Notes of Particle Physics, LIU Zuowei

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Contents

1 Introduction

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平时 20 作业 30 期末 50

References
– Mark Thomson "Modern Particle Physics"
– David Griffiths "Introduction to Elementary Particles"
– Francis Halzen & Alan D. Martin "Quarks & Leptons"

1.1 Elementary Particles

p – uud
n – ddu
u – upper quark, d – down quark
c – s –
t – top b –

Gen1 – $e^-/\nu_e/u/d$
Gen2 – $\mu^-/\nu_\mu/c/s$
Gen3 – $\tau^-/\nu_\tau/t/b$
all have spin 1/2
charge: u,c,t 2/3; d,s,b -1/3
mass: $m_t \approx 170 m_p$

Electromagnetic Interaction, EM – proton
Strong – gluon
Weak – $W^\pm/Z^0$
Gravity – Higgs boson
$m_g = 125GeV$, $m_Z = 91GeV$, $m_W = 80GeV$ – EW scale

1.2 Feynman Diagrams

EM $\alpha_{EM} = 1/137$
strong $\alpha_s = 1$
weak $\alpha_w = 1/30$

small α enables perturbative calculation
Feynman rules

2 Underlying Concepts

2.1 Natural Units

$[kg, m, s] \rightarrow [\mathrm{GeV}, \hbar, c]$
$\hbar = c = 1$ mass: GeV/c^2 – GeV
length: $\hbar c/$ GeV - GeV^{-1}
time: $\hbar/\mathrm{GeV}$ – GeV^{-1}

$E, \mathbf{p}, m \longrightarrow \mathrm{GeV}$
$t, l \longrightarrow \mathrm{GeV}^{-1}$
$\sigma \longrightarrow \mathrm{GeV}^{-2}$

e.g. $\left\langle r^2 \right\rangle^{1/2} = 4.1\,\mathrm{GeV}^{-1} =?$
$[L] = [E]^{-1}[\hbar][c]$
$\mathrm{GeV}^{-1} = 1.6 \times 10^{-10}\,\mathrm{J}^{-1} \cdot 3 \times 10^{8}\,\mathrm{m\,s}^{-1} \cdot 1.055 \times 10^{-34}\,\mathrm{J\,s} \approx 0.8\,\mathrm{fm}$

Heaviside-Lorentz Units

$$\epsilon_0 = \mu_0 = 1 \tag{2.1}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \Rightarrow \frac{e^2}{4\pi} = \frac{1}{137} \tag{2.2}$$

2.2 Special Relativity

2.2.1 The Lorentz Transformation

$$\mathbf{X}' = \mathbf{\Lambda}\mathbf{X} \tag{2.3}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{2.4}$$

where $\gamma = (1-\beta^2)^{-1/2}$.

$$\mathbf{X} = \mathbf{\Lambda}^{-1}\mathbf{X}' \tag{2.5}$$

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \tag{2.6}$$

$$\mathbf{\Lambda}\mathbf{\Lambda}^{-1} = \mathbf{I} \tag{2.7}$$

2.2.2 4-Vectors & Lorentz Invariant

contravariant 4-vector

$$X^\mu = (t, x, y, z) \tag{2.8}$$

when $\mu \rightarrow \mu'$,

$$X^{\mu'} = \Lambda^\mu_\nu X^\nu \tag{2.9}$$

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2 \tag{2.10}$$

covariant 4-vector $X_\mu = (t, -x, -y, -z)$
$X^\mu X_\mu = X^{\mu'} X_{\mu'}$ # Ein sum convention
thus $X^\mu X_\mu = \mathscr{L}.I.$ # Lorentz invariant

$$X_\mu = G^\nu_\mu X^\nu \tag{2.11}$$

$$G^\nu_\mu \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.12}$$

4-momentum

$$P^\mu = (E, p_x, p_y, p_z) \tag{2.13}$$

$$P^\mu P_\mu = E^2 - \mathbf{p}^2 = m^2 \tag{2.14}$$

total 4-momentum

$$P^\mu P_\mu = \left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2 \neq (\sum_i m_i)^2 \tag{2.15}$$

but also $\mathscr{L}$. I..

4-derivative

$$\begin{pmatrix} \partial_{t'} \\ \partial_{x'} \\ \partial_{y'} \\ \partial_{z'} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{2.16}$$

i.e.

$$\partial_{\mu'} = \Lambda^\nu_\mu \partial_\nu \tag{2.17}$$

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \tag{2.18}$$

2.2.3 Mandelstam Variables

$2 \rightarrow 2$ process

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned} \tag{2.19}$$

all are $\mathscr{L}$. I..
In CoM

$$s = (E_1^* + E_2^*)^2 \tag{2.20}$$

2.3 Non-relativistic QM

2.3.1 Wave Mechanics & Schrödinger Equation

2.3.2 Prob. Dens. & Prob. Current

$$\rho = \psi^* \psi \tag{2.21}$$

def

$$\frac{\partial}{\partial t} \int_V \rho \mathrm{d}V = -\int_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} \tag{2.22}$$

use div th.

$$\boldsymbol{\nabla} \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \tag{2.23}$$

