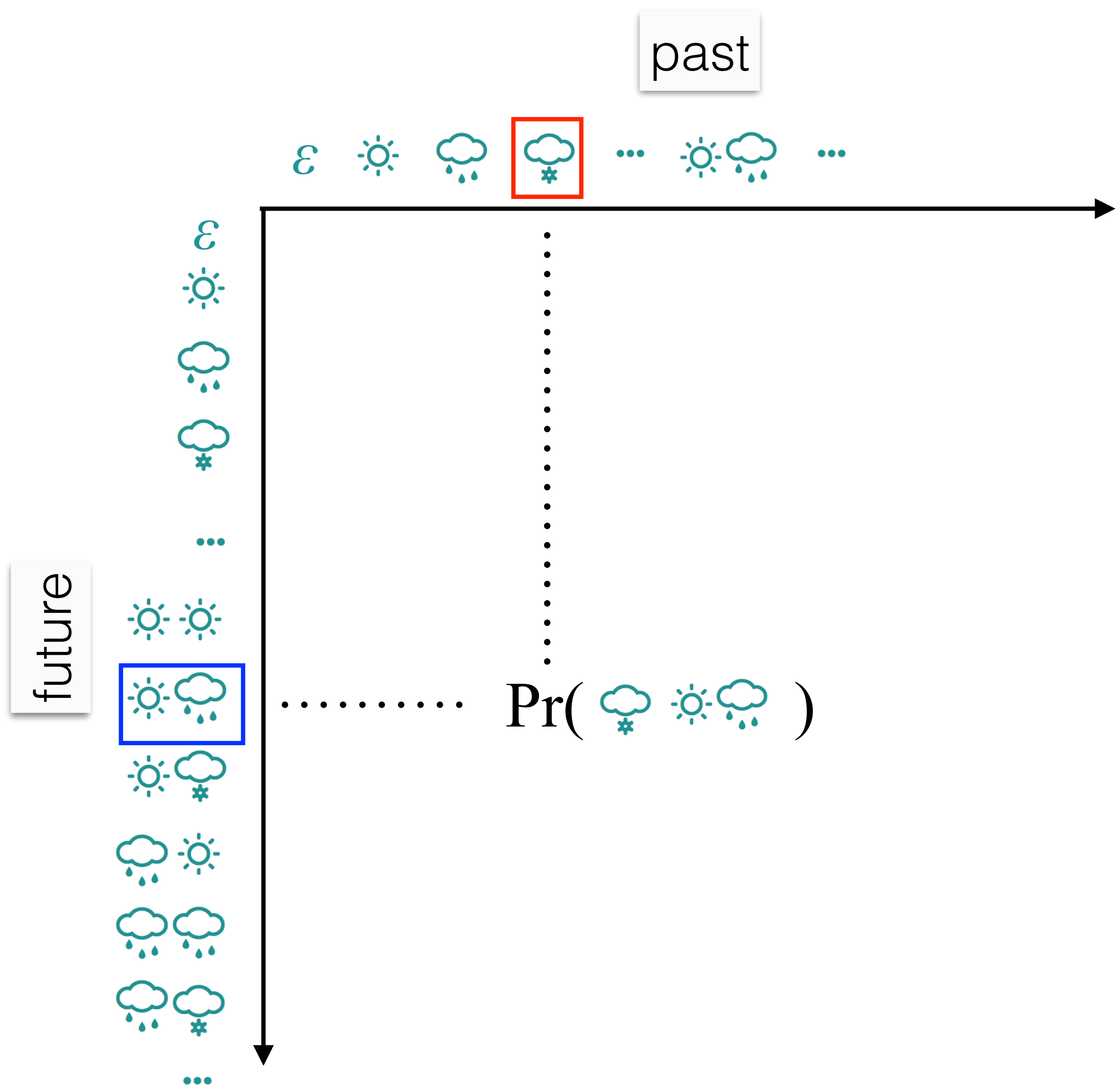
- But how to inject structure???

The system dynamics matrix M



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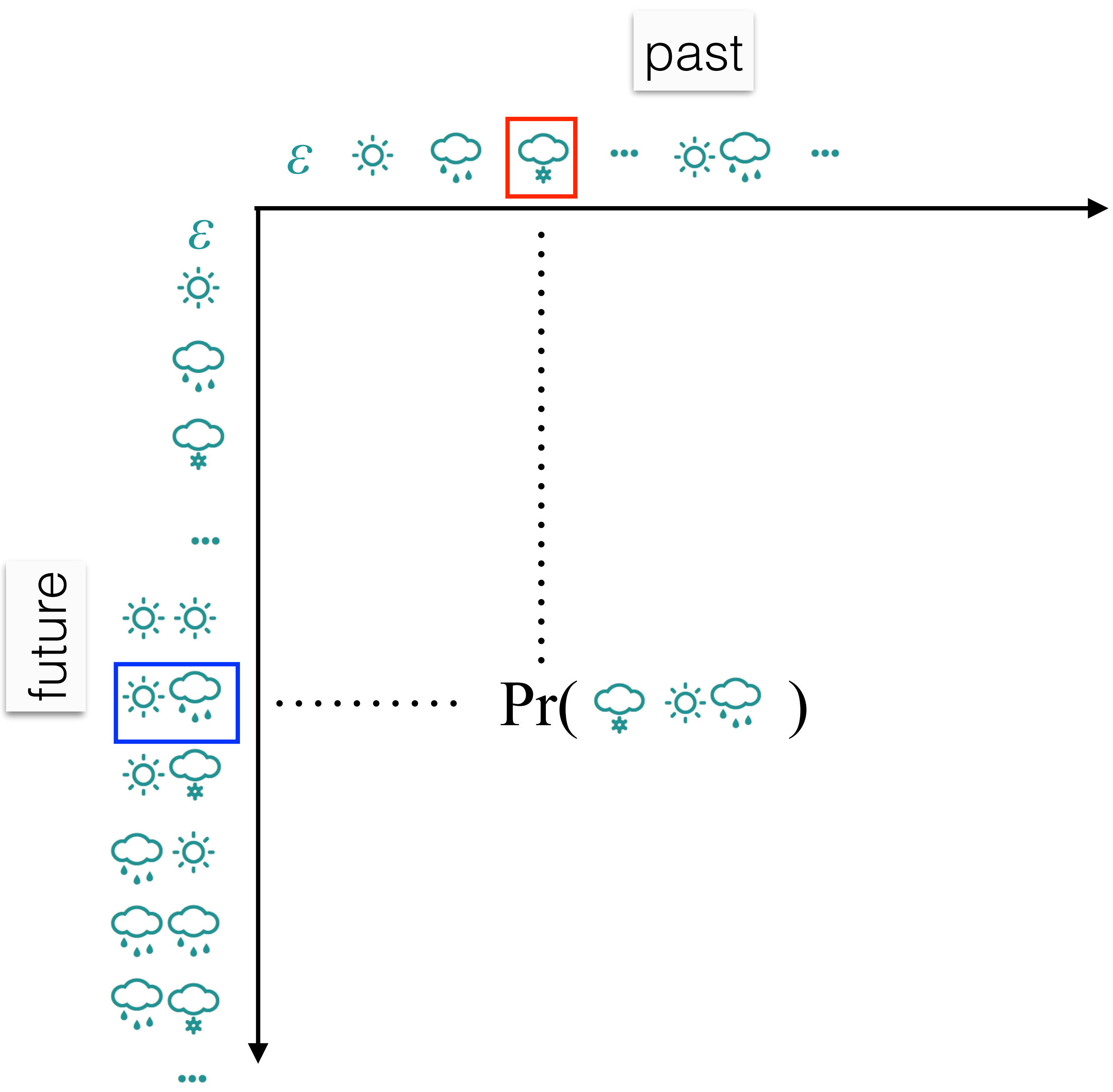

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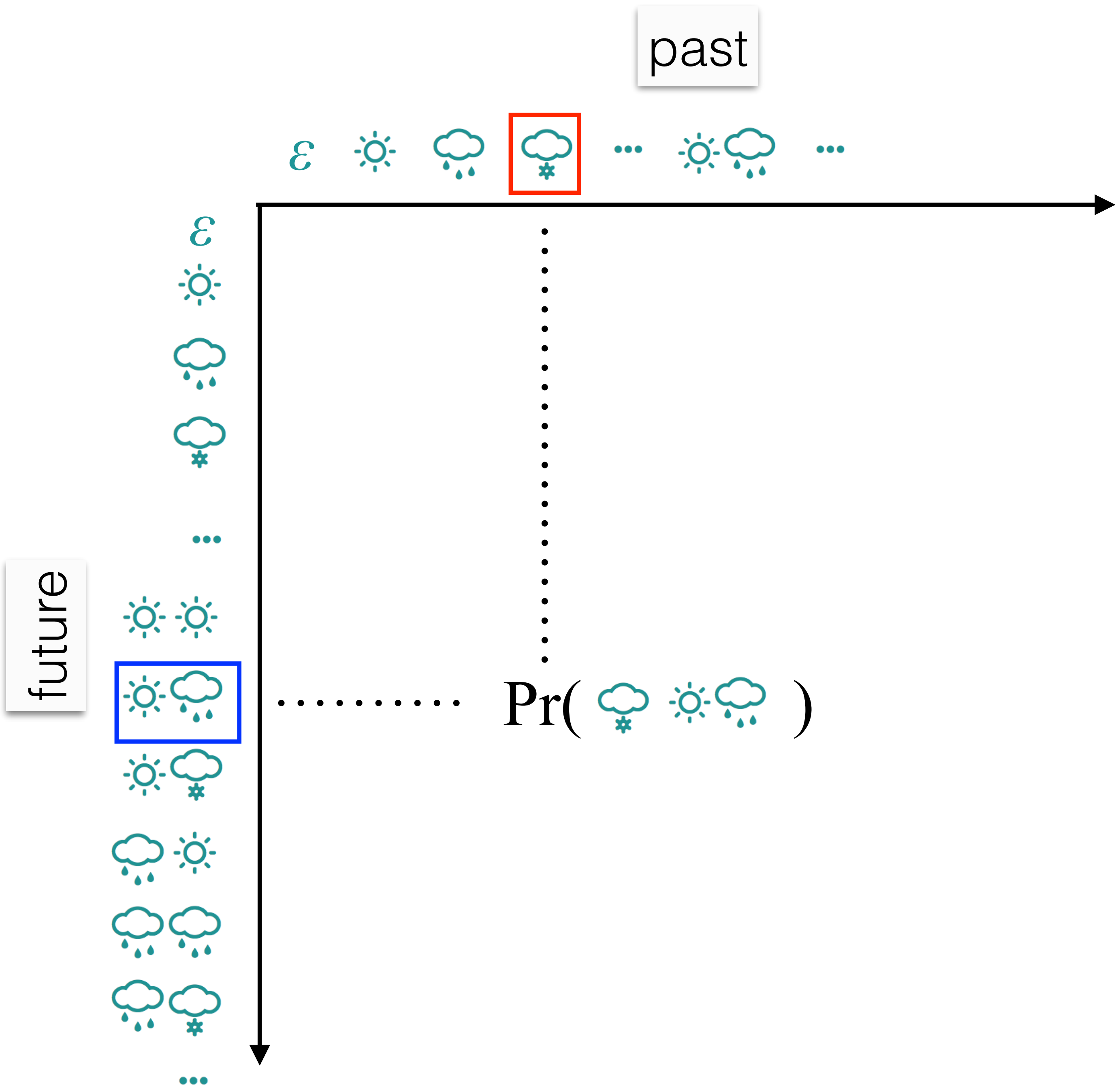

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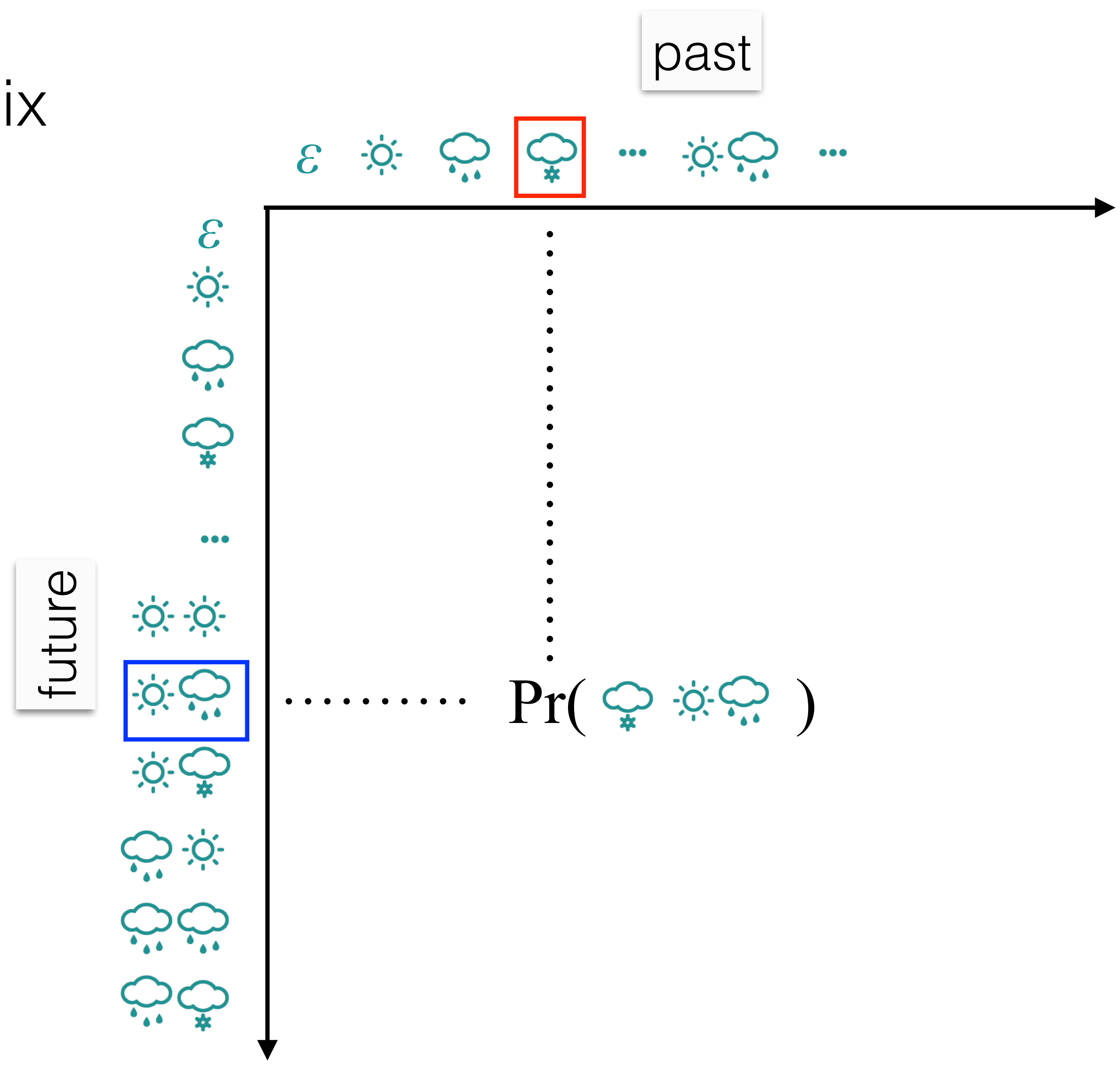

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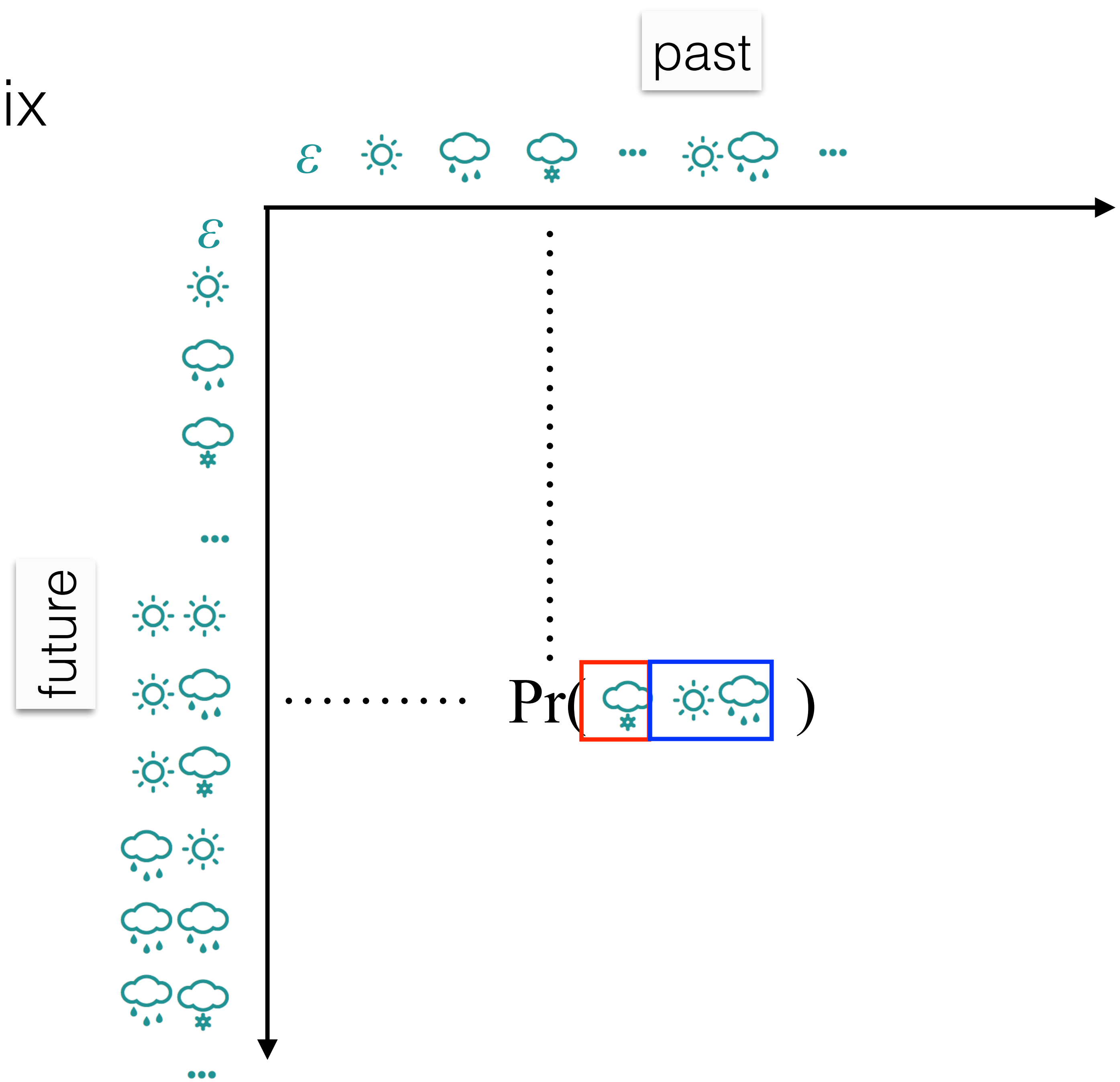

