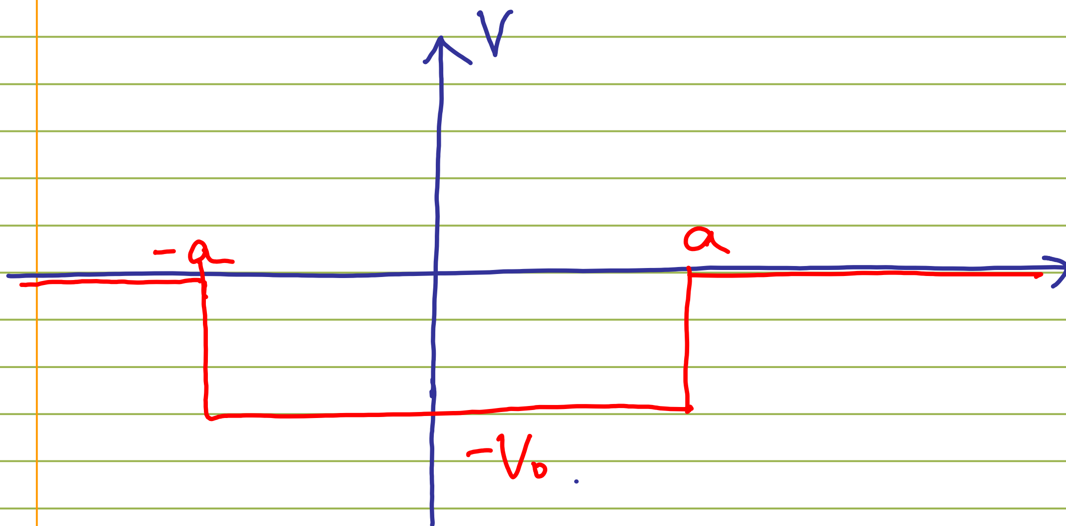
# Finite Square Well




$$V(x) = \begin{cases} -V_0, & |x| < a \\ 0, & |x| \geq a \end{cases}$$

BOUND STATES $\Leftrightarrow$ $E < 0$ $(= V(\infty), V(-\infty))$

Scattering states $\Leftrightarrow$ $E > 0$

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BOUND STATES

$x < -a$:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\frac{d^2}{dx^2}\psi(x) = +k^2\psi(x)$$

$$K^2 = -\frac{2mE}{\hbar^2}$$

$$\psi(x) = Ae^{+kx}$$

$x > a$ $\quad \psi(x) = Fe^{-kx}$

$-a < x < a$ : $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - V_0\psi(x) = E\psi(x)$

or $\quad \frac{d^2}{dx^2}\psi(x) = -\ell^2\psi(x)$

where $\quad \ell^2 = \frac{2m}{\hbar^2}(E+V_0) \geq 0$

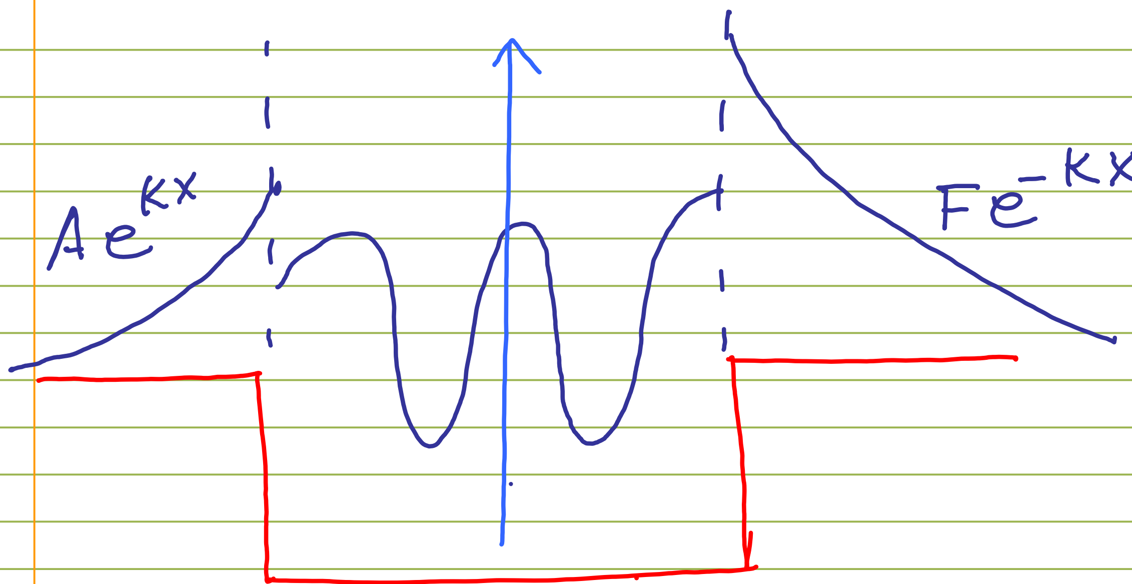
or $\quad \psi(x) = C\sin\ell x + D\cos\ell(x)$

Match $\psi(x)$, $\frac{d}{dx}\psi(x)$ at

$\underline{x=-a}$: ① $\quad Ae^{-ka} = -C\sin\ell a + D\cos\ell a$

② $\quad Ak e^{-ka} = +C\ell\cos\ell a + D\ell\sin\ell a$

$x = +a$:




$$C\sin lx + D\cos lx$$

$$③\ \ Fe^{-ka} = C\sin la + D\cos la$$

$$④\ \ -Fke^{-ka} = Cl\cos la - Dl\sin la$$

$$\left(\frac{d}{dx}\sin lx = l\cos lx,\ \ \frac{d}{dx}\cos lx = -l\sin lx\right)$$

$$①\cdot④ + (3)\cdot(2) \Rightarrow\ \ 0 = \left(-C^2 l\,\cancel{\sin}\cdot\cos - D^2 l\sin\cancel{\cos} + DCl\sin^2 + DCl\cos^2\right)$$
$$+\ C^2 l\sin\cancel{\cos} + D^2 l\cos\cancel{\ }\cdot\sin$$
$$+\ CDl\cos^2 l + DlC\sin^2$$

$$0 = CDl$$

$\Leftarrow$ Since $V(x) = V(-x)$

solutions are either odd.

