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1 Fundamental Conceptions

1.2 Vector Differential Operator & Laplace Operator

1.2.1 Vector Differential Operator ∇

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{1.1}$$

def:

1) gradient of function u:

$$\nabla u \equiv \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} \tag{1.2}$$

2) divergence of vector $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.3}$$

3) curl of vector $\mathbf{E}$:

$$\begin{aligned} \nabla \times \mathbf{E} &\equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \\ &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{k} \end{aligned} \tag{1.4}$$

Laplace Operator

$$\ldots \tag{1.5}$$

Relevant equations:

(4)

$$\nabla \cdot (u\mathbf{E}) = (\nabla u) \cdot \mathbf{E} + u(\nabla \cdot \mathbf{E}) \tag{1.6}$$

(5)

$$\nabla \times (u\mathbf{E}) = (\nabla u) \times \mathbf{E} + u(\nabla \times \mathbf{E}) \tag{1.7}$$

(6)

$$\nabla \cdot (\mathbf{E} \times \mathbf{F}) = (\nabla \times \mathbf{E}) \cdot \mathbf{F} - \mathbf{E} \cdot (\nabla \times \mathbf{F}) \tag{1.8}$$

Proof:

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \mathbf{F}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ E_x & E_y & E_z \\ F_x & F_y & F_z \end{vmatrix} \\
&= \frac{\partial}{\partial x}(E_yF_z - E_zF_y) + \frac{\partial}{\partial y}(E_zF_x - E_xF_z) + \frac{\partial}{\partial z}(E_xF_y - E_yF_x) \\
&= \frac{\partial E_y}{\partial x}F_z - \frac{\partial E_x}{\partial z}F_y + \frac{\partial E_z}{\partial y}F_x - \frac{\partial E_x}{\partial y}F_z + \frac{\partial E_x}{\partial z}F_y - \frac{\partial E_y}{\partial z}F_x \\
&\quad + \frac{\partial F_z}{\partial x}E_y - \frac{\partial F_y}{\partial x}E_z + \frac{\partial F_x}{\partial y}E_z - \frac{\partial F_z}{\partial y}E_x + \frac{\partial F_y}{\partial z}E_x - \frac{\partial F_x}{\partial z}E_y \\
&= F_x(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) + F_y(\frac{\partial E_x}{\partial z} - \frac{\partial E_x}{\partial z}) + F_z(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\
&\quad + E_x(\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y}) + E_y(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}) + E_z(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}) \\
&= (\nabla \times \mathbf{E}) \cdot \mathbf{F} - \mathbf{E} \cdot (\nabla \times \mathbf{F})
\end{aligned} \tag{1.9}$$

(7)

$$\nabla \times(\mathbf{E} \times \mathbf{F}) = (\mathbf{F} \cdot \nabla)\mathbf{E} - \mathbf{F}(\nabla \cdot \mathbf{E}) - (\mathbf{E} \cdot \nabla)\mathbf{F} + \mathbf{E}(\nabla \cdot \mathbf{F}) \tag{1.10}$$

Proof:

(8)

$$\nabla(\mathbf{E} \cdot \mathbf{F}) = (\mathbf{F} \cdot \nabla)\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{F} + \mathbf{F} \times(\nabla \times \mathbf{E}) + \mathbf{E} \times(\nabla \times \mathbf{F}) \tag{1.11}$$

Proof:

(9)

$$\nabla \times(\nabla u) = 0 \tag{1.12}$$

Proof:

$$\begin{aligned}
\nabla \times(\nabla u) &= \nabla \times(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} \\
&= 0
\end{aligned} \tag{1.13}$$

(10)

$$\nabla\cdot(\nabla\times\mathbf{E})=0 \tag{1.14}$$

Proof omitted.

(11)

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E} \tag{1.15}$$

Proof:

