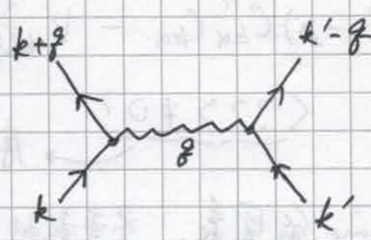
Lecture 9 超导. 参考: 教材 Chap. 12.

1) 有效相互作用. §12.2-3.

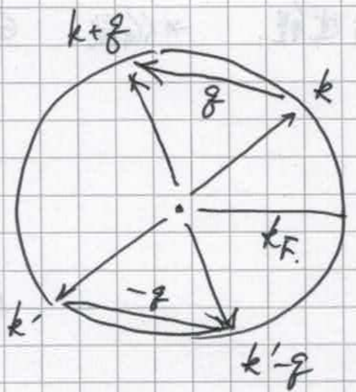
声子中介的电子-电子相互作用.

$$H = \frac{1}{2N}\sum_{kk'q} |M_q|^2 \frac{\omega_q}{(\epsilon_k-\epsilon_{k+q})^2-\omega_q^2}\, c^\dagger_{k+q}c^\dagger_{k'-q}c_{k'}c_k .$$



* 费米面上电子散射: (为什么能有效理论).

$\epsilon_k \approx \epsilon_{k'} \approx \epsilon_{k+q} \approx \epsilon_{k'-q} \approx \epsilon_F$; $\omega_q > 0$.



这样分母 $(\epsilon_k-\epsilon_{k+q})^2-\omega_q^2 < 0$

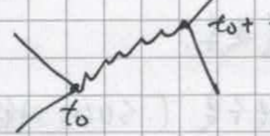
⇒ 吸引相互作用. $V<0$

* 频率依赖关系:

$$V = |M_q|^2 \frac{\omega_q}{\omega^2-\omega_q^2}.$$

w/ $\omega = \epsilon_k - \epsilon_{k+q} = \epsilon_{k'-q} - \epsilon_{k'}$: 能量交换.

$V=V(\omega)$ 相互作用不是在同一时刻发生



$V(t) = \int d\omega\, V(\omega)\, e^{-i\omega t}$

若 $V(\omega)$ 与 ω 无关 ⇒ $V(t) \propto \delta(t)$: 瞬时相互作用, 如 Coulomb $V = \frac{4\pi}{k^2}$.

$V(\omega)$ 与 ω 有关: 推迟相互作用.

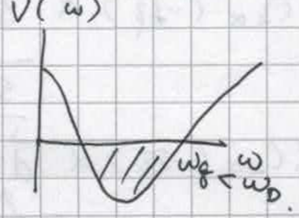
* 库仑 + 电声子:

$$V(\vec q,\omega) = \frac{V_c(\vec q)}{\epsilon(\vec q)} + \underbrace{|M_q|^2\frac{\omega_q}{\omega^2-\omega_q^2}}$$

$\epsilon(\vec q) > 0$: always 排斥.

$\omega < \omega_q$: 吸引

$\omega > \omega_q$: 排斥.



* 当 $\omega - \omega_q \to 0^-$ 时, $V(\vec q,\omega) < 0$.

* 重整化群:

$\frac{V_c(\vec q)}{\epsilon(\vec q)}$ 与 ω 无关: 被屏蔽.

虽然 bare 的相互作用 $V_c > |M_q|^2 \frac{\omega_q}{\omega^2-\omega_q^2}$

但屏蔽后 声子导致的相互作用更大.

* 声子能带: $\omega_q \le \omega_D$ ← Debye frequency.

简化假设: $\boxed{V = \begin{cases} -V_0, & \omega < \omega_D \\ 0, & \omega > \omega_D \end{cases}}$.

归一化因子: 这里 $V_0 \propto \frac{1}{N} \propto$ (系统体积)$^{-1}$

$H = -V \sum_{kk'q} c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k$.

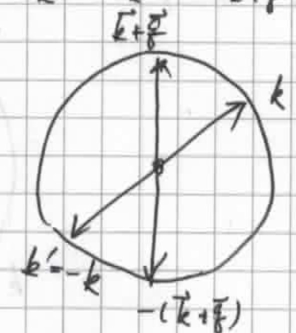
2) Cooper 对; 平均场序参量.

