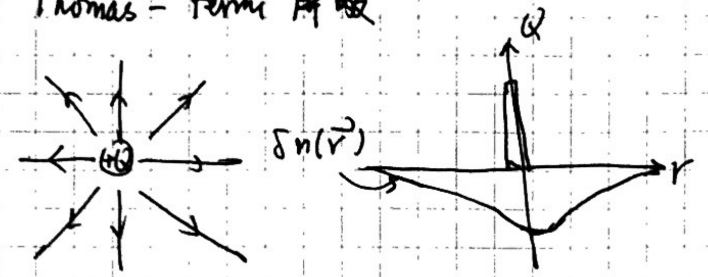
Lecture 4. 电子相互作用：屏蔽和等离激元

1) Thomas - Fermi 屏蔽



放置外加电荷：Q. $Q/e \ll 1$ (线性响应；微扰). $\phi = \frac{Q}{r}$.

$$V_{ext}(\vec r) = (-e)\phi = -e\frac{Q}{r}.$$

$$V_{ext} + \delta V = V_{eff} \longrightarrow \delta n(\vec r)$$

$V_{eff} \Rightarrow \delta n$: $n(\vec r) = \sum_{k\sigma} \frac{1}{1+e^{\beta(\epsilon_k-\mu)}}$ w/ $\epsilon_k = \frac{k^2}{2m} + V_{eff}(\vec r)$.

$$\frac{\partial n(\vec r)}{\partial V_{eff}(\vec r)} = -\underbrace{\frac{\partial n}{\partial \mu}}_{} \rightarrow \text{态密度 } N(0).$$

(零温下, $N(0) = \frac{m k_F}{\pi^2 \hbar^2}$) $\delta n(\vec r) = -N(0)\, V_{eff}(\vec r)$. $V_{eff} \ll 1$

$\delta n \Rightarrow \delta V$: $V_{eff} = V_{ext} + \delta V = -e(\phi_{ext} + \delta\phi) = -e\phi$.

ϕ 满足 Poisson 方程.

$$-\nabla^2\phi = 4\pi[Q\delta(\vec r) - e\delta n(\vec r)].$$

作 Fourier 变换. $\phi(\vec k) = \int d\vec r\, \phi(\vec r) e^{-i\vec k\cdot\vec r}$. $-\nabla^2 \rightarrow k^2$.

$$k^2\phi(\vec k) = 4\pi Q - 4\pi e\,\delta n(\vec k).$$

$$\phi(\vec k) = \frac{4\pi}{k^2}[Q - e\delta n(\vec k)]$$

$$V_{eff}(\vec k) = -e\phi(\vec k) = -\frac{4\pi e}{k^2}[Q - e\delta n(\vec k)].$$

$$\delta n(\vec k) = -N(0)\, V_{eff}(\vec k)$$

$$\therefore\ k^2 V_{eff}(\vec k) = -4\pi e Q + 4\pi e^2 \delta n(\vec k)$$

$$= -4\pi e Q - 4\pi e^2 N(0) V_{eff}(\vec k)$$

$$\therefore\ V_{eff}(\vec k) = -\frac{4\pi e Q}{k^2 + 4\pi e^2 N(0)}.$$

令 $K_{TF}^2 = 4\pi e^2 N(0)$ $\quad [K_{TF}] = [k] = [\vec r]$

$$V_{eff}(\vec k) = -e\frac{4\pi Q}{k^2 + K_{TF}^2}.$$

2) Fourier 变换，实空间的屏蔽库仑势.

* 量纲分析： $[k_{TF}] = [k] = [L^{-1}]$.

w/o k_{TF}: $V(\vec{k}) = -e\frac{4\pi Q}{k^2} \Rightarrow V(\vec{r}) = -e\frac{Q}{r}$

w/ k_{TF}: $V(\vec{k}) = -e\frac{4\pi Q}{k^2+k_{TF}^2} \Rightarrow V(\vec{r}) = -e\frac{Q}{r}e^{-k_{TF}/r}$.

