Lecture 3. 格林函数和线性响应.

Motivation: 格林函数是多体理论的主要描述工具. (可类比于统计力学中的关联函数).

一次量子化: 波函数.

二次量子化: 格林函数.

优点: 规范不变(与基底选择无关); ~~对于~~

直接与实验可观测量相关, 是沟通理论和实验的工具.

Outline: 1. 线性响应理论.

2. 涨落-耗散定理

3. K-K关系; 谱表示; 谱函数.

$\left.\begin{matrix}1\\2\\3\end{matrix}\right\}$ 玻色格林函数.

4. 自由电子格林函数; 编时格林函数; ~~Wick定理计算~~ 与之~~格林函数~~.

5. 用Wick定理计算格林函数.

$\left.\begin{matrix}4\\5\end{matrix}\right\} \to$ 费米子格林函数

1) 线性响应理论与格林函数的定义. 教材 §9.3.

外界微扰: $\hat{H} = \hat{H}_0 - \hat{A}\, h(t)$.

这里假设外界微扰是 $-\hat{A}\, h(t)$ 的特定形式. 实用中这种形式一般够用了.

响应: $y(t) = \langle \hat{B} \rangle (t) = \mathrm{tr}\,[\hat{\rho}(t) \hat{B}]$ w/ $\hat{\rho}(t)$ 密度矩阵.

密度矩阵, 描述一个热态(混合态). 平衡态 $\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}_0}$

$\hat{\rho}(t)$ 满足 Heisenberg 方程:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] + \frac{i}{\hbar}[\hat{A}, \hat{\rho}]\, h(t)$$

相互作用表象. $\hat{O}_I(t) = e^{i\hat{H}_0 t/\hbar}\, \hat{O}\, e^{-i\hat{H}_0 t/\hbar}$

$$\frac{d}{dt}\hat{O}_I(t) = \frac{i}{\hbar}\hat{H}_0\, \hat{O}_I(t) + e^{i\hat{H}_0 t/\hbar} \frac{d\hat{O}(t)}{dt} e^{-i\hat{H}_0 t/\hbar} - \frac{i}{\hbar}\hat{O}_I \hat{H}$$

$$= \frac{i}{\hbar}[\hat{H}_0, \hat{O}_I] + e^{i\hat{H}_0 t/\hbar} \frac{d\hat{O}(t)}{dt} e^{-i\hat{H}_0 t/\hbar}$$

$$\therefore \frac{d}{dt}\hat{\rho}_I = \cancel{\frac{i}{\hbar}[\hat{H}_0, \hat{O}_I]} - \cancel{\frac{i}{\hbar}[\hat{H}_0, \hat{O}_I]} + \frac{i}{\hbar} h(t) [\hat{A}_I, \hat{\rho}_I]$$

假设 $h(t \to -\infty) = 0$. $\Rightarrow \hat{\rho}(t \to -\infty) = \rho_0 = \frac{1}{Z} e^{-\beta \hat{H}_0}$

$$\hat{\rho}_I(t) \approx \hat{\rho}_0 + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, h(t') [\hat{A}_I, \hat{\rho}_0] + \underbrace{\cdots}_{h(t)\text{的高阶项}}$$

线性响应: $h(t) \ll 1$. 只保留线性项.

$$y(t) = \mathrm{tr}\,[\hat{\rho}(t) \hat{B}] = \mathrm{tr}\left\{\left(\hat{\rho}_0 + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, h(t') [\hat{A}_I(t'), \hat{\rho}_0]\right) \hat{B}_I(t)\right\}$$

$$= \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, h(t')\, \mathrm{tr}\left\{\hat{\rho}_0 [\hat{B}_I(t), \hat{A}_I(t')]\right\}$$

(假设 $\langle B \rangle_0 = \mathrm{tr}(\hat{\rho}_0 B) = 0$)

定义线性响应函数：

$$\chi^{ret}_{BA}(t,t') = \frac{i}{\hbar}\, tr\{\hat{\rho}_0 [\hat{B}(t), \hat{A}(t')]\}\, \theta(t-t')$$

$$= \frac{i}{\hbar}\langle[\hat{B}(t), \hat{A}(t')]\rangle\, \theta(t-t').$$

* $\theta(t-t')$：因果律. $h(t') \longrightarrow y(t)$ 只有 $t'<t$ 时 $h(t')$ 才会影响 $y(t)$.