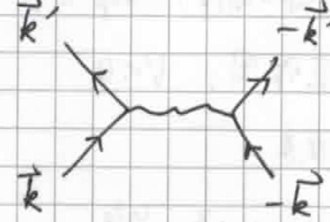
$$H = \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} - V\sum_{kk'q,\alpha\beta} c^\dagger_{k+q,\alpha}\, c^\dagger_{k'-q,\beta}\, c_{k'\beta}\, c_{k\alpha}.$$

平均场近似: $\langle ?? \rangle \neq 0$? $\longrightarrow$ 序参量选择.

Cooper pairing 的图象: 参考教材 §12.4

重要散射过程: *低能: $\epsilon_k \approx \epsilon_{k'} \approx \epsilon_{k+q} \approx \epsilon_{k'-q} \approx \epsilon_F$.





* $\langle c_{k\alpha}c_{-k\beta}\rangle \neq 0$ 或 $\langle c_{k\alpha}c_{k'\beta}\rangle \propto \delta_{k+k',0}$. Cooper 对总动量 $=0$.

FFLO态: $\langle c_{k\alpha}c_{k'\beta}\rangle \propto \delta_{k+k',\vec{Q}}$ 破缺平移不变性!

* 自旋依赖关系:

自旋旋转不变 (SU(2) 对称性) $\Rightarrow$ $\langle c_{k\alpha}c_{-k\beta}\rangle$ 自旋旋转不变.

$\frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1$.

① $\langle c_{k\alpha}c_{-k\beta}\rangle \propto \epsilon_{\alpha\beta}$: spin-singlet 配对

② $\langle c_{k\alpha}c_{-k\beta}\rangle \propto \vec{d}\cdot\vec{\sigma}_{\alpha\beta}$: spin-triplet 配对.

* 空间-依赖关系:

$\langle c_{k\alpha}c_{-k\beta}\rangle \propto \Delta(\vec{k})$. 整体为反对称.

① $\langle c_{k\alpha}c_{-k\beta}\rangle \propto \overset{\text{对称}}{\Delta(\vec{k})}\,\overset{\text{反对称}}{\epsilon_{\alpha\beta}}$: $\Delta(\vec{k}) = \Delta(-\vec{k})$

$\boxed{\text{s-wave}}$; d-wave; etc.

② $\langle c_{k\alpha}c_{-k\beta}\rangle \propto \underset{\text{反对称}}{\vec{d}(\vec{k})}\cdot\underset{\text{对称}}{\vec{\sigma}_{\alpha\beta}}$; $\vec{d}(\vec{k}) = -\vec{d}(-\vec{k})$.

s-wave: $\langle c_{k\alpha}c_{-k\beta}\rangle \propto \Delta_0\,\epsilon_{\alpha\beta}$

在费米面上打开均匀能隙, 能量最好

$\boxed{\text{平均场序参量: } \langle c_{k'\beta}c_{k\alpha}\rangle \propto \delta_{k'+k,0}\,\epsilon_{\beta\alpha}.}$

3) 平均场分解:

$$H = \underbrace{\sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha}}_{H_t} - \underbrace{V\sum_{\substack{kk'q\\ \alpha\beta}} c^\dagger_{k'-q,\beta}\,c^\dagger_{k+q,\alpha}\,c_{k\alpha}\,c_{k'\beta}}_{H_V}$$

$$H_V = -V\sum_{kk'q,\alpha\beta}\langle c^\dagger_{k'-q,\beta}\,c^\dagger_{k+q,\alpha}\rangle c_{k\alpha}\,c_{k'\beta} - V\sum_{kk'q,\alpha\beta} c^\dagger_{k'-q,\beta}\,c^\dagger_{k+q,\alpha}\langle c_{k\alpha}c_{k'\beta}\rangle$$

互为 h.c.

$$+\underbrace{V\sum_{kk'q,\alpha\beta}\langle c^\dagger_{k'-q,\beta}\,c^\dagger_{k+q,\alpha}\rangle\langle c_{k\alpha}c_{k'\beta}\rangle}$$

只影响基态能量，不影响激发谱.