* 推导： $V(\vec{k}) = -e\frac{4\pi Q}{k^2+k_{TF}^2} \Leftrightarrow \phi(\vec{k}) = \frac{4\pi Q}{k^2+k_{TF}^2}$.

$\phi(\vec{r}) = \int\frac{d\vec{k}}{(2\pi)^3}V(\vec{k})e^{i\vec{k}\cdot\vec{r}}$ （不失一般性假设 $\vec{r} = r\hat{e}_z$）.

$= \frac{1}{8\pi^3}\cancel{\int_0^{2\pi}d\varphi}^{\,2\pi}\int_{-1}^{1}d\cos\theta\int_0^\infty k^2dk\,\frac{4\pi Q}{k^2+k_{TF}^2}e^{ikr\cos\theta}$

$= \frac{1}{4\pi^2}\int_0^\infty k^2dk\,\frac{4\pi Q}{k^2+k_{TF}^2}\int_{-1}^{1}d\cos\theta\, e^{ikr\cos\theta}$

$= \frac{4\pi Q}{4\pi^2}\int_0^\infty\frac{k^2dk}{k^2+k_{TF}^2}\frac{e^{ikr\cos\theta}\big|_{-1}^{1}}{ikr}$

$= \frac{Q}{\pi}\int_0^\infty\frac{k\,dk}{k^2+k_{TF}^2}\frac{e^{ikr}-e^{-ikr}}{ir}$

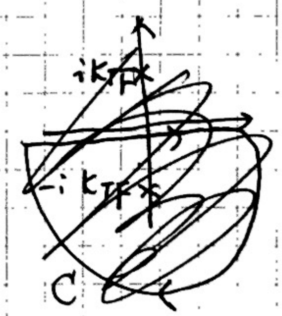
$= \frac{2Q}{r}\int_0^\infty\frac{dk}{2\pi i}\frac{k}{k^2+k_{TF}^2}\left(e^{ikr}-e^{-ikr}\right)$

考虑第二项 $-\frac{2Q}{r}\int_0^\infty\frac{dk}{2\pi i}\frac{k}{k^2+k_{TF}^2}e^{-ikr}$ 令 $k\to -k$

$= -\frac{2Q}{r}\int_{-\infty}^{0}\frac{dk}{2\pi i}\frac{-k}{k^2+k_{TF}^2}e^{ikr}$

$= \frac{2Q}{r}\int_{-\infty}^{0}\frac{dk}{2\pi i}\frac{k}{k^2+k_{TF}^2}e^{ikr}$

$\therefore \phi(r) = \frac{2Q}{r}\int_{-\infty}^{\infty}\frac{dk}{2\pi i}\frac{k}{k^2+k_{TF}^2}e^{ikr}$



$r>0$：从上半平面封闭围道

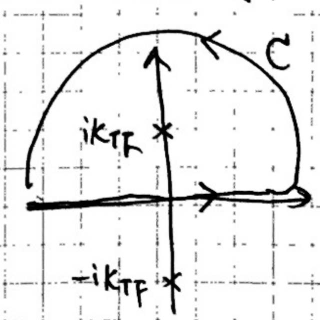
$\phi(r) = \frac{2Q}{r}\oint_C\frac{dk}{2\pi i}\frac{k}{k^2+k_{TF}^2}e^{ikr}$

极点 @ $k = ik_{TF}$.

$= \frac{\cancel{2}Q}{r}\frac{\cancel{ik_{TF}}}{\cancel{2}\cancel{ik_{TF}}}e^{-k_{TF}r}$

$= \frac{Q}{r}e^{-k_{TF}r}$

12.



3) 有限频率的动态响应；RPA 近似.