$(D=0)$ OR even $(C=0)$

EVEN $(C=0) \Rightarrow F = A$

$$A e^{-ka} = D \cos la$$

$$A k e^{-ka} = D l \sin la$$

$$k = l \tan la$$

$$D = A e^{-ka} / \cos la$$

EVEN

$$\psi(x) = \begin{cases} A e^{kx} & ; \; x < -a \\ \dfrac{A e^{-ka}}{\cos la} \cos lx & ; \; -a \le x \le a \\ A e^{-kx} & ; \; x > a \end{cases}$$

$\kappa = \ell \tan \ell a$ determines $E$

or $\boxed{\kappa a = \ell a \cdot \tan \ell a}$

"Solve" Graphically / Numerically

Recall:

$$\kappa^2 = -\frac{2mE}{\hbar^2}, \quad \ell^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$$\kappa^2 + \ell^2 = \frac{2mV_0}{\hbar^2}$$

Let

$$z = \ell \cdot a$$

$$z_0 = \sqrt{\frac{2mV_0}{\hbar^2}} \cdot a$$

then

$$(\kappa a)^2 + z^2 = z_0^2$$

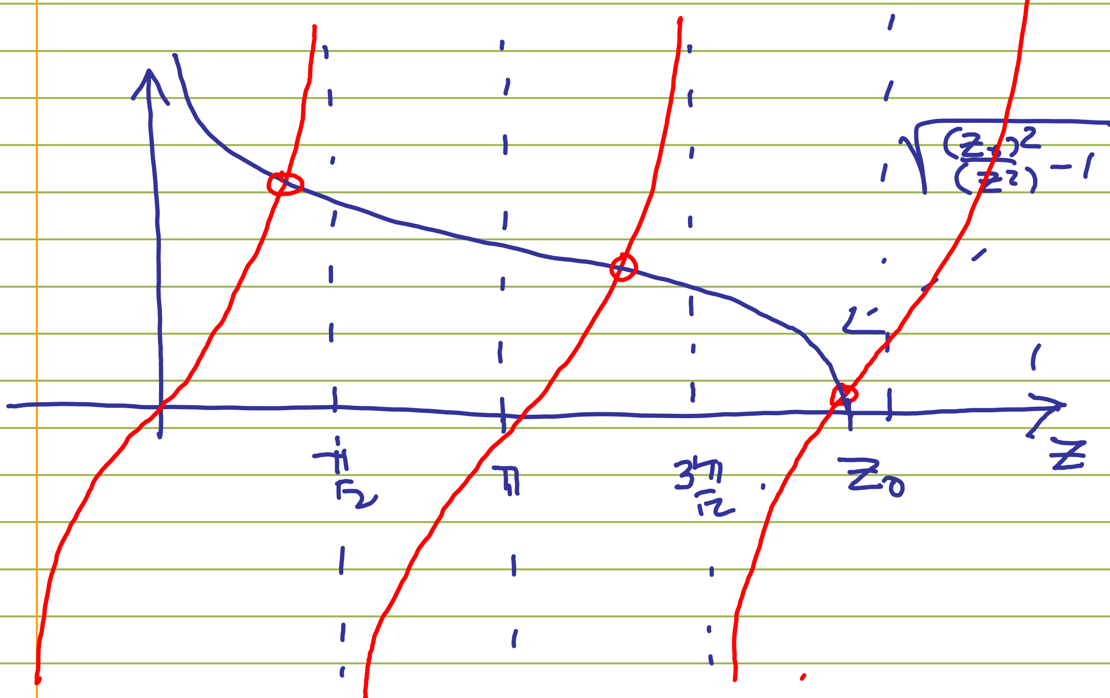
$$\kappa a = \sqrt{z_0^2 - z^2}$$

All together

$$\sqrt{z_0^2 - z^2} = z \tan z$$

or

$$\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} = \tan z$$




There are only finitely many solutions

Limit 1 Narrow, deep well

$$z_0 = \sqrt{\frac{2mV_0a^2}{\hbar^2}} \gg 1$$

$$\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} \approx \frac{z_0}{z} \gg 1$$

Requires $\tan z \gg 1$

$$\Rightarrow \quad z \approx \frac{n\pi}{2}, \quad n \text{ odd}$$

$$z = \ell \cdot a = \sqrt{\frac{2m(V_0+E)a^2}{\hbar^2}} = \frac{n\pi}{2}$$

$$\Rightarrow \quad E + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

as in even soln's of infinite square well (with bottom at $E+V_0=0$)

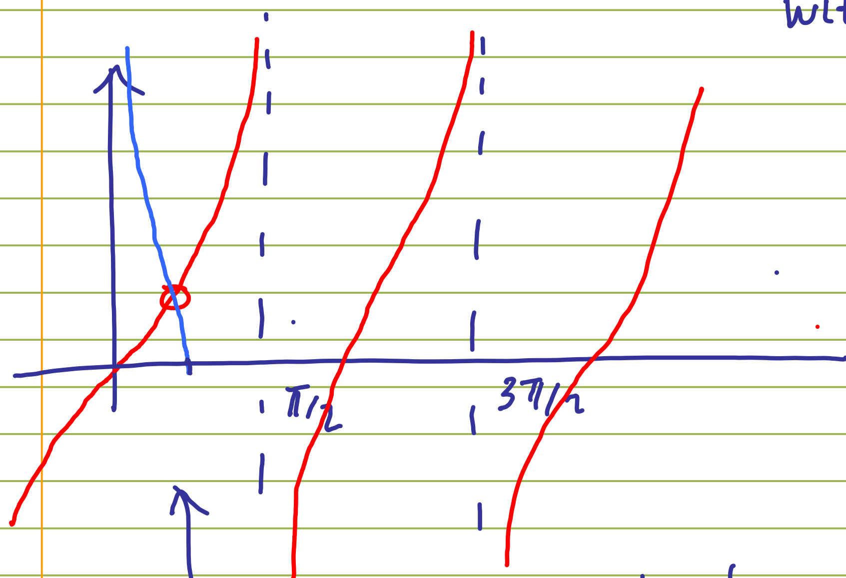
Limit 2

$\delta$-function potential

$$z_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\,a \to 0 \quad \Rightarrow \quad z \to 0$$

