Lecture 6

Ch 2.2 Infinite Square Well

$$V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$

Classical analogue: Ball on a Billiard table.

Want solutions to T.I.S.E

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

Corresponds to a stationary state

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

Outside: $\psi(x) = 0$ (otherwise $\langle V(x) \rangle = \infty$

Inside: T.I.S.E Becomes

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

or

$$\frac{d^2}{dx^2}\psi(x) = -k^2\,\psi(x)$$

where $k^2 = \frac{2mE}{\hbar^2}$, or $E = \frac{\hbar^2 k^2}{2m}$

Solutions to eqn:

$$\psi(x) = A \sin kx + B \cos kx$$

Boundary conditions:

We will see that whenever

$V(x)$

is continuous, Both

$\psi(x)$ and $\frac{d}{dx}\psi(x)$

need to be continuous.

We'll see later that, if

$V(x)$ has infinite discontinuity,

only $\psi(x)$ needs to be continuous.

So, at

$x=0$ and $x=a$

we require

$\psi(x)$

$\frac{d}{dx}\psi(x)$ are continuous

At $\underline{x=0}$:

$$0=\psi(0) = A\cdot \sin\phi + B\cos\phi$$

$$= B\cdot 1 = B \Rightarrow \underline{B=0}$$

At $x=a$:

$$0 = \psi(a) = A \cdot \sin k \cdot a$$

So:

$$k \cdot a = \cancel{0}, \cancel{\pm}\pi, \cancel{\pm}2\pi, \ldots$$

$k=0 \Rightarrow \psi(x)=0$
no particle

$\psi(x) \rightarrow -\psi(x)$
same solution
up to unobs.
phase.

Thus

$$k = \frac{\pi}{a}, \frac{2\pi}{a}, \ldots$$

Or

$$k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

So, the boundary conditions at $x=0$ and $x=a$ lead to quantization of energy

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

So:

$$\psi_n(x) = \begin{cases} 0, & \text{outside} \\ A_n \sin k_n x, & 0 \le x \le a \end{cases}$$

To find A_n, normalize $\psi_n(x)$:

$$1 = \int_{-\infty}^{\infty} |\psi_n(x)|^2 \, dx$$

$$1 = \int_0^a |A_n|^2 \cdot (\sin k_n x)^2 \, dx$$

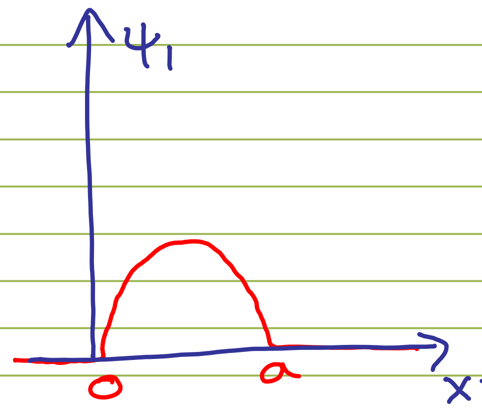
$$= |A_n|^2 \int_0^a \frac{1 - \cos 2k_n x}{2} \, dx$$

$$= |A_n|^2 \cdot \frac{a}{2}$$

$$\Rightarrow \quad A_n = \sqrt{\frac{2}{a}}$$

and

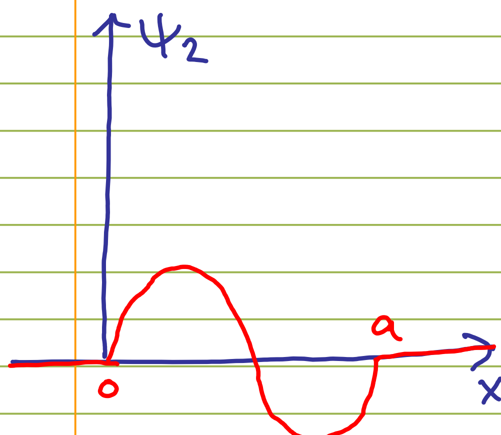
$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} \cdot x\right) \quad n = 1, 2, \ldots}$$



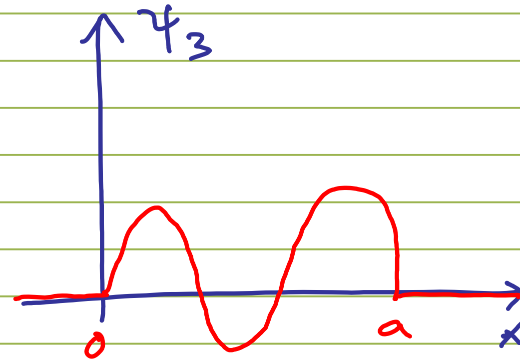
Lowest energy state or

GROUND STATE

$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$$



$$E_2 = 2^2 E_1$$



$$E_3 = 3^2 E_1$$

$$E_n = n^2 E_1$$

Note: With respect to center of well

$$\psi_n(x) = \begin{cases} \text{odd fn, } n \text{ odd} \\ \text{even fn, } n \text{ even} \end{cases}$$

Orthonormality

Spectrum is discrete and non-degenerate:

ψ_n's are mutually orthogonal

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \psi_n(x)\, dx = 0 \quad , \; n \neq m$$

Physically, this means that

for $n \neq m$

ψ_n and ψ_m } states of different energy

are "maximally different"

or "zero overlap"

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \psi_n(x)\, dx =$$

$$= \frac{2}{a} \int_0^a \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{a} x\right) dx$$

$$= \frac{1}{a} \int_0^a \left[\cos\left(\frac{m+n}{a} \pi x\right) - \cos\left(\frac{m-n}{a} \pi x\right) \right] dx$$

$$= \frac{1}{a}\left[\frac{a}{\pi(m+n)} \sin\left(\frac{m+n}{a}\pi x\right) - \frac{a}{\pi(m-n)} \sin\left(\frac{m-n}{a}\pi x\right)\right]_0^a$$

$$= 0$$

Combine orthogonality and normalization

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \psi_n(x)\, dx = \delta_{mn}$$

$$\delta_{mn} = \underset{\text{delta}}{\text{Kronecker}} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

ψ_n's are "orthonormal".

Note analogy with orthonormal bases of finite dimensional vector spaces.

<u>Completeness</u>:

Any function $f(x)$ that satisfies

$$f(x) = 0 \quad \text{for} \quad x \leq 0, \; x \geq a$$

can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$= \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} x\right)$$

where

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x) f(x)\, dx$$

This property will hold for all potentials we'll run into.

Remember: From these stationary solutions we get any solution to the full S.E:

1. Specify

$$\Psi(x,0)$$

2. Find $c_n = \int_0^a \psi_n^*(x) \Psi(x,0)\, dx$

then :

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

3. S.E. is solved By

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x)\, e^{-iE_n t/\hbar}$$

with $\psi_n(x)$, E_n as above.

4. Recall, normalization of solution to S.E. is time independent

$$1 = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\, \Psi(x,t)$$

$$= \sum_{n,m=1}^{\infty} c_n^* c_m \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\, \psi_m(x)$$

$$= \sum_{n,m=1}^{\infty} c_n^* c_m\, \delta_{mn} = \sum_{n=1}^{\infty} |c_n|^2$$

or

$$1 = \sum_{n=1}^{\infty} |c_n|^2$$

These sound like probabilities...

and they are:

$|c_n|^2 =$ Probability that, for a particle in state $\Psi(x,t)$ measurement of energy results in E_n.