1.2.2 Laplace Operator

in polar coordinates

$$r=\sqrt{x^2+y^2} \tag{1.16}$$

$$\theta=\arctan\frac{y}{x} \tag{1.17}$$

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right)\\
&=\frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r}+\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial x}\left(\frac{\partial}{\partial r}\right)^2+\frac{\partial\theta}{\partial x}\frac{\partial^2}{\partial\theta\partial r}\right)+\frac{\partial^2\theta}{\partial x^2}\frac{\partial}{\partial\theta}+\frac{\partial\theta}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial^2}{\partial\theta\partial r}+\frac{\partial\theta}{\partial x}\left(\frac{\partial}{\partial\theta}\right)^2\right)\\
&=\frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r}+\frac{\partial^2\theta}{\partial x^2}\frac{\partial}{\partial\theta}+\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial^2}{\partial r^2}+\left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial^2}{\partial\theta^2}+2\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2}{\partial\theta\partial r}
\end{aligned} \tag{1.18}$$

while

$$\frac{\partial r}{\partial x}=\frac{x}{r} \tag{1.19}$$

$$\frac{\partial^2 r}{\partial x^2}=\frac{\partial}{\partial x}\frac{x}{r}=\frac{r-x\frac{x}{r}}{r^2}=\frac{y^2}{r^3} \tag{1.20}$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{r^2} \tag{1.21}$$

$$\frac{\partial^2\theta}{\partial x^2}=2\frac{y}{r^3}\frac{\partial r}{\partial x}=\frac{2xy}{r^4} \tag{1.22}$$

Thus

$$\frac{\partial^2}{\partial x^2}=\frac{y^2}{r^3}\frac{\partial}{\partial r}+\frac{2xy}{r^4}\frac{\partial}{\partial\theta}+\frac{x^2}{r^2}\frac{\partial^2}{\partial r^2}+\frac{y^2}{r^4}\frac{\partial^2}{\partial\theta^2}-\frac{2xy}{r^3}\frac{\partial^2}{\partial\theta\partial r} \tag{1.23}$$

In the same way

$$\frac{\partial^2}{\partial y^2}=\frac{x^2}{r^3}\frac{\partial}{\partial r}-\frac{2xy}{r^4}\frac{\partial}{\partial\theta}+\frac{y^2}{r^2}\frac{\partial^2}{\partial r^2}+\frac{x^2}{r^4}\frac{\partial^2}{\partial\theta^2}+\frac{2xy}{r^3}\frac{\partial^2}{\partial\theta\partial r} \tag{1.24}$$

Thus

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1.25}$$

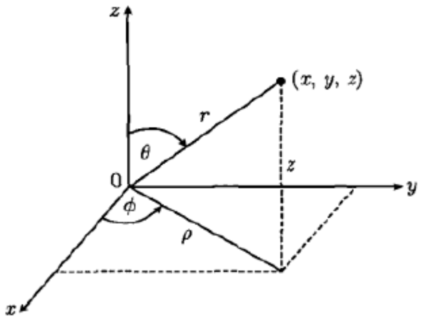
More succinctly

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1.26}$$

in cylindrical coordinates

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{1.27}$$

in spherical coordinates



$$x = r\cos\phi\sin\theta \tag{1.28}$$

$$y = r\sin\phi\sin\theta \tag{1.29}$$

$$z = r\cos\theta \tag{1.30}$$

As mentioned above

$$\rho = r\sin\theta,\ x = \rho\cos\phi,\ y = \rho\sin\phi \tag{1.31}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \tag{1.32}$$

Since

$$z = r\cos\theta,\ \rho = r\sin\theta \tag{1.33}$$

we have

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1.34}$$

According to (3.44)(1.34),

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r\sin\theta}\left(\frac{\partial r}{\partial \rho}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \rho}\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\end{aligned} \tag{1.35}$$

$\because\ \frac{\rho}{z} = \tan\theta$

$\therefore$

$$\frac{\partial \theta}{\partial \rho} = \frac{1}{1+\frac{z^2}{\rho^2}}\frac{1}{z} = \frac{z}{r^2} = \frac{\cos\theta}{r} \tag{1.36}$$

$\because\ \rho = r\sin\theta$

$\therefore$

$$1 = \frac{\partial r}{\partial \rho}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial\rho} \tag{1.37}$$

$$\frac{\partial r}{\partial \rho} = \sin\theta \tag{1.38}$$

Thus

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}
\end{aligned} \tag{1.39}$$

2 Fourier Series

2.1 Fourier Series of Periodic Functions

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x) \tag{2.1}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(t)\,\mathrm{d}t \tag{2.2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}t\mathrm{d}t \tag{2.3}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi}{l}t\mathrm{d}t \tag{2.4}$$

Discussion:

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{1}{2}(\pi - x) \quad (0 < x < 2\pi) \tag{2.5}$$

WSJ:

Bessel Inequality

$$|f(x)|^2 \geq a_0^2 + \sum_{k=1}^{n} a_k^2\cos\left|\frac{k\pi}{l}x\right|^2 + \sum_{k=1}^{n} b_k\left|\sin\frac{k\pi}{l}x\right|^2 \tag{2.6}$$

...