( why $\delta$-fn? $\delta(x) = \lim_{a\to 0} \begin{cases} 0, & |x|>a \\ \frac{1}{2a}, & |x|<a \end{cases}$

$V(x) = -\alpha\,\delta(x) \Leftrightarrow$ $a \to 0$ limit of potential with $V_0 = \frac{\alpha}{2a}$ )




Always at least one bound state.

$$\sqrt{\frac{z_0^2}{z^2} - 1} = \tan z \approx z$$

$$\sqrt{z_0^2 - z^2} \approx z^2 \quad \Rightarrow \quad z_0^2 \approx z^2 + z^4 \dots$$

$z \approx z_0$ Always ONE BOUND state

$$z^2 = z_0^2 - z_0^4 + \dots$$

$$\frac{2m(\cancel{V_0} + E)a^2}{\hbar^2} = \cancel{\frac{2mV_0a^2}{\hbar^2}} - \left(\frac{2mV_0a^2}{\hbar^2}\right)^2 + \dots$$

$$E \approx -\frac{2mV_0a^2}{\hbar^2} \cdot V_0 < 0$$

* Normalization $\Rightarrow$ Fixes $A$

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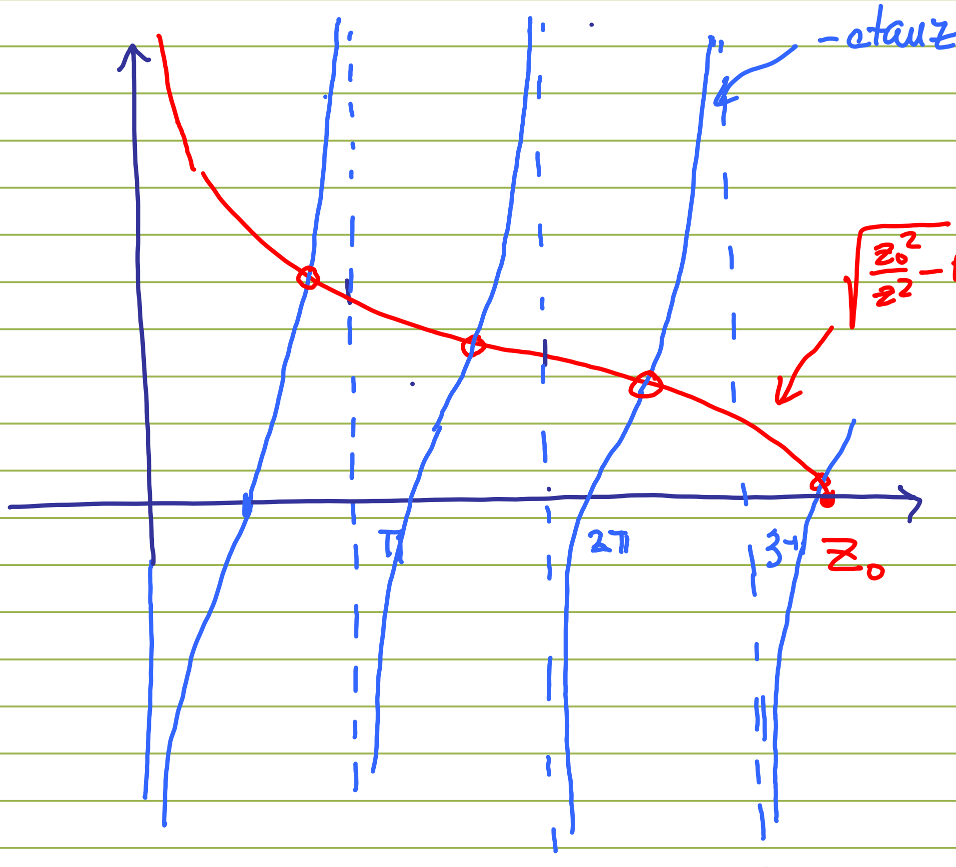
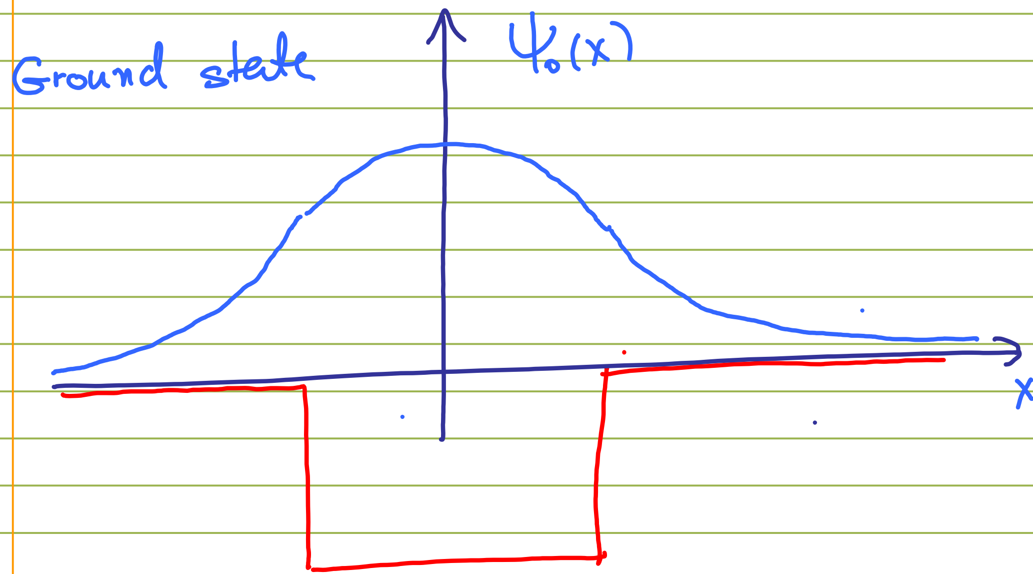
ODD $(D=0) \Rightarrow F=-A$