From free-particle TDSE

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} = -\frac{\nabla^2}{2m}\psi \qquad -\mathrm{i}\,\frac{\partial\,\psi^*}{\partial t} = -\frac{\nabla^2}{2m}\,\psi^* \tag{2.24}$$

$$-\frac{1}{2m}(\psi^*\,\nabla^2\,\psi - \psi\,\nabla^2\,\psi^*) = \mathrm{i}\left(\psi^*\,\frac{\partial \psi}{\partial t} + \psi\frac{\partial\,\psi^*}{\partial t}\right) \tag{2.25}$$

$$-\frac{1}{2m}\,\boldsymbol{\nabla} \cdot (\psi^*\,\boldsymbol{\nabla}\,\psi - \psi\,\boldsymbol{\nabla}\,\psi^*) = \mathrm{i}\,\frac{\partial}{\partial t}(\psi^*\,\psi) = \mathrm{i}\,\frac{\partial \rho}{\partial t} \tag{2.26}$$

thus

$$\mathbf{j} = \frac{-\,\mathrm{i}}{2m}(\psi^*\,\boldsymbol{\nabla}\,\psi - \psi\,\boldsymbol{\nabla}\,\psi^*) \tag{2.27}$$

suppose

$$\psi = N\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{2.28}$$

$$\mathbf{j} = N^2\,\frac{\mathbf{p}}{m} = n\mathbf{v} \tag{2.29}$$

2.3.3 TD & Conserved Quantities

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{A} \right\rangle = \mathrm{i}\left\langle [\widehat{\mathrm{H}}, \hat{A}] \right\rangle \tag{2.30}$$

2.3.4 Commutation relations & Compatible Observables

$$\Delta A \Delta B \leq \frac{1}{2}\left|\left\langle \mathrm{i}[\hat{A}, \hat{B}] \right\rangle\right| \tag{2.31}$$

where

$$\Delta A = \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2} \tag{2.32}$$

2.3.5 Angular Momentum

...

2.4 Fermi's Golden Rule

unperturbed SE

$$\widehat{\mathrm{H}}_0\,\psi_k = E_k\psi_k \tag{2.33}$$

add interaction Hamiltonian

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = [\widehat{\mathrm{H}}_0 + \widehat{\mathrm{H}}']\psi \tag{2.34}$$

basis set expd.

$$\psi = \sum_k c_k(t)\psi_k(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\,Et} \tag{2.35}$$

3 Decay Rates and Cross Sections

3.1 Fermi's Golden Rule ★

Transition rate

$$\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i) \tag{3.1}$$

3.2 Phase Space and Wavefxn Normalization ★

$\alpha \to 1 + 2$

1st order:

$$T_{fi} = \langle f \,|\, H' \,|\, i\rangle = \langle \psi_1^*\psi_2^* \,|\, H' \,|\, \psi_a\rangle \tag{3.2}$$

Born approx.:

$$\psi(\mathbf{x}, t) = A\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{3.3}$$

Normalization in a box of side a:

$$\langle \psi \,|\, \psi\rangle = 1 \Rightarrow A^2 = 1/a^3 \tag{3.4}$$

PBC gives

$$\mathbf{p} = \mathbf{n}\frac{2\pi}{a} \tag{3.5}$$

thus

$$\mathrm{d}^3\mathbf{p} = \mathrm{d}^3\mathbf{n}\,\frac{(2\pi)^3}{V} \tag{3.6}$$

$$\mathrm{d}n \equiv \mathrm{d}^3\mathbf{n}\mathrm{d}^3p\frac{V}{(2\pi)^3} = 4\pi p^2\mathrm{d}p\frac{V}{(2\pi)^3} \tag{3.7}$$

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{\mathrm{d}n}{\mathrm{d}p}\left|\frac{\mathrm{d}p}{\mathrm{d}E}\right| \tag{3.8}$$

Set $V = 1$, and there are N particles, thus $N - 1$ indep momenta

$$\mathrm{d}n = \prod_{i=1}^{N-1}\mathrm{d}n_i = \prod_{i=1}^{N-1}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3} \tag{3.9}$$

$$\tag{3.10}$$

3.2.1 LI Phase Space

$$\Gamma_{fi} = \frac{1}{2E_a}\int |\mathcal{M}_{fi}|^2 \left[(2\pi)^4\delta^4(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2)\right]\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \tag{3.11}$$

3.2.2 Fermi G R Revisited

$$\Gamma_{fi} = \tag{3.12}$$

In CoM, $E_a \to m_a$
next subsec.

3.3 Particle Decays ★

静止参考系中

$$p_a = (m_a, 0) \tag{3.13}$$

$\therefore$

$$\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2)\mathrm{d}^3\mathbf{p}_2 = \tag{3.14}$$

$$\Gamma = \frac{1}{2m_a}\int |\mathcal{M}|^2 (2\pi)^4 \delta(m_a - E_1 - E_2)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^6 4E_1E_2} \tag{3.15}$$

where $\mathbf{p}_2 = -\mathbf{p}_1$. thus $E_2 = \sqrt{m_2^2 + p_1^2}$.
and $\mathrm{d}^3\mathbf{p}_1 = p_1^2 \mathrm{d}p_1 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$

$$\Gamma = \frac{1}{8\pi^2 m_a}\int |\mathcal{M}|^2 \delta(m_a - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2})\frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{4E_1E_2} \tag{3.16}$$

Since

$$\int_{-\infty}^{\infty} \delta[f(x)]g(x)\mathrm{d}x = \sum_i \frac{g(x_i)}{|f'(x_i)|} \quad \mathrm{w}/ f(x_i) = 0 \tag{3.17}$$

thus

$$\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2}\int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega \tag{3.18}$$

where $m_a - \sqrt{m_1^2 + p^{*2}} - \sqrt{m_2^2 + p^{*2}} = 0$

3.4 Interaction Cross Sections ★

tot # of target particles

$$\delta N = n_b v \delta t A \tag{3.19}$$

interaction prob.