* $H_1 = -h(t)\hat{A}$ 只是假想的微扰：$\langle[\hat{B}(t), \hat{A}(t')]\rangle$ 是对 $\hat{H}=\hat{H}_0$ 计算的

因而 $\hat{A}_I(t)$ 就是 Heisenberg 表象 $\hat{A}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{A}e^{-\frac{i}{\hbar}\hat{H}t}$

ret：推迟（retarded）格林函数（Green's Function）.

2) 涨落-耗散定理.

定义 关联函数. $S_{BA}(t,t') = \langle\hat{B}(t)\hat{A}(t')\rangle - \langle\hat{B}(t)\rangle\langle\hat{A}(t')\rangle$

为简单起见，假设 $\langle\hat{B}(t)\rangle = \langle\hat{A}(t)\rangle = 0$.

（若不然，可定义 $\delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$）. $S_{BA}(t,t') = S_{BA}(t-t')$.

Fourier 变换：$S_{BA}(\omega) = \int_{-\infty}^{\infty} dt\, S_{BA}(t)\, e^{i\omega t}$.

$$\chi^{ret}_{BA}(\omega) = \int_{-\infty}^{\infty} dt\, \chi^{ret}_{BA}(t)\, e^{i\omega t} = \int_{0}^{\infty} dt\, \chi^{ret}_{BA}(t)\, e^{i\omega t}.$$

定理. 涨落-耗散定理.

$$\mathrm{Im}\chi^{ret}_{BA}(\omega) = \frac{1-e^{-\beta\hbar\omega}}{2\hbar} S_{BA}(\omega).$$

证明. 注意到 $\chi^{ret}_{BA}(t)$ 是实数

$$\therefore \mathrm{Im}\chi^{ret}_{BA}(\omega) = \frac{1}{2i}\left(\chi^{ret}_{BA}(\omega) - (\chi^{ret}_{BA}(\omega))^*\right)$$

$$= \frac{1}{2i}\left(\int_0^{\infty} dt\, \chi^{ret}_{BA}(t) e^{i\omega t} - \int_0^{\infty} dt\, \chi^{ret}_{BA}(t) e^{-i\omega t}\right)$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} dt\, \frac{i}{\hbar}\langle[B(t), A(0)]\rangle e^{i\omega t}$$

$$= \frac{1}{2\hbar}\int_{-\infty}^{\infty} dt\, \left(\langle B(t)A(0)\rangle - \langle A(0)B(t)\rangle\right) e^{i\omega t}$$

$$= \frac{1}{2\hbar}\int_{-\infty}^{\infty} dt\, \langle B(t)A(0)\rangle e^{i\omega t}$$

$$- \frac{1}{2\hbar}\int_{-\infty}^{\infty} dt\, \langle A(0)B(t)\rangle e^{i\omega t}$$

第一项 $= \frac{1}{2\hbar} S_{BA}(\omega)$.

第二项中 $\int_{-\infty}^{\infty} dt\, \langle A(0)B(t)\rangle e^{i\omega t}$

$$= \int_{-\infty}^{\infty} dt\, tr[e^{-\beta\hat{H}}\hat{A}e^{\frac{i}{\hbar}\hat{H}t}\hat{B}e^{-\frac{i}{\hbar}\hat{H}t}]e^{i\omega t}$$

$$\cancel{= \int_{-\infty}^{\infty} dt\, tr[e^{-\frac{i}{\hbar}\hat{H}t}e^{\beta\hat{H}}\hat{A}e^{\frac{i}{\hbar}\hat{H}t}e^{\beta\hat{H}}e^{-\beta\hat{H}}\hat{B}]e^{i\omega t}}$$