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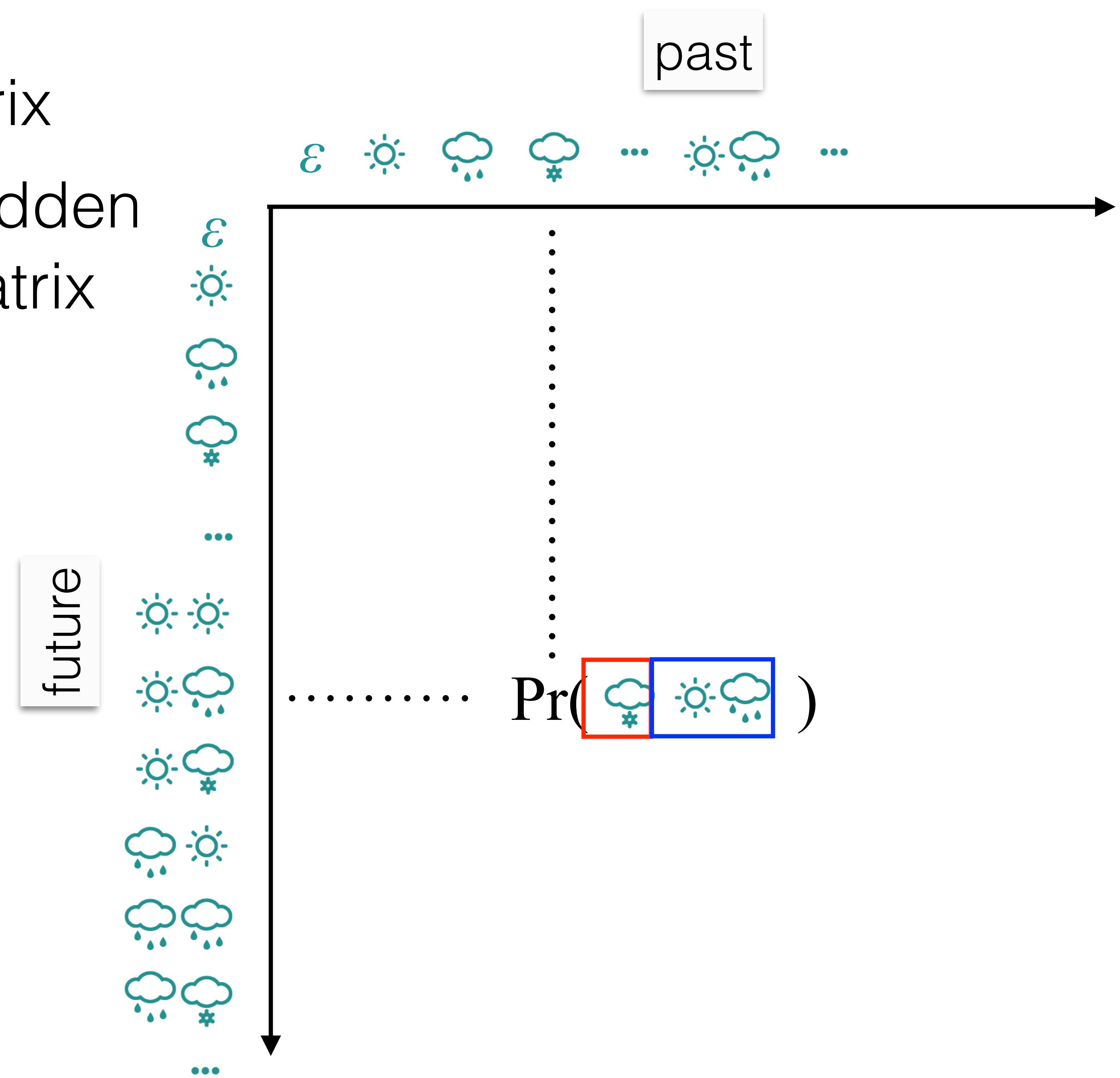

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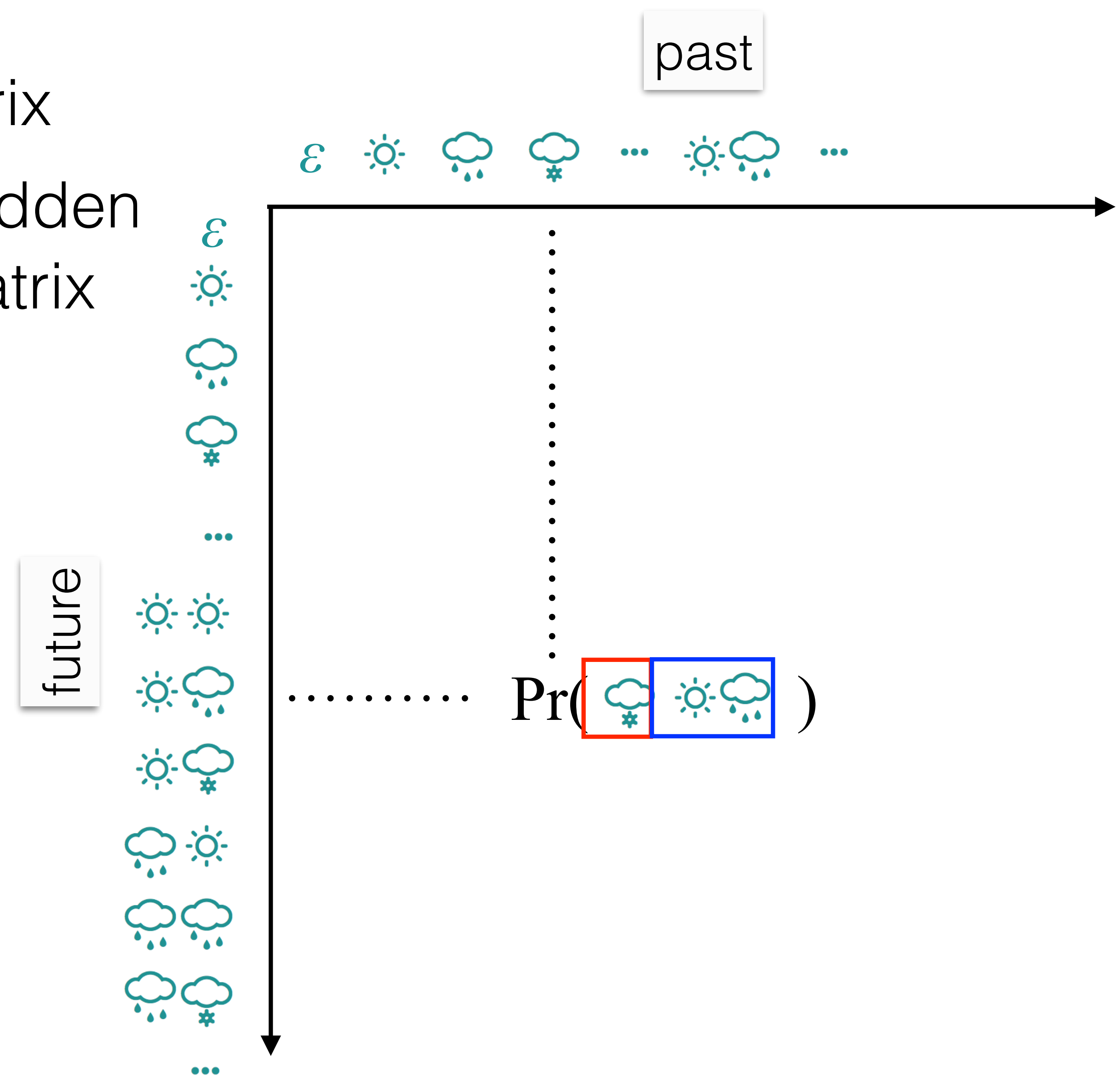

See project ref page for classical refs for PSRs
http://nanjiang.cs.illinois.edu/cs598project/