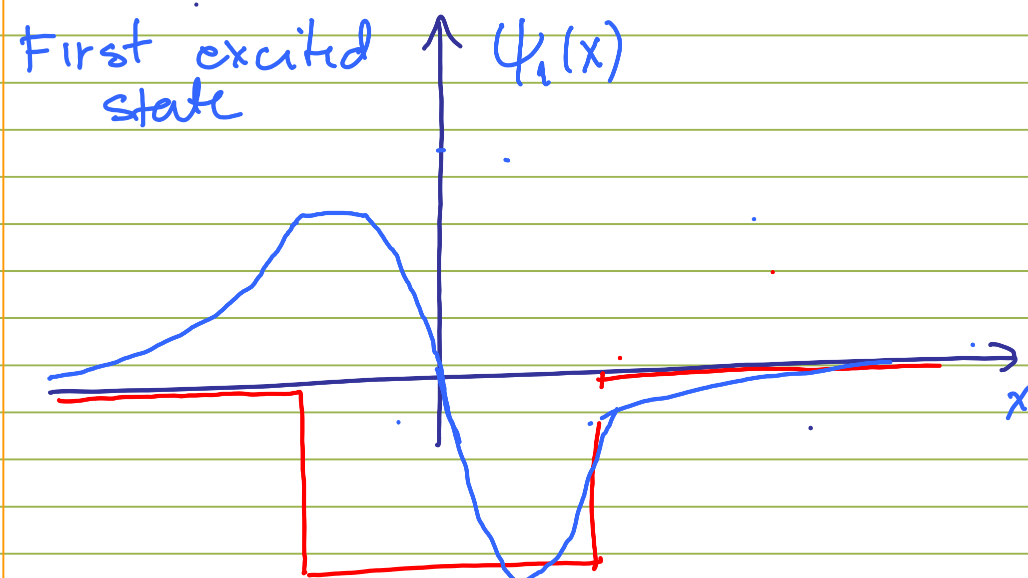
$$k = -\ell \cot \ell a$$

$$\sqrt{\frac{z_0^2}{z^2}-1} = -\cot z$$



Example



Number of even and odd bound states

One for every

$\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, $\frac{n\pi}{2}$

even odd even

Before $z_0$: Whole part of

$$\frac{z_0}{\pi/2} = \frac{2z_0}{\pi}$$

$$z_0 = \frac{\pi}{2} \otimes N + \text{remainder}$$

Scattering states

$E > 0$

$x < -a$: $\psi(x) = A e^{ikx} + B e^{-ikx}$

incoming reflected

$x > a$ $\psi(x) = F e^{ikx}$

transmitted

$-a < x < a \qquad \psi(x) = C e^{ilx} + D e^{-ilx}$

$$k^2 = \frac{2mE}{\hbar^2} > 0, \qquad l^2 = \frac{2m(E+V_0)}{\hbar^2}$$

Impose continuity of $\psi(x)$ and $\frac{d}{dx}\psi(x)$

at $x=-a$ and $x=a$:

4 equs for 5 variables

$\Rightarrow$ Lets you compute

$$R = \left|\frac{B}{A}\right|^2, \quad T = \left|\frac{F}{A}\right|^2$$

# Ch 3 Formalism

Lesson #13

QM requires new math language to describe physics

Classical:

State of system at time t = values of observables as fns of t.

Quantum:

state at t = $\psi(x,t)$

Observables = operators, e.g $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, acting on $\psi(x,t)$

For any fixed t, allowed states

$\psi(x,t)$

are elements of a vector space

= Space of square integrable functions

$$\int_{-\infty}^{\infty} dx \, |\psi(x,t)|^2 < \infty$$

- This is a vector space since if

$$\psi_1(x,t) \, , \, \psi_2(x,t)$$

are square integrable, so is

$$\alpha \psi_1(x,t) + \beta \psi_2(x,t)$$

- It has a norm which associates to $\psi_1(x,t)$, $\psi_2(x,t)$ a number (depending on $t$):

$$\int_{-\infty}^{\infty} \psi_1^*(x,t) \, \psi_2(x,t) \, dt$$

Space of square integrable functions is a vector space with a (Hermitian) norm:

"Hilbert Space"

Quantum states are elements of Hilbert space

- $\boxed{\text{observables are a special subset of linear operators acting on Hilbert space. "Hermitian operators".}}$

Physics is independent of the way we represent the vector space, e.g., the same state can be represented either as

$$\psi(x) \quad \text{or} \quad \tilde{\psi}(p)$$

where

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx \, e^{-i\frac{px}{\hbar}} \psi(x)$$

and linear algebra tools can be used to make this manifest.

## Vector space (finite dimensional)

An element of vector space is a vector

$$|a\rangle$$

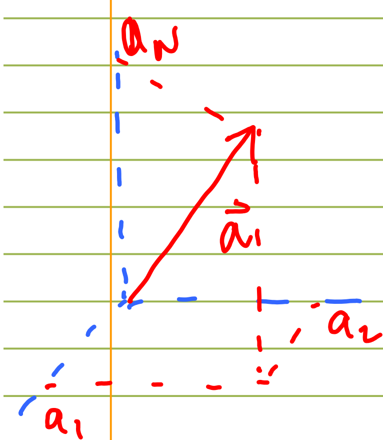
Given a set of Basis vectors

$$|1\rangle, \qquad , |N\rangle$$

We can write components

$$|a\rangle = a_1|1\rangle + \quad + a_N|N\rangle$$
$$= \sum_{i=1}^{N} a_i |i\rangle$$

Vector:




Usually, we'd package this as $\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$, but NOT here