$$\cancel{= \int_{-\infty}^{\infty} dt\, tr[e^{-\beta\hat{H}}\hat{B}e^{\frac{i}{\hbar}\hat{H}(-t+i\beta\hbar)}\hat{A}e^{-\frac{i}{\hbar}\hat{H}(-t+i\beta\hbar)}]e^{i\omega t}}$$

$$= \int_{-\infty}^{\infty} dt \, \mathrm{tr}\left[\hat{A} e^{-\beta\hat{H}} e^{\frac{i}{\hbar}\hat{H}t + \beta\hat{H}} \hat{B} e^{-\frac{i}{\hbar}\hat{H}t - \beta\hat{H}}\right] e^{i\omega t}$$

$$= \int_{-\infty}^{\infty} dt \, \mathrm{tr}\left[e^{-\beta\hat{H}} e^{\frac{i}{\hbar}\hat{H}(t - i\beta\hbar)} \hat{B} e^{-\frac{i}{\hbar}\hat{H}(t - i\beta\hbar)}\right] e^{i\omega t}$$

令 $\tau = t - i\beta\hbar$.

$$= \int_{-\infty - i\beta\hbar}^{\infty - i\beta\hbar} d\tau \, \mathrm{tr}\left[e^{-\beta\hat{H}} e^{\frac{i}{\hbar}\hat{H}\tau} \hat{B} e^{-\frac{i}{\hbar}\hat{H}\tau}\right] e^{i\omega\tau} e^{-\beta\hbar\omega}$$

$$= \int_{-\infty - i\beta\hbar}^{\infty - i\beta\hbar} dt \, \langle \hat{B}(t) \hat{A}(0) \rangle e^{i\omega t} e^{-\beta\hbar\omega}$$

$$= e^{-\beta\hbar\omega} S_{BA}(\omega).$$ (后面会看到, 被积函数为解析函数, 可以改变积分围道

$$\therefore \mathrm{Im}\, \chi_{BA}^{ret}(\omega) = \frac{1}{2\hbar}\left(1 - e^{-\beta\hbar\omega}\right) S_{BA}(\omega).$$

~~3) Kramers-Kronig 关系(略去)~~.

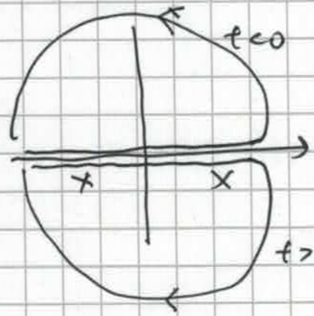
3. $\chi'(\omega)=?$ Kramers - Kronig relations. §15.6A. 4

$\chi(t)$ 中因果律的体现: $\chi(t)=0$, if $t<0$.

* Claim. $\chi(t)\propto\theta(t) \Rightarrow \chi(\omega)$ 在上半平面解析, $\chi(\omega)\lesssim\frac{1}{|\omega|}$.

证明. 考虑围道积分

$$\chi(t)=\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\,\chi(\omega).$$



$t<0$: 从上方封闭围道:

$\omega=+i\infty$ $\quad -i\omega t\to-\infty$, 积分收敛.

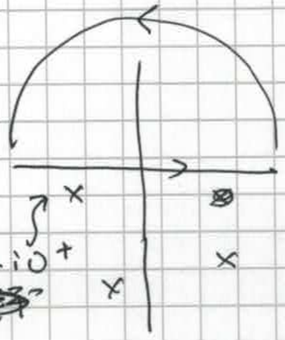
上半平面无奇点 $\Rightarrow \oint_C d\omega\, e^{-i\omega t}\chi(\omega)=0$, for $t<0$.

* K-K 关系:

考虑 $\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega+i0^+}d\omega'=0$.

奇点: 分子 $\chi(\omega')$ 在上半平面无奇点.

分母奇点为 $\omega'=\omega-i0^+$ 在下半平面.