In particular:

$$\langle H \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x,t) \, \hat{H} \, \psi(x,t)$$

$$\hat{H} \psi(x,t) = \sum_{n=1}^{\infty} c_n \hat{H} \psi_n(x) e^{-iE_n t/\hbar}$$

$$= \sum_{n=1}^{\infty} c_n E_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$\Rightarrow \langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

Expectation value of energy is independent of time in any state $\Psi(x,t)$, not just in stationary states

$\frac{d}{dt} E = 0$ $\Rightarrow$ $\frac{d}{dt}\langle H \rangle = 0$

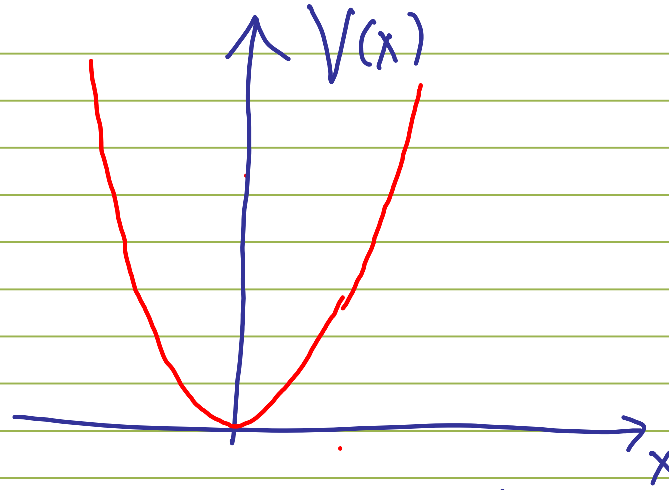
classical conservation of energy

expectation value of energy is indep. of time in any quantum state $\Psi(x,t)$

Harmonic Oscillator

Lecture 7

$V(x) = \frac{1}{2} m \omega^2 x^2$



Classically, corresponds to a linear force,

$$F = -\frac{d}{dx} V = -kx$$

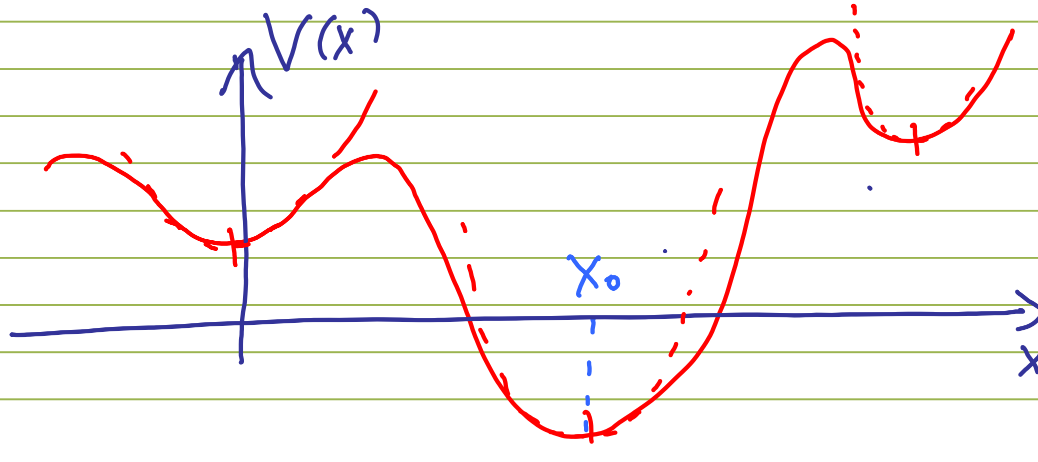
with $\quad k = m\omega^2 \quad$ Hooke's Law

E.o.m

$$m \frac{d^2}{dt^2} x = -\frac{d}{dx} V = -m\omega^2 x$$

$$\Rightarrow \quad x(t) = A \sin \omega t + B \cos \omega t .$$

Importance of H.O. potential :



An arbitrary potential looks like it, near its minimum!

$$V(x) = V(x_0) + \frac{d}{dx}V(x_0)\,(x-x_0) +$$
$$+ \frac{1}{2}\frac{d^2}{dx^2}V(x_0)\,(x-x_0)^2 + \dots$$

At a minimum

$$\frac{d}{dx}V(x_0) = 0, \qquad \frac{d^2}{dx^2}V(x_0) > 0$$

$\Rightarrow$ effective H.O. potential w/

$$k_{eff}^2 = \frac{d^2}{dx^2}V(x_0) > 0$$

$\Rightarrow$ In QM we get

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + m\omega^2 x^2 \psi(x) = E\psi(x)$$

as T.I.S.E

2 methods to solve eqn:

Brute force and algebraic

Algebraic

$$\hat{H}\psi(x) = E\psi(x)$$

where

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\hat{p} = -i\hbar\frac{d}{dx}$$

If these were numbers

$$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(\mp ip + m\omega x)$$

simpler form

$$H = \hbar\omega\, a_+ \cdot a_-$$

$$= \frac{\hbar\omega}{2m\hbar\omega}(-ip + m\omega x)(+ip + m\omega x)$$

$$= \frac{1}{2m}(p^2 + (m\omega x)^2)$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

Since $\hat{p}$ and $\hat{x}$ "do not commute"

$$\hat{x}\hat{p}\, f(x) \neq \hat{p}\hat{x}\, f(x)$$

$$\hat{x}\, f(x) = x\, f(x)$$

$f(x) =$ test function.

$$\hat{p}\, f(x) = -i\hbar \frac{d}{dx} f(x)$$

Instead:

$$\hat{x}\hat{p}\, f(x) = \hat{p}\hat{x}\, f(x) + i\hbar\, f(x)$$

It follows also

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(\mp i\hat{p} + m\omega\hat{x})$$

$$\hat{a}_-\hat{a}_+ f(x) \neq \hat{a}_+\hat{a}_- f(x)$$

instead

$$\hat{a}_-\hat{a}_+ f(x) = \hat{a}_+\hat{a}_- f(x) + f(x)$$

One usually drops test functions and works with operator relations:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

and

$$\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$$

For any pair of operators

$$\hat{A}, \hat{B},$$

we define a commutator

$$\boxed{[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}}$$

and then relations above read:

$$\boxed{\begin{aligned} [\hat{x}, \hat{p}] &= i\hbar \\ [\hat{a}_-, \hat{a}_+] &= 1 \end{aligned}}$$

Upshot:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

can be rewritten, using

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$$

as

$$\hat{H} = \frac{1}{2m}\left(-\frac{m\hbar\omega}{2}\right)(\hat{a}_+ - \hat{a}_-)^2$$

ORDER MATTERS!

$$+\frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(\hat{a}_+ + \hat{a}_-)^2$$

$$= -\frac{\hbar\omega}{4}\left(\cancel{\hat{a}_+^2} - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \cancel{\hat{a}_-^2}\right)$$

$$+\frac{\hbar\omega}{4}\left(\cancel{\hat{a}_+^2} + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \cancel{\hat{a}_-^2}\right)$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\right)$$

or

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)}$$

Claim:

Suppose state $\psi(x)$ satisfies

$$\hat{H}\,\psi(x) = E\,\psi(x)$$

Then:

- state $\hat{a}_\pm \psi(x)$ has energy

$$E \pm \hbar\omega$$

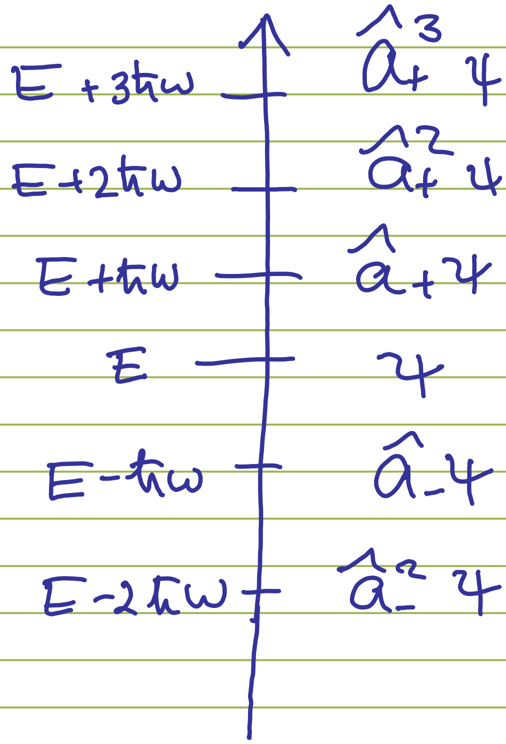
$$\hat{H}(\hat{a}_\pm \psi) = (E \pm \hbar\omega)(\hat{a}_\pm \psi)$$

In other words

$\hat{a}_+$ is a "raising" operator, since it increases energy by $\hbar\omega$

$\hat{a}_-$ is a "Lowering" operator, since it decreases energy by $\hbar\omega$.