$$\delta P = \frac{\delta N \sigma}{A} = n_b v \sigma \delta t \tag{3.20}$$

interaction rate per target particle

$$\Gamma_b = \frac{\mathrm{d}P}{\mathrm{d}t} = nv\sigma = \sigma\phi_a \tag{3.21}$$

$$\phi_a = \frac{\mathrm{d}N_a}{\mathrm{d}A\mathrm{d}t} = \frac{n\mathrm{d}Av\mathrm{d}t}{\mathrm{d}A\mathrm{d}t} = nv \tag{3.22}$$

total rate

$$\Gamma_a n_a V = (n_a v)(n_b V)\sigma = \text{flux} \times \#\ \text{target} \times \sigma \tag{3.23}$$

3.4.1 $\mathscr{L}$.I. Flux

a + b $\longrightarrow$ 1 + 2

$$\Gamma_{fi} = \frac{v_a + v_b}{V}\sigma \tag{3.24}$$

$$\sigma = \frac{1}{4E_aE_b(v_a+v_b)} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_a + p_b - p_1 - p_2) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \tag{3.25}$$

which is $\mathscr{L}$.I..
$\mathscr{L}$.I. flux factor $F = 4E_aE_b(v_a + v_b)$.

3.4.2 Scattering in the CoM frame

photo

$$\sigma = \frac{|\mathbf{p}_f|}{64\pi^2 s|\mathbf{p}_i|} \int |\mathcal{M}|^2 \mathrm{d}\Omega \tag{3.26}$$

3.5 Diff. Xsec. ★

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrm{d}\Omega \tag{3.27}$$

3.5.1 CoM Frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2} \frac{p_f^*}{p_i^*} |\mathcal{M}_{fi}|^2 \tag{3.28}$$

$$t = (p_i - p_3)^2 \quad \mathscr{L}.\mathrm{I}. \tag{3.29}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s} \frac{|\mathcal{M}_{fi}|^2}{p_i^{*2}} \quad \mathscr{L}.\mathrm{I}. \tag{3.30}$$

3.5.2 Lab Frame

$\mathrm{e}^- + \mathrm{p} \longrightarrow \mathrm{e}^- + \mathrm{p}$

$$\begin{aligned} p_1 &\approx (E_1, 0, 0, E_1) \\ p_2 &= (m_p, 0, 0, 0) \\ &... \end{aligned} \tag{3.31}$$

$$p_i^* \approx \frac{(s - m_p^2)^2}{4s} \tag{3.32}$$

where $s = (p_1 + p_2)^2 = m_p^2 + 2E_1 m_p$

...

4 The Dirac Equation

4.1 The Klein-Gordon Eq.

$$\widehat{\mathrm{H}}^2 = \widehat{\mathbf{p}}^2 + m^2 \tag{4.1}$$

where $\widehat{\mathrm{H}} = \mathrm{i}\,\dfrac{\partial}{\partial t}$, thus

$$\frac{\partial^2 \psi}{\partial t^2} = (\nabla^2 - m^2)\psi \tag{4.2}$$

in $\mathscr{L}$. I. form

$$(\partial^\mu \partial_\mu + m^2)\psi = 0 \tag{4.3}$$

4.2 The Dirac Eq. ★

$$E\psi = (\boldsymbol{\alpha} \cdot \widehat{\mathbf{p}} + \beta m)\psi \tag{4.4}$$

$$\begin{aligned} -\frac{\partial^2 \psi}{\partial t^2} &= (-\mathrm{i}\,\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m)^2 \psi \\ &= \ldots \end{aligned} \tag{4.5}$$

must satisfy

$$\begin{aligned} &\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \\ &\alpha_i \beta + \beta \alpha_i = 0 \\ &\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \end{aligned} \tag{4.6}$$

$$\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\beta) = 0 \tag{4.7}$$

$$\lambda_{\alpha_i} = \pm 1 \tag{4.8}$$

$$len(\alpha_i) = len(\beta) = \text{evennumber} \tag{4.9}$$

$$\alpha_i^\dagger = \alpha_i \quad \beta^\dagger = \beta \tag{4.10}$$

Dirac spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.11}$$

There are infinite choices of α and β.
Dirac-Pauli representation:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{4.12}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.13}$$

4.3 Prob. Dens. & Prob. Curr.

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} = -\,\mathrm{i}\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,\psi + m\beta\psi \tag{4.14}$$

$$-\,\mathrm{i}\,\frac{\partial \psi^\dagger}{\partial t} = \mathrm{i}\,\boldsymbol{\nabla}\,\psi^\dagger\cdot\boldsymbol{\alpha}^\dagger + m\psi^\dagger\beta^\dagger \tag{4.15}$$

$$\psi^\dagger(-\,\mathrm{i}\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,\psi + m\beta\psi) - \big(\mathrm{i}\,\boldsymbol{\nabla}\,\psi^\dagger\cdot\boldsymbol{\alpha}^\dagger + m\psi^\dagger\beta^\dagger\big)\psi = \mathrm{i}\,\psi^\dagger\frac{\partial \psi}{\partial t} + \mathrm{i}\,\frac{\partial \psi^\dagger}{\partial t}\psi \tag{4.16}$$

$$\boldsymbol{\nabla}\cdot(\psi^\dagger\boldsymbol{\alpha}\psi) + \frac{\partial}{\partial t}\big(\psi^\dagger\psi\big) = 0 \tag{4.17}$$

$$\boldsymbol{\nabla}\cdot\mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \tag{4.18}$$