Low-rankness of SDM

- Proof: for any past h and future t, let the current timestep be τ

$$
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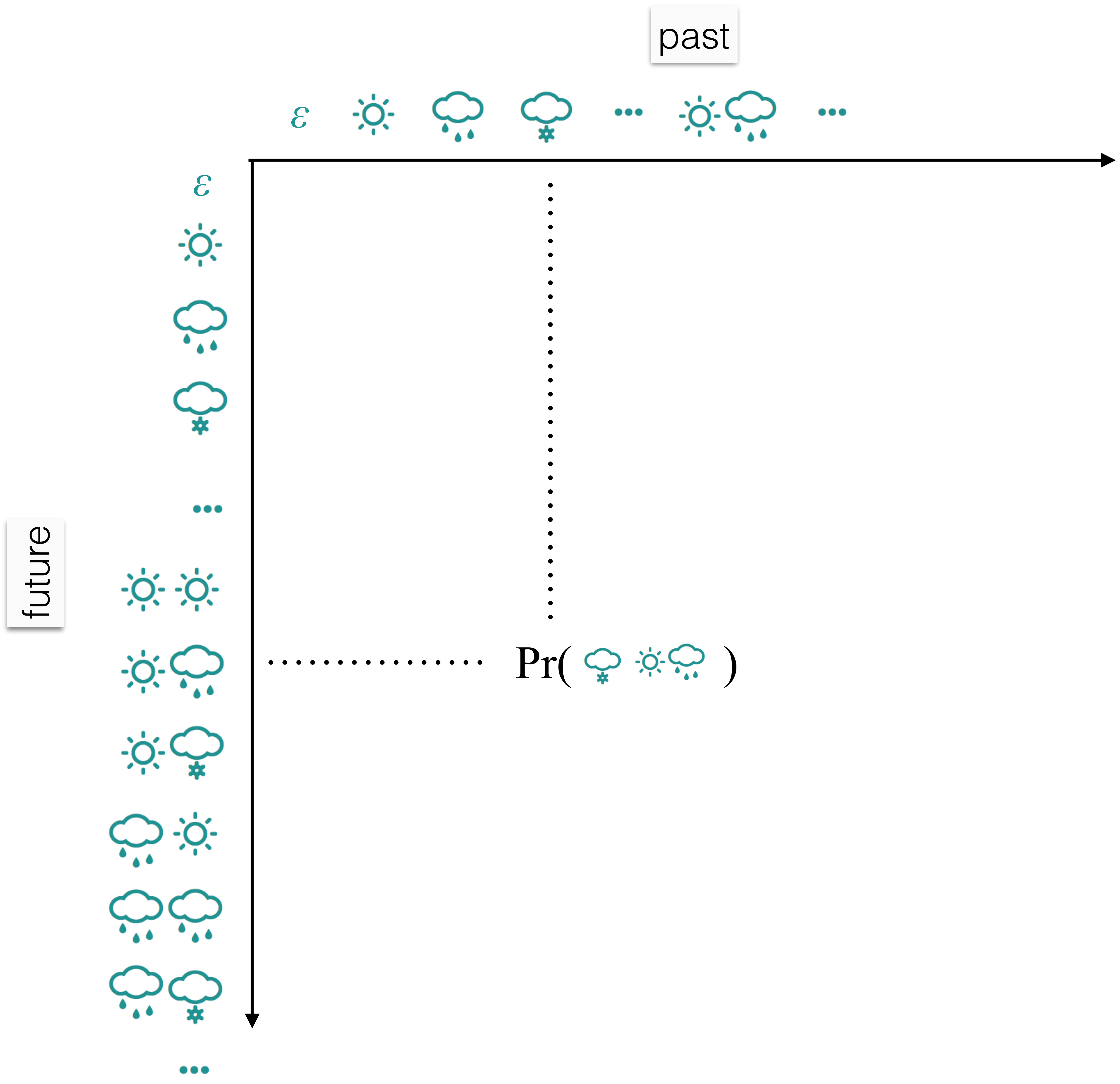
- rank of SDM is known as the *linear dimension* of the system
- Can we directly work with systems whose SDM has low-rank, instead of going through the latent variable route???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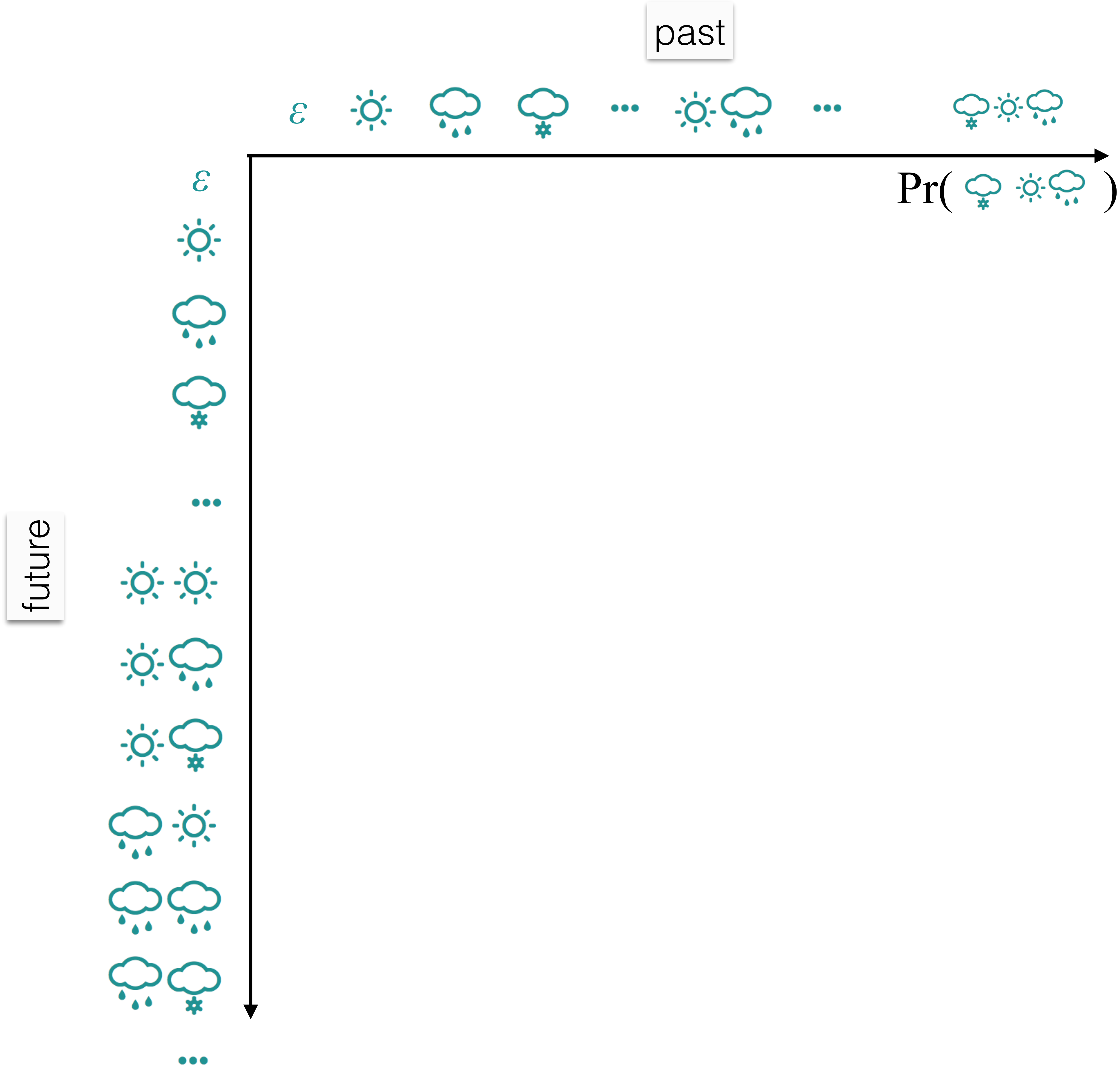
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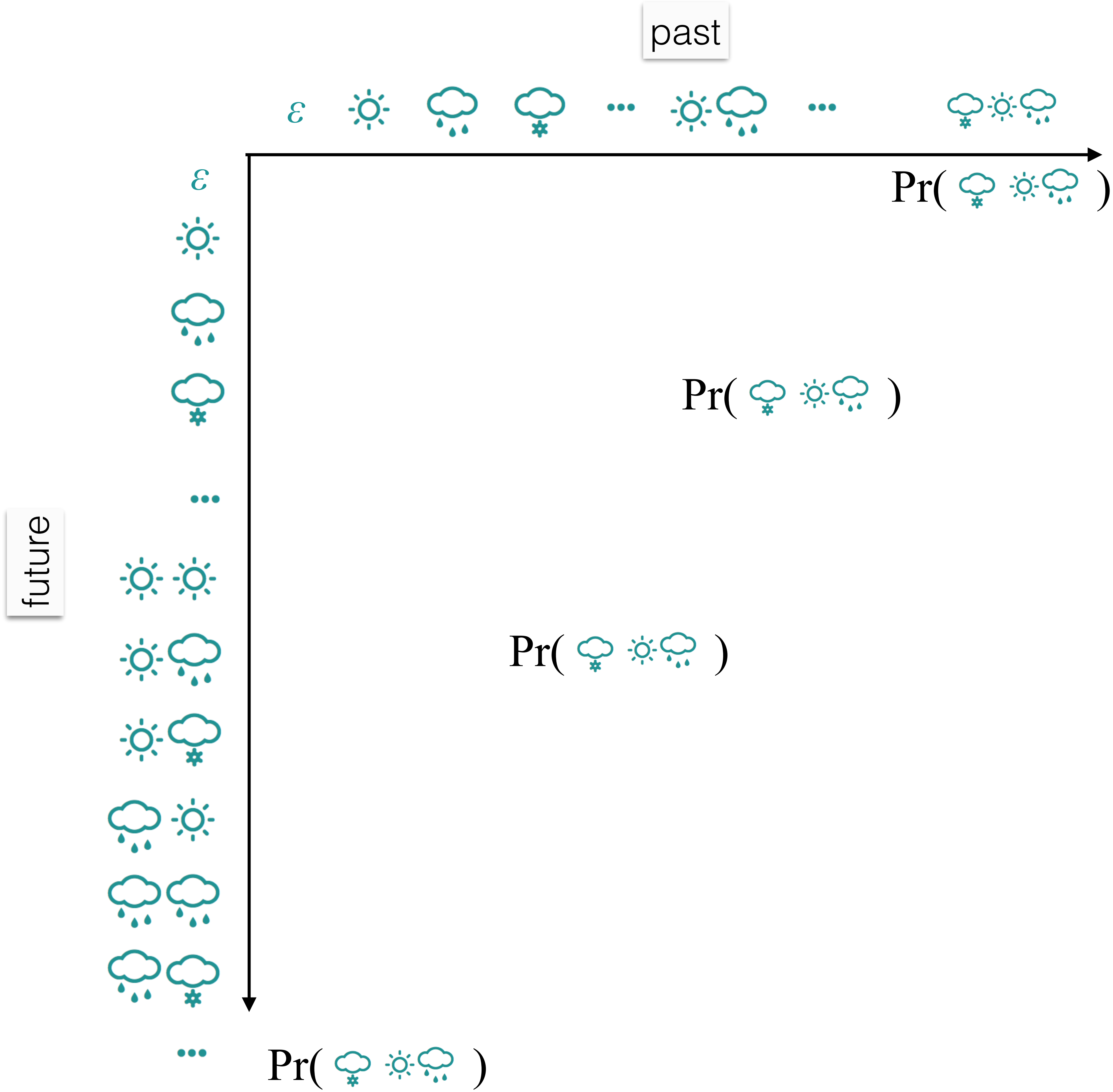
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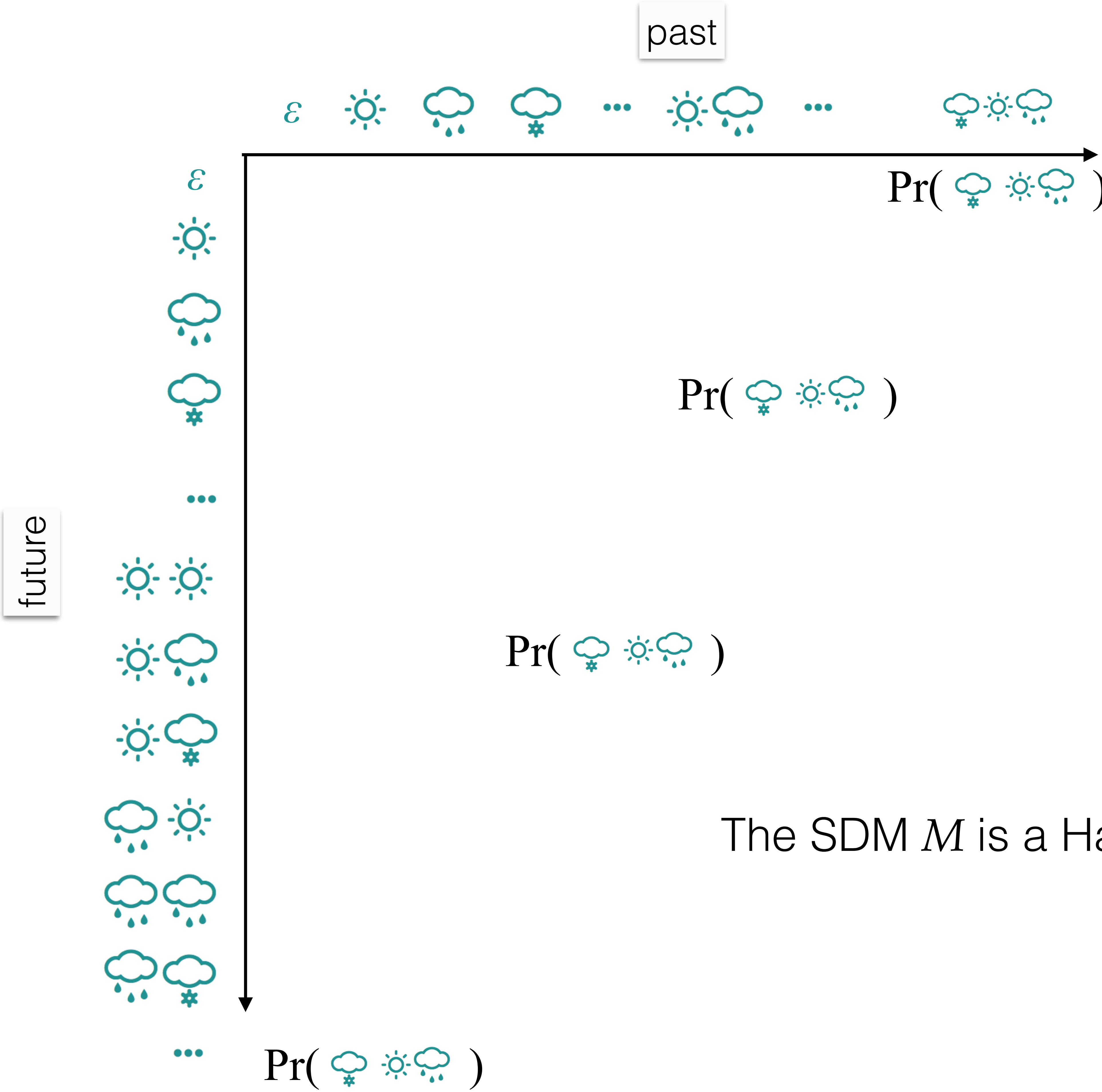
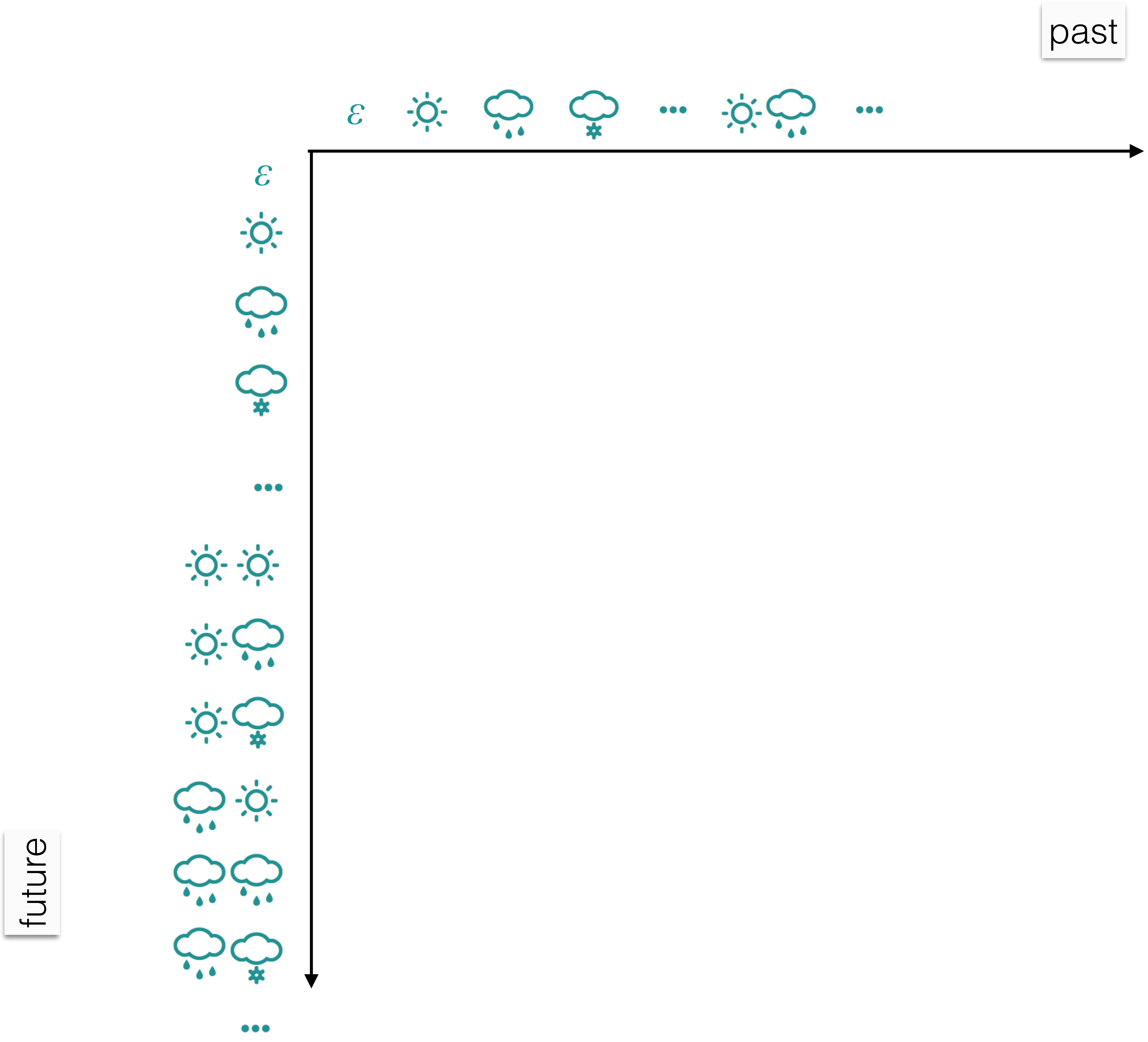


The SDM M is a Hankel matrix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ε
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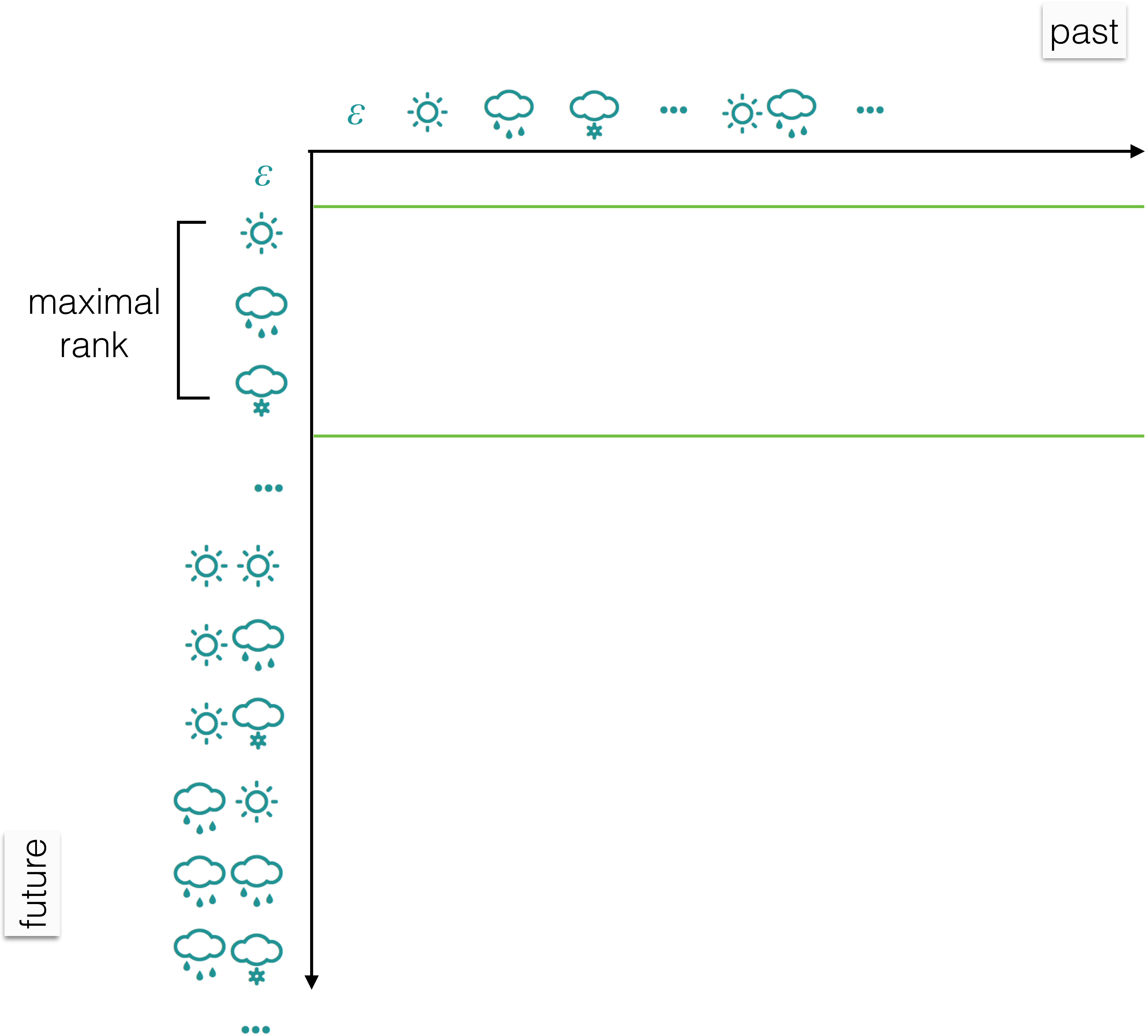
past
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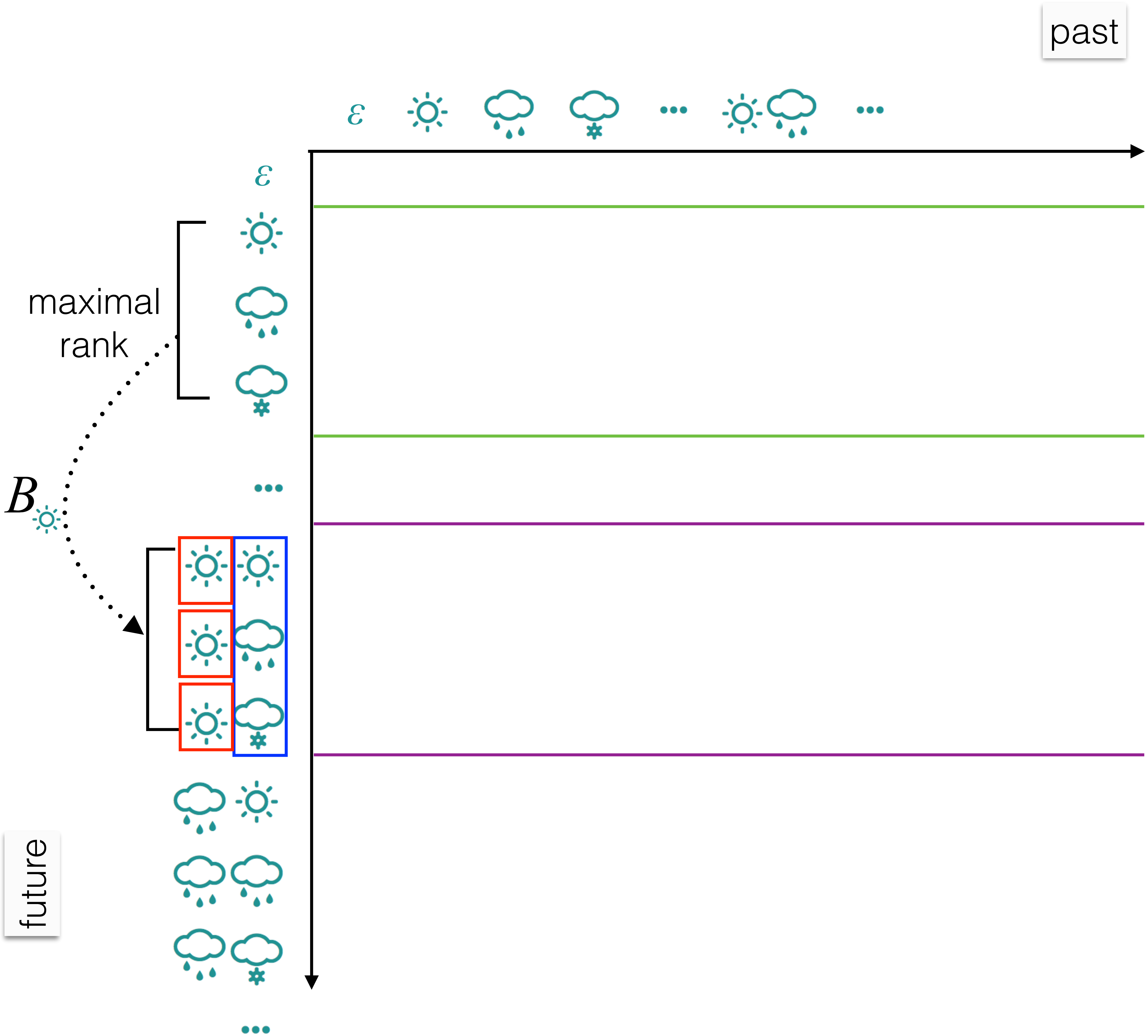
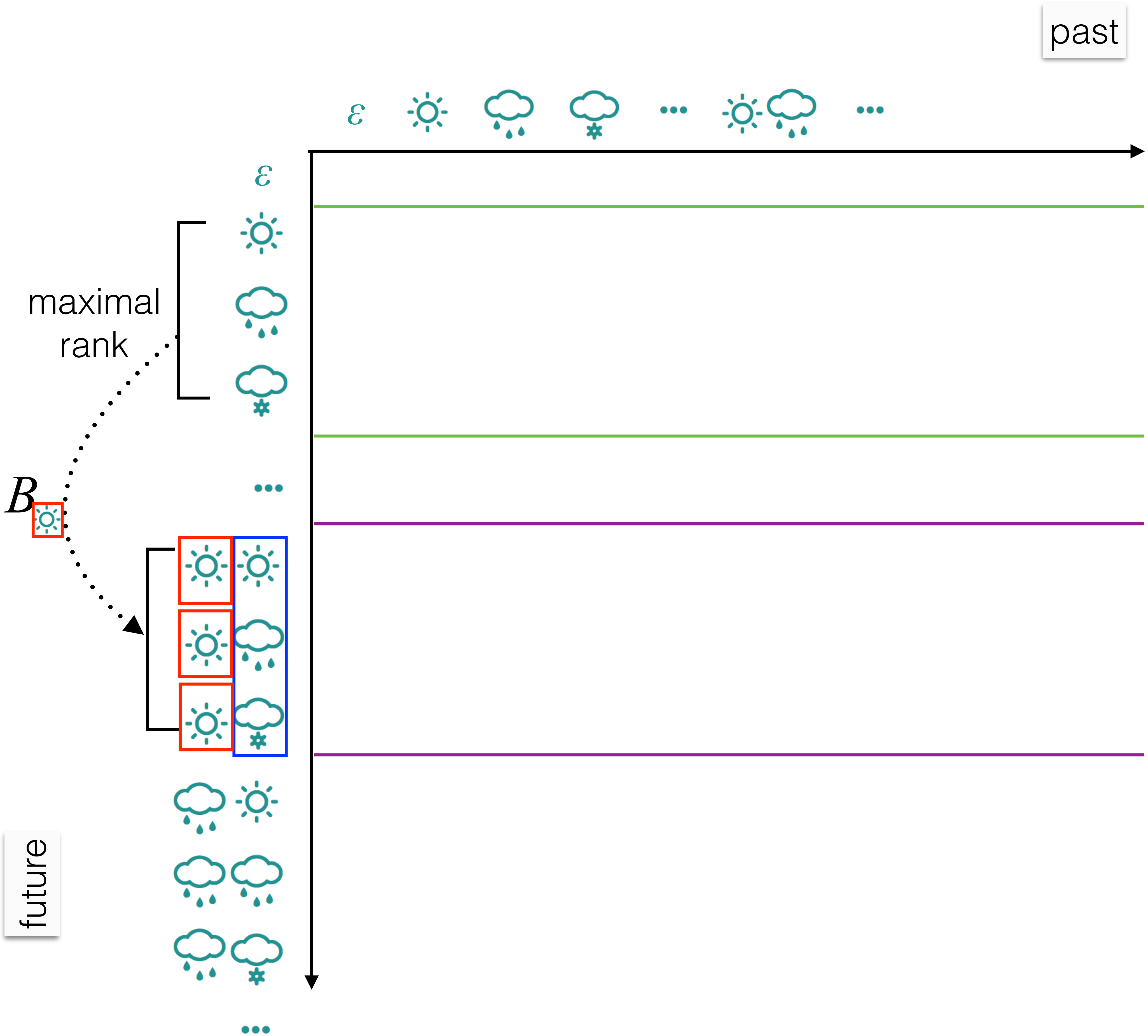


