# Notes of WU Shengjun MMP
# Part I: Complex Analysis

hebrewsnabla

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## 1 Complex Function

### 1.1 Complex Number

$$z = \rho\,\mathrm{e}^{\mathrm{i}\phi} \tag{1.1}$$

$\rho$ is the modulus of $z$. $\phi$ is the argument of $z$, namely $\operatorname{Arg} z$.
Def: $\arg z \in \{\operatorname{Arg} z\}$, $0 \le \arg z < 2\pi$
thus $\arg z$ is called the principle value of $Argz$.

$$\sqrt[n]{z} = \sqrt[n]{\rho}\,\mathrm{e}^{\mathrm{i}\frac{\phi+2k\pi}{n}} \tag{1.2}$$

$\sqrt[n]{z}$ have $n$ different values
Logarithm and exponential of complex numbers are defined as

$$\ln z = \ln\left(\rho\,\mathrm{e}^{\mathrm{i}\phi}\right) = \ln\rho + \mathrm{i}(\phi + 2k\pi) \tag{1.3}$$

$$z^s = \mathrm{e}^{s\ln z} = \mathrm{e}^{s(\ln\rho + \mathrm{i}(\phi+2k\pi))} \tag{1.4}$$

Specially

$$\ln \mathrm{i} = \mathrm{i}(\frac{\pi}{2} + 2k\pi) \tag{1.5}$$

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\cdot\mathrm{i}(\frac{\pi}{2}+2k\pi)} = -\frac{\pi}{2} + 2k\pi \tag{1.6}$$

### 1.2 Complex Function

#### 1.2.1 Definition

If $\forall z \in E \subseteq \mathbb{C}$, $\exists$ one or more complex number $\omega$ corresponds to $z$, we call $\omega$ as a complex function of $z$, namely

$$\omega = f(z) \tag{1.7}$$

### 1.2.2 Domain

Definitions:

#### Neighbourhood

Neighbourhood of $z_0$ is a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$

#### Interior Point

A point $z_0 \in S$ is said to be an interior point of $S$ if there exists a neighbourhood of $z_0$ which is contained in $S$.

#### Open Set

The set $S$ is said to be open if every point of $S$ is an interior point of $S$.

#### Exterior Point

A point $z_0$ is said to be an exterior point of $S$ if $z_0$ and all neighbourhood of $z_0$ are not contained in $S$.

#### Boundary Point

#### Connectivity

#### Domain

### 1.2.3 Examples

$$\sin z = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \tag{1.8}$$

$$\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \tag{1.9}$$

$$\begin{aligned}\sin z &= \frac{\mathrm{e}^{\mathrm{i}x-y} - \mathrm{e}^{-\mathrm{i}x+y}}{2\mathrm{i}} \\ &= \frac{\mathrm{e}^{-y}(\cos x + \mathrm{i}\sin x) - \mathrm{e}^{y}(\cos x - \mathrm{i}\sin x)}{2\mathrm{i}} \\ &= \frac{\mathrm{e}^{-y}(-\mathrm{i}\cos x + \sin x) + \mathrm{e}^{y}(\mathrm{i}\cos x + \sin x)}{2} \\ &= \frac{\mathrm{e}^{y} + \mathrm{e}^{-y}}{2}\sin x + \mathrm{i}\frac{\mathrm{e}^{y} - \mathrm{e}^{-y}}{2}\cos x\end{aligned} \tag{1.10}$$

$$|\sin z| = \frac{1}{2}\sqrt{\mathrm{e}^{2y} + \mathrm{e}^{-2y} + 2(\sin^2 x - \cos^2 x)} \tag{1.11}$$

$|\sin z|$ and $|\cos z|$ can $> 1$.
$\sin z$ and $\cos z$ have period $2\pi$.

$$\sinh z = \frac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z}) \tag{1.12}$$

$$\cosh x = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}) \tag{1.13}$$

$\mathrm{e}^z,\ \sinh z\ \cosh z$ have period $2\pi\mathrm{i}$.

$$\ln z = \ln|z| + \mathrm{i}\,\mathrm{Arg}\, z \tag{1.14}$$

### 1.2.4 Derivatives

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta x \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{1.15}$$

suppose $\Delta y = 0,\ \Delta z = \Delta x \to 0$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \tag{1.16}$$

suppose $\Delta x = 0,\ \Delta z = \mathrm{i}\Delta y \to 0$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} \tag{1.17}$$

Cauchy-Riemann (C-R) condition: $\dfrac{\partial u}{\partial x} + \mathrm{i}\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - \mathrm{i}\dfrac{\partial u}{\partial y}$, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \tag{1.18}$$

which is the necessary condition for $f(z)$ being differentiable at $z$.
The sufficient condition: $\exists$ continuous $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y}$, which satisfy C-R condition.