4.4 Spin ★

$$[\widehat{\mathrm{H}}_D, \widehat{\mathbf{L}}] = [\boldsymbol{\alpha}\cdot\widehat{\mathbf{p}}, \widehat{\mathbf{r}}\times\widehat{\mathbf{p}}] = -\,\mathrm{i}\,\boldsymbol{\alpha}\times\widehat{\mathbf{p}} \neq= 0 \tag{4.19}$$

$$\widehat{\mathbf{S}} = \frac{1}{2}\,\widehat{\boldsymbol{\Sigma}} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \tag{4.20}$$

$$[\alpha_i, \Sigma_j] = \begin{pmatrix}0 & [\sigma_i, \sigma_j]\\ [\sigma_i, \sigma_j] & 0\end{pmatrix} \tag{4.21}$$

$$[\widehat{\mathrm{H}}_D, \widehat{\boldsymbol{\Sigma}}] = 2\,\mathrm{i}(\boldsymbol{\alpha}\times\widehat{\mathbf{p}}) \Rightarrow [\widehat{\mathrm{H}}_D, \widehat{\mathbf{S}}] = \mathrm{i}(\boldsymbol{\alpha}\times\widehat{\mathbf{p}}) \tag{4.22}$$

def

$$\widehat{\mathbf{J}} = \widehat{\mathbf{L}} + \widehat{\mathbf{S}} \tag{4.23}$$

$$[\widehat{\mathrm{H}}_D, \widehat{\mathbf{J}}] = 0 \tag{4.24}$$

$\widehat{\mathbf{J}}$ is a conserved quantity.

$$\widehat{\mathbf{S}}^2 = \tag{4.25}$$

EM – magnetic moment ★

$$E = -\boldsymbol{\mu}\cdot\mathbf{B} \tag{4.26}$$

$$E\psi = (\boldsymbol{\alpha}\cdot\widehat{\mathbf{p}} + \beta m)\psi \tag{4.27}$$

minimal substitution

$$\begin{aligned} E &\to E - q\phi \\ \widehat{\mathbf{p}} &\to \widehat{\mathbf{p}} - q\mathbf{A} \end{aligned} \tag{4.28}$$

where $A = (\phi, \mathbf{A})$

$$(E - q\phi)\psi = [\boldsymbol{\alpha}\cdot(\widehat{\mathbf{p}} - q\mathbf{A}) + \beta m]\psi \tag{4.29}$$

$$\begin{aligned} (E - q\phi - m)\psi_A &= \boldsymbol{\sigma}\cdot(\widehat{\mathbf{p}} - q\mathbf{A})\psi_B \\ (E - q\phi + m)\psi_B &= \boldsymbol{\sigma}\cdot(\widehat{\mathbf{p}} - q\mathbf{A})\psi_A \end{aligned} \tag{4.30}$$

Non-Rela limit: $E \approx m >> q\phi$

$$2m\psi_B = \boldsymbol{\sigma}\cdot(\widehat{\mathbf{p}} - q\mathbf{A})\psi_A \quad \psi_B << \psi_A \tag{4.31}$$

$$
\begin{aligned}
(E - q\phi - m)\psi_A &= \frac{1}{2m}[\boldsymbol{\sigma}\cdot(\widehat{\mathbf{p}} - q\mathbf{A})]^2\psi_A \\
&= \frac{1}{2m}[(\widehat{\mathbf{p}} - q\mathbf{A})^2 + \mathrm{i}\,\sigma\cdot(\widehat{\mathbf{p}} - q\mathbf{A})\times(\widehat{\mathbf{p}} - q\mathbf{A})]\psi_A \\
&= \frac{1}{2m}[(\widehat{\mathbf{p}} - q\mathbf{A})^2 + q\sigma\cdot(\boldsymbol{\nabla}\times\mathbf{A} + \mathbf{A}\times\boldsymbol{\nabla})]\psi_A
\end{aligned}
\tag{4.32}
$$

$$
(\boldsymbol{\nabla}\times\mathbf{A} + \mathbf{A}\times\boldsymbol{\nabla})\psi_A = \mathbf{B}\psi_A \tag{4.33}
$$

$$
E\psi_A = \left[m + \frac{1}{2m}(\widehat{\mathbf{p}} - q\mathbf{A})^2 + q\phi - \frac{q}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}\right]\psi_A \tag{4.34}
$$

From (4.26),

$$
\boldsymbol{\mu} = \frac{q}{2m}\boldsymbol{\sigma} = \frac{q}{m}\widehat{\mathbf{S}} \tag{4.35}
$$

which is spin magnetic moment.
Or,

$$
\boldsymbol{\mu} = g\frac{q}{2m}\widehat{\mathbf{S}} \tag{4.36}
$$

Dirac Eq. explained $g = 2$.
orbital magnetic moment

$$
\boldsymbol{\mu} = \frac{q}{2m}\widehat{\mathbf{L}} \tag{4.37}
$$

In fact, Schwinger gave a correction on g in 1948

$$
a = \frac{g-2}{2} = \frac{\alpha}{2\pi} \tag{4.38}
$$

4.5 Covariant Form of the Dirac Eq.

$$
\beta(\mathrm{i}\frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\widehat{\mathbf{p}} - \beta m)\psi = 0 \tag{4.39}
$$

def

$$
\gamma^0 = \beta \quad \gamma^1 = \beta\alpha_x \quad \gamma^2 = \beta\alpha_y \quad \gamma^3 = \beta\alpha_z \tag{4.40}
$$

and since $\beta^2 = I$

$$
(\mathrm{i}\,\gamma^\mu\partial_\mu - m)\psi = 0 \tag{4.41}
$$

where $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
Note: γ is not usual 4-vector, but (4.41) is Lorentz covariant.