$$= -V\sum_{kk'q,\alpha\beta} c^\dagger_{k'-q,\beta}\,c^\dagger_{k+q,\alpha}\,\langle c_{k\alpha}c_{-k\beta}\rangle\,\delta_{k',-k}\,\delta_{\beta,-\alpha} + h.c. + E_0$$

$$= -V\sum_{kk',\alpha}\langle c_{k\alpha}c_{-k-\alpha}\rangle\, c^\dagger_{-k',-\alpha}\,c^\dagger_{k'\alpha} + h.c. + E_0$$

$$= -V\sum_{kk'}\left(\langle c_{k\uparrow}c_{-k\downarrow}\rangle c^\dagger_{-k'\downarrow}c^\dagger_{k'\uparrow} + \langle c_{k\downarrow}c_{-k\uparrow}\rangle c^\dagger_{-k'\uparrow}c^\dagger_{k'\downarrow}\right) + h.c. + E_0.$$

这里 $\langle c_{k\uparrow}c_{-k\downarrow}\rangle = -\langle c_{k\downarrow}c_{-k\uparrow}\rangle$.

$$\therefore H_V = -V\sum_{kk'}\langle c_{k\uparrow}c_{-k\downarrow}\rangle\left(c^\dagger_{-k'\downarrow}c^\dagger_{k'\uparrow} - c^\dagger_{-k'\uparrow}c^\dagger_{k'\downarrow}\right) + h.c. + E_0.$$

令 $\Delta = -V\sum_k\left(\langle c_{k\uparrow}c_{-k\downarrow}\rangle - \langle c_{k\downarrow}c_{-k\uparrow}\rangle\right) = -V\sum_k\langle c_{k\alpha}c_{-k\beta}\rangle$

$$= -2V\sum_k\langle c_{k\uparrow}c_{-k\downarrow}\rangle.$$

$$H_V = \frac{\Delta}{2}\sum_{k'}\left(c^\dagger_{-k'\downarrow}c^\dagger_{k'\uparrow} - c^\dagger_{-k'\uparrow}c^\dagger_{k'\downarrow}\right) + h.c. + E_0$$

$$= \Delta\sum_k c^\dagger_{-k\downarrow}c^\dagger_{k\uparrow} + \Delta^*\sum_k c_{k\uparrow}c_{-k\downarrow} + E_0.$$

注: $-\sum_{k'} c^\dagger_{-k'\uparrow}c^\dagger_{k'\downarrow} = \sum_{k'} c^\dagger_{k'\downarrow}c^\dagger_{-k'\uparrow} = \sum_k c^\dagger_{-k\downarrow}c^\dagger_{k\uparrow}$ $(-k'\to k)$.

$$\therefore H_{MF} = \sum_{k\alpha}\xi_k c^\dagger_{k\alpha}c_{k\alpha} + \Delta\sum_k c^\dagger_{-k\downarrow}c^\dagger_{k\uparrow} + \Delta^*\sum_k c_{k\uparrow}c_{-k\downarrow} + E_0.$$

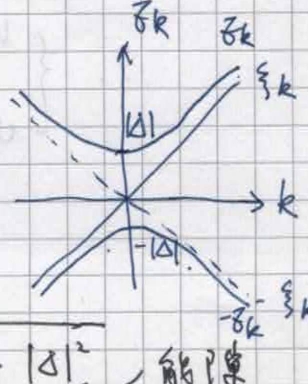
4) 能谱; Bogoliubov 变换.

$$H_{MF} = \sum_{k}\xi_k\left(c^\dagger_{k\uparrow}c_{k\uparrow} + c^\dagger_{k\downarrow}c_{k\downarrow}\right) + \Delta\sum_k c^\dagger_{-k\downarrow}c^\dagger_{k\uparrow} + \Delta^*\sum_k c_{k\uparrow}c_{-k\downarrow} + E_0$$

$$= \sum_k\left(c^\dagger_{k\uparrow},\ c_{-k\downarrow}\right)\begin{pmatrix}\xi_k & -\Delta\\ -\Delta^* & -\xi_k\end{pmatrix}\begin{pmatrix}c_{k\uparrow}\\ c^\dagger_{-k\downarrow}\end{pmatrix}.$$

注: $\xi_k = \xi_{-k}$ b/c $\epsilon_k = \epsilon_{-k}$.