静态 → 动态：$\phi_{ext}(\vec{r}) = \dfrac{Q}{r} \Rightarrow \phi_{ext}(\vec{r}, t)$.

$$\phi_{ext} + \delta\phi = \phi_{eff} \xrightarrow{\text{(自由电子)}} \delta n$$

* $\phi_{ext} \Rightarrow \delta n$：线性响应

$$\hat{H} = \hat{H}_0 + \hat{H}', \quad \hat{H}' = \int d\vec{r}\, \hat{n}(\vec{r}, t)\, (-e)\, \phi_{ext}(\vec{r}, t)$$

$$= \int d\vec{r}\, \hat{n}(\vec{r}, t)\, V_{ext}(\vec{r}, t) \text{ —— 注意这里是 + 号}$$

$$\langle \delta \hat{n}(\vec{r}, t) \rangle = \int_{-\infty}^{t} dt'\, V_{ext}(\vec{r}', t')\, \chi^{ret}_{nn}(\vec{r} - \vec{r}', t - t')$$

$$\chi^{ret}_{nn}(\vec{r} - \vec{r}', t - t') = -\frac{i}{\hbar}\theta(t - t') \langle [\delta\hat{n}(\vec{r}, t), \delta\hat{n}(\vec{r}', t')] \rangle$$

多了一个负号；与上节课符号相反，与书上符号相同.
这个负号会导致 $\chi(\omega \to 0) < 0$

Fourier 变换之后，

$$\langle \delta\hat{n}(\vec{k}, \omega) \rangle = \chi^{ret}_{nn}(\vec{k}, \omega)\, V_{ext}(\vec{k}, \omega).$$

* 屏蔽效应，RPA 近似.

考虑 δn 产生的屏蔽势：$\delta V(\vec{k}, \omega) = \dfrac{4\pi e^2}{k^2} \langle \delta n(\vec{k}, \omega) \rangle$

这个关系与之间前推导的静态屏蔽中相同.

$$-\nabla^2 \delta\phi = 4\pi(-e)\langle\delta\hat{n}\rangle = -4\pi e \langle \delta\hat{n} \rangle$$

$$\therefore\ k^2 \delta\phi(\vec{k}) = -4\pi e \langle \delta\hat{n} \rangle \Rightarrow \delta\phi(\vec{k}) = -e\frac{4\pi}{k^2}\langle\delta\hat{n}\rangle$$

$$\delta V(\vec{k}) = -e\,\delta\phi(\vec{k}) = \frac{4\pi e^2}{k^2}\langle \delta n(\vec{k}) \rangle$$

$$V_{eff} = V_{ext} + \delta V$$

$$V_{eff}(\vec{k}, \omega) = V_{ext}(\vec{k}, \omega) + \delta V(\vec{k}, \omega)$$

$$= \chi^{ret}_{nn}(\vec{k}, \omega)^{-1} \langle \delta\hat{n}(\vec{k}, \omega) \rangle + \frac{4\pi e^2}{k^2} \langle \delta n(\vec{k}, \omega) \rangle$$

$$= \left[\chi^{ret}_{nn}(\vec{k}, \omega)^{-1} + \frac{4\pi e^2}{k^2}\right] \langle \delta n(\vec{k}, \omega) \rangle.$$

RPA：近似地认为 $V_{eff} \Rightarrow \langle\delta\hat{n}\rangle$ 与自由电子响应相同.

$$V_{eff}(\vec{k}, \omega) = \chi^{0, ret}_{nn}(\vec{k}, \omega)^{-1} \langle \delta n(\vec{k}, \omega) \rangle.$$

$$\therefore\ \chi^{0, ret}_{nn}(\vec{k}, \omega)^{-1} = \chi^{ret}_{nn}(\vec{k}, \omega)^{-1} + \frac{4\pi e^2}{k^2} \Rightarrow \chi^{ret}_{nn}(\vec{k}, \omega) = \frac{1}{\chi^{0, ret}_{nn}(\vec{k}, \omega)^{-1} - \frac{4\pi e^2}{k^2}}$$

$$\boxed{\chi^{ret}_{nn}(\vec{k}, \omega) = \frac{\chi^{0, ret}_{nn}(\vec{k}, \omega)}{1 - \frac{4\pi e^2}{k^2}\chi^{0, ret}_{nn}(\vec{k}, \omega)}} \leftarrow \text{RPA}$$

= Random Phase Approximation.

下面，我们计算自由电子 $\chi_{nn}^{0}(\vec{k},\omega)$. 我们介绍 Matsubara（松原）频率格林函数这一理论工具.