4.5.1

4.6 Sol. to the Dirac Eq. ★

Suppose

$$
\psi(\mathbf{r}, t) = u(E, \mathbf{p})\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - Et)} \tag{4.42}
$$

where u is a 4-component Dirac spinor
with (4.41)

$$
(\beta E - \beta\boldsymbol{\alpha}\cdot\mathbf{p} - m)u(E, \mathbf{p})\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - Et)} = 0 \tag{4.43}
$$

$$
(\gamma^\mu P_\mu - m)u = 0 \tag{4.44}
$$

4.6.1 Particles at Rest

Since $\mathbf{p} = 0$

$$(E\beta - m)u = 0 \tag{4.45}$$

$$E\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = m\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} \tag{4.46}$$

the eigenvectors are

$$u_1 = N\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} \quad u_2 = N\begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix} \tag{4.47}$$

with $E = m$

$$u_3 = N\begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix} \quad u_4 = \tag{4.48}$$

with $E = -m$

How to explain negative energy?
Feynman-Stueckelberg:

$$\mathrm{e}^{-\mathrm{i}\,Et} \equiv \mathrm{e}^{\mathrm{i}\,E(-t)} \tag{4.49}$$

E.g. holes in solid state physics. Holes ”move” oppositely as the electrons move, like propagating ”backward” in time.
or anti-particles, like positron.

4.6.2 General Free-particle Sol.

$$(\gamma^\mu P_\mu - m)u = 0 \tag{4.50}$$

$$\left[\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}E - \begin{pmatrix}0 & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0\end{pmatrix} - m\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\right]u = 0 \tag{4.51}$$

$$\begin{pmatrix}(E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p}\\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = 0 \tag{4.52}$$

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}u_B u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}u_A \tag{4.53}$$

$$u_A = .. = u_A \tag{4.54}$$

thus we can choose $u_A = \begin{pmatrix}1\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\end{pmatrix}$

4.7 Antiparticles ★

4.7.1

4.7.2 the Feynman-Stückelberg Interpretation

$$(\gamma^\mu P_\mu + m)v = 0 \tag{4.55}$$

$$\psi = v(p)\,\mathrm{e}^{\mathrm{i}\,Et - \mathrm{i}\,\mathbf{p}\cdot\mathbf{x}} \tag{4.56}$$

...

$$v_1 = N\begin{pmatrix} ... \\ ... \\ 0 \\ 1 \end{pmatrix} \quad v_2 = N\begin{pmatrix} ... \\ ... \\ 1 \\ 0 \end{pmatrix} \tag{4.57}$$

kick out v_3, v_4, u_3, u_4 whose energy are negative.

4.7.3 Normalization

$$\rho = \psi^\dagger\psi = u_1^\dagger u_1 = |N|^2\frac{2E}{E+m} \tag{4.58}$$

def

$$\rho = 2E \quad N = \sqrt{E+m} \tag{4.59}$$

4.7.4 Operators for Antiparticles

$$\widehat{\mathrm{H}}^{(v)} = -\,\mathrm{i}\,\frac{\partial}{\partial t} \quad \widehat{\mathbf{p}}^{(v)} = +\,\mathrm{i}\,\boldsymbol{\nabla} \tag{4.60}$$

thus

$$\widehat{\mathbf{L}}^{(v)} = -\,\widehat{\mathbf{L}}^{(u)} \quad \widehat{\mathbf{S}}^{(v)} = -\,\widehat{\mathbf{S}}^{(u)} \tag{4.61}$$

4.7.5 Charge Conjugation

$$E \to E - q\phi \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A} \tag{4.62}$$

$$p_\mu \to p_m u - qA_\mu \text{ i.e. } \mathrm{i}\,\partial_\mu \to \mathrm{i}\,\partial_\mu - qA_\mu \tag{4.63}$$

for electron, $q = -e$

$$\gamma^\mu(\partial_\mu - \mathrm{i}\,eA_\mu)\psi + \mathrm{i}\,m\psi = 0 \tag{4.64}$$

take complex conj and pre-multiply $-\,\mathrm{i}\,\gamma^2$

$$-\,\mathrm{i}\,\gamma^2(\gamma^\mu)^*(\partial_\mu + \mathrm{i}\,eA_\mu)\psi^* - m\gamma^2\psi^* = 0 \tag{4.65}$$

since

$$\gamma^2\gamma^\mu = -\gamma^\mu\gamma^2 \tag{4.66}$$

for $\mu \neq 2$

4.8 Spin and Helicity States ★

$\{u_1, u_2, v_1, v_2\}$ are not $\widehat{\mathbf{S}}_z$ eigenstates except $\mathbf{p} = \pm p\,\widehat{\mathbf{k}}$
and $[\widehat{\mathrm{H}}_D, \widehat{\mathbf{S}}_z] \neq 0$, thus we can't find a basis of simultaneous eigenstates.
def: helicity operator

$$\widehat{\mathrm{h}} \equiv \frac{\widehat{\mathbf{S}} \cdot \mathbf{p}}{p} \tag{4.67}$$

$$\widehat{\mathrm{h}} = \frac{1}{2p}\begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix} \tag{4.68}$$

$$[\widehat{\mathrm{h}}, \widehat{\mathrm{H}}_D] = 0 \tag{4.69}$$

suppose $\widehat{\mathrm{h}}\, u = \lambda u$

$$\frac{1}{2p}\begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix} u = \lambda u \tag{4.70}$$

$$(\boldsymbol{\sigma} \cdot \mathbf{p}) u_A = 2p\lambda u_A \tag{4.71}$$

since $(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = p^2$

$$\lambda = \pm\frac{1}{2} \tag{4.72}$$

according to (4.53)

$$(\boldsymbol{\sigma} \cdot \mathbf{p}) u_A = (E + m) u_B \tag{4.73}$$

$\therefore$

$$u_B = \frac{2\lambda p}{E + m} u_A \tag{4.74}$$

To solve (4.71), let

$$\mathbf{p} = \ldots \tag{4.75}$$

$$\widehat{\mathrm{h}} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{-\,\mathrm{i}\,\phi} \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\,\phi} & -\cos\theta \end{pmatrix} \tag{4.76}$$

assume $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\frac{b}{a} = \mathrm{e}^{\mathrm{i}\,\phi} \tan\frac{\theta}{2} \tag{4.77}$$

thus

$$u_\uparrow = N \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{\mathrm{i}\,\phi} \sin\frac{\theta}{2} \\ \ldots \\ \ldots \end{pmatrix} \tag{4.78}$$