So: if there is a state of energy E, then by applying raising and lowering operators we get



Proof:

we'll show
$$\begin{cases} [\hat{H}, \hat{a}_+] = \hbar\omega\,\hat{a}_+ \\ [\hat{H}, \hat{a}_-] = -\hbar\omega\,\hat{a}_- \end{cases}$$

so that, if $\hat{H}\psi = E\psi$,

$$\hat{H}\hat{a}_+\psi = (\hat{a}_+\hat{H} + \hbar\omega\,\hat{a}_+)\psi$$
$$= (\hat{a}_+E + \hbar\omega\,\hat{a}_+)\psi$$

and
$$= (E+\hbar\omega)\psi$$

$$\hat{H}\hat{a}_-\psi = \left(\hat{a}_- H - \hbar\omega \hat{a}_-\right)\psi$$

$$= \left(\hat{a}_- E - \hbar \quad \hat{a}_-\right)\psi$$

$$= (E - \hbar\omega)\hat{a}_-\psi$$

To prove above, we calculate

$$[\hat{H}, \hat{a}_+] = \hbar\omega\left[\hat{a}_+\hat{a}_- + \frac{1}{2}, \hat{a}_+\right]$$

expanding

$$= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ - \right.$$

$$\left. - \hat{a}_+\hat{a}_+\hat{a}_-\right)$$

$$\hat{a}_-\hat{a}_+ = \hat{a}_+\hat{a}_- + 1$$

$$= \hbar\omega\left(\hat{a}_+\hat{a}_+\hat{a}_- + \hat{a}_+\right.$$

$$\left. - \hat{a}_+\hat{a}_+\hat{a}_-\right)$$

$$= \hbar\omega\hat{a}_+$$

Physical Expectation

There should exist a state ψ_0 of

Lowest energy

for which

$$\hat{a}_- \psi_0 = 0,$$

since we cannot lower energy any further. Otherwise, energy would be unbounded from below – we would have an unstable system.

The state ψ_0 of lowest energy is called the

"ground state".

Ground state

For such a state

$$\hat{H}\psi_0 = \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\psi_0$$

$$= \tfrac{1}{2}\hbar\omega\,\psi_0 = E_0\psi_0$$

$\Rightarrow$ $E_0 = \frac{1}{2}\hbar\omega$ $\leftarrow$ ground state energy

From its existence, we get the entire spectrum:

$\psi_1 \propto \hat{a}^+\psi_0$ of

energy $E_1 = E_0 + \hbar\omega = \frac{3}{2}\hbar\omega$

$\psi_n \propto (\hat{a}_+)^n \psi_0$ of energy

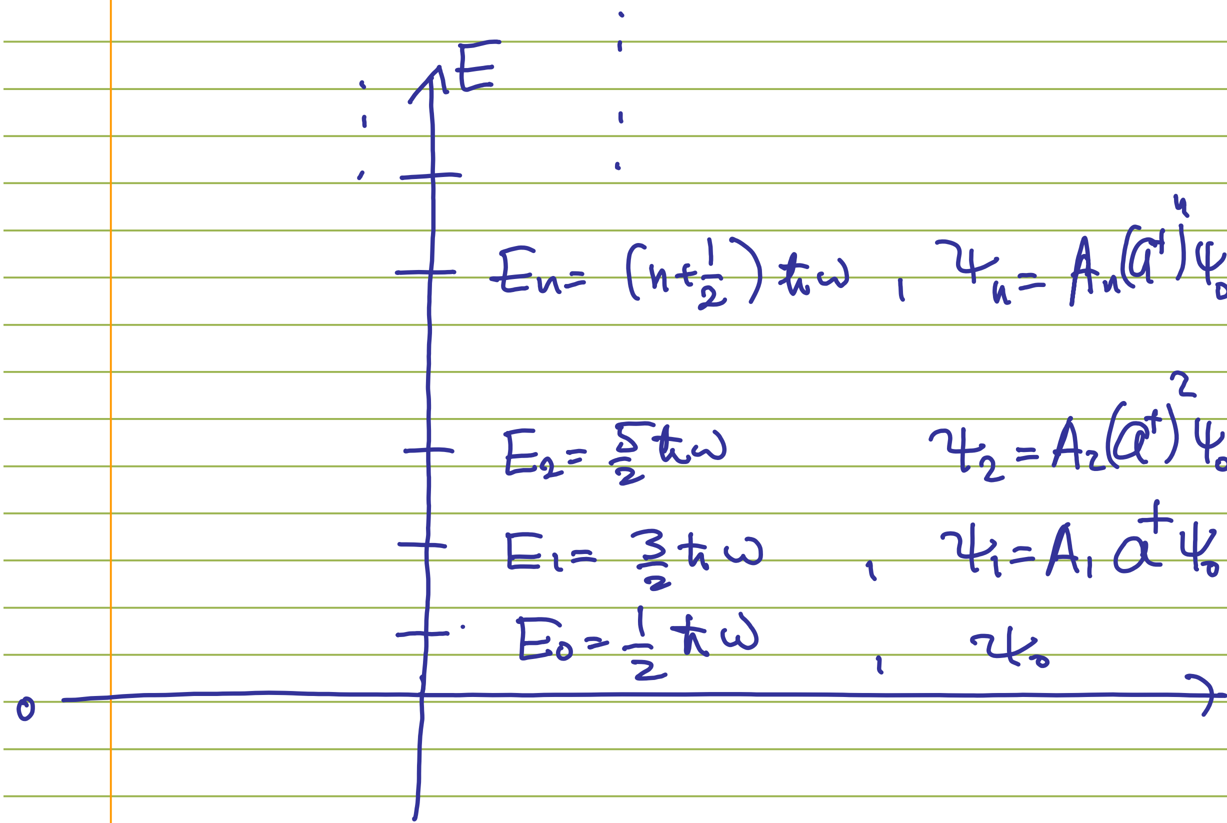
$$E_n = E_0 + n\hbar\omega$$
$$= (n + \tfrac{1}{2})\hbar\omega$$

Q: How do we know these are all the energy eigenstates?

A: Existance of a state not of this form would imply spectrum not Bounded from Below. Since By acting on it w/ $\hat{a}_-$ we could keep lowering the energy Below ψ_0.

In summary:

Energy spectrum of harmonic oscillator



We can find from this explicit form of

$$\psi_n(x)$$

as well. Begin with ψ_0;

Ground state

Per definition

$$\hat{a}_- \psi_0 = 0$$

Since

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar\omega m}} \left(i\hat{p} + m\omega\hat{x} \right)$$

with $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

This is an eqn

$$\left(\hbar \frac{\partial}{\partial x} + m\omega x \right) \psi_0(x) = 0$$

Solved By

$$\psi_0(x) = A_0\, e^{-\frac{m\omega}{2\hbar}x^2}$$

Normalization constant (see exam)

$$A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

So

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

From this, get

$$\psi_n(x) = A_n\,(\hat{a}_+)^n\,\psi_0(x)$$

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar\omega m}}\left(-i\hat{p} + m\omega\hat{x}\right)$$

$$= \frac{1}{\sqrt{2\hbar\omega m}}\left(-\hbar\frac{d}{dx} + m\omega x\right)$$

$$\psi_n(x) = A_n \left(\hat{a}_+\right)^n \psi_0(x)$$

$$= \frac{A_n}{\sqrt{(2\hbar\omega m)^n}} \left(-\hbar \frac{d}{dx} + m\omega x\right)^n \psi_0(x)$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

As explicit form as you will ever need

All we need to do is find suitabe

A_n:

want to know the constant

$$\psi_n = A_n (a_+)^n \psi_0$$

So that ψ_n is normalized to 1.

Claim $A_n = \frac{1}{\sqrt{n!}}$

Work this out: Let

$$a_+ \psi_n = c_n \psi_{n+1}$$

$$a_- \psi_n = d_n \psi_{n-1}$$

If we know c_n's and d_n's

we know A_n's

It turns out

$$\int_{-\infty}^{\infty} f^* (a_\pm g)\, dx = \int_{-\infty}^{\infty} (a_\mp f)^* g\, dx$$

$a_\pm$ is hermitian conjugate of $a_\mp$ "

$$a_\pm = \frac{1}{\sqrt{2\hbar\omega m}}\left(\mp ip + m\omega x\right)$$

$$= \frac{1}{\sqrt{2\hbar\omega m}}\left(\pm \hbar \frac{d}{dx} + m\omega x\right)$$

Above follows by integraly by parts

$$\int f^* \frac{d}{dx} g \, dx \overset{I.P.}{=} -\int \left(\frac{d}{dx} f^* \, g\right) dx$$

We can use this as follows,

$$a_- \psi_n = d_n \psi_{n-1}$$

$$(a_- \psi_n)^* = d_n^* \psi_{n-1}^*$$

It follows

$$|d_n|^2 = \int (d_n \psi_{n-1})^* (d_n \psi_{n-1})\, dx$$

$$= \int (a_- \psi_n)^* a_- \psi_n \, dx$$

$$= \int \psi_n^* a_+ a_- \psi_n \, dx$$

Using $a_+ a_- \psi_n = n \psi_n$

$$\Rightarrow \quad |d_n|^2 = n \quad ; \quad d_n = \sqrt{n}$$

Simw , using

$$a_- a_+ \psi_n = (a_+ a_- + 1) \psi_n$$

$$= (n+1) \psi_n$$

It follows $|c_n|^2 = n+1$; $c_n = \sqrt{n+1}$.