Denote inner product of $|a\rangle$ and $|b\rangle$

By

$$\langle b|a\rangle$$

Natural inner product is Hermitian

$$\langle b | a \rangle = \langle a | b \rangle^*$$

If the basis is orthonormal

$$\langle i | j \rangle = \delta_{ij}$$

then

$$a_i = \langle i | a \rangle$$

and

$$a_i^* = \langle a | i \rangle$$

It follows

$$\langle a | = \sum_{i=1}^{N} a_i^* \langle i |$$

So

$$\langle b | a \rangle = \sum_{i=1}^{N} b_i^* a_i$$

## Linear operators

$$T : |a\rangle \to T|a\rangle$$

$$|i\rangle \to T|i\rangle \equiv \sum_j T_{ji} |j\rangle$$

where

$$T_{ji} = \langle j|T|i\rangle$$

It follows

$$|b\rangle = T|a\rangle$$

is a vector with components

$$b_j = \sum_{i=1}^{N} T_{ji} a_i$$

$$T|a\rangle = T\left(\sum_{i=1}^{N} a_i |i\rangle\right)$$

$$= \sum_{i=1}^{N} a_i T|i\rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i T_{ji} |j\rangle$$

$$= \sum_{j=1}^{N} b_j |j\rangle$$

<u>Vector space ($\infty$ dimensional)</u>

$$|i\rangle \longrightarrow |x\rangle \quad , \quad x \in [\alpha, \infty]$$

$$|f\rangle = \int_{-\infty}^{\infty} dx \; f(x) \; |x\rangle$$

$$\langle j|i\rangle = \delta_{ij} \rightarrow \langle x'|x\rangle = \delta(x - x')$$

Then

$$\langle x|f\rangle = \int_{-\infty}^{\infty} dx' \; f(x') \langle x'|x\rangle$$

$$= \int_{-\infty}^{\infty} dx' \; f(x') \; \delta(x' - x)$$

$$= f(x)$$

Function $f(x)$ is a component of vector $|f\rangle$ i position basis

$$f(x) = \langle x | f \rangle$$

Then

$$\langle f | x \rangle = f^*(x)$$
$$\left( = \langle x | f \rangle^* \right)$$

It follows

$$\langle f | = \int_{-\infty}^{\infty} dx \, f^*(x) \langle x |$$

and

$$\langle f | g \rangle = \int dx \, dx' \, f^*(x) \, g(x') \, \delta(x - x')$$

$$= \int_{-\infty}^{\infty} dx \, f^*(x) \, g(x)$$

which is Hermitian

$$\langle f|g\rangle^* = \langle g|f\rangle$$

Note the definition of norm requires

$$\langle f|g\rangle = \int_{-\infty}^{\infty} dx\, f^*(x)\, g(x) < \infty$$

(For this to make sense $f(x)$, $g(x)$ need to be square integrable. Then $\langle f|g\rangle$ is finite, since

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$$

Schwarz inequality. "=" holds if and only if $|f\rangle = c|g\rangle$ for some $c$.)

We can pick any basis of square integrable functions, not necessarily one corresponding to

$$|x\rangle, \quad \langle x'|x\rangle = \delta(x-x')$$

$$\{|f_n\rangle\}_{n=1}^{\infty} \quad \text{or in components} \quad \{f_n(x)\}_{n=1}^{\infty}$$

Which lets us write

$$|f\rangle = \sum_{n=1}^{\infty} c_n |f_n\rangle$$

or

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

If $\langle f_n | f_m \rangle = \delta_{nm}$

or

$$\int dx \, f_n^*(x) f_m(x) = \delta_{nm}$$

Then

$$c_n = \langle f_n | f \rangle$$

$$= \int dx\, f_n^*(x)\, f(x)$$

## 3.2 Observables

Lesson #14

Observables are linear operators acting on Hilbert space

Matrix elements

$$\langle i | Q j \rangle = Q_{ij}$$

generalize to

$$\langle f | Q g \rangle = \int dx\, f^*(x)\, \hat{Q}\, g(x)$$

where $\hat{Q}\, g(x)$ = action of $\hat{Q}$ on $g(x)$

Eg. $\hat{p}\, g(x) = -i\hbar \frac{\partial}{\partial x} g(x)$

$\hat{x}\, g(x) = x\, g(x)$.

Outcomes of all measurements, and hence expectation values, have to be real so

$$\boxed{\langle f | Q f \rangle = \langle Q f | f \rangle}$$

for any square integrable $f$.

Recall:
$$\langle f | g \rangle = \langle g | f \rangle^*$$

so reality of expectation values

$$\langle f | Q f \rangle = \langle f | Q f \rangle^* = \langle Q f | f \rangle$$

(Above property is equivalent to

$$\langle f | Q g \rangle = \langle Q f | g \rangle$$

which in turn is the standard definition of hermitian matrix

$$Q_{ij} = Q_{ji}^* .$$

$i = f$, $j = g$ )

$\boxed{\text{Observables are represented by Hermitian operators.}}$

Ex:

$$\langle f | \hat{p} g \rangle = \int_{-\infty}^{\infty} dx \, f^* \left(-i\hbar \frac{\partial}{\partial x} g\right)$$

$$= \int_{-\infty}^{\infty} dx \, \cancel{\left(-i\hbar \frac{\partial}{\partial x}\right)\left(f^* g\right)} \to 0$$

$$- \int_{-\infty}^{\infty} dx \left(-i\hbar \frac{\partial}{\partial x} f^*\right) g$$

$$= \int_{-\infty}^{\infty} \left(-i\hbar \frac{\partial}{\partial x} f\right)^* g$$

$$= \langle \hat{p} f | g \rangle$$

Note, we use that $f, g$ being square integrable

$$f^* g \longrightarrow 0 \quad \text{at} \quad x \to \pm\infty$$

Note: $\frac{d}{dx}$ is not hermitian.

Claim:

Measurement of $\hat{Q}$ in a state $|\Psi\rangle$ zero uncertainty if and only if $|\Psi\rangle$ is an eigenstate of $\hat{Q}$, and then:

## Determinate states

To measure observable Q,
prepare an ensamble of
particles, all in state $|\psi\rangle$.
Measurements in general do not
all give the same value.

