Notes of WU Shengjun MMP
Part I: Complex Analysis

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1 Complex Function

1.1 Complex Number

$$z = \rho\,\mathrm{e}^{\mathrm{i}\phi} \tag{1.1}$$

$$\sqrt[n]{z} = \sqrt[n]{\rho}\,\mathrm{e}^{\mathrm{i}\frac{\phi+2k\pi}{n}} \tag{1.2}$$

$\sqrt[n]{z}$ have n different values

Logarithm and exponential of complex numbers are defined as

$$\ln z = \ln\left(\rho\,\mathrm{e}^{\mathrm{i}\phi}\right) = \ln\rho + \mathrm{i}(\phi + 2k\pi) \tag{1.3}$$

$$z^s = \mathrm{e}^{s\ln z} = \mathrm{e}^{s(\ln\rho + \mathrm{i}(\phi+2k\pi))} \tag{1.4}$$

Specially

$$\ln\mathrm{i} = \mathrm{i}(\frac{\pi}{2} + 2k\pi) \tag{1.5}$$

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\cdot\mathrm{i}(\frac{\pi}{2}+2k\pi)} = -\frac{\pi}{2} + 2k\pi \tag{1.6}$$

1.2 Complex Function

1.2.1 Definition

1.2.2 Domain

1.2.3 Examples

$$\sin z = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \tag{1.7}$$

$$\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \tag{1.8}$$

$$
\begin{aligned}
\sin z &= \frac{\mathrm{e}^{\mathrm{i}x-y}-\mathrm{e}^{-\mathrm{i}x+y}}{2\mathrm{i}} \\
&= \frac{\mathrm{e}^{-y}(\cos x+\mathrm{i}\sin x)-\mathrm{e}^{y}(\cos x-\mathrm{i}\sin x)}{2\mathrm{i}} \\
&= \frac{\mathrm{e}^{-y}(-\mathrm{i}\cos x+\sin x)+\mathrm{e}^{y}(\mathrm{i}\cos x+\sin x)}{2} \\
&= \frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\sin x+\mathrm{i}\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2}\cos x
\end{aligned} \tag{1.9}
$$

$$
|\sin z| = \frac{1}{2}\sqrt{\mathrm{e}^{2y}+\mathrm{e}^{-2y}+2(\sin^2 x-\cos^2 x)} \tag{1.10}
$$

$|\sin z|$ and $|\cos z|$ can > 1.

$|\sin z|$ and $|\cos z|$ have period 2π.

$$
\sinh z = \frac{1}{2}(\mathrm{e}^{z}-\mathrm{e}^{-z}) \tag{1.11}
$$

$$
\cosh x = \frac{1}{2}(\mathrm{e}^{z}+\mathrm{e}^{-z}) \tag{1.12}
$$

e^{z}, $\sinh z$ $\cosh z$ have period $2\pi\mathrm{i}$.

1.2.4 Derivatives

$$
\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\Delta x\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \tag{1.13}
$$

suppose $\Delta z = \Delta x$

$$
\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x} \tag{1.14}
$$

suppose $\Delta z = \mathrm{i}\Delta y$

$$
\tag{1.15}
$$

2 Integral

2.1 Introduction

2.2 Cauchy Theorem

单连通

$$
\oint_l f(z)\mathrm{d}z = 0 \tag{2.1}
$$

复连通

$$
\oint_l f(z)\mathrm{d}z+\sum_{i=1}^{n}\oint_{l_i} f(z)\mathrm{d}z = 0 \tag{2.2}
$$

2.3 不定积分

Complex Newton-Lebniz

$$F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta, \quad F'(z) = f(z) \tag{2.3}$$

Consider integral

$$I = \int_a^b z^n \mathrm{d}z,\ n \in \mathbb{Z} \tag{2.4}$$

1) $n \neq -1$

2) $n = -1$

$$I = \ln b - \ln a = \ln\left|\frac{b}{a}\right| + \mathrm{i}(\operatorname{Arg} b - \operatorname{Arg} a) \tag{2.5}$$