4) ~~Matsubara~~ Matsubara-frequency 格林函数.（编时格林函数）.

定义. $G_{AB}(\tau_1,\tau_2) = -\frac{1}{Z}T_\tau\langle A(\tau_1)B(\tau_2)\rangle$.

[对比 $G_{AB}^{ret}(t_1,t_2) = -\frac{i}{\hbar}\langle[A(t_1),B(t_2)]\rangle\theta(t_1-t_2)$]

T_τ：编时算符. $T_\tau\langle A(\tau_1)B(\tau_2)\rangle = \begin{cases} A(\tau_1)B(\tau_2): & \tau_1>\tau_2 \\ \eta B(\tau_2)A(\tau_1) & \tau_1<\tau_2 \end{cases}$

$\eta = \begin{cases} +1: & A,B\text{为玻色算符} \\ -1, & A,B\text{为费米算符} \end{cases}$

$\boxed{\begin{array}{l}\tau = it\ \text{虚时} \\ A(\tau) = e^{H\tau}Ae^{-H\tau}\end{array}}$

定理. $G_{AB}(\tau) = \eta G_{AB}(\tau+\beta)$

证明. [若有时间平移不变性：$G_{AB}(\tau_1,\tau_2) = G_{AB}(\tau_1-\tau_2)$].

$G_{AB}(\tau) = -T_\tau\langle A(\tau)B(0)\rangle = -T_\tau\, tr[\frac{1}{Z}e^{-\beta H}e^{H\tau}Ae^{-H\tau}B]$.

$\cancel{= -\frac{1}{Z}T_\tau\, tr[\eta B e}$

$= -\frac{1}{Z}T_\tau\, tr[\eta e^{-H\tau}B\, e^{H(\tau-\beta)}A]$

$= -\frac{1}{Z}T_\tau\, tr[e^{-\beta H}e^{H(\beta-\tau)}B\,e^{H(\tau-\beta)}A]$

$= \eta\, G_{BA}(\beta-\tau)$

~~反过来，$G_{AB}(\tau) = \eta G_{BA}(\tau+\beta)$~~

~~这表明 $G_{AA}(\tau)$ 为周期/反周期 边界条件~~

~~玻色~~ ~~费米~~

$\therefore\ \eta G_{AB}(\beta+\tau) = G_{BA}(-\tau)$

$= -T_\tau\langle B(-\tau)A(0)\rangle$

$= -T_\tau\langle B(0)A(\tau)\rangle$ ← 时间平移不变.

$= -\eta T_\tau\langle A(\tau)B(0)\rangle$

$= \eta G_{AB}(\tau)$.

$\therefore\ G_{AB}(\tau) = \eta G_{AB}(\tau+\beta)$ □

这表明 $G_{AB}(\tau)$ 为周期/反周期函数.

因此，选用离散 Fourier 变换.

$$G_{AB}(\omega_n) = \int_0^\beta d\tau\, G_{AB}(\tau)\, e^{i\omega_n\tau}$$

$$G_{AB}(\tau) = \frac{1}{\beta}\sum_n G_{AB}(\omega_n)\, e^{-i\omega_n\tau}.$$

费米子 $\omega_n = \frac{\pi}{\beta}(2n+1)$

玻色子 $\omega_n = \frac{2\pi n}{\beta}$.

5) Matsubara ~~表示~~. 格林函数的谱展开；解析延拓.

考虑谱展开. 取多体 Hamiltonian 的本征态, $H|\alpha\rangle = \xi_\alpha |\alpha\rangle$.