Q: Under which conditions do
all measurements give same answer
$q$?

A: If $|\psi\rangle$ is eigenvector of
$\hat{Q}$ w/ eigenvalue $q$

$$\hat{Q}|\psi\rangle = q|\psi\rangle$$

Example: $\hat{Q} = \hat{H}$, then $q = E$

and $|\psi\rangle$ is a stationary state

Measure of spread of different values is uncertainty:

$$\sigma_Q^2 = \langle \hat{Q}^2 \rangle - \langle Q \rangle^2$$

Thm:

Uncertainty vanishes

$$\sigma_Q = 0$$

if and only if $|\psi\rangle$ is an eigenvector.

Proof:

One direction is easy: In eigenstate

$$\hat{Q}|\psi\rangle = q\,|\psi\rangle, \quad \hat{Q}^2|\psi\rangle = q^2|\psi\rangle$$

so $\langle \hat{Q}^2 \rangle = q = \langle Q \rangle^2$

The other direction:

Schwartz inequality

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$$

for any $|f\rangle$ and $|g\rangle$,

where "=" holds if and only if $|f\rangle = c \cdot |g\rangle$

Use above with

$$|f\rangle = |\Psi\rangle$$
$$|g\rangle = |Q\Psi\rangle$$

If $\sigma_Q = 0$ in state $\Psi$, then

$$\langle Q^2\rangle = \langle Q\rangle^2$$

It follows

$$\underbrace{\langle Q\Psi|Q\Psi\rangle}_{\langle Q^2\rangle}\langle\Psi|\Psi\rangle = \underbrace{|\langle\Psi|Q\Psi\rangle|}_{\langle Q\rangle^2}$$

since $\langle\Psi|Q^2\Psi\rangle = \langle Q\Psi|Q\Psi\rangle$

by hermiticity.

By schwartz inequality "="

if and only if $|Q\Psi\rangle = q\cdot|\Psi\rangle$

for some constant $q$.

Collection of all eigenvalues
of operator $\hat{Q}$ = "spectrum of Q",
in analogy to energy spectrum

Note: Every linear operator has a trivial eigenvector which vanishes identically. We exclude this, since corresponding eigenvalue can be any number.

- Zero eigenvalues are OK.

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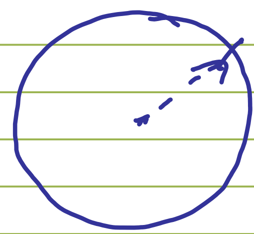
Spectrum is degenerate if a single eigenvalue corresponds to more than one eigen-vector.

Example

$$\hat{Q} = i \frac{d}{dx}$$

acting on functions on a circle



$x = 0 \Leftrightarrow x = 2\pi R$

$$f(x) = f(x + 2\pi R)$$

Eigenvalues

$$\hat{Q} f(x) = i \frac{d}{dx} f(x) = q f(x)$$

$$f(x) = e^{-iqx}$$

Require $\quad f(x) = f(x + 2\pi R)$

$$\Rightarrow \quad e^{iq \cdot 2\pi R} = 1$$

or $\quad q = \frac{n}{R}, \quad n = 0, \pm 1, \pm 2, \ldots$

$\hat{Q}$ is hermitian

$$\langle f|\hat{Q} g\rangle = \langle \hat{Q} f| g\rangle$$

$$: \int_0^{2\pi R} dx\, f^*(x) \left(i \frac{d}{dx}\, g(x)\right) =$$

$$= \int_0^{2\pi R} dx \quad i\frac{d}{dx}\left(f^* g\right)$$

$$- \int_0^{2\pi R} i\frac{d}{dx} f^* \; g$$

$$= \overset{\to 0}{\cancel{f^* g \Big|_0^{2\pi R}}} + \int_0^{2\pi} \left(i \frac{d}{dx} f\right)^* g$$

- Spectrum of $\hat{Q}$ is non-degenerate

- spectrum of $\hat{Q}^2 = -\frac{d^2}{dx^2}$ is degenerate

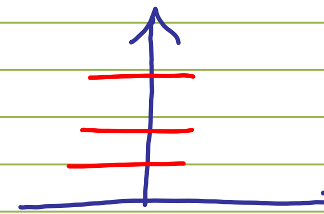
$$e^{+inx/R}, \quad e^{-inx/R}$$

correspond to same eigenvalue $\frac{n^2}{R^2}$.

## 3.3 Eigenvalues of a hermitian operator

Eigenvalues:

discrete



continuous

Continuous case is more subtle since eigenvectors are not normalizable in a standard sense. So, we will focus for now at discrete spectrum.

### Discrete Eigenvalues

Consider eigenvalue eqn

$$\hat{Q}|\psi_q\rangle = q\,|\psi_q\rangle$$

where $\hat{Q}$ is Hermitian,

and $\langle \psi | \psi \rangle < \infty$.

- $|\psi\rangle$ and $c \cdot |\psi\rangle$ both solve eigenvalue eqn for any $c \neq 0$. Since $\langle \psi | \psi \rangle < \infty$, we can choose normalualar so

$$\langle \psi | \psi \rangle = 1.$$

Theorem 1

Eigenvalues $q$ are real.

Proof:

If $|\hat{Q} f\rangle = q\,|f\rangle$ then

$$\langle \hat{Q} f | = q^* \langle f |.$$

Since $\hat{Q}$ is hermitian

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle$$

so $\quad (q - q^*) \langle f | f \rangle = 0 \Rightarrow q = q^*$

Lesson # 15

**Thm 2** Eigenvectors corresponding to distinct eigenvalues are orthogonal

Proof:

If
$$\hat{Q}|f\rangle = q|f\rangle$$
$$\hat{Q}|g\rangle = q'|g\rangle$$

and $\hat{Q}$ is hermitian

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

$$\Rightarrow \quad q'\langle f|g\rangle = q^*\langle f|g\rangle = q\langle f|g\rangle$$

Since $q \neq q' \Rightarrow \langle f|g\rangle = 0.$

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What we saw for energy eigenvalues is far more general!