What about

$$I = \oint_l (z-\alpha)^n \mathrm{d}z \tag{2.6}$$

1) α is external, $I = 0$

2) α is internal

2.1) $n \geqslant 0$

2.2) $n < 0$, let $z - \alpha = R\,\mathrm{e}^{\mathrm{i}n\phi}$

$$\begin{aligned} I &= \int_0^{2\pi} R^n\,\mathrm{e}^{\mathrm{i}n\phi}\,\mathrm{d}\big(\alpha + R\,\mathrm{e}^{\mathrm{i}n\phi}\big) \\ &= \mathrm{i}R^{n+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)\phi}\,\mathrm{d}\phi \end{aligned} \tag{2.7}$$

2.2.1) $n \neq -1$ 2.2.2) $n = -1$

$$I = \int_C (|z| - \mathrm{e}^z \sin z)\mathrm{d}z \tag{2.8}$$

where C is

2.4 Cauchy Equation

$f(z)$在闭单联通区域B上解析, $l$is boundary of B, $\alpha \in B$.

$$f(\alpha) = \frac{1}{2\pi\mathrm{i}}\oint_l \frac{f(z)}{z-\alpha}\mathrm{d}z \tag{2.9}$$

Discussion

1)

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_l \tag{2.10}$$

2) 无界推广

3) derivatives

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \oint_l \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \tag{2.11}$$

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_l \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta \tag{2.12}$$

3 Power Series

3.1 Complex Series

3.1.1 Introduction

$$\sum_{k=1}^{\infty} w_k = \sum_{k=1}^{\infty} u_k + \mathrm{i} \sum_{k=1}^{\infty} v_k \tag{3.1}$$

3.1.2 Convergence Test

Cauchy's Convergence Test

$\forall \varepsilon > 0,\ \exists N, \text{ s.t. when } n > N,\ \forall p \in \mathbb{N}$

$$\left| \sum_{k=n+1}^{n+p} w_k \right| < \varepsilon \tag{3.2}$$

Absolute Convergence

3.1.3 Function Series

Convergence Test

Cauchy's

3.1.4 Uniform Convergence

3.2 Power Series

3.2.1 Definition

3.2.2 Convergence and Divergence Test

D'Alembert's Test

Root

Convergence Circle

3.2.3 Analytical Features

3.3 Taylor Expansion

3.4 解析延拓

3.5 Laurent Expansion

3.5.1 Bilateral Power Series

$$\cdots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{3.3}$$

Positive part: convergence radius $= R_1$
Negative: denote $\zeta = \dfrac{1}{z-z_0}$
conv radius $= \dfrac{1}{R_2}$ thus, bilateral power series is abs and uniform conv when

$$R_2 < |z-z_0| < R_1 \tag{3.4}$$

which is called convergence ring.

3.5.2 Laurent Expansion Th.

pos part:
aka canonical part
neg part:
aka 主部

Laurent Expansion is unique. Proof omitted.

Attention 1) $z = z_0$ may be a singularity or not.
2) Although Laurent Expansion looks the same as Taylor Expansion,

$$a_k \neq \frac{F^{(k)}(z_0)}{k!} \tag{3.5}$$

no matter whether z_0 is singularity.
3)

3.6 Isolated Singularity

4 留数定理

4.1

4.2 计算实变函数定积分

type I

type II

Suppose

$$I = \lim_{\substack{R_1 \to \infty \\ R_2 \to \infty}} \int_{-R_1} R_2 f(x)\,\mathrm{d}x \tag{4.1}$$

exists

when $R_1 = R_2 \to \infty$, I is called principle (主值) of the integral above, namely

$$\mathscr{P} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \tag{4.2}$$

Th

$$\mathscr{P} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i} \sum_k \operatorname{Res} f(b_k) \quad \text{(upper semi-plane)} \tag{4.3}$$

type III

$$\int_0^{\infty} F(x) \cos mx \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} F(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x \tag{4.4}$$

$$\int_0^{\infty} G(x) \sin mx \mathrm{d}x = \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} G(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x \tag{4.5}$$

Jordan's Lemma (约当引理)

$$\lim_{R \to \infty} \int_{C_R} F_z\,\mathrm{e}^{\mathrm{i}mz}\,\mathrm{d}z = 0 \tag{4.6}$$

when $m > 0$, C_R is a semi-circle on upper semi-plane,
or $m < 0$, C_R is a semi-circle on lower semi-plane.