past
ε
maximal rank
$B_{☼}$
future



past
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rank
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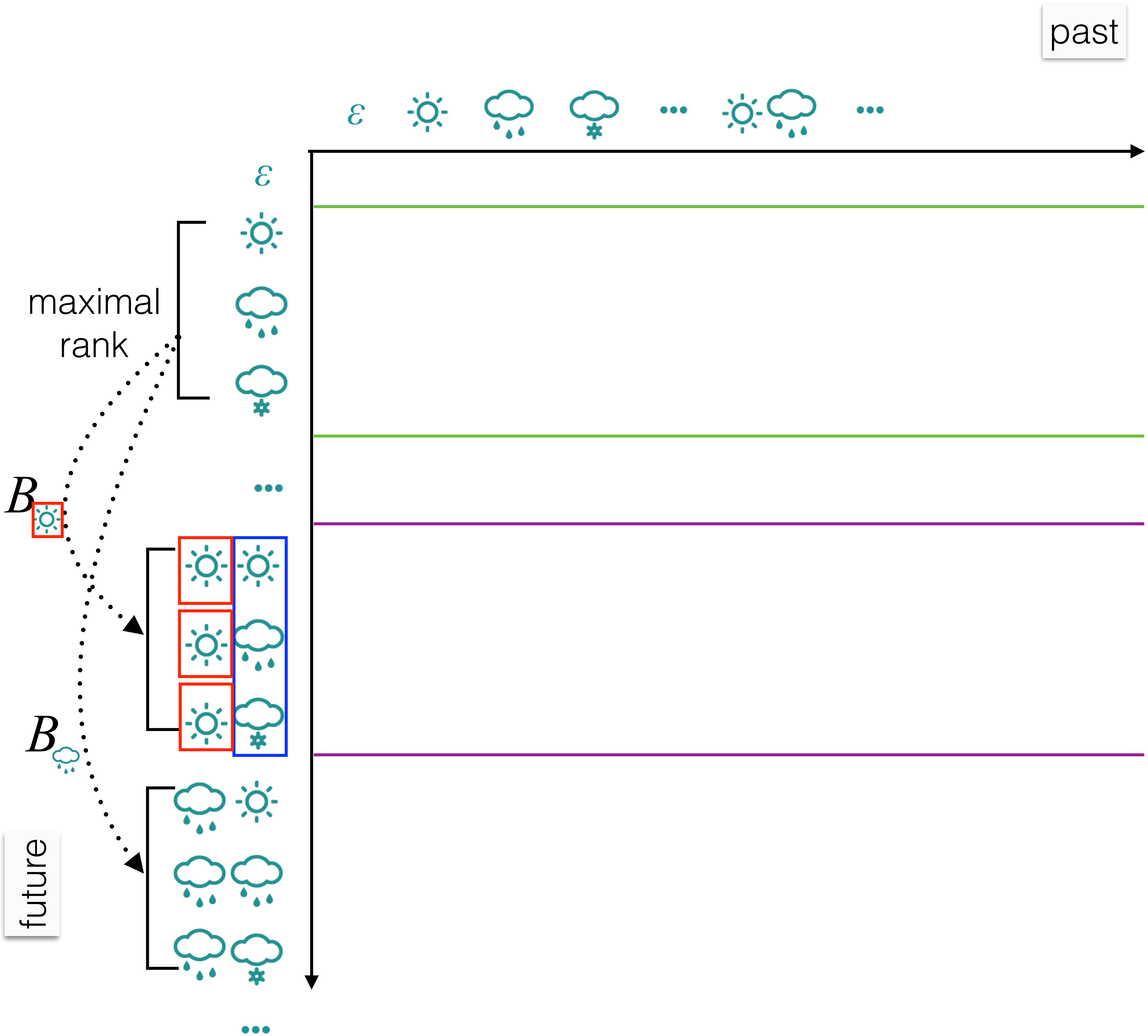
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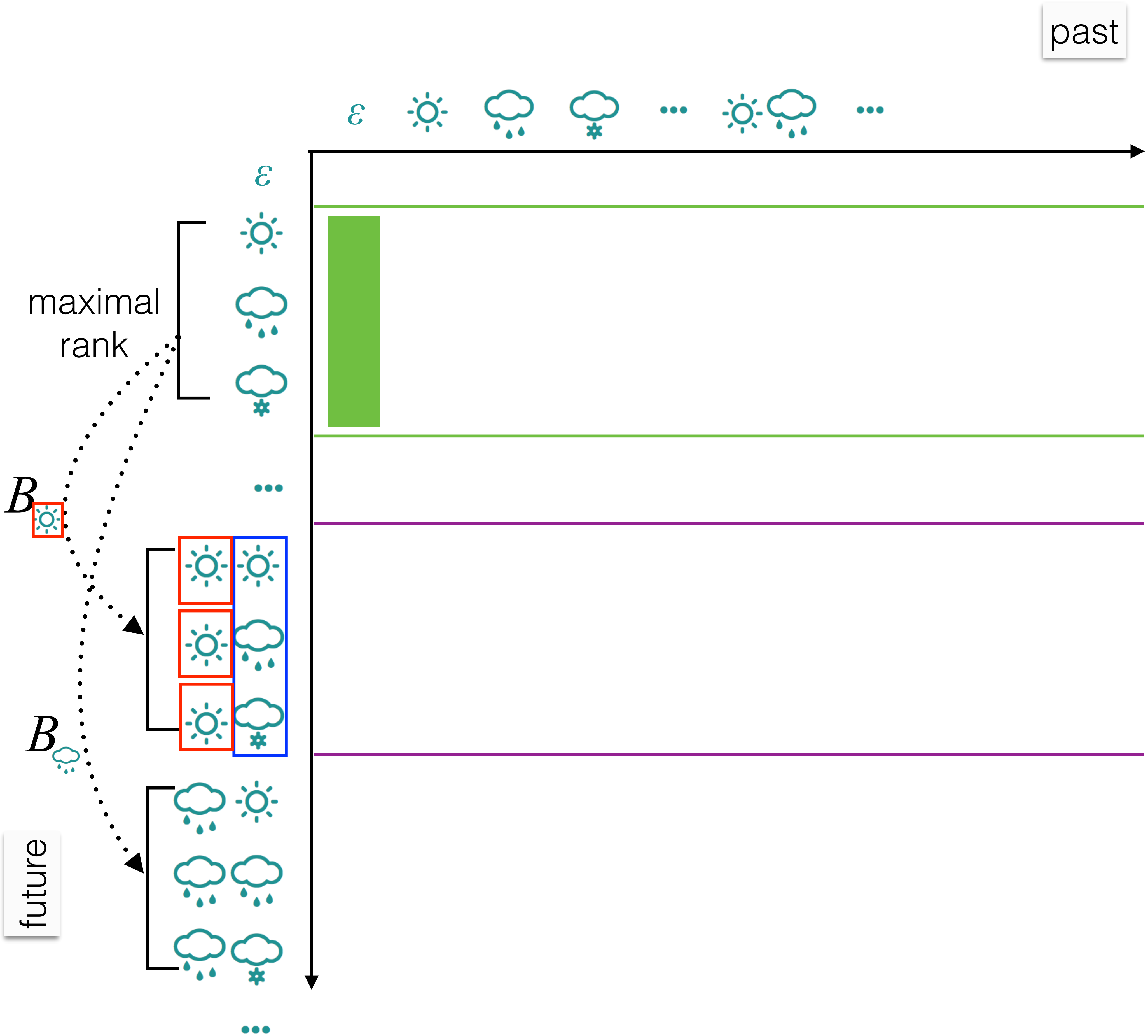
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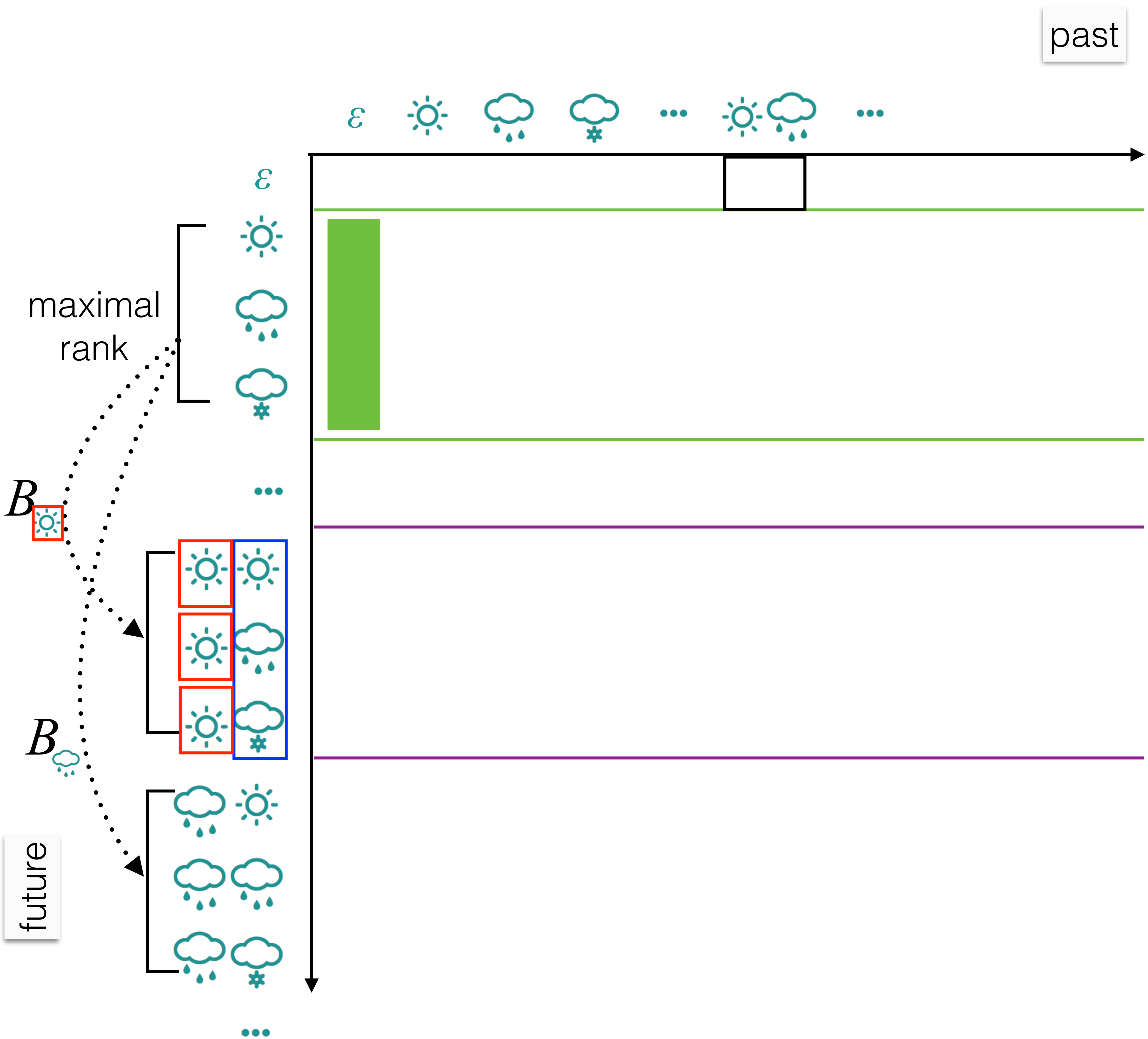