# Degenerate states

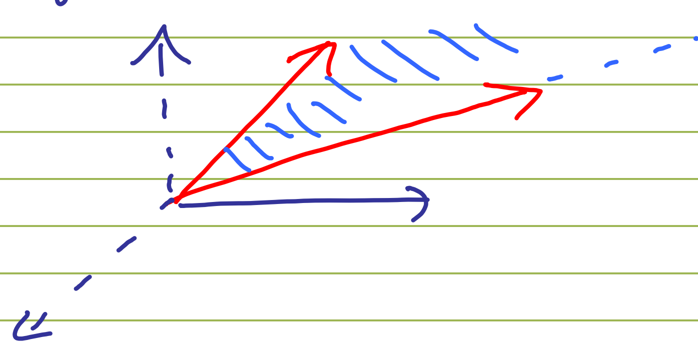
Suppose $|f\rangle$ and $|g\rangle$ are linearly independent

$$|f\rangle \neq c\,|g\rangle$$

for some $c \in \mathbb{C}$, But Both correspond to eigenvectors of $Q$ w/ eigenvalue $q$:

$$Q|f\rangle = q \cdot |f\rangle$$
$$Q|g\rangle = q\,|g\rangle$$

$\Rightarrow$ This means the eigen-space w/ eigenvalue $q$ has dimension greater than 1



Can use Gram-Schmidt orthogonaliz. procedure to construct two orthogonal eigenvectors

$$|g\rangle \to |g\rangle - \frac{\langle f|g\rangle}{\langle f|f\rangle}|f\rangle$$
$$= |g'\rangle$$

Then

$$\langle f|g'\rangle = 0$$

and

$$|f\rangle, \ |g'\rangle$$

are a pair of linearly indep. orthogonal eigenvectors of $\hat{Q}$.

Let

$$|f_{q,A}\rangle$$

be set of orthogonal, linearly indep. eigenfunctions of $\hat{Q}$, with eigenvalue $q$, normalized to

$$\langle f_{q,A} | f_{q',B} \rangle = \delta_{qq'} \delta_{AB}$$

Axiom / expectation

For a physical observable

$$\hat{Q}$$

the set of all linearly indep. eigenvectors spans the Hilbert space.

Then, if $|\psi\rangle$ is any state in Hilbert-space

$$|\psi\rangle = \sum_{q,A} c_{q,A} |f_{q,A}\rangle$$

with $\quad c_{q,A} = \langle f_{q,A} | \psi \rangle$

It follows

$$\sum_{q,A} |f_{q,A}\rangle\langle f_{q,A}| = \mathbb{1}$$

acts as identity in Hilbert space

Above restates the completeness axiom, and provides "resolution" of identity.

Continuous spectra

Wave functions are not normalizable, so; they do not live in the Hilbert space.

Still... in a sense, we still have reality, orthogonality, completeness

What replaces orthonormality

$$\langle f_q | f_{q'} \rangle = \delta_{qq'}$$

in discrete case is $\delta$-function type orthogonality in continuous case

$$\langle f_q | f_{q'} \rangle = \delta(q-q'),$$

so that

$$|f\rangle = \int dq \, c(q) \, |f(q)\rangle$$

where $\quad c(q) = \langle f(q) | f \rangle$

This can be restated as

$$\int dq \, |f(q)\rangle\langle f(q)| = 1$$

since acting on any $|f\rangle$ in Hilbert space, it gives (*).

Ex:

$$\hat{p} = -i\hbar \frac{d}{dx}$$

Has eigenfunctions

$$\hat{p}|q\rangle = q|q\rangle$$

In position basis

$$\langle x|\hat{p}|f\rangle = -i\hbar \frac{d}{dx}\langle x|f\rangle$$

for any $f$, so eigenfunction eqn says

$$-i\hbar \frac{d}{dx}\langle x|q\rangle = q\,\langle x|q\rangle$$

Solution:

$$\langle x|q\rangle \equiv f_q(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{+iqx/\hbar}$$

Eigenvalues are real, $q \in \mathbb{R}$

- These are not square integrable

But they are "$\delta$-function normalizable"

$$\langle q|q'\rangle = \int_{-\infty}^{\infty} f_q^*(x)\, f_{q'}(x)\, dx = \int_{-\infty}^{\infty} e^{i(q'-q)x/\hbar} \frac{dx}{2\pi\hbar}$$

$$= \delta(q-q')$$

<u>Ex</u>: Position operator $\hat{x}$

$$\hat{x}|y\rangle = y|y\rangle$$

has real eigenvalues & cont. spectrum

$$\langle x|y\rangle = \delta(x-y) \quad \left(\text{c.f. } \langle i|j\rangle = \delta_{ij}\right)$$

which satisfy completeness

$$\int_{-\infty}^{\infty} dy\, |y\rangle\langle y| = 1$$

or $\quad |f\rangle = \int_{-\infty}^{\infty} dy\, f(y)\, |y\rangle$ , where

$$f(x) = \langle x | f \rangle .$$

Since

$$\langle x | f \rangle = \int_{-\infty}^{\infty} dy \, f(y) \langle x | y \rangle$$

$$= \int_{-\infty}^{\infty} dy \, f(y) \, \delta(x-y)$$

$$= f(x) .$$

## 3.4 Generalized Stat. interpretation:

If you measure observable

$$Q(x,p)$$

on a particle in state $|\Psi\rangle$, the outcome is one of its eigenvalues

$$q_n$$

with probability

$$|c_n|^2 .$$

where

$$c_n = \langle f_n | \psi \rangle$$

and

$$\hat{Q} | f_n \rangle = q_n | f_n \rangle$$

This assumes non-degenerate spectrum

and $\langle f_n | f_m \rangle = \delta_{nm}$.

