Notes of WU Shengjun MMP
Part II: Mathematical Physical Equation

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7.1 Math Model

7.1.1 Oscillation of String

$$\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{F(x,t)}{\rho} \tag{7.1}$$

7.2 D'Alembert Eq.

7.2.1 Infinite String

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = 0 \tag{7.2}$$

Denote

$$\xi = x + at \quad \eta = x - at \tag{7.3}$$

thus

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \tag{7.4}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \tag{7.5}$$

$$\frac{\partial^2}{\partial t^2} = a^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) \tag{7.6}$$

(7.2) can be rewritten as

$$-4a^2\frac{\partial^2}{\partial \xi \partial \eta}u(\xi,\eta) = 0 \tag{7.7}$$

i.e.

$$\frac{\partial^2}{\partial \xi \partial \eta}u(\xi,\eta) = 0 \tag{7.8}$$

initial condition:

$$u|_{t=0} = \phi(x) \quad \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \tag{7.9}$$

suppose

$$u(x,t) = f_1(x+at) + f_2(x-at) \tag{7.10}$$

thus

$$\begin{cases} \phi(x) = f_1(x) + f_2(x) \\ \psi(x) = a(f_1'(x) - f_2'(x)) \end{cases} \tag{7.11}$$

$$f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(\xi)\mathrm{d}\xi + f_1(x_0) - f_2(x_0) \tag{7.12}$$

thus

$$\begin{cases} f_1(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(\xi)\mathrm{d}\xi + \frac{1}{2}[f_1(x_0) - f_2(x_0)] \\ f_2(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_{x_0}^{x} \psi(\xi)\mathrm{d}\xi - \frac{1}{2}[f_1(x_0) - f_2(x_0)] \end{cases} \tag{7.13}$$

$$\begin{cases} f_1(x+at) = \frac{1}{2}\phi(x+at) + \frac{1}{2a}\int_{x_0}^{x+at} \psi(\xi)\mathrm{d}\xi + \frac{1}{2}[f_1(x_0) - f_2(x_0)] \\ f_2(x-at) = \frac{1}{2}\phi(x-at) - \frac{1}{2a}\int_{x_0}^{x-at} \psi(\xi)\mathrm{d}\xi - \frac{1}{2}[f_1(x_0) - f_2(x_0)] \end{cases} \tag{7.14}$$

General solution (D'Alembert Eq.):

$$\begin{aligned} u(x,t) &= f_1(x+at) + f_2(x-at) \\ &= \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi \end{aligned} \tag{7.15}$$

7.2.2 Half infinite string

Odd Continuation Boundary condition: $x \geq 0,\ u|_{x=0} = 0$.
Odd continuation: $\phi x \to \Phi(x),\ \psi(x) \to \Psi(x)$

$$\Phi(x) = \begin{cases} \phi(x) & x \geq 0 \\ -\phi(-x) & x \leq 0 \end{cases} \qquad \Psi(x) = \begin{cases} \psi(x) & x \geq 0 \\ -\psi(-x) & x \leq 0 \end{cases} \tag{7.16}$$

thus

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\Phi(x+at) + \Phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)\mathrm{d}\xi \\ &= \begin{cases} \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi & t \leq \frac{x}{a} \\ \frac{1}{2}[\phi(x+at) - \phi(-x+at)] + \frac{1}{2a}\int_{-x+at}^{x+at} \psi(\xi)\mathrm{d}\xi & t \geq \frac{x}{a} \end{cases} \end{aligned} \tag{7.17}$$

Even Continuation ...

8 Separation of Variables

Boundary conditions:
Type I (Dirichlet): $u(x,t)|_s = f_1$
Type II (Neumann): $\frac{\partial u}{\partial x}|_s = f_2$
Type III (Robin):

8.1

String oscillation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (0 < x < \ell, t > 0) \\ u|_{x=0} = 0,\ u|_{x=\ell} = 0 \quad (t > 0) \\ u|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \quad (0 \le x \le \ell) \end{cases} \tag{8.1}$$

Suppose

$$u(x,t) = X(x)T(t) \tag{8.2}$$

thus

$$X(x)T''(t) = a^2 X''(x)T(t) \tag{8.3}$$

8.2 非齐次

8.2.1 Fourier

8.2.2 冲量定理法

受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) u|_{x=0} = 0,\ u|_{x=\ell} = 0 u|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \tag{8.4}$$

decomposition:

$$u(x,t) = u_I(x,t) + u_{II}(x,t) \tag{8.5}$$

which satisfies

$$\frac{\partial^2 u_I}{\partial t^2} - a^2 \frac{\partial^2 u_I}{\partial x^2} = 0 u_I|_{x=0} = 0,\ u_I|_{x=\ell} = 0 u_I|_{t=0} = \phi(x),\ \frac{\partial u_I}{\partial t}|_{t=0} = \psi(x) \tag{8.6}$$

$$\frac{\partial^2 u_{II}}{\partial t^2} - a^2 \frac{\partial^2 u_{II}}{\partial x^2} = f(x,t) u_{II}|_{x=0} = 0,\ u_{II}|_{x=\ell} = 0 u_{II}|_{t=0} = 0,\ \frac{\partial u_{II}}{\partial t}|_{t=0} = 0 \tag{8.7}$$

u_I goto 8.1
u_{II} (冲量定理法)