4.9 Intrinsic Parity of Dirac Fermions ★

parity transformation

$$t' = t \quad x' = -x \tag{4.79}$$

parity operator

$$\psi' = \widehat{\mathbf{P}}\, \psi\, \widehat{\mathbf{P}}^2 = I \tag{4.80}$$

$$\tag{4.81}$$

$$\gamma^0\, \widehat{\mathbf{P}} \propto I \tag{4.82}$$

5 Interaction by Particle Exchange

5.1 1st and 2nd Order Perturbation Theory

5.1.1 Time-Ordered Perturbation Theory

5.2 Feynman Diagrams and Virtual Particles

5.3 Intro to QED

Dirac Eq. + minimal substitution

$$\partial_\mu \to \partial_\mu + \mathrm{i}\, q A_\mu \tag{5.1}$$

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} + \mathrm{i}\,\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}\,\psi - q\gamma^0\gamma^\mu A_\mu \psi - m\gamma^0\psi = 0 \tag{5.2}$$

$$\widehat{\mathrm{H}} = (m\gamma^0 - \mathrm{i}\,\gamma^0\boldsymbol{\gamma} \cdot \boldsymbol{\nabla}) + q\gamma^0\gamma^\mu A_\mu \tag{5.3}$$

这一节的例子可以不看。

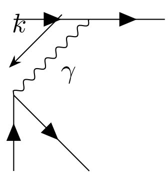
5.4 Feynman Rules for QED

$\mathrm{e}^- \, \mathrm{e}^- \longrightarrow \mathrm{e}^- \, \mathrm{e}^-$

$\mathrm{e}^- \, \mathrm{e}^+ \longrightarrow \, ^- \, ^+$

$$-\mathrm{i}\,\mathcal{M} = \tag{5.4}$$

6 e-p Annihilation



计算 matrix element

6.1 Perturbation Theory

6.2 e-p Annihilation

$$\mathcal{M} = \ldots \tag{6.1}$$

6.3 Spin Sums

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \tag{6.2}$$

6.4 Helicity Amplitudes

6.5 The μ and e Currents

6.6 Cross Section

$$\mathcal{M}_{RL\to RL} = e^2(1+\cos\theta) \tag{6.3}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = e^4(1+\cos^2\theta) \tag{6.4}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4(1+\cos^2\theta)}{64\pi^2 s} \tag{6.5}$$

$$\sigma = \frac{4\pi\alpha^2}{3s} \tag{6.6}$$

total spin $= \pm 1 \Rightarrow$ contribution to helicity amplitude. (because photon has spin 1)

6.7

6.8 Spin in e-p Anni.

$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1+\cos\theta)\,|1,+1\rangle \tag{6.7}$$

thus

$$\mathcal{M}_{RL\to RL} \propto \langle 1,+1\,|\,1,+1\rangle_\theta = \frac{1}{2}(1+\cos\theta) \tag{6.8}$$

6.9 Chirality ★

def

$$\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.9}$$

in the limit $E >> m$

$$\gamma^5\,u_\uparrow = +u_\uparrow \quad \gamma^5\,u_\downarrow = -u_\downarrow \tag{6.10}$$

def: left- and right-handed chiral states

$$\gamma^5\,u_R = +u_R \quad \gamma^5\,u_L = -u_L \tag{6.11}$$

$$u_R = N\begin{pmatrix} c \\ s\,\mathrm{e}^{\mathrm{i}\,\phi} \\ c \\ s\,\mathrm{e}^{\mathrm{i}\,\phi} \end{pmatrix} \quad u_L = \tag{6.12}$$

chiral states are identical to massless helicity states.

6.9.1 Chiral Projection Operators

$$P_R = \frac{1}{2}(1+\gamma^5)$$
$$P_L = \frac{1}{2}(1-\gamma^5) \tag{6.13}$$

features

$$P_R + P_L = 1 \quad P_R P_R = 1 \quad P_R P_L = 0 \tag{6.14}$$

$$P_R u_R = u_R \quad ... \tag{6.15}$$

$$P_L u_R = 0 \quad P_L u_L = u_L \tag{6.16}$$

thus any spinor can be decomposed with

$$u = a_R u_R + a_L u_L = P_R u + P_L u \tag{6.17}$$

6.9.2 Chirality in QED

$$\bar{\psi}\,\gamma^\mu\,\phi \tag{6.18}$$

7 E-p Elastic Scattering

7.1 Probing ...

7.2 Rutherford and Mott Scattering ★

$$\mathcal{M}_{fi} = \frac{e^2}{q^2}\bar{u}_3\,\gamma^\mu\,u_1 g_{\mu\nu}\bar{u}_4\,\gamma^\nu\,u_2 \tag{7.1}$$

Helicity states

$$u_\uparrow = ... \qquad u_\downarrow = ... \tag{7.2}$$

where

$$\kappa = \frac{p}{E+m_e} = \frac{\beta_e\gamma_e}{\gamma_e+1} \tag{7.3}$$

NR: $\kappa \to 0$, R: $\kappa \approx 1$
init e$^-$: $(\theta, \phi) = (0, 0)$
fin e$^-$: $(\theta, \phi) = (\theta, 0)$

$$\begin{aligned} j^e_{\uparrow\uparrow} &= (E+m_e)[(\kappa^2+1)c, 2\kappa s, 2\,\mathrm{i}\,\kappa s, 2\kappa c] \\ &= \end{aligned} \tag{7.4}$$