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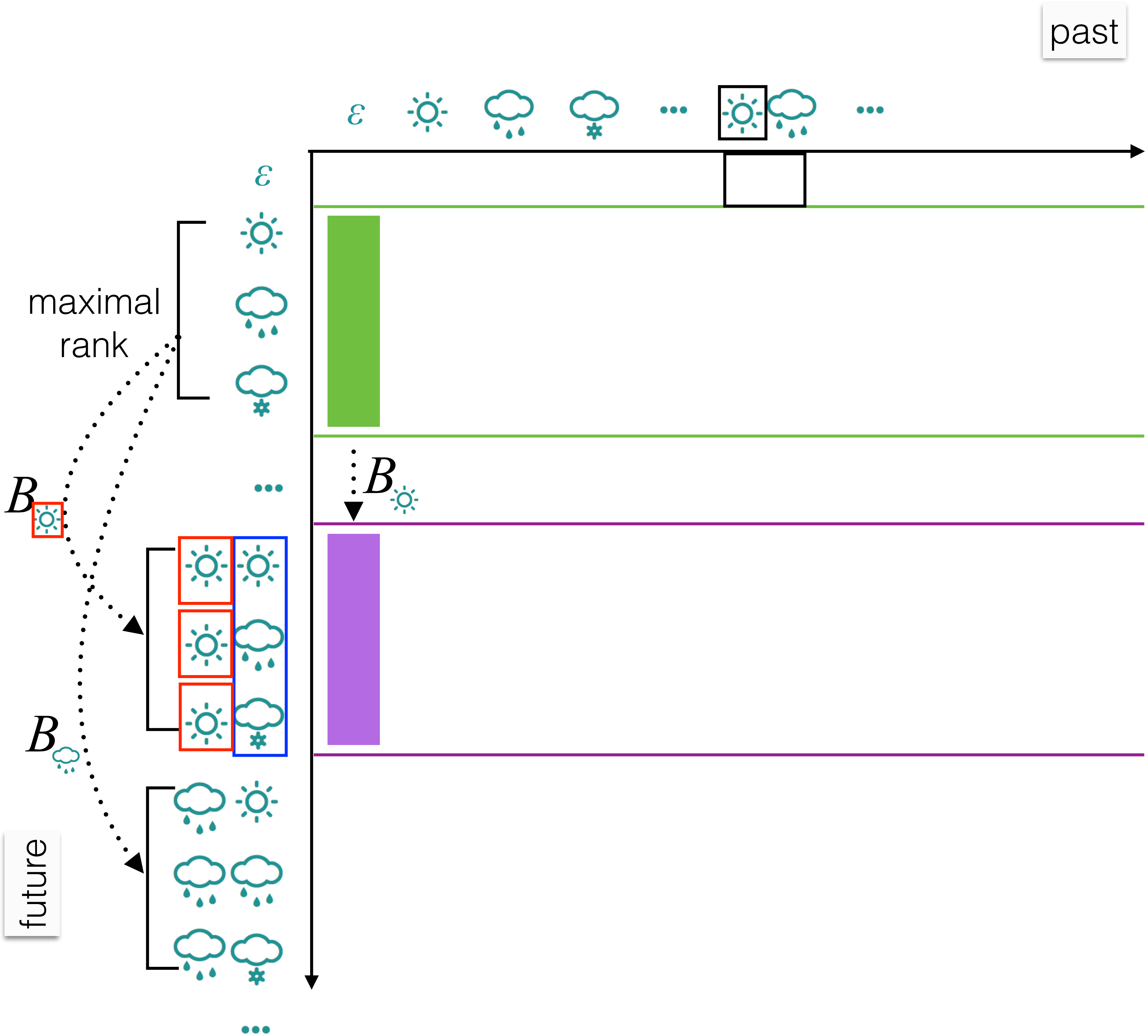

past
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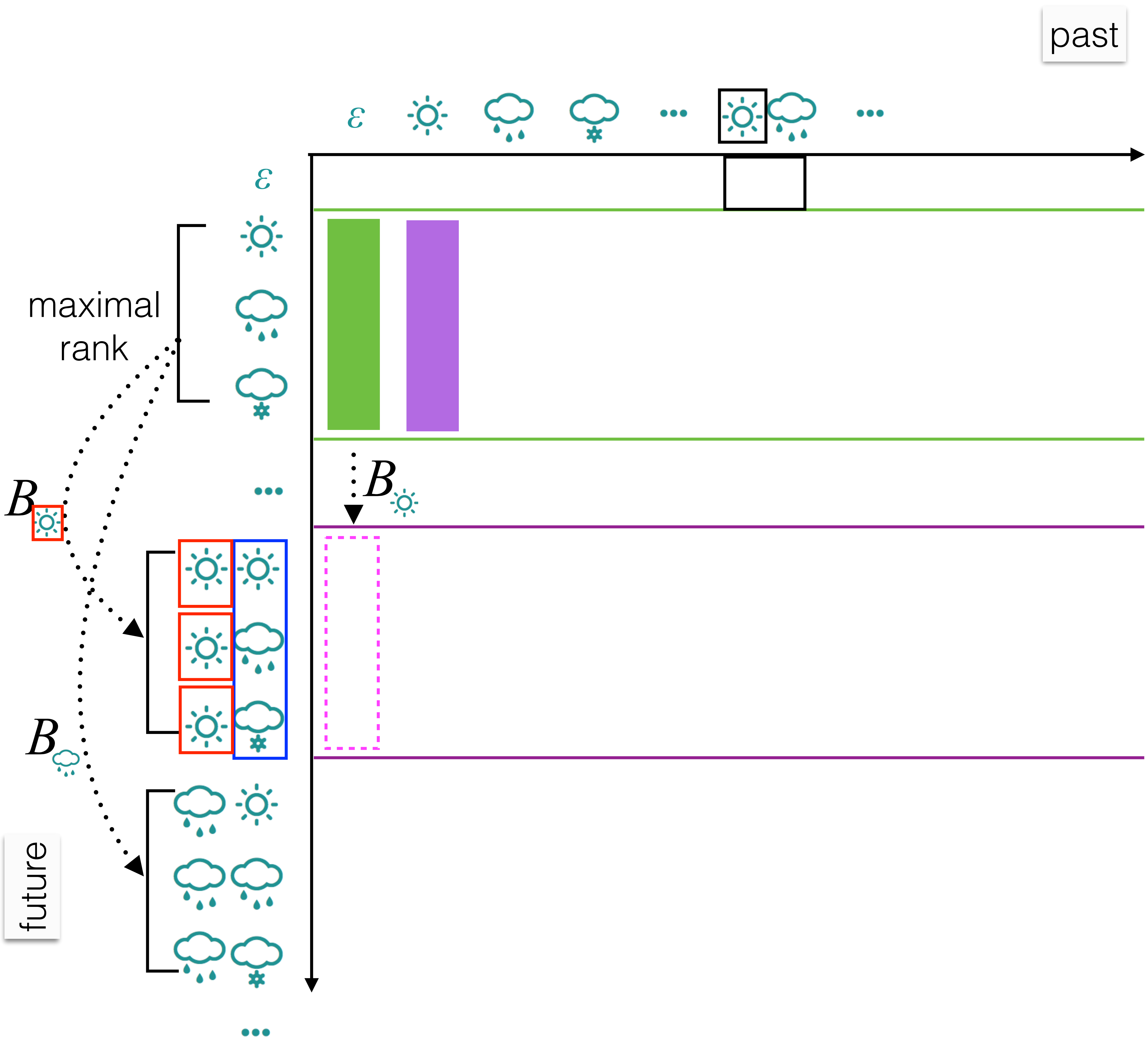

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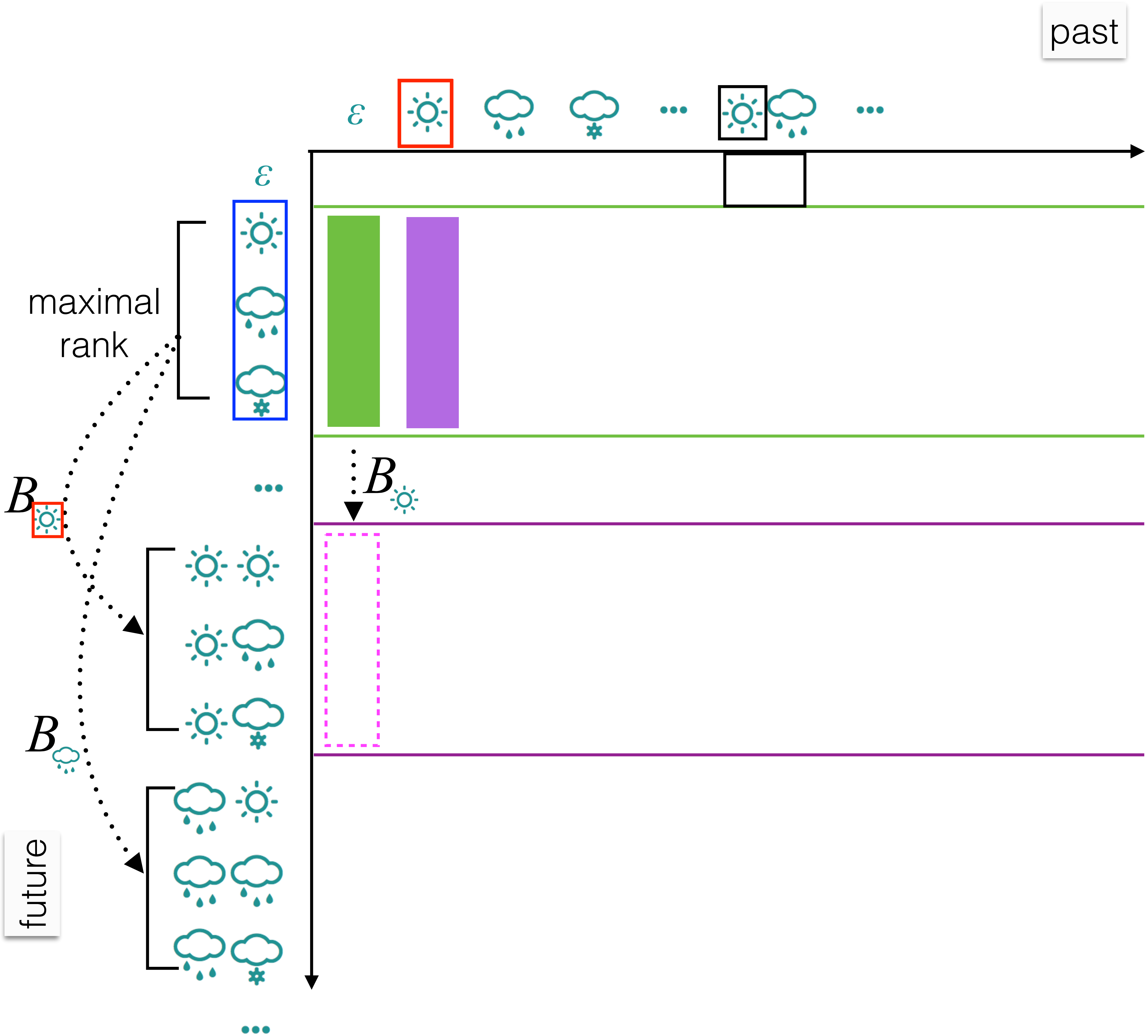

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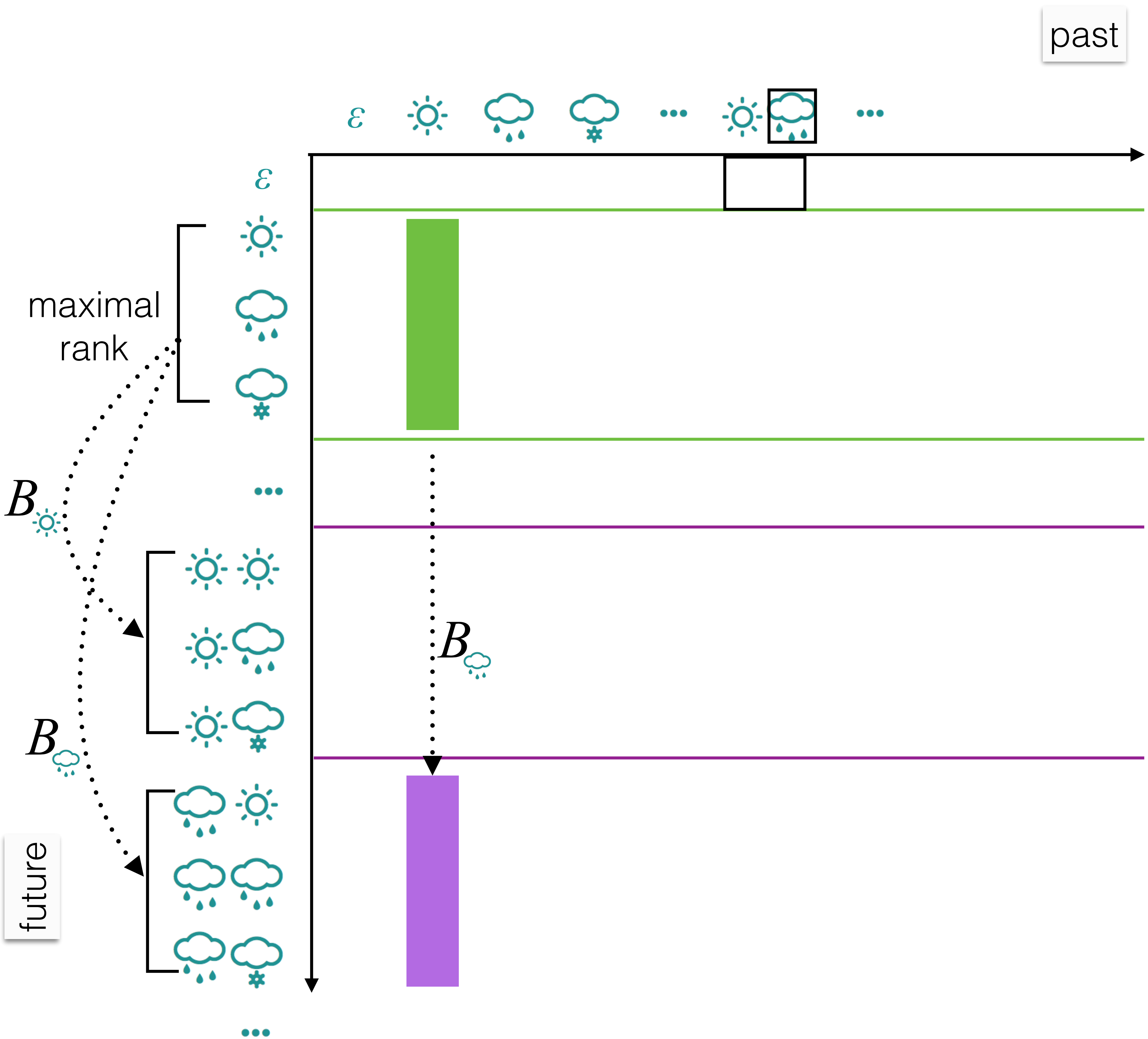
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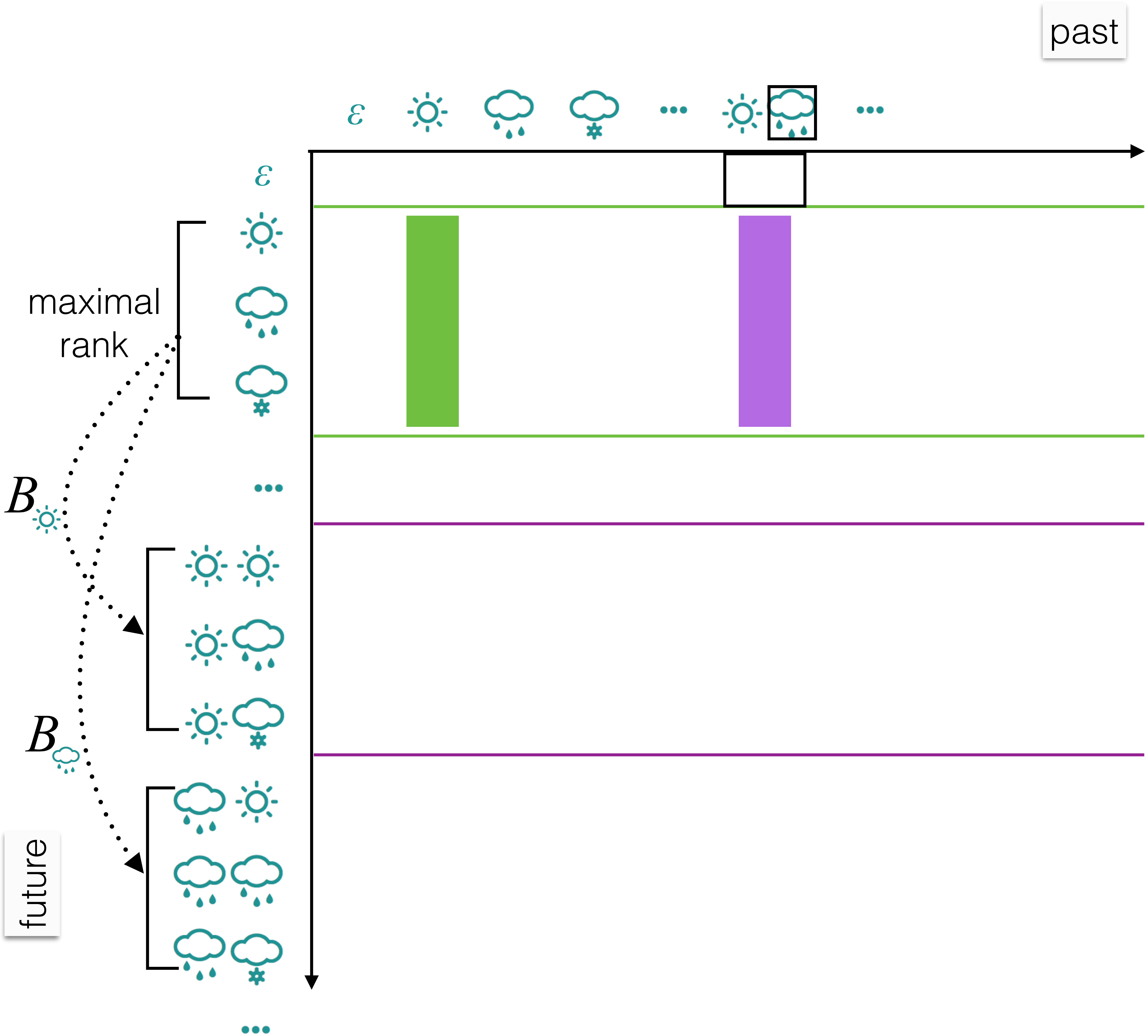
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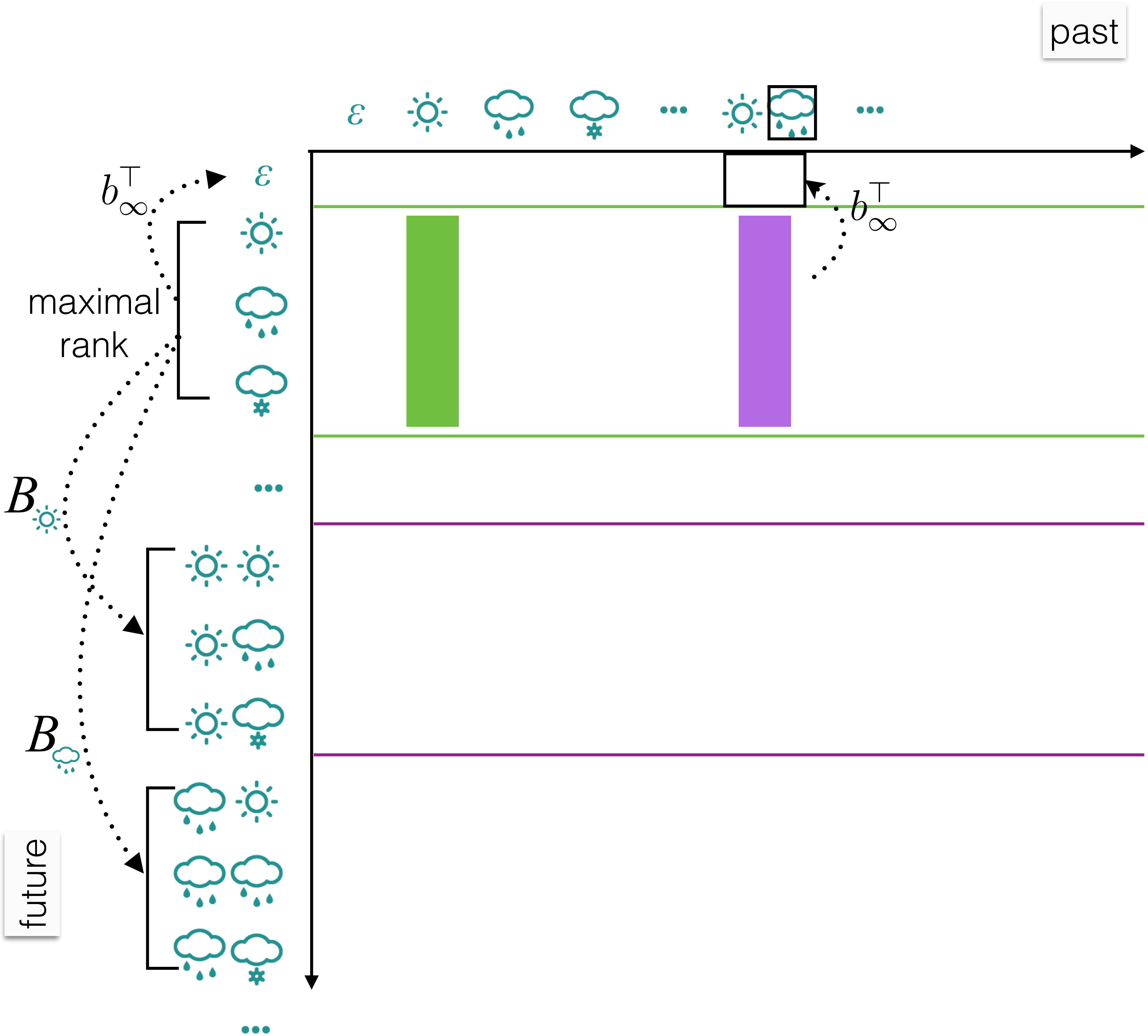
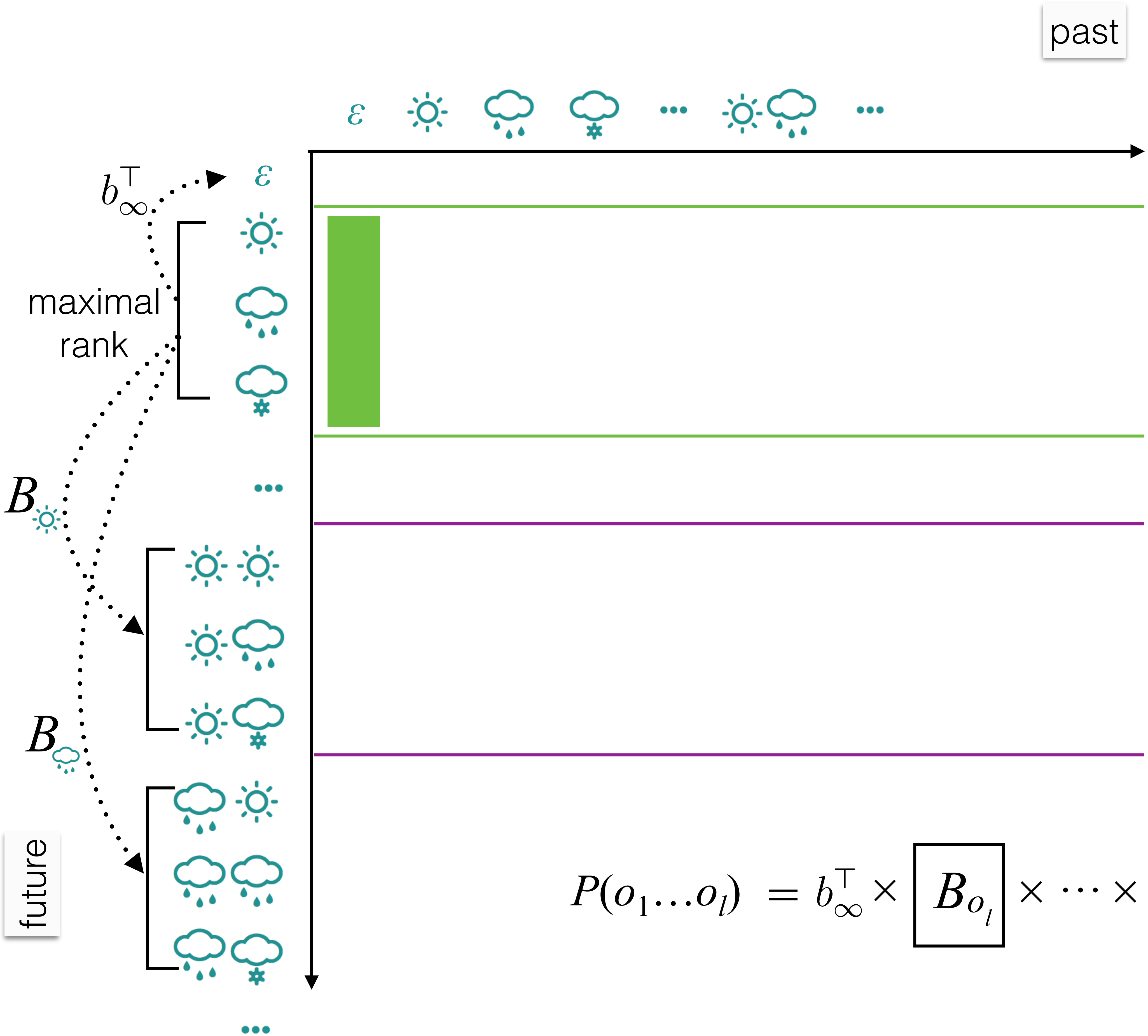