Parseval Equality

Dirichlet Theorem

2.2 Half-range Fourier Series

半幅傅里叶级数

For $0 < x < l$

sine expansion:

$$\phi(x) = \sum_{\mathrm{n=1}}^{\infty} c_n \sin\frac{n\pi}{l}x \tag{2.7}$$

$$c_n = \frac{2}{l}\int_0^l \phi(t)\sin\frac{n\pi}{l}\mathrm{d}t \tag{2.8}$$

cosine expansion:

$$\phi(x) = d_0 + \sum_{\mathrm{n=1}}^{\infty} d_n \cos\frac{n\pi}{l}x \tag{2.9}$$

$$d_0 = \frac{2}{l}\int_0^l \phi(t)\mathrm{d}t \tag{2.10}$$

$$d_n = \frac{2}{l}\int_0^l \phi(t)\cos\frac{n\pi}{l}\mathrm{d}t \tag{2.11}$$

variants:

$$\phi(x) = \sum_{\mathrm{n=1}}^{\infty} c'_n \sin\frac{(2n+1)\pi}{2l}x \tag{2.12}$$

$$\phi(x) = \sum_{\mathrm{n=1}}^{\infty} d'_n \cos\frac{(2n+1)\pi}{2l}x \tag{2.13}$$

where

$$c'_n = \int_0^l \phi(t)\sin\frac{(2n+1)\pi}{2l}t\mathrm{d}t \tag{2.14}$$

$$d'_n = \int_0^l \phi(t)\cos\frac{(2n+1)\pi}{2l}t\mathrm{d}t \tag{2.15}$$

WSJ:

奇延拓，偶延拓

Complex Fourier Expansion ...

2.3 Fourier Integral

$f(x)$ is absolutely integrable in $(-\infty, \infty)$ $\quad\Leftrightarrow\quad$ $\displaystyle\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x < \infty$

$$\Rightarrow \quad f(x) \to 0 \quad \text{when} \quad x \to \pm\infty$$

Consider an absolutely integrable periodic function with period l, whose Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x) \tag{2.16}$$

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(t)\, \mathrm{d}t \tag{2.17}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{n\pi}{l} t \mathrm{d}t \tag{2.18}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(t) \sin \frac{n\pi}{l} t \mathrm{d}t \tag{2.19}$$

To convert $f(x)$ to a non-periodic function in $(-\infty, \infty)$, we can let $l \to \infty$, thus

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(t)\, \mathrm{d}t \to 0 \tag{2.20}$$

Moreover, we can let $\omega_n = \dfrac{n\pi}{l}$, thus

$$\delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{l} \to 0 \tag{2.21}$$

so we can replace $\delta\omega$ with $\mathrm{d}\omega$
i.e.

$$\sum_{n=1}^{\infty} \cdots \Delta\omega \xrightarrow{l\to\infty} \int_{0}^{\infty} \cdots \mathrm{d}\omega \tag{2.22}$$

thus

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x &= \sum_{n=1}^{\infty} \frac{1}{l} \left[\int_{-l}^{l} f(t) \cos \frac{n\pi}{l} t \mathrm{d}t \right] \cos \frac{n\pi}{l} x \\
&= \sum_{n=1}^{\infty} \frac{\Delta\omega}{\pi} \left[\int_{-l}^{l} f(t) \cos \omega_n t \mathrm{d}t \right] \cos \omega_n x \\
&\xrightarrow{l\to\infty} \int_{0}^{\infty} \mathrm{d}\omega \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \mathrm{d}t \right] \cos \omega x
\end{aligned} \tag{2.23}$$

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x \xrightarrow{l\to\infty} \int_{0}^{\infty} \mathrm{d}\omega \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \mathrm{d}t \right] \sin \omega x \tag{2.24}$$

Now we can rewrite (2.15) as Fourier integral expression

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\mathrm{d}\omega \tag{2.25}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\mathrm{d}t \tag{2.26}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\mathrm{d}t \tag{2.27}$$

Fourier Integral Theorem ...

Discussion WSJ:

1) Fourier Integral can be rewritten as

$$f(x) = \int_0^\infty C(\omega)\cos[\omega x - \phi(\omega)]\mathrm{d}\omega \tag{2.28}$$

where

$$C(\omega) = \sqrt{A^2(\omega) + B^2(\omega)} \tag{2.29}$$

$$\phi(\omega) = \arctan\frac{B(\omega)}{A(\omega)} \tag{2.30}$$

2) for odd functions

...