Very useful!

$$\boxed{\begin{aligned} \hat{a}_+ \psi_n &= \sqrt{n+1}\, \psi_{n+1} \\ \hat{a}_- \psi_n &= \sqrt{n}\, \psi_{n-1} \end{aligned}}$$

Iteraly:

$$a_+ \psi_0 = \psi_1$$

$$a_+ \psi_1 = \sqrt{2}\, \psi_2$$

$$a_+ \psi_{n-1} = \sqrt{n}\, \psi_n$$

$$\Rightarrow \quad (a_+)^n \psi_0 = \sqrt{n!}\, \psi_n$$

or

$$\psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0$$

Orthogonality

Since $\quad E_m \neq E_n$

for $\quad m \neq n$

Expect:

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \psi_n(x)\, dx = \delta_m^n$$

We can prove this explicity here

$$\boxed{\hat{a}_+\hat{a}_-\psi_n = n\psi_n}$$

$$\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\hat{a}_+\hat{a}_-\psi_n = \sqrt{n}\,\hat{a}_+\psi_{n-1}$$

$$= \sqrt{n}\cdot\sqrt{n-1+1}\,\psi_n$$

$$= n\psi_n$$

$$n\int_{-\infty}^{\infty}\psi_m^*\psi_n\,dx = \int_{-\infty}^{\infty}\psi_m^*\hat{a}_+\hat{a}_-\psi_n\,dx$$

$$= \int_{-\infty}^{\infty}(\hat{a}_-\psi_m)^*\,\hat{a}_-\psi_n\,dx$$

$$= \int_{-\infty}^{\infty}(\hat{a}_+\hat{a}_-\psi_m)^*\,\psi_n\,dx$$

$$= m\int_{-\infty}^{\infty}\psi_m^*\psi_n\,dx$$

$$\Rightarrow \quad (m-n)\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = 0$$

$$\Rightarrow \text{ If } \quad m \neq n, \quad \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = 0.$$

Expectation values

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \cdot (\hat{a}_+ + \hat{a}_-)$$

$\Rightarrow$ For any of eigenstates

$$\langle \hat{p} \rangle_n = \int \psi_n^* \, \hat{p} \, \psi_n \, dx$$

$$= 0$$

Since $\hat{p}\psi_n$ = Linear combination of ψ_{n+1} and ψ_{n-1}

Similarly

$$\langle \hat{x} \rangle_n = 0$$

But $\langle \hat{p}^2 \rangle_n$, $\langle \hat{x}^2 \rangle$

do not vanish:

$$\hat{p}^2 \psi_n = -\left(\frac{m\hbar\omega}{2}\right)\big(\hat{a}_+^2 - \hat{a}_-\hat{a}_+$$
$$- \hat{a}_+\hat{a}_- + \hat{a}_-^2\big)\psi_n$$

$$= -\left(\frac{m\hbar\omega}{2}\right)\big(\underbrace{-2\hat{a}_+\hat{a}_- - 1}_{+\text{ vanishes}}\big)\psi_n$$

$[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{a}_-\hat{a}_+ = \hat{a}_+\hat{a}_- + 1$

$$= (m\hbar\omega)\left(n + \frac{1}{2}\right)\psi_n$$

$$\Rightarrow \quad \langle \hat{p}^2 \rangle_n = m\hbar\omega\left(n + \frac{1}{2}\right)$$

or $$\left\langle \frac{\hat{p}^2}{2m} \right\rangle_n = \frac{1}{2}\hbar\omega\left(n + \frac{1}{2}\right)$$

Similarly

$$\langle \hat{x}^2 \rangle_n = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

and $$\left\langle \frac{m\omega^2 \hat{x}^2}{2} \right\rangle = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)$$

$$\Rightarrow \quad \langle \hat{T}_{kin} \rangle_n = \langle V(\hat{x}) \rangle_n$$

$$= \frac{1}{2}\langle \hat{H} \rangle_n$$

Using this we can also prove the spectrum of $\hat{H}$ is bounded from below. This follows since

$$\hat{H} = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

where operator

$$\hat{a}_+ \hat{a}_-$$

has spectrum bounded from below by 0. Let f be an eigenf.

$$\hat{a}_+ \hat{a}_- f = \nu f$$

$$\int f^* \hat{a}_+ \hat{a}_- f = \nu \int f^* f \, dx$$

$\Rightarrow$

$$= \int (a_- f)^* a_- f \, dx$$

$$= \int |a_- f|^2 dx$$

It follows

$$\nu = \frac{\int |a_- f|^2 dx}{\int |f|^2 dx}$$

Either $a_- f = 0$ so $\nu = 0$

or $a_- f = g \neq 0$ and ν

is a ratio of 2 positive numbers

$\nu > 0$

Since

$$\hat{H} = \hbar\omega\left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\right)$$

it follows eigenvalues of $\hat{H}$

are also Bom from below,

by $\hbar\omega/2$.

The Free Particle Ch 2.4

$$V(x) = 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

Let $\quad k^2 = \frac{2mE}{\hbar^2} \iff E = \frac{\hbar^2 k^2}{2m}$

$$\frac{d^2}{dx^2}\psi(x) = -k^2\psi(x)$$

$$\Rightarrow \quad \psi(x) = A e^{ikx} + B e^{-ikx}$$

$$\Downarrow$$

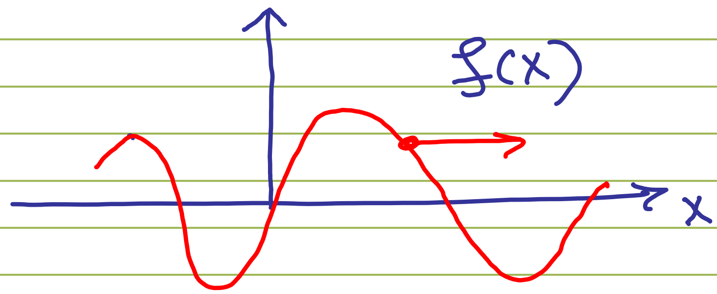
$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

$$= A e^{ik\left(x - \frac{\hbar k}{2m}t\right)} +$$

$$+ B e^{-ik\left(x + \frac{\hbar k}{2m}t\right)}$$

Recall $\quad f(x \pm vt)$ represents a

wave, whose shape at $t=0$ is



propagating in $\pm x$ direction with speed v. (Any point on wave propagates with same speed)

$$\psi(x,t) = A e^{ik\left(x - \frac{\hbar k}{2m} t\right)} + B e^{-ik\left(x + \frac{\hbar k}{2m} t\right)}$$

is Sum of waves moving to and right with speed

$$v = \frac{\hbar k}{2m}, \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

Let's just let k run from $-\infty$ to $+\infty$ and write

$$\psi_k(x,t) = A e^{ik\left(x - \frac{\hbar k}{2m} t\right)}, \qquad |k| = \frac{\sqrt{2mE}}{\hbar}$$

$k > 0 \Leftrightarrow$ right moving

$k < 0 \Leftrightarrow$ left moving

Classically, a free particle with energy E has speed

v_{class}

$$E = \frac{m v_{class}^2}{2}$$

or $$v_{class} = \sqrt{\frac{2E}{m}}$$

Here, we apparently find

$$v = \frac{\hbar k}{2m} = \frac{\hbar}{2m} \frac{\sqrt{2mE}}{\hbar} = \sqrt{\frac{E}{2m}}$$

So: $$v = \frac{1}{2} v_{class} \; !$$

There is another problem:

$$|\Psi_k(x,t)|^2 = |A|^2$$

$\Psi_k(x,t)$ is not normalizable: we cannot find a const. $|A|^2$ so

$$1 = \int_{-\infty}^{\infty} |\Psi_k(x,t)|^2 dx = |A|^2 \cdot \int_{-\infty}^{\infty} 1\, dx$$

$$= |A|^2 \cdot \infty$$

$\Downarrow$

Free particle with definite energy and momentum does not exist.