8.3 (非齐次) Boundary Condition

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) \\ &u|_{x=0} = \mu(t),\ u|_{x=\ell} = \nu(t) \\ &u|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x) \end{aligned} \tag{8.8}$$

Choose

8.4 Poisson Equation

$$\begin{cases} \nabla^2 u = f(\mathbf{r}) \\ u|_\Sigma = \phi(M) \end{cases} \tag{8.9}$$

choose $v(\mathbf{r})$, s.t.

$$\nabla^2 v = f(\mathbf{r}) \tag{8.10}$$

Let $u = v + w$

$$\begin{cases} \nabla^2 w = 0 \\ w|_\Sigma = \phi(M) - v|_\Sigma \end{cases} \tag{8.11}$$

9 Solving 2°ODE with Series

9.1

spheric coord

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \tag{9.1}$$

9.1.1 Laplace Equation

$$\nabla^2 u = 0 \tag{9.2}$$

Spheric Coords

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \tag{9.3}$$

Let

$$u(r,\theta,\phi) = R(r)Y(\theta,\phi) \tag{9.4}$$

Cylindric Coords

9.1.2 波动方程

9.1.3 输运方程

9.1.4 Helmholtz Equation

Spheric Coords

Cylindric Coords

9.2 常点邻域上的级数解法

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + p(z)\frac{\mathrm{d}w}{\mathrm{d}z} + q(z)w = 0 \tag{9.5}$$

9.2.1 (常点) and Singularity

9.2.2 常点邻域上的级数解

9.2.3 Solving Legendre Eq. with Series

9.3 Serial Solution in Neighborhood of Canonical Singularity

9.3.1

9.3.2

9.3.3 Bessel Equation

9.3.4 Imaginary Bessel Equation

9.4 Sturm–Liouville Eigenvalue Problem

9.4.1 Sturm–Liouville Eigenvalue Problem

Sturm–Liouville Equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] - q(x)y + \lambda\rho(x)y = 0 \quad (a \leq x \leq b) \tag{9.6}$$

9.4.2 Eigenvalue Problem

9.4.3 Generalized Fourier Series

10 Spheric Function

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}+\ell(\ell+1)Y=0 \tag{10.1}$$

10.1 axis-symmetric

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\ell(\ell+1)\Theta=0 \tag{10.2}$$

Let $x=\cos\theta$

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}-2x\frac{\mathrm{d}\Theta}{\mathrm{d}x}+\ell(\ell+1)\Theta=0 \tag{10.3}$$

10.1.1 Legendre Polynomial

$$P_\ell(x)=\sum_{k=0}^{[\ell/2]}(-1)^k\frac{(2\ell-2k)!}{k!2^\ell(\ell-k)!(\ell-2k)!}x^{\ell-2k} \tag{10.4}$$

Differential representation

...

Integral representation

...

10.1.2 Second Legendre Function

10.1.3 Orthogonality

10.1.4 Normality

10.1.5 Generalized Fourier Series

$$f(x)=\sum_{\ell=0}^{\infty}f_\ell P_\ell(x) \tag{10.5}$$

10.1.6 Axis-symmetric Solution of Laplace Function

10.2 Associated Legendre Polynomial

Associated Legendre Equation

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\Theta}{\mathrm{d}x} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \tag{10.6}$$

10.2.1 Associated Legendre Function

Expression ...

$$P_\ell^m(x) = (1-x^2)^{m/2} P_\ell^{[m]}(x) \tag{10.7}$$

where

$$P_\ell^{[m]}(x) = \frac{\mathrm{d}^m P_\ell(x)}{\mathrm{d}x^m} \tag{10.8}$$

and

$$m = 0, 1, 2, \cdots, \ell \tag{10.9}$$

Differential

Integral

10.2.2 Orthogonality

$$\int_{-1}^{1} P_k^m(x) P_\ell^m(x) \mathrm{d}x = 0 \quad (k \neq \ell) \tag{10.10}$$

or

$$\int_0^\pi P_k^m(\cos\theta) P_\ell^m(\cos\theta) \sin\theta \mathrm{d}\theta = 0 \tag{10.11}$$

10.2.3 Module

$$(N_\ell^m)^2 = \int_{-1}^{1} [P_\ell^m(x)]^2 \mathrm{d}x = \sqrt{\frac{(\ell+m)! \cdot 2}{(\ell-m)!(2\ell+1)}} \tag{10.12}$$

10.2.4 Generalized Fourier Series

$$\begin{cases} f(x) = \displaystyle\sum_{\ell=m}^{\infty} f_\ell P_\ell^m(x) \\ f_\ell = \dfrac{(\ell-m)!(2\ell+1)}{(\ell+m)! \cdot 2} \displaystyle\int_{-1}^{1} f(x) P_\ell^m(x) \mathrm{d}x \end{cases} \tag{10.13}$$

10.2.5 递推