$$G_{AB}(\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\left(-T_\tau\langle A(\tau)B(0)\rangle\right)$$

$$= -\int_0^\beta d\tau\, e^{i\omega_n\tau}\langle A(\tau)B(0)\rangle \quad \leftarrow \text{积分区域内 } \tau>0.$$

$$= -\frac{1}{Z}\int_0^\beta d\tau\, e^{i\omega_n\tau}\,\mathrm{tr}\left(e^{-\beta H}e^{H\tau}Ae^{-H\tau}B\right)$$

$$= -\frac{1}{Z}\sum_{\alpha\beta}\int_0^\beta d\tau\, e^{i\omega_n\tau}\langle\alpha|e^{-\beta\xi_\alpha}e^{\xi_\alpha\tau}A|\beta\rangle\langle\beta|e^{-\xi_\beta\tau}B|\alpha\rangle$$

$$= -\frac{1}{Z}\sum_{\alpha\beta} e^{-\beta\xi_\alpha}\int_0^\beta d\tau\, e^{(i\omega_n+\xi_\alpha-\xi_\beta)\tau}\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle$$

$$= -\frac{1}{Z}\sum_{\alpha\beta} e^{-\beta\xi_\alpha}\frac{1}{i\omega_n+\xi_\alpha-\xi_\beta}e^{(i\omega_n+\xi_\alpha-\xi_\beta)\tau}\Big|_0^\beta\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle$$

费米子: $e^{i\omega_n\beta} = e^{i\pi(2n+1)} = -1$

玻色子: $e^{i\omega_n\beta} = e^{i2\pi n} = +1$

$$\Big\}\ e^{i\omega_n\beta} = \eta.$$

$$\therefore G_{AB}(\omega_n) = -\frac{1}{Z}\sum_{\alpha\beta} e^{-\beta\xi_\alpha}\frac{1}{i\omega_n+\xi_\alpha-\xi_\beta}\left[\eta e^{\beta(\xi_\alpha-\xi_\beta)}-1\right]\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle$$

$$= \frac{1}{Z}\sum_{\alpha\beta} e^{-\beta\xi_\alpha}\left[1-\eta e^{\beta(\xi_\alpha-\xi_\beta)}\right]\frac{\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle}{i\omega_n+\xi_\alpha-\xi_\beta}$$

$$= \frac{1}{Z}\sum_{\alpha\beta}\left(e^{-\beta\xi_\alpha}-\eta e^{-\beta\xi_\beta}\right)\frac{\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle}{i\omega_n+\xi_\alpha-\xi_\beta}.$$

~~对比谱函数~~

$\cancel{A_{AB}(\omega) = \frac{1}{Z}\sum_{\alpha\beta}}$

对比推迟格林函数的谱展开:

$$G^{ret}_{AB}(\omega) = \frac{1}{Z}\sum_{\alpha\beta}\left(e^{-\beta\xi_\alpha}-\eta e^{-\beta\xi_\beta}\right)\frac{\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle}{\omega+\xi_\alpha-\xi_\beta+i0^+}$$

$G^{ret}_{AB}(\omega)$ 可由 $G_{AB}(\omega_n)$ 作解析延拓 $i\omega_n\to\omega+i0^+$ 得到.

也可以对比谱函数

$$A_{AB}(\omega) = \frac{1-\eta e^{-\beta\omega}}{Z}\sum_{\alpha\beta} e^{-\beta\xi_\alpha}\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle\,\delta(\omega+\xi_\alpha-\xi_\beta).$$

$$= \frac{1}{Z}\sum_{\alpha\beta}\left(e^{-\beta\xi_\alpha}-\eta e^{-\beta\xi_\beta}\right)\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle\,\delta(\omega+\xi_\alpha-\xi_\beta)$$

$$\therefore\ G_{AB}(\omega_n) = \int d\omega'\,\frac{A_{AB}(\omega')}{i\omega_n-\omega'}.$$

$$\text{vs}\ G^{ret}_{AB}(\omega) = \int d\omega'\,\frac{A_{AB}(\omega')}{\omega-\omega'+i0^+}.$$

$i\omega_n \to \omega+i0^+.$

6) 计算 χ^0_{nn}.

自由费米子: $A(\vec{k},\omega)=\delta(\omega-\xi_k)$, $\xi_k=\epsilon_k-\mu=\frac{k^2}{2m}-\mu$.

$\therefore\ G_\sigma(\vec{k},i\omega_n)=\frac{1}{i\omega_n-\xi_k}$.