3) symmetrical Fourier Transform pair

3 Fourier Transformation

Def: **Integral transform** is to convert function $f(t)$ to $F(\beta)$ by integral calculation

$$F(\beta) = \int_a^b f(t)K(\beta, t)\mathrm{d}t \tag{3.1}$$

where $K(\beta, t)$ is called kernel function or nucleus.

3.1 Introduction

3.1.1 definition of FT

Consider function $f(x)$ defined in $(-\infty, \infty)$, whose Fourier integral is

$$
\begin{aligned}
f(x) &= \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\mathrm{d}t\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)\cos\omega(x-t)\mathrm{d}t\right)\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(t)(\mathrm{e}^{\mathrm{i}\omega(x-t)} + \mathrm{e}^{-\mathrm{i}\omega(x-t)})\mathrm{d}t\right]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_0^\infty f(t)(\mathrm{e}^{\mathrm{i}\omega(x-t)} + \mathrm{e}^{-\mathrm{i}\omega(x-t)})\mathrm{d}\omega\right]\mathrm{d}t
\end{aligned}
\tag{3.2}
$$

Considering

$$
\int_0^\infty f(t)\,\mathrm{e}^{-\mathrm{i}\omega(x-t)}\,\mathrm{d}\omega = \int_{-\infty}^0 f(t)\,\mathrm{e}^{\mathrm{i}\omega(x-t)}\,\mathrm{d}\omega \tag{3.3}
$$

we have

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\,\mathrm{e}^{\mathrm{i}\omega(x-t)}\,\mathrm{d}\omega\right]\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right]\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\
&\equiv \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)\,\mathrm{e}^{i\omega x}\,\mathrm{d}\omega
\end{aligned}
\tag{3.4}
$$

i.e.

$$
\boxed{
\begin{aligned}
F(\omega) &= \int_{-\infty}^\infty f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \qquad (3.5)\\
f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \qquad (3.6)
\end{aligned}
}
$$

$F(\omega)$ is defined as a Fourier transform of $f(x)$, and $f(x)$ is defined as an inverse Fourier transform of $F(\omega)$, i.e.

$$
F(\omega) = \mathcal{F}\{f(x)\} \tag{3.7}
$$

$$
f(x) = \mathcal{F}^{-1}\{F(\omega)\} \tag{3.8}
$$

or

$$
F(\omega) \longleftrightarrow f(x) \tag{3.9}
$$

In (3.7), $F(\omega)$ is called 象函数, $f(x)$ is called 原函数.

The process from $f(x)$ to $F(\omega)$ is called Fourier analysis. The inverse process is called 反

演.
Discussion:
In (4.5), let $\omega = 0$, we have

$$F(0) = \int_{-\infty}^{\infty} f(x)\mathrm{d}x \tag{3.10}$$

In (4.6), let $x = 0$, we have

$$f(0) == \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\mathrm{d}\omega \tag{3.11}$$

3.1.2 Features of FT

1) Linear
...
If $f(x) \longleftrightarrow F(\omega)$, we have the following theorems
2) Differential Theorem I

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \longleftrightarrow \mathrm{i}\omega F(\omega) \tag{3.12}$$

Proof:

$$\begin{aligned}\frac{\mathrm{d}f(x)}{\mathrm{d}x} \longleftrightarrow \int_{-\infty}^{\infty} \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \mathrm{d}f(x)\,\mathrm{e}^{-\mathrm{i}\omega x} \\ &= \left[f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\right]_{-\infty}^{\infty} + \mathrm{i}\omega \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \mathrm{i}\omega F(\omega)\end{aligned} \tag{3.13}$$

Likewise

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x) \longleftrightarrow (\mathrm{i}\omega)^n F(\omega) \tag{3.14}$$

3) Differential Theorem II

$$xf(x) \longleftrightarrow \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\omega}F(\omega) \tag{3.15}$$

Proof:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}F(\omega) &= \frac{\mathrm{d}}{\mathrm{d}\omega}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\mathrm{e}^{-\mathrm{i}\omega x}\right)\mathrm{d}x \\ &= -\mathrm{i}\int_{-\infty}^{\infty} xf(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= -\mathrm{i}\mathcal{F}\{xf(x)\}\end{aligned} \tag{3.16}$$

Likewise

$$x^n f(x) \longleftrightarrow \mathrm{i}^n \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n F(\omega) \tag{3.17}$$