对角化矩阵 $\begin{pmatrix}\xi_k & -\Delta\\ -\Delta^* & -\xi_k\end{pmatrix}$: Bogoliubov 变换.



本征值: $\det\begin{pmatrix}\xi_k-\lambda & -\Delta\\ -\Delta^* & -\xi_k-\lambda\end{pmatrix} = 0$

$\lambda^2 - \xi_k^2 - |\Delta|^2 = 0 \Rightarrow E_k = \pm\sqrt{\xi_k^2+|\Delta|^2}$ 能隙.

5) ~~※~~ Bogoliubov 变换.

$$\begin{pmatrix} \gamma_{k\uparrow} \\ \gamma_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^{+} \end{pmatrix}$$

不妨假设 $\Delta \in \mathbb{R}$; $\Delta = \Delta^*$ 故可以取正交矩阵.

若 Δ 为复数, 此处应为 $-v_k^*$

正交矩阵: $u_k^2 + v_k^2 = 1$. $u_k ; v_k \in \mathbb{R}$. $u_k = u_{-k}$; $v_k = v_{-k}$.

$$\begin{cases} \gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c_{-k\downarrow}^{+} \\ \gamma_{k\downarrow} = \cancel{v_k c_{k\downarrow} + u_k c_{-k\uparrow}^{+}}\ u_k c_{k\downarrow} + v_k c_{-k\uparrow}^{+} \end{cases}$$

$$\begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix}^{-1} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix}$$

$$\therefore \begin{cases} c_{k\uparrow} = u_k \gamma_{k\uparrow} + v_k \gamma_{-k\downarrow}^{+} \\ c_{k\downarrow} = u_k \gamma_{k\downarrow} - v_k \gamma_{-k\uparrow}^{+} \end{cases}$$

$$\cancel{\begin{pmatrix} \xi_k & -\Delta \\ -\Delta & \xi_k \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} \xi_k & -\Delta \\ -\Delta & \xi_k \end{pmatrix} \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix}}$$

令 $M_k = \begin{pmatrix} \xi_k & -\Delta \\ -\Delta & \xi_k \end{pmatrix}$; $U_k = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix}$. $C_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^{+} \end{pmatrix}$ $\gamma_k = \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma_{-k\downarrow}^{+} \end{pmatrix}$.

$$H = C_k^{+} M_k C_k$$

$$\gamma_k = U_k C_k \quad ; \quad C_k = U_k^{-1} \gamma_k = U_k^{+} \gamma_k$$

$$\therefore \quad H = \gamma_k^{+} U_k M_k U_k^{+} \gamma_k = (\gamma_{k\uparrow}^{+} \quad \gamma_{-k\downarrow}) \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma_{-k\downarrow}^{+} \end{pmatrix}.$$

$\therefore$ 我们希望 $M_k \begin{pmatrix} u_k \\ -v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ -v_k \end{pmatrix}$. ; $M_k \begin{pmatrix} v_k \\ u_k \end{pmatrix} = -E_k \begin{pmatrix} v_k \\ u_k \end{pmatrix}$.

$$\begin{pmatrix} \xi_k & -\Delta \\ -\Delta & \xi_k \end{pmatrix} \begin{pmatrix} u_k \\ -v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ -v_k \end{pmatrix}$$

$$\therefore \quad \xi_k u_k + \Delta v_k = E_k u_k. \Rightarrow \frac{u_k}{v_k} = \frac{\Delta}{E_k - \xi_k} = \frac{E_k + \xi_k}{\Delta} = \sqrt{\frac{E_k + \xi_k}{E_k - \xi_k}}.$$

$$\therefore \begin{cases} u_k = \sqrt{\dfrac{E_k + \xi_k}{2E_k}} \\ v_k = \sqrt{\dfrac{E_k - \xi_k}{2E_k}}. \end{cases}$$

6) 自洽方程.