past
ε
$b_\infty^\top$
ε
$b_\infty^\top$
maximal rank
$B_{☼}$
$B_{☁}$
future



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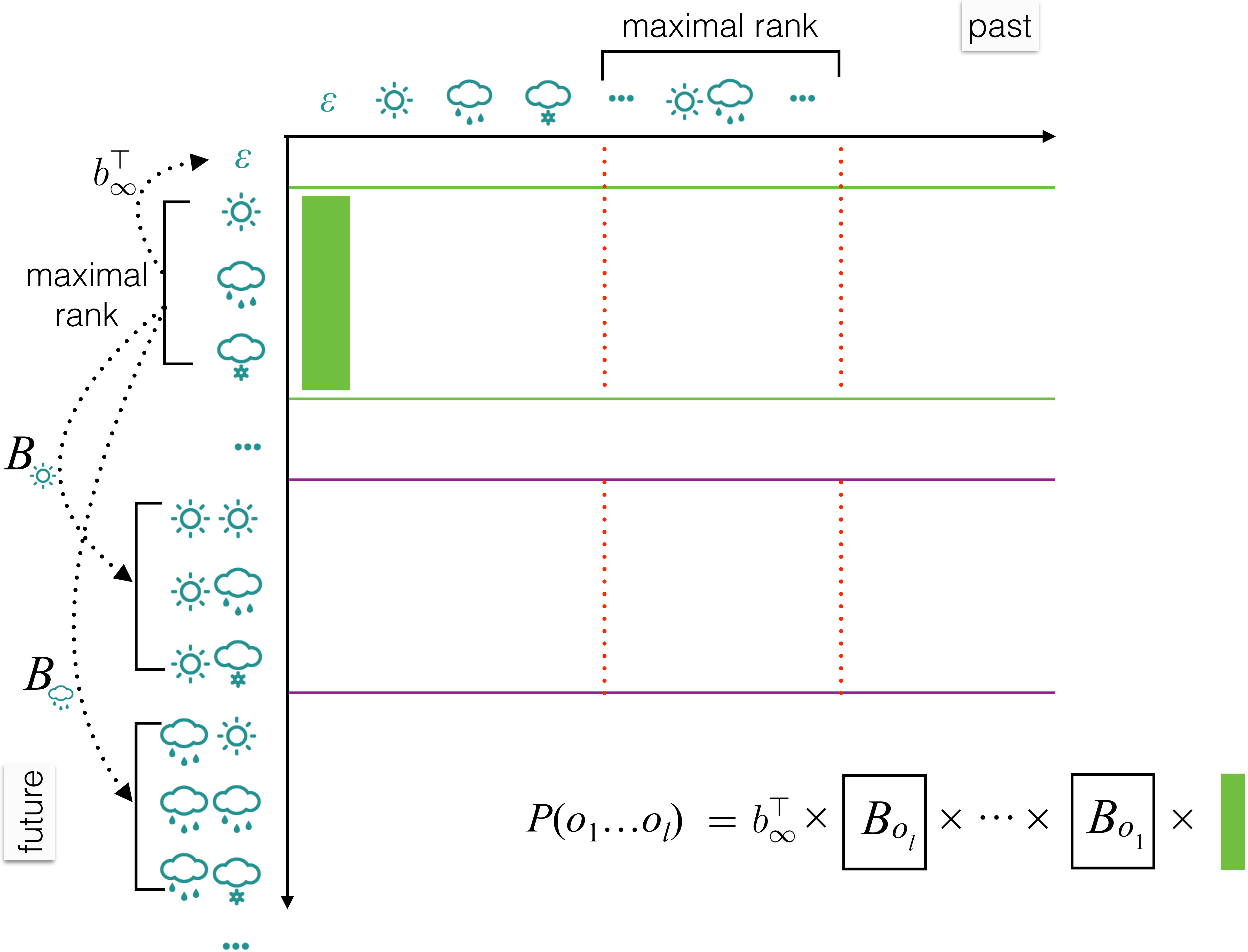
maximal rank
past
ε
$b_\infty^\top$
maximal rank
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B
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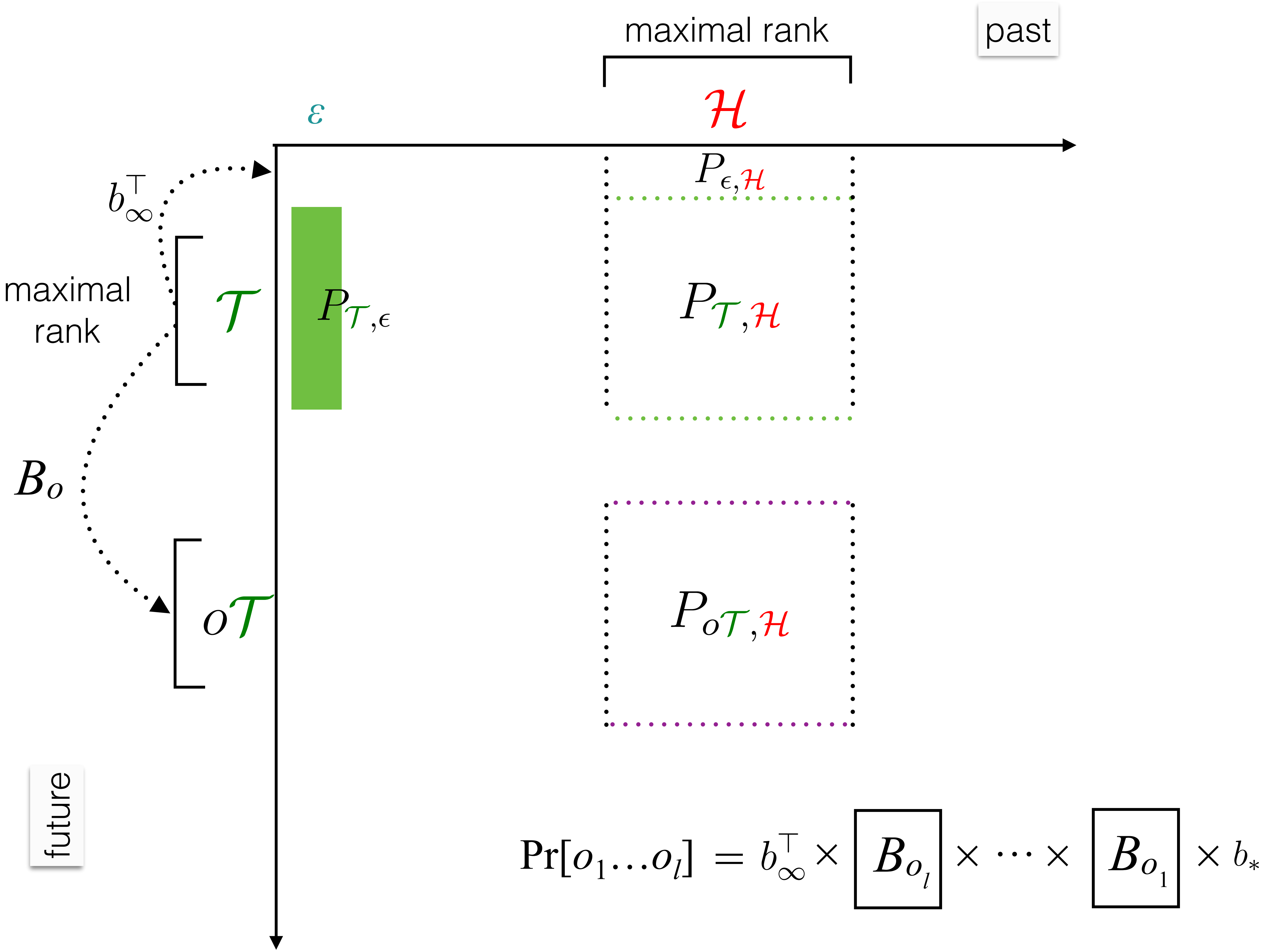

past
maximal rank
$\mathcal{H}$
ε
$P_{\epsilon,\mathcal{H}}$
$b_\infty^\top$
maximal rank
$\mathcal{T}$
$P_{\mathcal{T},\epsilon}$
$P_{\mathcal{T},\mathcal{H}}$
B_o
$o\mathcal{T}$
$P_{o\mathcal{T},\mathcal{H}}$
future
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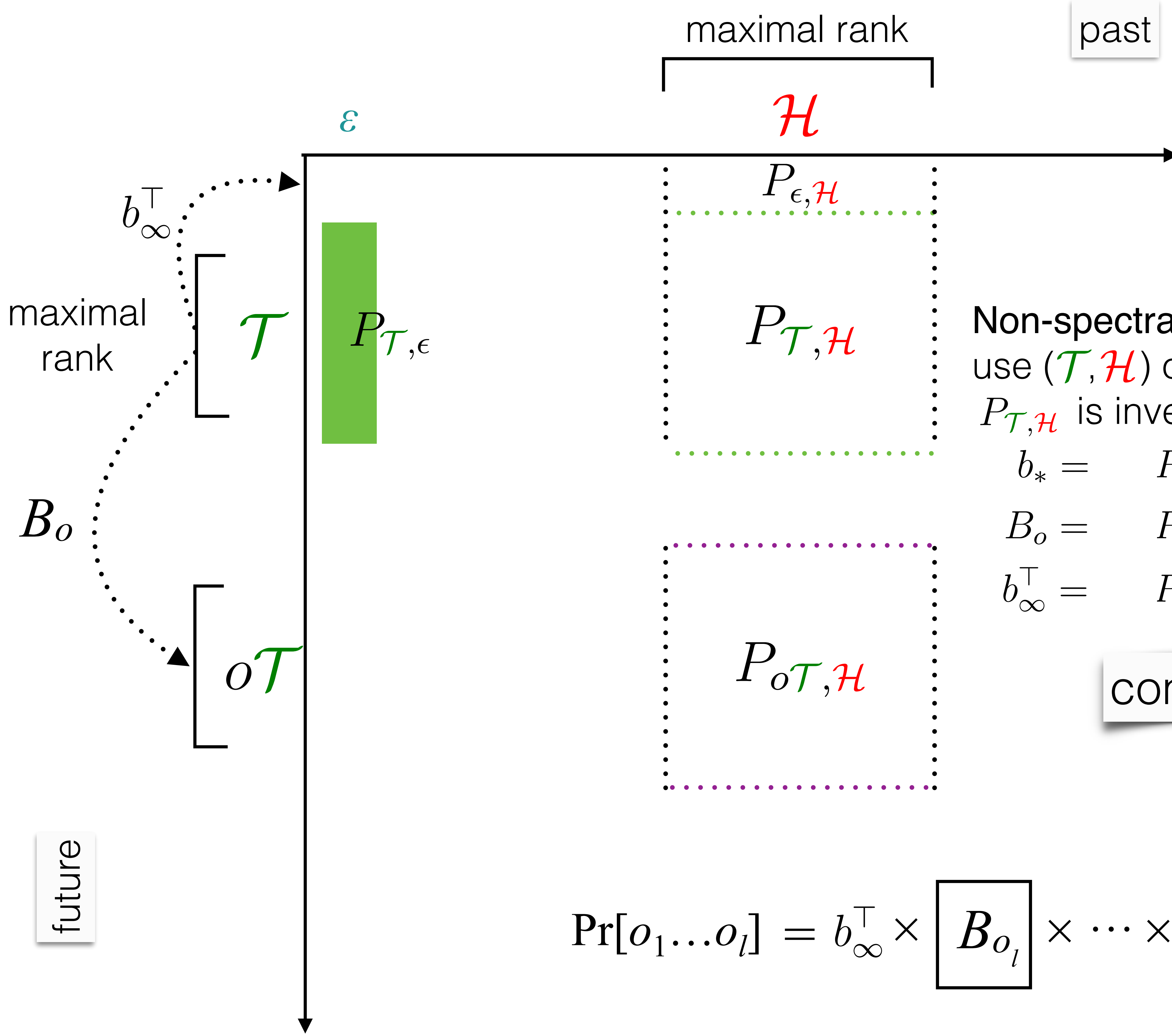