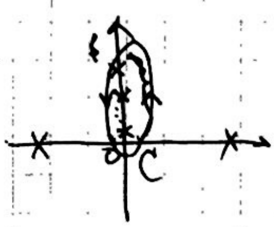
$\cancel{\chi^0_{nn}(\vec{k})}$

$$
\begin{aligned}
\chi^0_{nn}(\vec{q},i\omega_n) &= -T_\tau\langle\delta n(\vec{q},i\omega_n)\,\delta n(-\vec{q},-i\omega_n)\rangle\\
&= -T_\tau\int d\vec{r}\,d\tau\, e^{i(\omega_n\tau-\vec{q}\cdot\vec{r})}\langle\delta n(\vec{r},\tau)\,\delta n(0,0)\rangle\\
&= -T_\tau\int d\vec{r}\,d\tau\, e^{i(\omega_n\tau-\vec{q}\cdot\vec{r})}\langle\psi^\dagger_\sigma(\vec{r},\tau)\psi_\sigma(\vec{r},\tau)\,\psi^\dagger_\sigma(0,0)\psi_\sigma(0,0)\rangle\\
&\qquad -\langle\psi^\dagger_\sigma(\vec{r},\tau)\psi_\sigma(\vec{r},\tau)\rangle\langle\psi^\dagger_\sigma(0,0)\psi_\sigma(0,0)\rangle\\
&= \sum_\sigma T_\tau\int d\vec{r}\,d\tau\, e^{i(\omega_n\tau-\vec{q}\cdot\vec{r})}\langle\psi_\sigma(0,0)\psi^\dagger_\sigma(\vec{r},\tau)\rangle\langle\psi_\sigma(\vec{r},\tau)\psi^\dagger_\sigma(0,0)\rangle\\
&\cancel{=\frac{1}{\beta V}T_\tau\int d\vec{r}\,d\vec{r}'\,d\tau\,d\tau'\,e^{i\omega_n(\tau-\tau')-i\vec{q}\cdot(\vec{r}-\vec{r}')}}\\
&\qquad\cancel{\langle\psi(\vec{r},\tau)}\\
&= \sum_\sigma\int d\vec{r}\,d\tau\, e^{i(\omega_n\tau-\vec{q}\cdot\vec{r})}\left[-T_\tau\langle\psi_\sigma(0,0)\psi^\dagger_\sigma(\vec{r},\tau)\rangle\right]\left[-T_\tau\langle\psi_\sigma(\vec{r},\tau)\psi^\dagger_\sigma(0,0)\rangle\right]\\
&= \sum_\sigma\int d\vec{r}\,d\tau\, e^{i(\omega_n\tau-\vec{q}\cdot\vec{r})}G_\sigma(-\vec{r},-\tau)\,G_\sigma(\vec{r},\tau).\\
&= \sum_\sigma\sum_k\frac{1}{\beta}\sum_{\nu_n}G_\sigma(\vec{k}+\vec{q},\omega_n+\nu_n)\,G_\sigma(\vec{k},\nu_n).\\
&= 2\sum_k\frac{1}{\beta}\sum_{\nu_n}\frac{1}{i\omega_n+i\nu_n-\xi_{k+q}}\cdot\frac{1}{i\nu_n-\xi_k}.
\end{aligned}
$$

计算 $\sum_{\nu_n}$: 围道积分的技巧.

$i\nu_n=i\frac{\pi}{\beta}(2n+1)$ 为 $f(z)=\frac{1}{e^{\beta z}+1}$ 的极点. $e^{\beta z}+1=0\Rightarrow z=\frac{i\pi}{\beta}(2n+1)$.

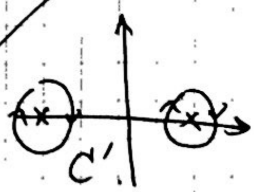
$\cancel{\therefore\ \frac{1}{\beta}\sum_{\nu_n}F(i\nu_n)=\oint_C\frac{dz}{2\pi i}f(z)F(z)}$



$$
\begin{aligned}
\operatorname*{Res}_{z=\frac{i\pi(2n+1)}{\beta}} f(z) &= \lim_{z\to i\nu_n}\frac{z-i\nu_n}{e^{\beta z}+1}\\
&= \frac{1}{(e^{\beta z}+1)'}\Big|_{z=i\nu_n}\\
&= \frac{1}{\beta e^{\beta z}|_{z=i\nu_n}}=-\frac{1}{\beta}.
\end{aligned}
$$

⇩ 围道变形.