$$\Delta = -V\sum_k \langle c_{k\uparrow} c_{-k\downarrow}\rangle$$

$$= -V\sum_k \langle (u_k\gamma_{k\uparrow} + v_k\gamma^\dagger_{-k\downarrow})(u_k\gamma_{-k\downarrow} - v_k\gamma^\dagger_{k\uparrow})\rangle.$$

$$= -V\sum_k u_k v_k\left(\langle\gamma^\dagger_{-k\downarrow}\gamma_{-k\downarrow}\rangle - \langle\gamma_{k\uparrow}\gamma^\dagger_{k\uparrow}\rangle\right). \quad (\langle\gamma\gamma\rangle = \langle\gamma^\dagger\gamma^\dagger\rangle = 0).$$

$$= -V\sum_k u_k v_k\left(\langle\gamma^\dagger_{-k\downarrow}\gamma_{-k\downarrow}\rangle + \langle\gamma^\dagger_{k\uparrow}\gamma_{k\uparrow}\rangle - 1\right)$$

$$= -V\sum_k u_k v_k\left(2n_F(E_k) - 1\right) \qquad w/\ n_F(\epsilon) = \frac{1}{e^{\beta\epsilon}+1}.$$

$$= V\sum_k \frac{\Delta}{2E_k}\left[1 - \frac{2}{e^{\beta E_k}+1}\right]$$

$$= V\sum_k \frac{\Delta}{2E_k}\,\frac{e^{\beta E_k}-1}{e^{\beta E_k}+1}.$$

两边消去Δ，得到自洽方程，

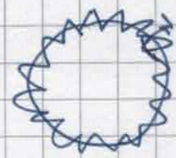
$$1 = V\sum_k \frac{1}{2E_k}\,\frac{e^{\beta E_k}-1}{e^{\beta E_k}+1} \qquad w/\ E_k = \sqrt{\xi_k^2+\Delta^2}.$$

7) T=0：零温下的能隙.

$T=0$: $n_F(E_k)=0$.

$$1 = V\sum_k \frac{1}{2E_k} = V\int d\epsilon\, N(\epsilon)\,\frac{1}{2\sqrt{\epsilon^2+\Delta^2}}$$

$$\approx \frac{VN(0)}{2}\int_{-\omega_D}^{\omega_D} d\epsilon\,\frac{1}{\sqrt{\epsilon^2+\Delta^2}} \qquad (\text{假定 } \omega_D \ll E_F).$$


ω_D.

$$= VN(0)\int_0^{\omega_D} d\epsilon\,\frac{1}{\sqrt{\epsilon^2+\Delta^2}} = N(0)V\sinh^{-1}\left(\frac{\omega_D}{\Delta}\right)$$

当 $\Delta \ll \omega_D$ 时：

$$1 \approx N(0)V\ln\frac{2\omega_D}{\Delta}$$

$$\Delta = 2\omega_D\, e^{-\frac{1}{N(0)V}}$$

能量尺度 $\propto \omega_D$

非微扰：Δ不能写成V的Taylor展开.
$N(0)V$：指数依赖关系.
$V \propto \frac{1}{\text{系统体积}}$；$N(0)V$是强度量.

8) $\Delta = 0$：计算 T_c.

ⓐ T_c： $\Delta \to 0$.

$$1 = \lim_{\Delta\to 0} V \sum_k \frac{1}{2\sqrt{\xi_k^2+\Delta^2}} \frac{e^{\beta E_k}-1}{e^{\beta E_k}+1}$$

$$= V \sum_k \frac{1}{2|\xi_k|} \frac{e^{\beta|\xi_k|}-1}{e^{\beta|\xi_k|}+1}$$

$$\approx N(0) V \int_0^{\omega_D} d\epsilon \frac{1}{\epsilon} \frac{e^{\beta\epsilon}-1}{e^{\beta\epsilon}+1}.$$

$$\approx N(0) V \int_{k_BT}^{\omega_D} \frac{d\epsilon}{\epsilon} = N(0) V \ln \frac{\omega_D}{k_BT}.$$

$$\therefore \quad k_BT = \omega_D e^{-\frac{1}{N(0)V}}$$

完整计算： $k_BT = 1.14\, \omega_D e^{-\frac{1}{VN(0)}}$.

比较 $\Delta(T=0) = 2\omega_D e^{-\frac{1}{VN(0)}}$

$$\Rightarrow \quad \frac{2\Delta}{T} \simeq 3.5 \qquad (\text{BCS ratio}).$$