$\kappa = 1 \Rightarrow j^e_{\uparrow\downarrow} = j^e_{\downarrow\uparrow} = 0$, the same with last week.
Helicity is effectively conserved at vertices in high energy limits.

init p: $\theta = 0, \phi = 0, \kappa = 0$
fin p: $\theta = \eta, \phi = 0, \kappa \approx 0$

$$\begin{aligned} j^p_{\uparrow\uparrow} &= j^p_{\downarrow\downarrow} = 2m_p[c_\eta, 0, 0, 0] \\ j^p_{\uparrow\downarrow} &= j^p_{\downarrow\uparrow} = -2m_p[s_\eta, 0, 0, 0] \end{aligned} \tag{7.5}$$

all NR.

$$\begin{aligned}\left\langle|\mu|^2\right\rangle = \frac{1}{4}\sum\left|\mathcal{M}_{fi}^2\right| &= \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}[(1-\kappa^2)^2+4\kappa^2c^2] \\ &= \frac{16m_p^2m_e^2e^4}{q^4}\left[1+\beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\right]\end{aligned} \tag{7.6}$$

$$q^4 = ... \tag{7.7}$$

i) e^- is NR, $1+\beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\approx 1$ (Rutherford)

ii) e^- is R, $1+\beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\approx \beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}$ (Mott)

7.2.1 Rutherford Scattering

$$\left\langle|\mu|^2\right\rangle = \frac{m_p^2m_e^2e^4}{p^4\sin^4(\theta/2)} \tag{7.8}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p+E_1-E_1\cos\theta}\right)\left\langle|\mu|^2\right\rangle \tag{7.9}$$

Since

$$E_1 \sim m_e << m_p \tag{7.10}$$

we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = ... \tag{7.11}$$

7.2.2 Mott Scattering

$E\approx p_e$

$$\left\langle|\mu|^2\right\rangle = \frac{m_p^2e^4}{E^2\sin^4(\theta/2)}\cos^2\frac{\theta}{2} \tag{7.12}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = ... \tag{7.13}$$

7.3 Form Factors ★

形状因子
charge density $Q\rho(\mathbf{r}')$, where $\int\rho(\mathbf{r}')=1$
potential

$$V(\mathbf{r}) = \int\frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \tag{7.14}$$

In the Born approx.,

$$\psi_i = \mathrm{e}^{\mathrm{i}(\mathbf{p}_1\cdot\mathbf{r}-Et)} \qquad \psi_f = \mathrm{e}^{\mathrm{i}(\mathbf{p}_3\cdot\mathbf{r}-Et)} \tag{7.15}$$

$$\begin{aligned}\mathcal{M}_{fi} = \langle\psi_f\,|\,V(\mathbf{r})\,|\,\psi_i\rangle &= \int\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}}\int\frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r} \\ &= \int\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{R}}\int\frac{Q}{4\pi|\mathbf{R}|}\mathrm{d}^3\mathbf{R}\int\rho(\mathbf{r}')\,\mathrm{e}^{\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}'}\,\mathrm{d}^3\mathbf{r}' \\ &\equiv \mathcal{M}_{fi}^{\mathrm{pt}}F(\mathbf{q}^2)\end{aligned} \tag{7.16}$$

pt: point-like
Form factor

$$F(\mathbf{q}^2) = \int \rho(\mathbf{r}')\, e^{i\,\mathbf{q}\cdot\mathbf{r}'}\, d^3\mathbf{r}' \tag{7.17}$$

i.e. Fourier transform of chg density distribution.

$$F(0) = 1 \quad F(\infty) = 0 \tag{7.18}$$

Note 7.8 留数定理 ★

7.4 Relativistic e-p Elastic Scattering

$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 & \end{aligned} \tag{7.19}$$

$$q^2 = -2E_1E_3(1-\cos\theta) \tag{7.20}$$

def $Q^2 = -q^2$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2}\left(\frac{E_3}{m_pE_1}\right)^2 \left\langle |\mathcal{M}_{fi}|^2 \right\rangle \\ &= \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right) \end{aligned} \tag{7.21}$$

when $Q^2 << m_p^2$ and $E_3 \approx E_1$, reduced to Mott.

7.5 the Rosenbluth Formula

7.5.1 Measuring G

$$\frac{d\sigma}{d\Omega}\Bigg/\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2} \tag{7.22}$$

G can be fit by a dipole function

$$G_E^2(Q^2) = \frac{1}{(1+Q^2/Q_0^2)^2} \tag{7.23}$$

$$G_M^2(Q^2) = 2.79 G_E^2(Q^2) \tag{7.24}$$

charge distr.

$$\rho(r) = \rho_0\, e^{-r/a} \tag{7.25}$$

$a = 0.24fm$

7.5.2 Elastic Scattering at high Q^2

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7.8 Q_0 数据错了

8 Deep Inelastic Scattering (DIS)

8.1 E-p Inelastic Scattering

$\mathrm{e}^- + p \to \mathrm{e}^- + \mathrm{X}$
invariant mass of the hadronic system X: $W^2 = p_4^2$
elastic: $W^2 = m_p^2$