Instead, we need to look for more complicated solutions to S.E., those which are superpositions of

$\Psi_k(x,t)$

Since any k is allowed

$$\sum_k c_k \psi_k(x,t)$$

$$\downarrow$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \phi(k) \, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}$$

General solution to free particle

S.E:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t)$$

Really, we want

$$\phi(k)$$

which correspond to initial

condition $\psi(x,0)$

I.e:

$$\Psi(x,0) \overset{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ik\cdot x}\, dk$$

$\rightarrow$: Fourier transform

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ik\cdot x}\, dx$$

This is Because

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k-k')$$

$$\delta(k) = \text{Dirac } \delta\text{-function}$$

$$\int_{-\infty}^{\infty} \delta(k-k')\, f(k)\, dk = f(k')$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ik\cdot x}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx' \, \overbrace{\psi(x',0)\, e^{-ikx'}}^{\phi(k)} \cdot e^{ikx}$$

$$= \int_{-\infty}^{\infty} dx' \, \delta(x-x')\, \psi(x',0)$$

$$= \psi(x,0) \checkmark$$

Here, we want to choose a normalizable $\psi(x,0)$, so that

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = 1$$

Then,

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$$

for all t. This is equivalent to:

$$\int_{-\infty}^{\infty} |\phi(k)|^2 dk = 1$$

$$\left(= \int_{-\infty}^{\infty} dk \int \frac{dx\, dx'}{2\pi} \psi^*(x)\, \psi(x')\, e^{-ik(x-x')} \right)$$

$$= \int dx\, dx' \, \psi^*(x)\, \psi(x')\, \delta(x-x') = \int dx\, |\psi(x)|^2$$

A free quantum mechanical particle

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk$$

in one of states above, with normalizable

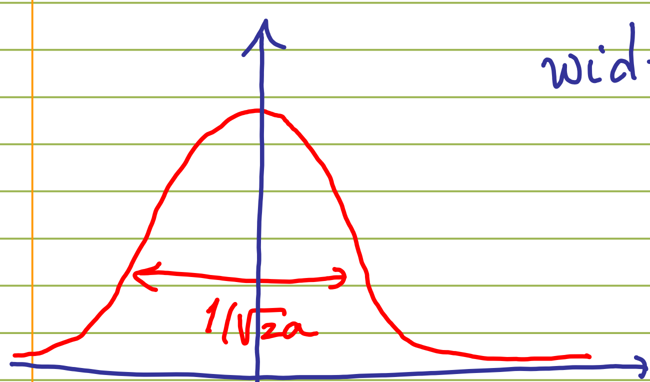
$\phi(k) \Rightarrow$ "Wave packet"

Ex:

Gaussian Wave packet

$$\psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

width $\propto 1/\sqrt{2a}$



$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2 - ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \cdot \int_{-\infty}^{\infty} e^{-a\left(x+\frac{ik}{2a}\right)^2} dx \; e^{-\frac{k^2}{4a}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} \; e^{-\frac{k^2}{4a}}$$

$$\int_{-\infty}^{\infty} e^{-\frac{\alpha x^2}{2}} dx = \sqrt{\frac{2\pi}{\alpha}}$$

$$= \underbrace{\left(\frac{1}{2\pi a}\right)^{1/4} \cdot e^{-\frac{k^2}{4a}}}_{}$$

Gaussian wave packet in k-space

with width $\propto \sqrt{2a}$

$x, \; a \leftrightarrow \frac{1}{4a}, \; k$

One can compute

$$\delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} = \frac{1}{\sqrt{4a}}$$

$$\delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\omega\hbar}{2}} = \sqrt{a}\cdot\hbar$$

To show: $\delta x \cdot \delta p = \hbar/2$

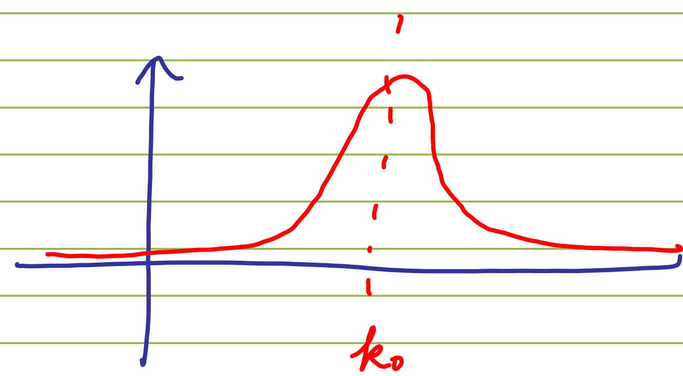
$$\sqrt{\frac{\hbar}{2m\omega}} \equiv a$$

Let's now try to understand what happens to velocity.

Suppose $\phi(k)$ is narrowly peaked around $k = k_0$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega(k)t)}\, dk$$

$$\omega(k) = \frac{\hbar k^2}{2m}$$



Taylor expand

$$\omega(k) = \omega(k_0) + \omega_0'(k-k_0) + \dots$$

$$\omega_0' = \frac{d}{dk}\omega \Big|_{k=k_0}$$

Let $k - k_0 = s$ or $k = s + k_0$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \, \phi(k_0+s) e^{i(k_0+s)x} \times$$
$$\times e^{-i(\omega(k_0) + \omega_0' s)t}$$

At $t=0$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \, \phi(k_0+s) e^{i(k_0+s)x}$$

At later times

$$\Psi(x,t) = [\dots] \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \, \phi(k_0+s) e^{i(k_0+s)(x-\omega_0' t)}$$

$$= [\dots] \cdot \Psi(x - \omega_0' t, 0)$$

$$[\dots] = e^{-i\omega_0 t + i k_0 \omega_0' t}$$ ← Phase factor; does not affect $|\Psi|^2$

So, the wave packet propagates with speed

$$\omega_0' = \frac{d}{dk}\omega\Big|_{k_0}$$

"group velocity" $v_{group} = \frac{d}{dk}\omega\Big|_{k_0}$

while individual components propagate with

"phase velocity" $v_{phase} = \frac{\omega}{k}\Big|_{k_0}$

For us $\omega = \frac{\hbar k^2}{2m}$

$$v_{phase} = \frac{\hbar k}{2m} = \frac{1}{2} v_{classical}$$

$$v_{group} = \frac{\hbar k}{m} = v_{class}$$

When

$$\omega(k) = \underset{\substack{\uparrow \\ \text{some constant}}}{c} \times k$$

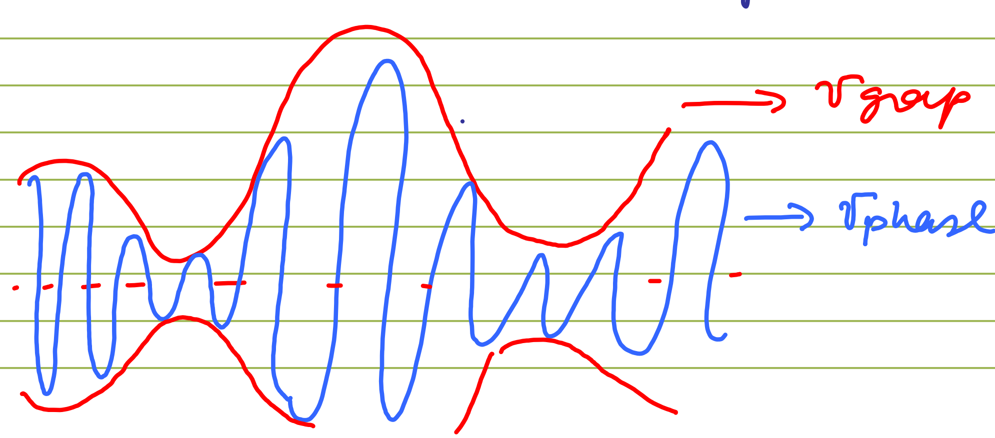
$$v_{group} = \frac{d}{dk}\omega = \frac{\omega}{k} = v_{phase}$$

But otherwise it is not.