---

Note $\langle f_n | \psi \rangle = c_n$

is the same as

$$| \psi \rangle = \sum_n c_n | f_n \rangle$$

since $I = \sum_n | f_n \rangle \langle f_n |$

The measurement that results in eigenvalue $q_n$ projects

$$| \psi \rangle = \sum_n c_n | f_n \rangle$$

to corresp. eigenstate $|f_n\rangle$.
and this occurs w/ probability $|c_n|^2$.

$|c_n|^2$ = prob. that measuring Q one gets $q_n$

$c_n$ = how much $|f_n\rangle$ is in $|\Psi\rangle = \langle f_n|\Psi\rangle$

Consistency:

Check: $\sum_n |c_n|^2 = 1$

= "with probability 1 get some eigenvalue of Q"

This follows from $\langle\Psi|\Psi\rangle = 1$

Proof:

$$1 = \langle \Psi | \Psi \rangle = \sum_{n,m} c_m^* c_n \langle f_n | f_m \rangle$$

$$= \sum_{n,m} c_m^* c_n \delta_{nm} =$$

$$= \sum_n |c_n|^2$$

Check 2

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

Proof:

$$\langle Q \rangle = \langle \Psi | \hat{Q} | \Psi \rangle$$

$$= \sum_{n,m} c_m^* c_n \langle f_m | \hat{Q} f_n \rangle$$

$$= \sum_{n,m} c_m^* c_n q_n \langle f_m | f_n \rangle$$

$$= \sum_{nm} c_m^* c_n q_n \delta_{nm}$$

$$= \sum_n |c_n|^2 q_n$$

- For degenerate spectra

$$|c_n|^2 \to \sum_{\substack{n \\ \text{with} \\ \text{same } q_n}} |c_n|^2$$

For continuous spectra!

If

$$Q|f_z\rangle = z|f_z\rangle$$

with $\langle f_z | f_{z'} \rangle = \delta(z-z')$

probability of getting a result between $z$ and $z+dz$

$$|c_z|^2 dz$$

where $c_z = \langle f_z | \psi \rangle$

(= Continuous limit of

$$\sum |c_n|^2$$

over all

$q_n$ in $q_n, q_n + \Delta q$ )

Check 3 $\quad Q(x,p) = x$

Should get.

probability of measuring position of particle to be between

$x$ and $x + dx$

$$= |\psi(x,t)|^2 \, dx$$

where $\psi(x,t) = \langle x | \psi(t) \rangle$

Indeed

$$\hat{x} |y\rangle = y |y\rangle$$

and $\langle x | y \rangle = \delta(x - y)$

so $|\psi(t)\rangle = \int dy\, \psi(y)\, |y\rangle$

with

$$\langle x|\psi(t)\rangle = \int dy\, \psi(y)\langle x|y\rangle$$

$$= \int dy\, \psi(y)\, \delta(x-y)$$

$$= \psi(x).$$

<u>Momentum</u>

Probability of measuring momentum between

$p$ and $p+dp$

is $|C(p)|^2 dp$

where

$$C(p) = \langle p|\psi(t)\rangle$$

Inserting,

$$1 = \int dx\, |x\rangle\langle x|$$

and using

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$c(p) = \int dx \, \langle p | x \rangle \langle x | \psi(t) \rangle$$

$$= \int dx \, \underbrace{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \psi(x,t)}_{\text{Fourier transform of } \psi(x,t)}$$

Really also time-dep.

$$c(p) \equiv \tilde{\psi}(p,t)$$

Inverse

$$\underbrace{\psi(x,t)}_{} = \underbrace{\int \frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \psi(x,t)}_{}$$

$$\langle x | \psi(t) \rangle = \int dp \, \langle x | p \rangle \langle p | \psi(t) \rangle$$

since $\int dp \, |p\rangle\langle p| = \mathbb{1}$.

Prob. of finding $p$ betw $p_1$ and $p_2$

$$= \int_{p_1}^{p_2} |\psi(p,t)|^2 dp$$

## Uncertainty Principle

Time to prove it.

General statement:

Let $A, B$ be observ

(= corresponding Hermitian operators)

Then

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [A,B] \rangle \right|$$

Note: If $x = A$, $p = B$,

$$[x, p] = i\hbar$$

so get

$$\sigma_x \sigma_p \geq \left| \frac{1}{2i} \; i\hbar \right| = \frac{\hbar}{2}$$

<u>Proof</u>

$$\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$$

$$= \langle (A - \langle A \rangle)^2 \rangle$$

$$= \langle (A - \langle A \rangle)\psi \,|\, (A - \langle A \rangle)\psi \rangle$$

$$= \langle f | f \rangle$$

with $|f\rangle = (\hat{A} - \langle A \rangle)|\psi\rangle$

Similarly

$$\sigma_B^2 = \langle g | g \rangle$$

with $|g\rangle = (\hat{B} - \langle B \rangle)|\psi\rangle$

Schwarz inequality

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

Let $\langle f|g\rangle = z$, $z^* = \langle g|f\rangle$

$$|z|^2 = z_R^2 + z_I^2 \geq z_I^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

so $$\sigma_A^2 \sigma_B^2 \geq \left[\frac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)\right]^2$$

$$\langle f|g\rangle - \langle g|f\rangle =$$

$$= \langle \psi|(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)|\psi\rangle - \langle \psi|(\hat{B} - \langle B\rangle)(\hat{A} - \langle A\rangle)|\psi\rangle$$

$$= \langle \psi|\hat{A}\hat{B} - \hat{B}\hat{A}|\psi\rangle$$

$$= \langle \psi|[\hat{A}, \hat{B}]|\psi\rangle$$

$$= \langle [\hat{A}, \hat{B}]\rangle$$

everything but products of operators trivially commutes & cancels

Upshot:

For every pair of operators

$A, B$

that do not commute

$$[A, B] \neq 0$$

uncertainty principle. For such observables

$$\sigma_A \sigma_B > 0$$

in all states, preventing existence of common eigenfunctions

If $|\psi\rangle$ is eigenvector of $A$, $\sigma_A = 0$, but then $\sigma_B \to \infty$.