#### C-R Eq. in Polar Coordinates

$$\begin{aligned} \frac{\partial u}{\partial \rho} &= \frac{1}{\rho}\frac{\partial v}{\partial \phi} \\ \frac{1}{\rho}\frac{\partial u}{\partial \phi} &= -\frac{\partial v}{\partial \rho} \end{aligned} \tag{1.19}$$

# 2 Integral

## 2.1 Introduction

## 2.2 Cauchy Theorem

单连通

$$\oint_l f(z)\mathrm{d}z = 0 \tag{2.1}$$

复连通

$$\oint_l f(z)\mathrm{d}z + \sum_{i=1}^{n} \oint_{l_i} f(z)\mathrm{d}z = 0 \tag{2.2}$$

## 2.3 不定积分

Complex Newton-Lebniz

$$F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta, \quad F'(z) = f(z) \tag{2.3}$$

Consider integral

$$I = \int_a^b z^n \mathrm{d}z,\ n \in \mathbb{Z} \tag{2.4}$$

1) $n \neq -1$

2) $n = -1$

$$I = \ln b - \ln a = \ln\left|\frac{b}{a}\right| + \mathrm{i}(\operatorname{Arg} b - \operatorname{Arg} a) \tag{2.5}$$

What about

$$I = \oint_l (z-\alpha)^n \mathrm{d}z \tag{2.6}$$

1) $\alpha$ is external, $I = 0$

2) $\alpha$ is internal

2.1) $n \geqslant 0$

2.2) $n < 0$, let $z - \alpha = R\,\mathrm{e}^{\mathrm{i}n\phi}$

$$\begin{aligned} I &= \int_0^{2\pi} R^n\,\mathrm{e}^{\mathrm{i}n\phi}\,\mathrm{d}\big(\alpha + R\,\mathrm{e}^{\mathrm{i}n\phi}\big) \\ &= \mathrm{i}R^{n+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)\phi}\,\mathrm{d}\phi \end{aligned} \tag{2.7}$$

2.2.1) $n \neq -1$ 2.2.2) $n = -1$

$$I = \int_C (|z| - \mathrm{e}^z \sin z)\mathrm{d}z \tag{2.8}$$

where $C$ is

## 2.4 Cauchy Equation

$f(z)$在闭单联通区域$B$上解析, $l$is boundary of $B$, $\alpha \in B$.

$$f(\alpha) = \frac{1}{2\pi \mathrm{i}} \oint_l \frac{f(z)}{z - \alpha} \mathrm{d}z \tag{2.9}$$

Discussion

1)

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_l \tag{2.10}$$

2) 无界推广

3) derivatives

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \oint_l \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \tag{2.11}$$

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_l \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta \tag{2.12}$$

# 3 Power Series

## 3.1 Complex Series

### 3.1.1 Introduction

$$\sum_{k=1}^{\infty} w_k = \sum_{k=1}^{\infty} u_k + \mathrm{i} \sum_{k=1}^{\infty} v_k \tag{3.1}$$

### 3.1.2 Convergence Test

#### Cauchy's Convergence Test

$\forall \varepsilon > 0$, $\exists N$, s.t. when $n > N$, $\forall p \in \mathbb{N}$

$$\left| \sum_{k=n+1}^{n+p} w_k \right| < \varepsilon \tag{3.2}$$

#### Absolute Convergence

### 3.1.3 Function Series

#### Convergence Test

#### Cauchy's

### 3.1.4 Uniform Convergence

## 3.2 Power Series

### 3.2.1 Definition

### 3.2.2 Convergence and Divergence Test

#### D'Alembert's Test

#### Root

#### Convergence Circle

### 3.2.3 Analytical Features

## 3.3 Taylor Expansion

## 3.4 解析延拓

## 3.5 Laurent Expansion

### 3.5.1 Bilateral Power Series

$$\cdots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{3.3}$$

Positive part: convergence radius $= R_1$
Negative: denote $\zeta = \frac{1}{z-z_0}$
conv radius $= \frac{1}{R_2}$ thus, bilateral power series is abs and uniform conv when

$$R_2 < |z-z_0| < R_1 \tag{3.4}$$

which is called convergence ring.

### 3.5.2 Laurent Expansion Th.

pos part:
aka canonical part
neg part:
aka 主部

Laurent Expansion is unique. Proof omitted.

Attention 1) $z = z_0$ may be a singularity or not.
2) Although Laurent Expansion looks the same as Taylor Expansion,

$$a_k \neq \frac{F^{(k)}(z_0)}{k!} \tag{3.5}$$

no matter whether $z_0$ is singularity.
3)

### 3.6 Isolated Singularity

# 4 留数定理

## 4.1

## 4.2 计算实变函数定积分

type I

type II
Suppose

$$I = \lim_{\substack{R_1 \to \infty \\ R_2 \to \infty}} \int_{-R_1} R_2 f(x)\,\mathrm{d}x \tag{4.1}$$

exists
when $R_1 = R_2 \to \infty$, I is called principle (主值) of the integral above, namely

$$\mathscr{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \tag{4.2}$$

Th

$$\mathscr{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_k \mathrm{Res}\, f(b_k) \;\; \text{(upper semi-plane)} \tag{4.3}$$

type III

$$\int_0^{\infty} F(x)\cos mx\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} F(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x \tag{4.4}$$

$$\int_0^{\infty} G(x)\sin mx\mathrm{d}x = \frac{1}{2\mathrm{i}}\int_{-\infty}^{\infty} G(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x \tag{4.5}$$

Jordan's Lemma (约当引理)

$$\lim_{R \to \infty} \int_{C_R} F_z \, \mathrm{e}^{\mathrm{i}mz} \, \mathrm{d}z = 0 \tag{4.6}$$

when $m > 0$, $C_R$ is a semi-circle on upper semi-plane,
or $m < 0$, $C_R$ is a semi-circle on lower semi-plane.