v_{group} = velocity with which pattern as a whole moves

$$\Psi(x,0) \longrightarrow \Psi(x - v_{group}\, t, 0)$$

v_{phase} = speed of "ripples" within pattern



Unless

$\omega(k) = c \cdot k \Rightarrow v_{group} \neq v_{phase}$

waves w/ different k travel with different velocity

There is another effect

$\Rightarrow$ spreading of the wave packet

since

True only approximately

$$|\psi(x,t)|^2 \simeq |\psi(x - v_g t, 0)|^2$$

Gaussian example

At later time

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \phi(k) \, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}$$

with $\phi(k) = \left(\frac{1}{2\pi a}\right)^{1/4} \cdot e^{-\frac{k^2}{4a}}$

(Peaked at $k_0 = 0 \Rightarrow v_{class} = 0$)

Compute integral, using

$$\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}x^2 + \beta x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}(x - \beta/\alpha)^2} dx \cdot e^{+\frac{\beta^2}{2\alpha}}$$

$$= \sqrt{\frac{2\pi}{\alpha}}\; e^{\beta^2/2\alpha}$$

A: $\psi(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{2\pi a}\right)^{1/4}$

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + ikx - i\frac{k^2\hbar}{2m}t}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\pi a}\right)^{1/4}$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}\left[\frac{1}{2a} + i\frac{\hbar t}{m}\right]} \times e^{+ikx}\, dx$$

$\beta = ix$

$\alpha = \left[\frac{1}{2a} + i\frac{\hbar t}{m}\right]$

$$= \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/\left[1 + i2a\frac{\hbar t}{m}\right]}}{\sqrt{1 + i2a\frac{\hbar t}{m}}}$$

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\frac{ax^2}{[1+i2a\hbar t/m]}}}{\sqrt{1+i2a\frac{\hbar t}{m}}}$$

Using.

$$\frac{1}{\alpha+i\beta} = \frac{\alpha-i\beta}{\alpha^2+\beta^2}, \quad \left|\frac{1}{\alpha+i\beta}\right|^2 = \frac{1}{\alpha^2+\beta^2}$$

$$|\Psi(x,t)|^2 = \left(\frac{2a}{\pi}\right)^{1/2} \frac{e^{-2ax^2\left(\frac{1}{1+(2a\frac{\hbar t}{m})^2}\right)}}{\sqrt{1+(2a\frac{\hbar t}{m})^2}}$$

Gaussian, normalized to 1 at all times ... but with width

$$\sigma_x = \sqrt{\frac{1+\left(2a\frac{\hbar t}{m}\right)^2}{4a}}$$

In momentum space the width stays fixed

$$\sigma_p = \sqrt{a}\cdot\hbar$$

so uncertainty

$$\sigma_x \cdot \sigma_p = \frac{\hbar}{2}\sqrt{1+\left(\frac{2a\hbar t}{m}\right)^2}$$

increases with time.

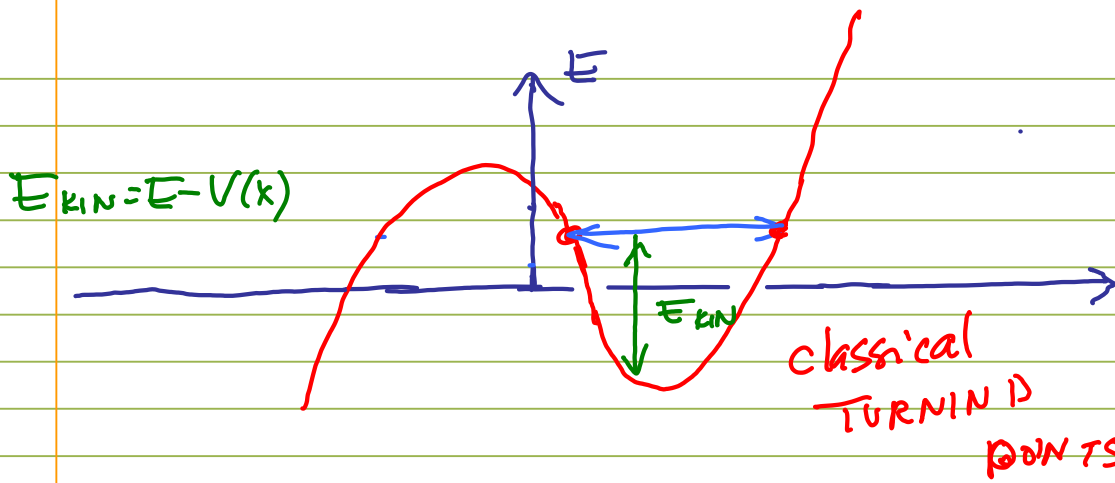
2.5 Delta-Function Potential

First:

BOUND STATES VS SCATTERING

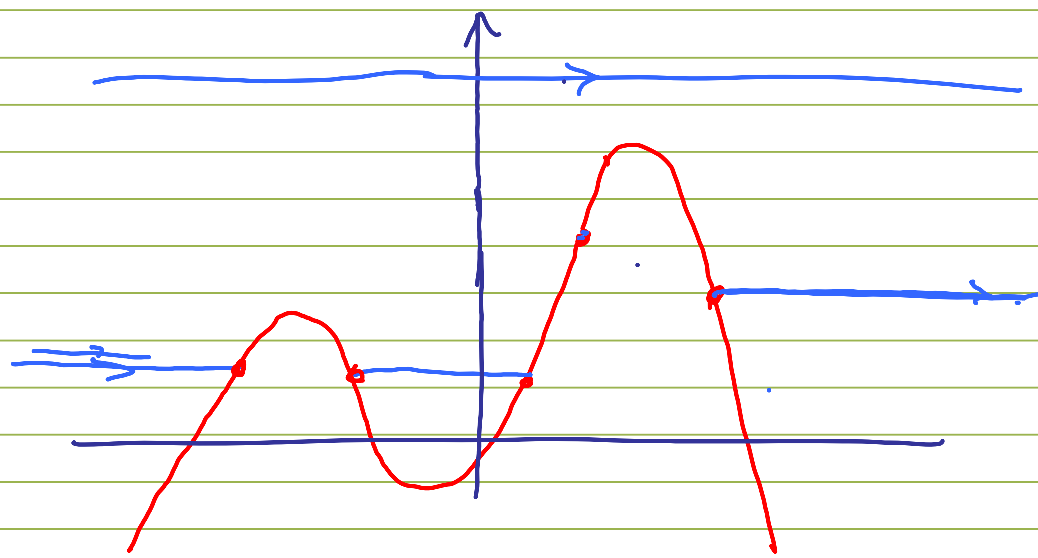
CLASSICALY :

BOUND STATE $\Longleftrightarrow$ PARTICLE CANT GET TO $\pm\infty$; BACK AND FORTH BETWEEN TURNING PTS



SCATTERING STATE: - FROM $\pm\infty$ TO TURNING PT AND BACK

- FROM + TO $-\infty$

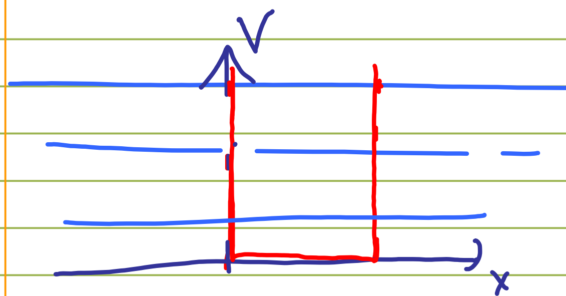


NOTE: CLASSICALY, FOR FIXED E BOTH TYPES OF STATES CAN EXIST.

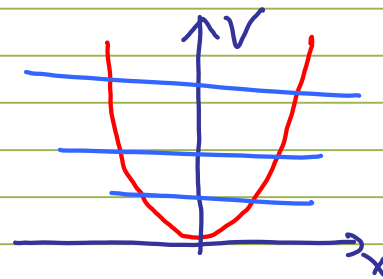
WHAT HAPPENS IN QUANTUM CASE:

INFINITE SQ. WELL

HARMONIC OSCILLATOR

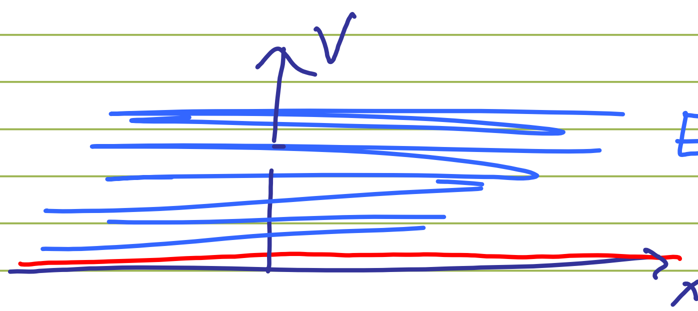


$E_n \propto n^2$



$E_n \propto n$

BOUND STATES CLASSICALLY $\Longleftrightarrow$ DISCRETE ENERGIES



$E \propto k^2$

SCATTERING STATES $\Longleftrightarrow$ CONTINUOUS SPECTRUM

GENERAL CASE:

ONLY DEPENDS ON E,

$V(\infty)$, $V(-\infty)$

GET BOUND STATES FOR

$$E < V(+\infty), V(-\infty)$$

Discrete energy spectrum

T.I.S.E HAS SOLN'S FOR
DISCRETE VALUES OF E,
WHICH ARE NORMALIZABLE

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$\Rightarrow$ NORMALIZABLE SOLNS of S.E.