$\therefore\ \oint_C\frac{dz}{2\pi i}f(z)F(z)=-\frac{1}{\beta}\sum_n F(i\nu_n).$



$$
\begin{aligned}
\frac{1}{\beta}\sum_{\nu_n}F(i\nu_n) &= -\oint_C\frac{dz}{2\pi i}f(z)F(z).\\
&= -\oint_{C'}\frac{dz}{2\pi i}f(z)F(z)
\end{aligned}
$$

$\therefore\ \chi^0_{nn}(\vec{q},i\omega_n)=2\sum_k-\oint_{C'}\frac{dz}{2\pi i}\frac{1}{z+i\omega_n-\xi_{k+q}}\frac{1}{z-\xi_k}f(z)$

C' 内包围奇点: $z=\xi_k$ 和 $z=\xi_{k+q}-i\omega_n$. 顺时针包围.

$$
=2\sum_k\cancel{f(\xi_k)}\left[\frac{f(\xi_k)}{\xi_k+i\omega_n-\xi_{k+q}}+\frac{f(\xi_{k+q}+i\omega_n)}{\xi_{k+q}-i\omega_n-\xi_k}\right]
$$

$f(\xi_{k+q}+i\omega_n)=\frac{1}{e^{\beta\xi_{k+q}}e^{\beta i\omega_n}+1}=\frac{1}{e^{\beta\xi_{k+q}}+1}=f(\xi_{k+q}).$

$\therefore\ \chi^0_{nn}(\vec{q}, \omega_n) = 2\sum_k \frac{f(\xi_k) - f(\xi_{k+q})}{i\omega_n + \xi_k - \xi_{k+q}}$

解析延拓：$\omega_n \longrightarrow \omega + i0^+$

$$\chi^{0,ret}_{nn}(\vec{q}, \omega) = 2\sum_k \frac{f(\xi_k) - f(\xi_{k+q})}{\omega + \xi_k - \xi_{k+q} + i0^+}.$$

7) 极限情形：TF 屏蔽.

① $\omega = 0$, $\vec{k} \to 0$：静态；长波极限.

$$\chi^{0,ret}_{nn}(\vec{q}\to 0, \omega=0) = 2\sum_k \frac{f(\xi_k) - f(\xi_{k+q})}{\xi_k - \xi_{k+q}} = 2\sum_k \frac{\partial f(\xi_k)}{\partial \xi_k}$$

$$= -2\sum_k \frac{\partial f(\xi_k)}{\partial \mu} = -2\frac{\partial n}{\partial \mu} = -N(0).$$

回到 Thomas-Fermi 屏蔽：简记 $\chi^{ret}(\vec{q}=0, \omega=0) = \chi$

$$\chi^0 = -N(0)$$

$$\chi = \frac{\chi^0}{1 - \frac{4\pi e^2}{k^2}\chi^0} = -\frac{N(0)}{1 + \frac{4\pi e^2}{k^2}N(0)}$$

定义 $\quad V_{eff}(\vec{k}, \omega) = \frac{1}{\epsilon(\vec{k}, \omega)} V_{ext}(\vec{k}, \omega)$

RPA: $\quad$ ~~$\delta n = \chi\ V_{ext} = \chi^0$~~

$$\delta n(\vec{k}, \omega) = \chi^{ret}_{nn}(\vec{k}, \omega) V_{ext}(\vec{k}, \omega) = \chi^{0,ret}_{nn}(\vec{k}, \omega) V_{eff}(\vec{k}, \omega)$$

$$\therefore\ \epsilon(\vec{k}, \omega) = \frac{V_{ext}(\vec{k}, \omega)}{V_{eff}(\vec{k}, \omega)} = \frac{\chi^{0,ret}_{nn}(\vec{k}, \omega)}{\chi^{ret}_{nn}(\vec{k}, \omega)}$$

$$= 1 - \frac{4\pi e^2}{k^2}\chi^{0,ret}_{nn}(\vec{k}, \omega).$$

$\vec{k} \to 0$, $\omega = 0$：$\quad \epsilon(\vec{k}, \omega = 0) = 1 + \frac{4\pi e^2}{k^2} N(0).$