Non-spectral algorithm

use $(\mathcal{T},\mathcal{H})$ of just size so that $P_{\mathcal{T},\mathcal{H}}$ is invertible.

$$b_* = P_{\mathcal{T},\epsilon}$$

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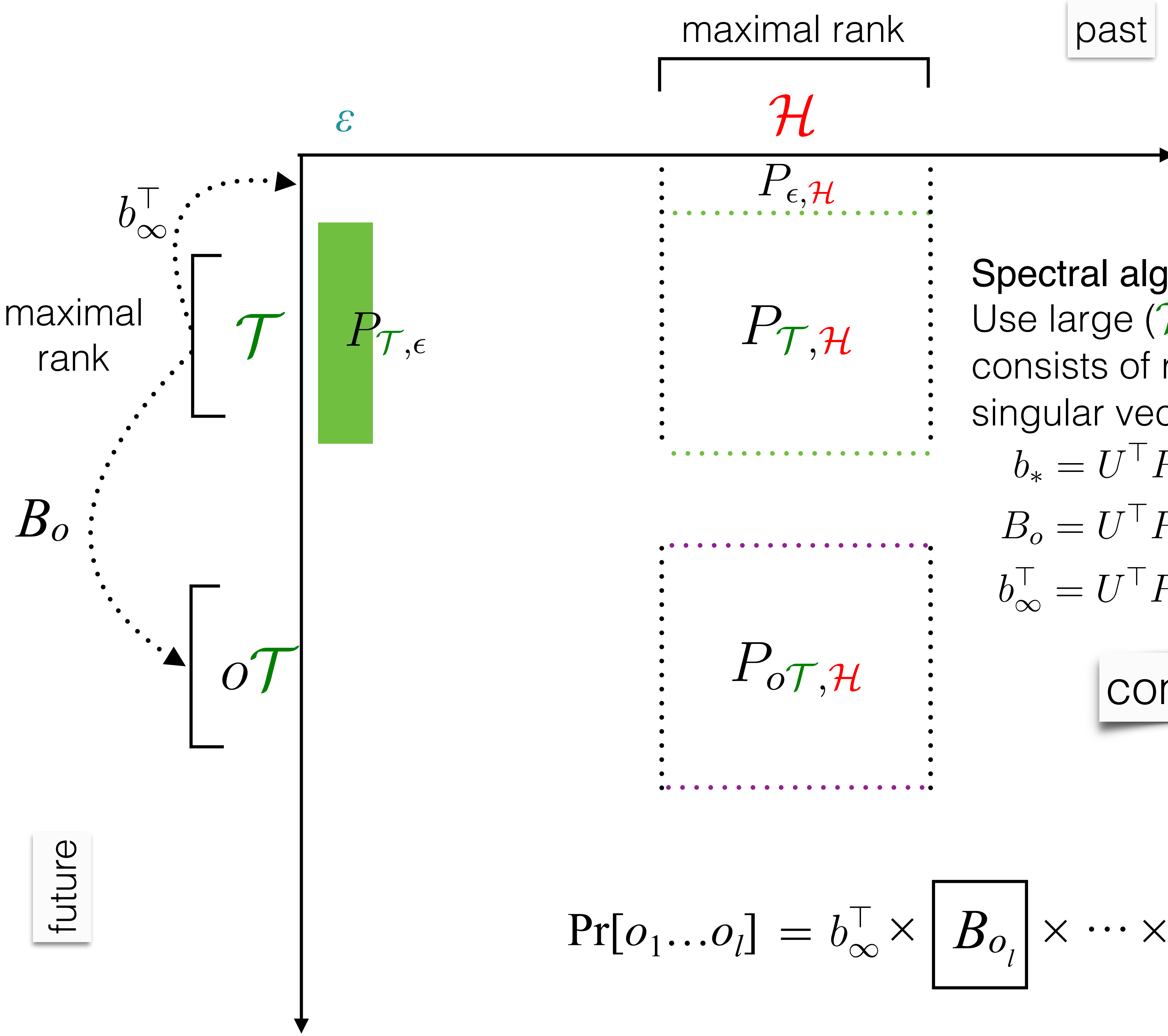

Spectral algorithm

Use large $(\mathcal{T},\mathcal{H})$, and let U consists of rank(M) leading left singular vectors of $P_{\mathcal{T},\mathcal{H}}$

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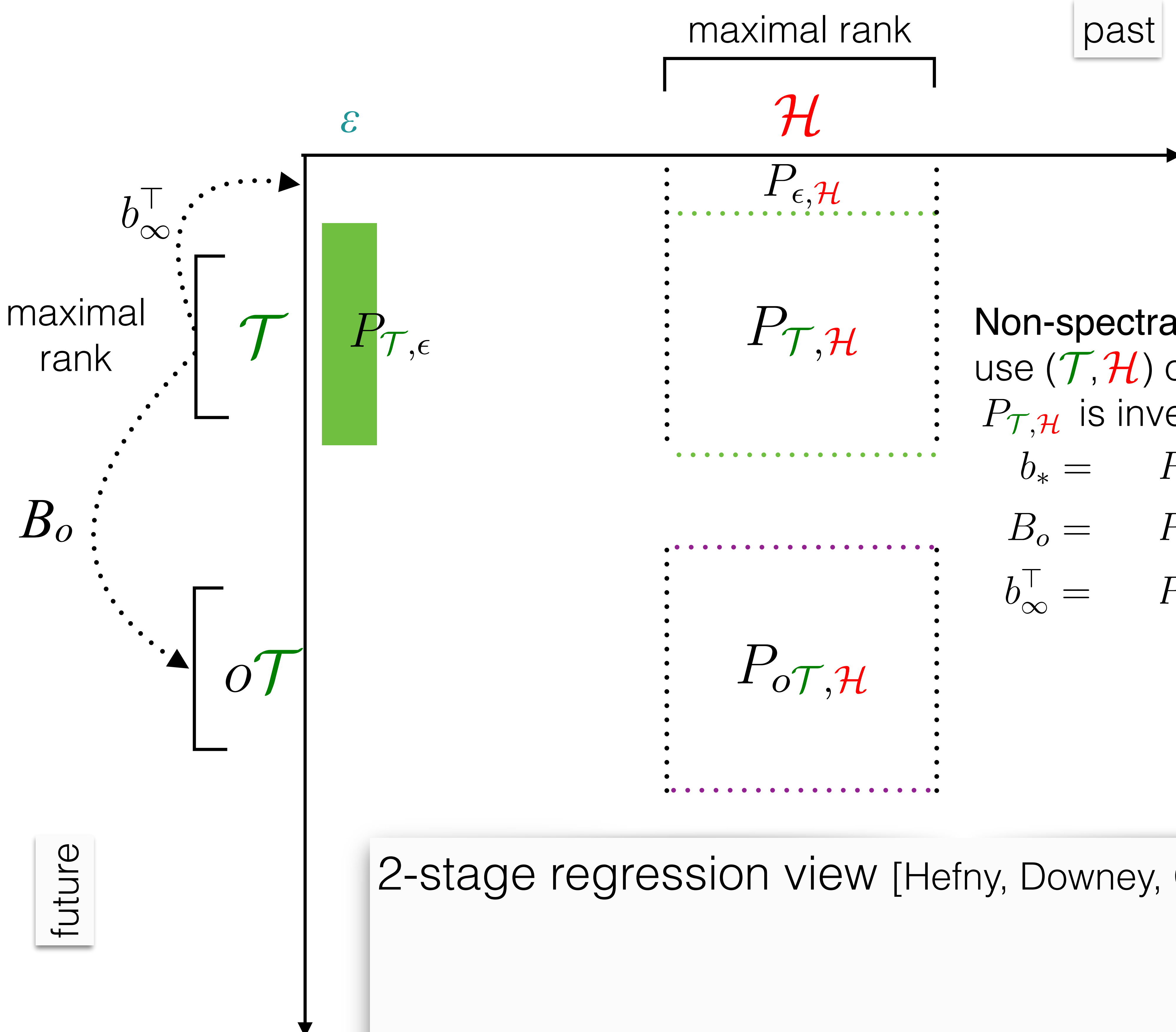
- PSR: when system has certain low-rank structure, the infinite-dimensional object is uniquely determined by a subset of its coordinates, which is tractable.



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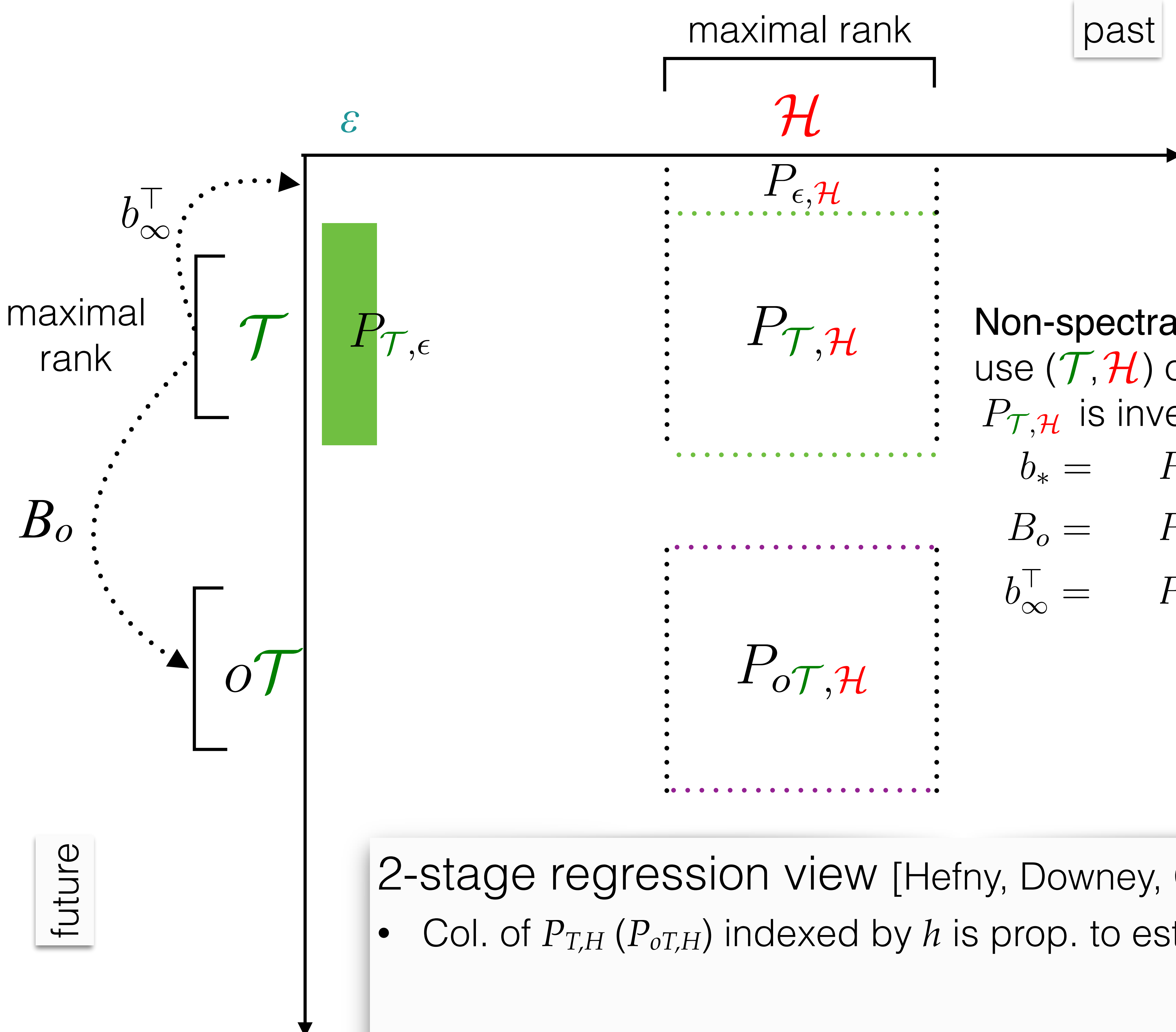
2-stage regression view [Hefny, Downey, Gordon 2015]



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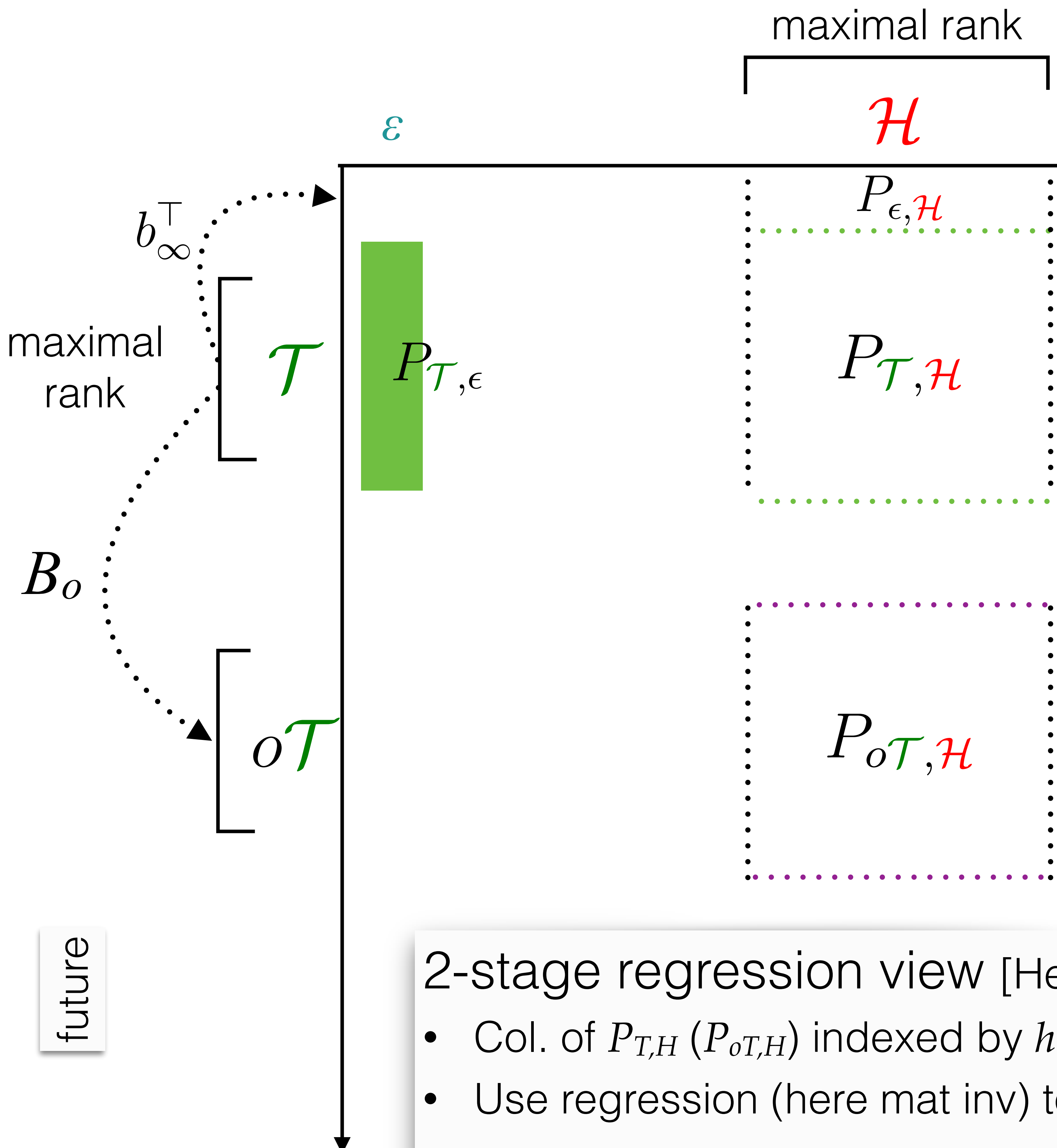