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

Continuous energy spectra

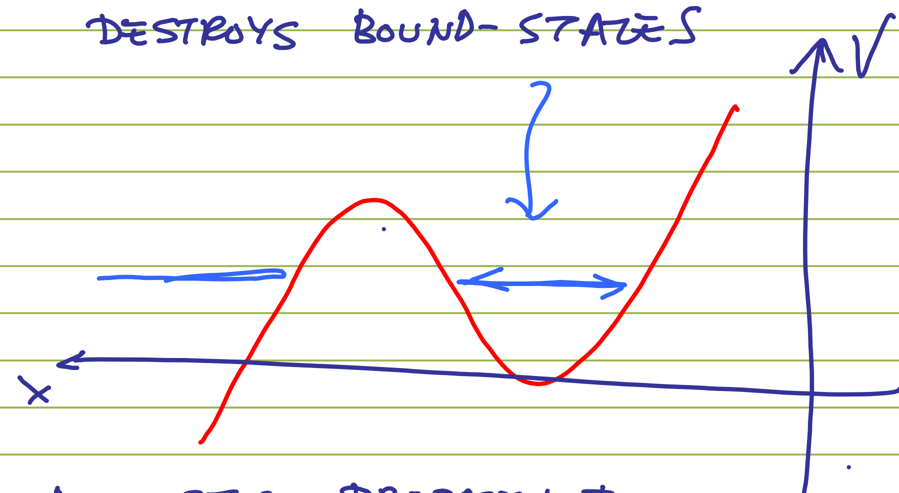
$$E > V(+\infty) \quad \text{or} \quad E > V(-\infty)$$

T.I.S.E HAS SOLN'S FOR ALL VALUES OF E, WHICH ARE NOT NORMALIZABLE

$\Rightarrow$ NORMALIZABLE SOLN'S TO S.E. ARE WAVE PACKETS

EXAMPLE $V(x)=0$

PHENOMENON OF "TUNNELING" DESTROYS BOUND-STATES



NON-ZERO PROBABILITY

Lecture 11

2.5 Delta-function potential

Dirac Delta-function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

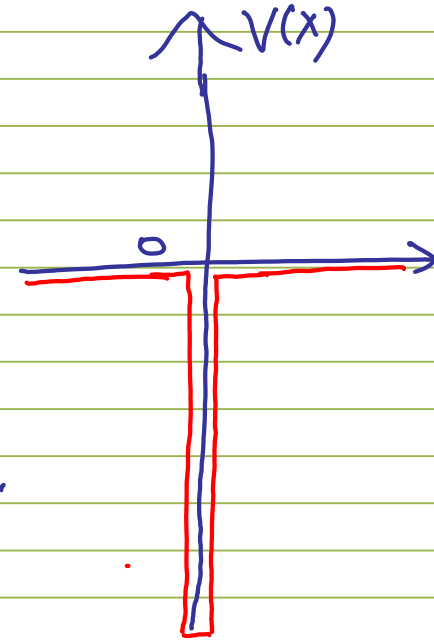
$\delta(x)$ is an example of a "distribution", since it is not a function in standard terms. One can obtain it as a limit of a sequence of normal functions

E.g:

$$g_a(x) = \begin{cases} 0, & |x| > a \\ \frac{1}{2a}, & |x| < a \end{cases}$$

$$\lim_{a \to 0} g_a(x) = \delta(x)$$

δ- Fn Potential



$\alpha > 0$

$$V(x) = -\alpha\,\delta(x)$$

Think narrow, steep well.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)$$

BOUND STATES $\Leftrightarrow$ $E < 0 = V(\pm\infty)$

SCATERING STATES $\Leftrightarrow$ $E \geq 0$

LOOK FOR BOUND STATES $(E<0)$

$x>0$ $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi$$

or $$\frac{d^2}{dx^2}\psi(x) = k^2\psi$$

$$k^2 = -2mE/\hbar^2 > 0$$

$$\psi(x) = A e^{-kx} + B e^{+kx}$$

NOT NORM.

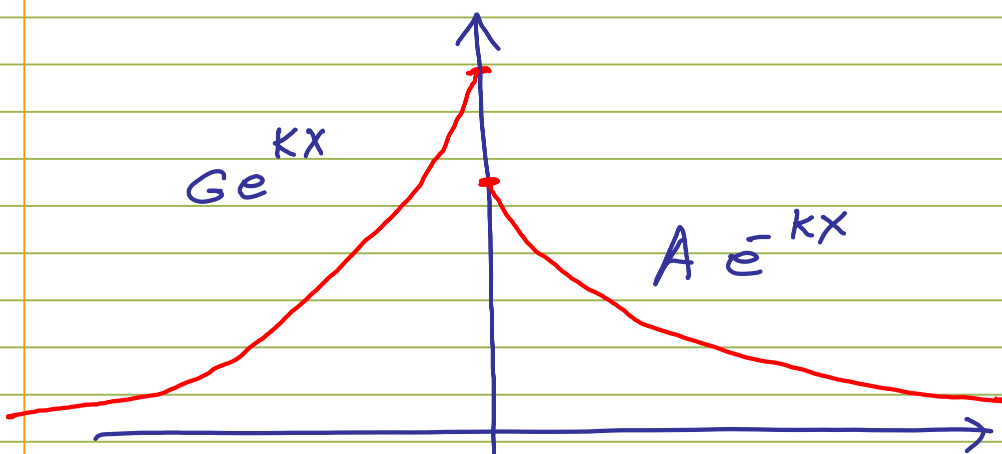
$$x>0 \Rightarrow \quad \psi(x) = A e^{-kx} \qquad (B=0)$$

$x<0$ SAME EQN, SOL

$$\psi(x) = \cancel{F e^{-kx}} + G e^{kx}$$

$$x<0 \Rightarrow \quad \psi(x) = G e^{kx}$$

STITCH TOGETHER AT $x=0$

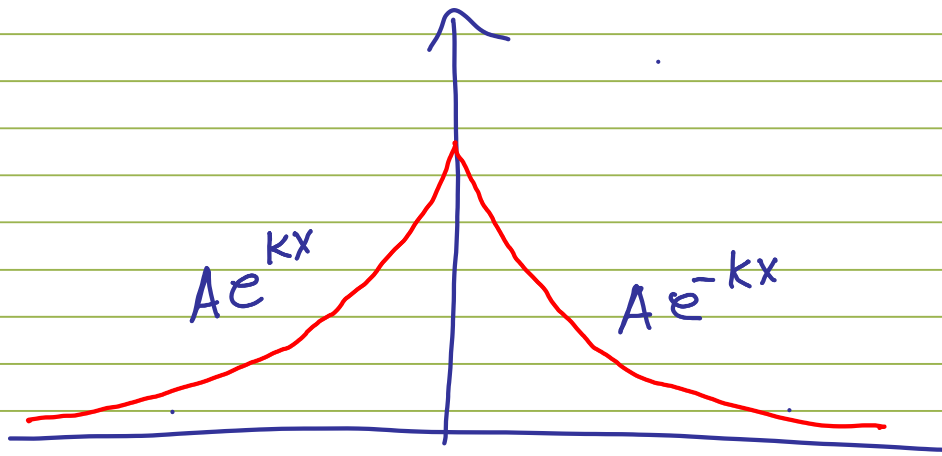


- REQUIRE $\psi(x)$ CONT. AT $x=0$

$$\Rightarrow \quad A = G$$

$$\psi(x) = \begin{cases} A e^{-kx} & x>0 \\ A e^{+kx} & , x<0 \end{cases}$$

- $\frac{d}{dx}\psi(x)$ DOES NOT NEED TO BE CONT'S SINCE $V(x)$ HAS ∞ DISCONTINUITY



THE DISCONTINUITY IN

$$\frac{d}{dx}\psi$$

AT $x=0$ IS DETERMINED BY EQUATION.