8.1.1 Kinematic Variables

$$Q^2 = -(p_1 - p_3)^2 = -(2m_e^2 - 2p_1p_3) = -2m_e^2 + 2E_1E_3 - 2\mathbf{p}_1 \cdot \mathbf{p}_3 \tag{8.1}$$

inelastic $->$ high E, thus

$$Q^2 = 2E_1E_3(1 - \cos\theta) = \tag{8.2}$$

def

$$x \equiv \frac{Q^2}{2p_2q} \quad y \equiv \frac{p_2q}{p_2p_1} \quad \nu \equiv \frac{p_2q}{m_p} \tag{8.3}$$

$$W^2 = (q + p_2)^2 = m_p^2 + q^2 + 2p_2q \tag{8.4}$$

$$W^2 + Q^2 - m_p^2 = 2p_2q \tag{8.5}$$

$$x \equiv \frac{Q^2}{Q^2 + W^2 - m_p^2} = \frac{Q^2}{2p_2q} \tag{8.6}$$

elastic: $x = 1$
inelastic: $0 \leq x \leq 1$, since ...
x expresses the elasticity.

$$y = \frac{p_2q}{p_2p_1} = ... = 1 - \frac{E_3}{E_1} \tag{8.7}$$

y expresses the inelasticity, or fractional energy loss of e^-.

$$\nu = ... = E_1 - E_3 \tag{8.8}$$

the kinematics of inelastic scattering can be described by any two of the Lorentz-invariant quantities x, Q^2, y and ν, except $\{y, \nu\}$.

8.1.2 IS at Low Q^2

$$\begin{aligned} W^2 &= m_p^2 + q^2 + 2p_2q = m_p^2 + (p_1 - p_3)^2 + 2p_2(p_1 - p_3) \\ &= m_p^2 - 2E_1E_3(1 - \cos\theta) + 2m_p(E_1 - E_3) \\ &= [m_p^2 + 2m_pE_1] - [2m_p + 2E_1(1 - \cos\theta)]E_3 \end{aligned} \tag{8.9}$$

...

8.2 Deep Inelastic Scattering ★

Elastic Rosenbluth formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = ... \tag{8.10}$$

Using def of Q^2 and y

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right) + \frac{1}{2}y^2 G_M^2\right] \tag{8.11}$$

def: $f_1(Q^2) = ...,\ f_2(Q^2) = ...$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right) f_2(Q^2) + \frac{1}{2}y^2 f_1(Q^2)\right] \tag{8.12}$$

8.2.1 Structure Func.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x}Q^2 = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{8.13}$$

deep IS: $Q^2 >> m_p^2 y^2$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x}Q^2 = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{8.14}$$

SLAC's 2 striking features Bjorken scaling

$$F_1(x,Q^2) \to F_1(x) \qquad F_2(x,Q^2) \to F_2(x) \tag{8.15}$$

almost independent of Q^2. point-like constituents within the proton.
Callan-Gross relation

$$F_2(x) = 2xF_1(x) \tag{8.16}$$

8.3 Elec-quark Scattering

[e-p scat. cross sect.] = [pdf(slow)] × [e-q scat. cross sect.(fast)]
pdf: parton distribution function ★

e-q scat. in CoM frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Q_q^2 e^4}{8\pi^2 s}\frac{1 + \frac{1}{4}(1+\cos\theta)^2}{1-\cos\theta)^2} \tag{8.17}$$

$\mathscr{L}$.I. form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left|\frac{\mathrm{d}\Omega}{\mathrm{d}q^2}\right| = ... \tag{8.18}$$

8.4 The Quark Model

Infinite Momentum Frame $\quad E_p >> m_p$
the struck quark

$$p_q = \xi p_2 = (\xi E_2, 0, 0, \xi E_2) \tag{8.19}$$

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8.2 8.3 8.6 8.8

$$(\xi p_2 + q)^2 = m_q^2 \quad (\xi p_2)^2 = m_q^2 \tag{8.20}$$

$$\xi = \frac{-q^2}{2p_2 q} = x \tag{8.21}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2 Q_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right] \tag{8.22}$$

where $s_q = (p_1 + xp_2)^2 = xs$
And

$$y_q = \frac{xp_2 q}{xp_2 p_1} = y \quad x_q = 1\text{(e-q is elastic)} \tag{8.23}$$

$$\frac{q^2}{s_q} = \frac{-(s_q - m_q^2)x_q y_q}{s_q} = -y \tag{8.24}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2 Q_q^2}{q^4}\left[1 + (1-y)^2\right] \tag{8.25}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2 Q_q^2}{Q^4}\left[1 - y + \frac{y^2}{2}\right] \tag{8.26}$$

8.4.1 PDF

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x}Q^2 = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_i Q_i^2 q_i(x) \tag{8.27}$$

which predicts Bjorken scaling and CG relation.

8.4.2 Determination of the PDFs

But PDFs cannot be computed in QFT.

$$F_2^{ep} = x\left(\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x) + \frac{4}{9}\bar{u}^p(x) + \frac{1}{9}\bar{d}^p(x)\right) \tag{8.28}$$

valence quark: $p = uud, n = udd$
sea quark: any flavor can be produced via gluon, so anti-quark be taken into the eq above. ★

$$F_2^{en} = x\left(\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x) + \frac{4}{9}\bar{u}^n(x) + \frac{1}{9}\bar{d}^n(x)\right) \tag{8.29}$$

From symm

$$u^n(x) = d^p(x) \tag{8.30}$$

thus, def

$$F_2^{ep} = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)\right) \tag{8.31}$$

$$F_2^{en} = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x)\right) \tag{8.32}$$

$$\int_0^1 F_2^{ep}(x)\mathrm{d}x = ... \equiv \frac{4}{9}f_u + \frac{1}{9}f_d \tag{8.33}$$

$$\int_0^1 F_2^{en}(x)\mathrm{d}x \equiv \frac{4}{9}f_d + \frac{1}{9}f_n \tag{8.34}$$

By experiment

$$f_u = 0.36 \quad f_d = 0.18 \tag{8.35}$$

8.4.3 Valence and Sea Quarks

$x \to 0$, $R \to 1$
$x \to 1$, $R \to 1/4$, while $R_{naive} = 2/3$.
why?