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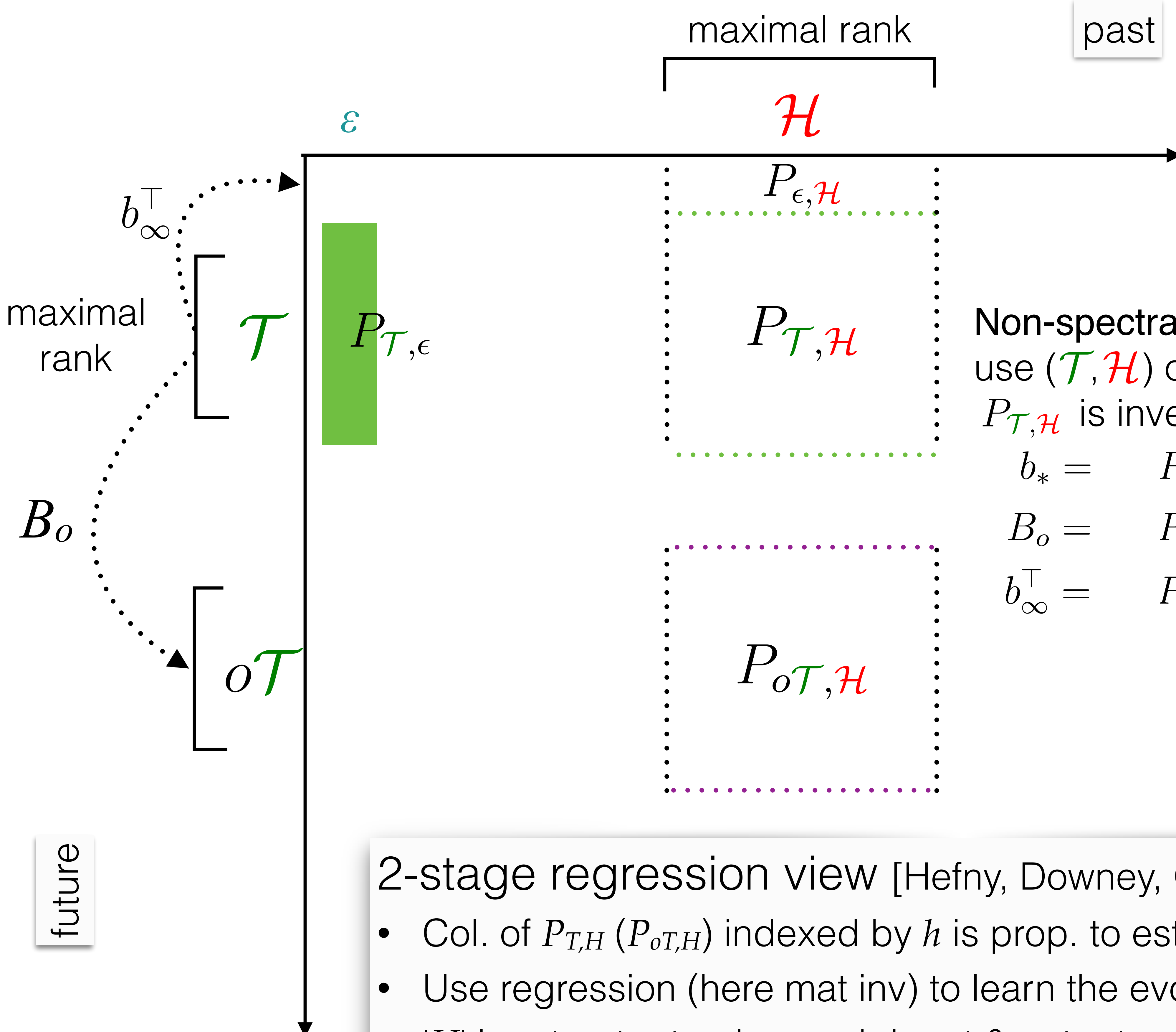

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- $|H|$ input-output pairs, each input & output are vectors in $R^{|T|}$

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  - Also known under the name Weighted Finite Automata (WFA)

Example: Markov Chain

Let f be the one-hot encoding of the last observation for an MC. Assume the transition matrix of the MC, T, is invertible. Define $\mathcal{T}$ as the set of length-1 sequences, then .

$$f(h) = T^{-1} P_{\mathcal{T}|h}$$

$$\begin{array}{cc} & o \\ o' & \begin{bmatrix} & \vdots & & \\ \cdots\cdots & P(o'|o) & & \\ & & T & \\ & & & \end{bmatrix} \end{array}$$

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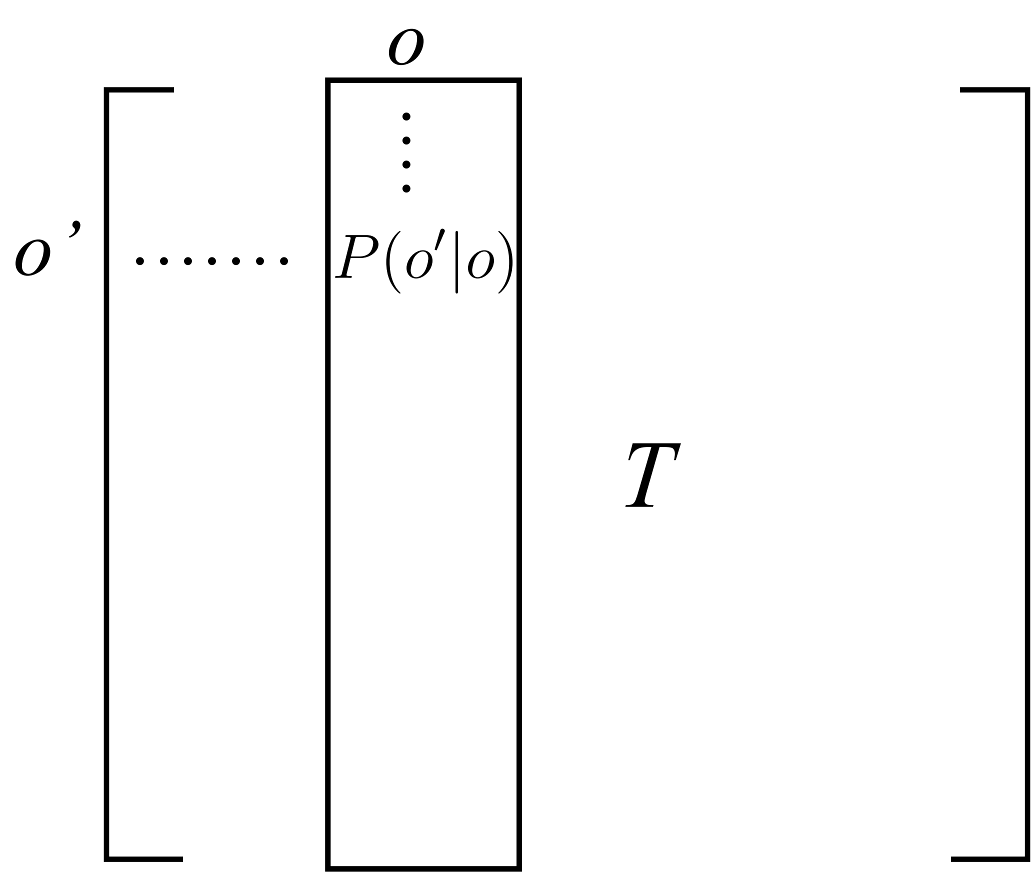

$P_{\color{green}{\mathcal{T}}|h}$

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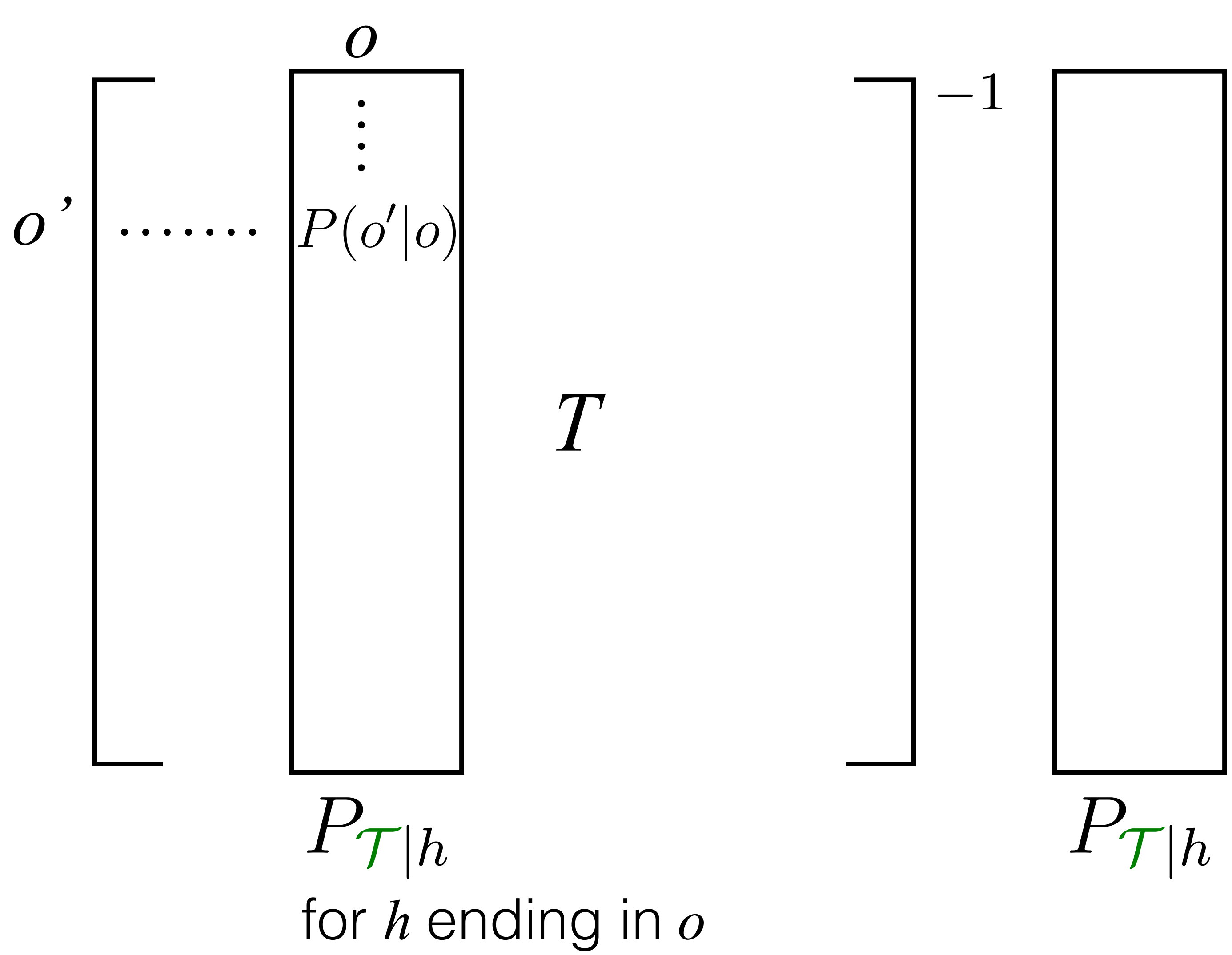

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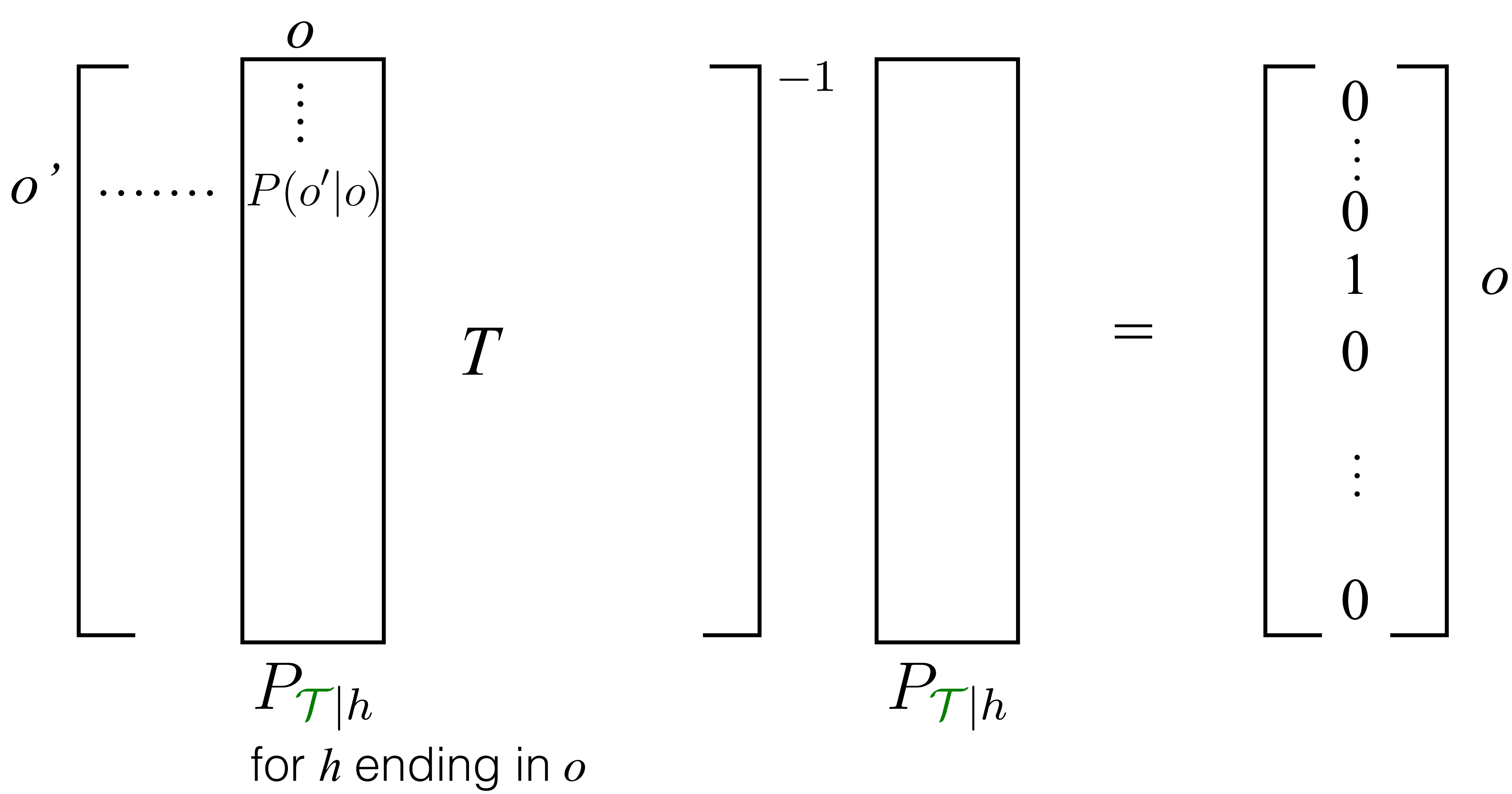

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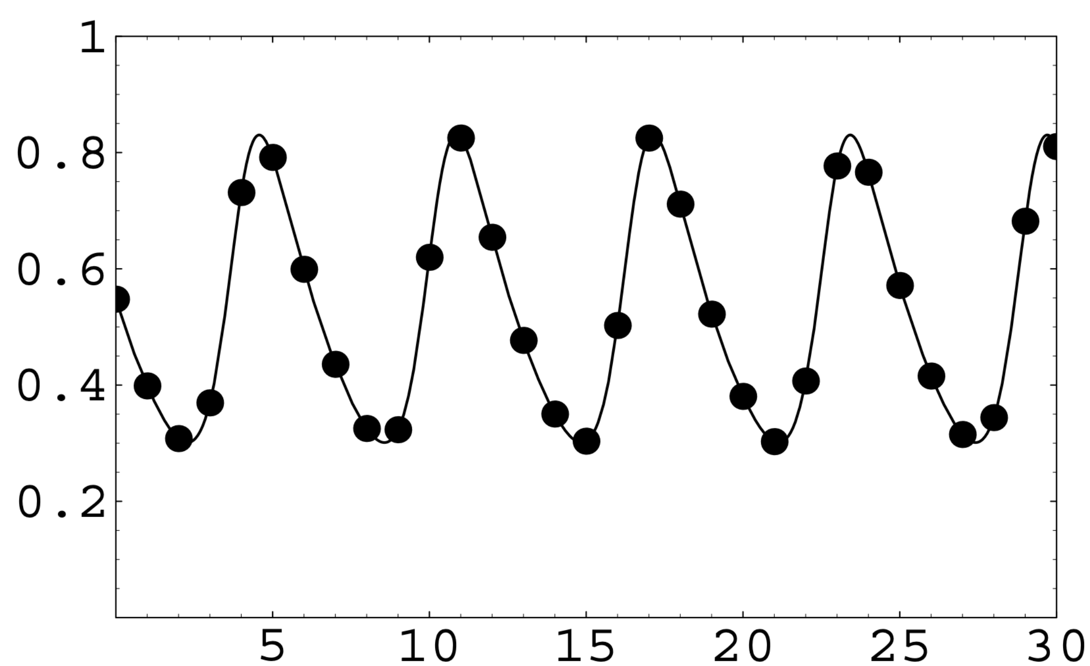
  - "Probability lock": 0-1 sequence where the probability of 1 appearing next goes like a sine wave sampled at an interval that is not a rational multiple of the wave's period; see Jaeger [2000] for details



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- Conditional: $\Pr[\mathrm{obs}(t) \mid h \mid\mid \mathit{do}\ \mathrm{act}(t)]$

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    e.g., off-policy eval with unknown behavior policy
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- When combined with planning, the approach is model-based RL (which isn't working quite well yet in the era of deep RL)