4) Integral Theorem

$$\forall x_0,\ \int_{x_0}^{x} f(x)\,\mathrm{d}x \longleftrightarrow \frac{F(\omega)}{\mathrm{i}\omega} \tag{3.18}$$

5) Displacement Theorem

$$\forall \xi,\ f(x+\xi) \longleftrightarrow \mathrm{e}^{\mathrm{i}\omega\xi}\,F(\omega) \tag{3.19}$$

Proof:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x+\xi)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x &\xlongequal{y=x+\xi} \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}\omega(y-\xi)}\,\mathrm{d}y \\ &= \mathrm{e}^{\mathrm{i}\omega\xi}\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y \\ &= \mathrm{e}^{\mathrm{i}\omega\xi}\,F(\omega)\end{aligned} \tag{3.20}$$

6) Convolution (卷积) Theorem

Def:

$f_1(x),\ f_2(x)$ defined at $(-\infty,\infty)$

Their convolution is defined as

$$f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi)\,\mathrm{d}\xi \tag{3.21}$$

Convolution Theorem:

$$f_1(x) * f_2(x) \longleftrightarrow F_1(\omega)F_2(\omega) \tag{3.22}$$

Proof:

$$\begin{aligned}\mathcal{F}\{f_1(x) * f_2(x)\} &= \int_{-\infty}^{\infty} f_1(x) * f_2(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\xi)f_2(x-\xi)\,\mathrm{d}\xi\right]\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_1(\xi)\left[\int_{-\infty}^{\infty} f_2(x-\xi)\,\mathrm{e}^{-\mathrm{i}\omega(x-\xi)}\,\mathrm{d}x\right]\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} f_1(\xi)F_2(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi \\ &= F_2(\omega)\int_{-\infty}^{\infty} f_1(\xi)\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi \\ &= F_1(\omega)F_2(\omega)\end{aligned} \tag{3.23}$$

$\therefore$

$$f_1(x) * f_2(x) \longleftrightarrow F_1(\omega)F_2(\omega) \tag{3.24}$$

Discussion:

6.1) Commutative property

$$f_1(x) * f_2(x) = f_2(x) * f_1(x) \tag{3.25}$$

6.2) For even fxn:

$$f(x) * \cos\omega x = F(\omega)\cos\omega x \tag{3.26}$$

$$f(x) * \sin\omega x = F(\omega)\sin\omega x \tag{3.27}$$

For odd fxn:

$$f(x) * \cos\omega x = \mathrm{i}F(\omega)\sin\omega x \tag{3.28}$$

$$f(x) * \sin\omega x = -\mathrm{i}F(\omega)\cos\omega x \tag{3.29}$$

Proof:

For even fxn

$$\begin{aligned}
f(x) * \cos\omega x &= \int_{-\infty}^{\infty} f(\xi)\cos\omega(x-\xi)\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} f(\xi)\left(\cos\omega x\cos\omega\xi + \sin\omega x\sin\omega\xi\right)\mathrm{d}\xi \\
&= \cos\omega x\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\mathrm{d}\xi \\
&= \cos\omega x\int_{-\infty}^{\infty} f(\xi)\left(\cos\omega\xi - \mathrm{i}\sin\omega\xi\right)\mathrm{d}\xi \\
&= \cos\omega x\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi \\
&= F(\omega)\cos\omega x
\end{aligned} \tag{3.30}$$

...

WSJ:

1) 相似性定理

2) 延迟定理

3) 位移定理

3.1.3 n-D Fourier Integral

...

3.2 Dirac δ Function

3.2.1 Definition

$$\delta(x-x_0)=\begin{cases} 0 & (x\neq x_0) \\ \infty & (x=x_0) \end{cases} \tag{3.31}$$

$$\int_{-\infty}^{\infty}\delta(x-x_0)\,\mathrm{d}x=1 \tag{3.32}$$

Features:
WSJ:
阶跃函数:

$$H(x)\equiv\begin{cases}0 & x<0\\ 1 & x>0\end{cases}=\int_{-\infty}^{x}\delta(t)\mathrm{d}t \tag{3.33}$$

i.e.

$$\delta(x)=\frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{3.34}$$

1) 筛选性质
$\forall$ continuous fxn $f(x)$,

$$\int_{-\infty}^{\infty}f(x)\delta(x-x_0)\,\mathrm{d}x=f(x_0) \tag{3.35}$$

Proof:

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\delta(x-x_0)\,\mathrm{d}x &\xlongequal{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}f(x)\delta(x-x_0)\,\mathrm{d}x\\ &=f(x_0)\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)\,\mathrm{d}x\\ &=f(x_0)\end{aligned} \tag{3.36}$$

Attention　不是严格证明，不连续情况下不能用积分中值定理。
2)

$$\delta(x)*f(x)=\int_{-\infty}^{\infty}f(\xi)\delta(x-\xi)\mathrm{d}\xi=f(x) \tag{3.37}$$

$$\delta(x-a)*f(x)=\int_{-\infty}^{\infty}f(\xi)\delta(x-a-\xi)\mathrm{d}\xi=f(x-a) \tag{3.38}$$

$$\delta(x-a)*\delta(x-b)=\delta(x-a-b) \tag{3.39}$$

3) eigenfunction of operator x

$$(x-x_0)\delta(x-x_0)=0 \tag{3.40}$$

$$x\delta(x) = 0 \tag{3.41}$$

正交归一性：

$$\int_{-\infty}^{\infty} \delta(x-x_1)\delta(x-x_2)\,\mathrm{d}x = \delta(x_1 - x_2) \tag{3.42}$$

完备性：

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(\xi - x)\mathrm{d}\xi \tag{3.43}$$

4) FT of delta fxn

$$\mathcal{F}\{\delta(x-x_0)\} = \int_{-\infty}^{\infty} \delta(x-x_0)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\omega x_0} \tag{3.44}$$

when $x_0 = 1$

$$\mathcal{F}\{\delta(x)\} = 1 \tag{3.45}$$

Thus

$$\begin{aligned}\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}\omega\end{aligned} \tag{3.46}$$

∴

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \tag{3.47}$$

as a result

$$\mathcal{F}\{\delta(x)\} = 1 \tag{3.48}$$

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \tag{3.49}$$

thus 1 and $\delta(x)$ compose a Fourier transformation pair (傅里叶变换对).

Discussion:

1)

(3.44) can be rewritten as

$$\begin{aligned}\delta(x) = &= \frac{1}{2\pi a}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{a}(a\omega)x}\,\mathrm{d}(a\omega) \\ &\xlongequal{p=a\omega} \frac{1}{2\pi a}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{a}px}\,\mathrm{d}p\end{aligned} \tag{3.50}$$

Let $x = p - p'$

$$\tag{3.51}$$

2) momentum eigenfunction:

$$\psi_p(x) = c\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}px} \tag{3.52}$$

where

$$c=\frac{1}{\sqrt{2\pi\hbar}} \tag{3.53}$$

正交归一性：

$$\begin{aligned}\int_{-\infty}^{\infty}\psi_{p'}^{*}(x)\psi_p(x)\,\mathrm{d}x &= |c|^2\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p-p')}\,x\,\mathrm{d}x\\ &= |c|^2\cdot 2\pi\hbar\delta(p-p')\\ &= \delta(p-p')\end{aligned} \tag{3.54}$$

4 Laplace Transformation

f(t)不绝对可积

suppose

$$g(t)=\mathrm{e}^{-\sigma t}f(t)H(t) \tag{4.1}$$

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t=\frac{1}{2\pi}\int_{0}^{\infty}f(t)\,\mathrm{e}^{-(\sigma+\mathrm{i}\omega)t}\,\mathrm{d}t \tag{4.2}$$

let $p=\sigma+\mathrm{i}\omega$, $\bar{f}(p)=2\pi G(\omega)$, we have

$$\bar{f}(p)=\int_0^{\infty}f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t \tag{4.3}$$

which is called Laplace Transformation.

Reverse L Transformation:

$$f(t)=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\,\mathrm{e}^{pt}\,\mathrm{d}p \tag{4.4}$$

namely

$$\boxed{\begin{aligned}\mathcal{L}[f(t)]=\bar{f}(p)&=\int_0^{\infty}f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t & (4.5)\\ \mathcal{L}^{-1}[\bar{f}(p)]=f(t)&=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\bar{f}(p)\,\mathrm{e}^{pt}\,\mathrm{d}p & (4.6)\end{aligned}}$$

Ex.

$$\mathcal{L}[t]=\frac{1}{p^2} \tag{4.7}$$

$$\mathcal{L}[\mathrm{e}^{st}]=\frac{1}{p-s}\qquad(\mathrm{Re}\,p>\mathrm{Re}\,s) \tag{4.8}$$