$\therefore$ 对于 $V_{ext}(\vec{k}) = \frac{4\pi Q}{k^2}$

$$V_{eff}(\vec{k}) = \frac{1}{1 + \frac{4\pi e^2}{k^2}N(0)} \cdot \frac{4\pi Q}{k^2} = \frac{4\pi Q}{k^2 + 4\pi e^2 N(0)} = \frac{4\pi Q}{k^2 + k_{TF}^2}.$$

8) 极限情形：高频，集体激元.

② $\omega \to \infty$：$\quad \chi^{0,ret}_{nn}(\vec{q}, \omega) \approx 2\sum_k [f(\xi_k) - f(\xi_{k+q})] \cdot \left[\frac{1}{\omega} - \frac{\xi_k - \xi_{k+q}}{\omega^2} + \cdots\right]$

$\frac{1}{\omega}$-阶：$= \frac{2}{\omega}\sum_k [f(\xi_k) - f(\xi_{k+q})] = 0 \quad$ b/c $\sum_k f(\xi_k) = \sum_k f(\xi_{k+q})$.

$\frac{1}{\omega^2}$-阶：$= -\frac{2}{\omega^2}\sum_k [f(\xi_k) - f(\xi_{k+q})](\xi_k - \xi_{k+q})$. $\quad$ 换元：$\vec{k} \to \vec{k} - \vec{q}$.

$$= -\frac{2}{\omega^2}\left[\sum_k f(\xi_k)(\xi_k - \xi_{k+q}) - \sum_k f(\xi_k)(\xi_{k-q} - \xi_k)\right]$$

$$= -\frac{2}{\omega^2}\sum_k f(\xi_k)\frac{2k^2 - (\vec{k}+\vec{q})^2 - (\vec{k}-\vec{q})^2}{2m}$$

$$= \frac{2q^2}{m\omega^2}\sum_k f(\xi_k) = \frac{q^2}{m\omega^2} n_e.$$

$$\therefore\ \epsilon(\vec{k}, \omega \to \infty) = 1 - \frac{4\pi e^2}{\cancel{k^2}} \cdot \frac{\cancel{k^2}}{m\omega^2} n_e = 1 - \frac{4\pi e^2 n_e}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

w/ Plasma 频率 $\quad \omega_p^2 = \frac{4\pi e^2 n_e}{m}.$

① $\omega = \omega_p$: $\epsilon(\vec{k}, \omega_p) = 0$.

$$\chi^{ret}_{nn}(\vec{k}, \omega) = \frac{\chi^{0,ret}_{nn}(\vec{k}, \omega)}{\epsilon(\vec{k}, \omega)}$$

② $\omega = \omega_p$: $\epsilon(\vec{k}, \omega) = 0$ 是 $\chi^{ret}_{nn}(\vec{k}, \omega)$ 的极点.

格林函数极点 $\Rightarrow$ 激发模式.

例如: 自由费米子 $G(\vec{k}, \omega) = \frac{1}{\omega - \xi_k + i0^+}$.

谱表示: $G(\vec{k}, \omega) = \int d\omega' \frac{A(\vec{k}, \omega')}{\omega - \omega' + i0^+}$.

$A(\vec{k}, \omega') \neq 0 \Leftrightarrow G(\vec{k}, \omega)$ 为在 ω' 处有极点.

$\therefore$ $\epsilon(\vec{k}, \omega) = 0$ 对应集体激发模式 $\omega_k = \omega_p$.

无色散关系: $\omega_k = \omega_p$ 与 k 无关. 因为我们一直令光速 $c = \infty$.

考虑 $c < \infty$: $\omega_k^2 = c^2 k^2 + \omega_p^2$.

教材 9.2节.