LET

$$\Delta\psi' = \lim_{\varepsilon\to 0}\left(\psi'(\varepsilon) - \psi'(-\varepsilon)\right)$$

$$= \text{DISCONTINUITY IN } \psi'.$$

TO FIND WHAT IT NEEDS TO BE

INTEGRATE

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)$$

FROM $-\varepsilon$ to ε FOR SMALL $\varepsilon \to 0$

$$\underbrace{-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\psi(x)\,dx}_{\Delta\psi'} + \int_{-\varepsilon}^{\varepsilon}\overbrace{V(x)\psi(x)\,dx}^{-\alpha\psi(0)} = \int_{-\varepsilon}^{\varepsilon}\cancelto{0}{E\,\psi(x)\,dx}$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}E\,\psi(x)\,dx = \lim_{\varepsilon\to 0}E\cdot\psi(0)\cdot 2\varepsilon = 0$$

= disc. in $E\cdot\psi$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}V(x)\psi(x)\,dx = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}(-\alpha\,\delta(x)\,\psi(x))\,dx$$

$$= -\alpha\,\psi(0)$$

For $V(x)$ with at most finite discontinuities, RHS vanishes.

we would get

$$\Delta \psi' = 0 \iff \psi' \text{ is continuous}$$

As is, we get

$$-\frac{\hbar^2}{2m} \Delta \psi'(0) - \alpha \psi(0) = 0$$

OR: $\boxed{\Delta \psi'(0) = -\frac{2m\alpha}{\hbar^2} \psi(0)}$

THIS DETERMINES K:

$$\psi'(x) = \begin{cases} \frac{d}{dx} A e^{-Kx} = -K \cdot A e^{-Kx}, & x>0 \\ \frac{d}{dx} A e^{Kx} = K A e^{Kx}, & x<0 \end{cases}$$

$$\Rightarrow \Delta \psi'(0) = -K \cdot A - (KA) = -2KA$$

$$\psi(0) = A$$

$\Rightarrow$ ONLY ONE BOUND STATE

$$K = \frac{m\alpha}{\hbar^2}, \quad E = -\frac{\hbar^2 K^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

NORMALIZE

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$

$$= \int_{-\infty}^{0} |A|^2 e^{2kx} dx + \int_{0}^{\infty} |A|^2 e^{-2kx} dx$$

$$= |A|^2 \frac{1}{2k} \left\{ e^{2kx} \Big|_{-\infty}^{0} - e^{-2kx} \Big|_{0}^{\infty} \right\}$$

$$= \frac{|A|^2}{2k} \left[+1 - 0 - (0-1) \right\}$$

$$= \frac{|A|^2}{2k} \Rightarrow A = \sqrt{2k}$$

SINGLE BOUND STATE

$$\Rightarrow \psi(x) = \frac{1}{\sqrt{2k}} e^{-|kx|}, \quad k = \frac{m\alpha}{\hbar^2}.$$

SCATTERING STATES $(E<0)$

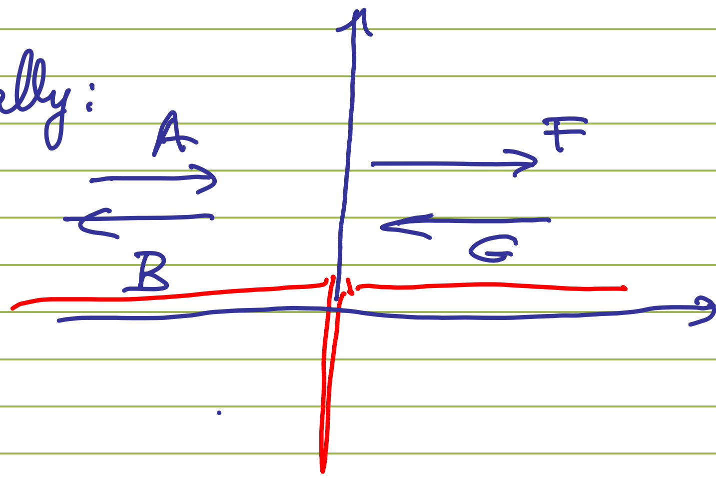
$x<0$: $$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

$$k^2 = \frac{2mE}{\hbar^2}, \quad E = \frac{\hbar^2 k^2}{2m}$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$x>0$ $$\psi(x) = F e^{ikx} + G e^{-ikx}$$

Physically:



FOR A PARTICLE COMING IN FROM LEFT $G=0$ (ONLY RIGHT-MOVING WAVE FOR $X>0$).

A = amplitude of incident wave

B = amplitude of reflected wave

F = —"— of transmitted wave

STITCH SOLUTIONS AT $x=0$

ψ IS CONT $\Rightarrow A+B=F$

JUMP IN $\frac{d}{dx}\psi = -\frac{2m\alpha}{\hbar^2}\psi(0) = -\frac{2m\alpha}{\hbar^2}F$

$\frac{d}{dx}\psi\Big|_+ = ikF \quad , \quad \frac{d}{dx}\psi\Big|_- = ikA - ikB$

$$\Delta\psi' = \frac{d}{dx}\psi\Big|_+ - \frac{d}{dx}\psi\Big|_- = ik(F-A+B)$$

$$= -\frac{2m\alpha}{\hbar^2}F$$

GET 2 EQNS FOR 3 UNKNOWNS

$$A+B=F$$

$$ik(F-A+B) = -2\frac{m\alpha}{\hbar^2}F$$

Let $\beta = \frac{m\alpha}{\hbar^2 k}$

$$F = A+B$$

$$(1-i2\beta)F = A-B$$

$$\Rightarrow \quad F = \frac{1}{1-i\beta}A \; ; \quad B = \frac{i\beta}{1-i\beta}A$$

Relative PROBABILITY OF REFLECTION

"REFLECTION COEFF"
$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$$

(FRACTION OF PARTICLES IN REFLECTED BEAM)

Relative PROB. OF TRANSMISSION

"TRANSMISS. COEFF"
$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

$$R + T = 1$$

R, T ARE FNS OF

$$\beta = \frac{2\alpha}{\hbar^2 k} = \frac{\alpha}{\hbar}\sqrt{\frac{2}{mE}}$$

HIGHER E $\Leftrightarrow$ SMALLER β

T GROWS, R FALLS.

CLASSICALLY R=0, T=1.

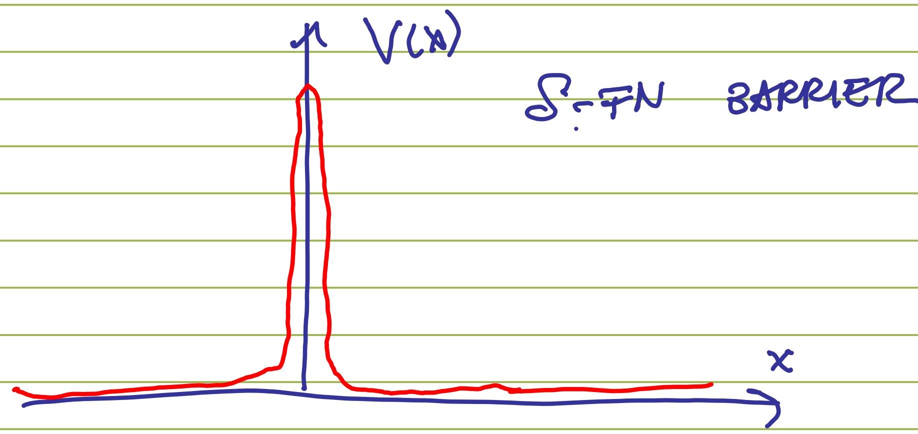
QUANTUM : NON-ZERO PROB W/

REFLECTION.

WE DID THIS ALL W/ STATIONARY STATES, WHICH ARE NOT PARTICLES

TO GET PHYSICAL PARTICLES, NEED WAVE PACKETS. ABOVE IS A GOOD APPROX. FOR A WAVE PEAKED NARROWLY AT k

δ-fn BARRIER ($\alpha \to -\alpha$)



* NO BOUND STATES :

* CLASSICALLY $T=0$, $R=1$

* QUANTUM ANSWER IS FN OF α^2

DOES NOT CHANGE