故从上半平面封闭围道 $\Rightarrow \oint=0$.

$$0=\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega+i0^+}d\omega'$$

$$=\int_{-\infty}^{\infty}d\omega'\,\chi(\omega')\left[\mathcal{P}\frac{1}{\omega'-\omega}-i\pi\delta(\omega'-\omega)\right]$$

考虑 $\epsilon\to0^+$.

$$\frac{1}{\omega'-\omega+i\epsilon}=\frac{\omega'-\omega-i\epsilon}{(\omega'-\omega)^2+\epsilon^2}=\frac{\omega'-\omega}{(\omega'-\omega)^2+\epsilon^2}-i\frac{\epsilon}{(\omega'-\omega)^2+\epsilon^2}$$

$$\to\mathcal{P}\frac{1}{\omega'-\omega}-i\pi\delta(\omega'-\omega).$$

$$=\int_{-\infty}^{\infty}d\omega'\left[\chi'(\omega')+i\chi''(\omega')\right]\left[\mathcal{P}\frac{1}{\omega'-\omega}-i\pi\delta(\omega'-\omega)\right].$$

实部: $0=\int_{-\infty}^{\infty}d\omega'\,\chi'(\omega')\,\mathcal{P}\frac{1}{\omega'-\omega}+\pi\int_{-\infty}^{\infty}\chi''(\omega')\,\delta(\omega'-\omega)$

$$\Rightarrow\chi''(\omega)=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}d\omega'\,\frac{\chi'(\omega')}{\omega'-\omega}\,d\omega'.$$

虚部: $0=i\int_{-\infty}^{\infty}d\omega'\,\chi''(\omega')\,\mathcal{P}\frac{1}{\omega'-\omega}-i\pi\int_{-\infty}^{\infty}d\omega'\,\chi'(\omega')\,\delta(\omega'-\omega)$

$$\Rightarrow\chi'(\omega)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}d\omega'\,\frac{\chi''(\omega')}{\omega'-\omega}\,d\omega'.$$

(4) 谱表示.

假设我们知道 $\hat{H}$ 的一组本征态:

$$\hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle.$$

(令 $\hbar=1$):

$$\langle[B(t),A(0)]\rangle = tr\left\{\frac{1}{Z}e^{-\beta\hat{H}}[B(t),A(0)]\right\}$$

$$= \frac{1}{Z}\sum_\alpha e^{-\beta E_\alpha}\langle\alpha|B(t)A(0)-A(0)B(t)|\alpha\rangle$$

$$= \frac{1}{Z}\sum_\alpha e^{-\beta E_\alpha}\{\langle\alpha|e^{iE_\alpha t}Be^{-iHt}A|\alpha\rangle - \langle\alpha|Ae^{iHt}Be^{-iE_\alpha t}|\alpha\rangle\}$$

$$= \frac{1}{Z}\sum_{\alpha\beta}e^{-\beta E_\alpha}\{\langle\alpha|e^{iE_\alpha t}Be^{-iE_\beta t}|\beta\rangle\langle\beta|A|\alpha\rangle - \langle\alpha|A|\beta\rangle\langle\beta|e^{iE_\beta t}Be^{-iE_\alpha t}|\alpha\rangle\}$$

$$= \frac{1}{Z}\sum_{\alpha\beta}e^{-\beta E_\alpha}\{e^{i(E_\alpha-E_\beta)t}\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle - e^{i(E_\beta-E_\alpha)t}\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle\}$$

$$= \frac{1}{Z}\sum_{\alpha\beta}(e^{-\beta E_\alpha}-e^{-\beta E_\beta})\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle e^{i(E_\alpha-E_\beta)t}$$

$$\therefore \operatorname{Im}\chi^{ret}_{BA}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{\hbar}\frac{1}{Z}\sum_{\alpha\beta}(e^{-\beta E_\alpha}-e^{-\beta E_\beta})\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle e^{i\frac{(E_\alpha-E_\beta)}{\hbar}t}e^{i\omega t}dt$$

$$= \frac{\pi}{\hbar}\frac{1}{Z}\sum_{\alpha\beta}(e^{-\beta E_\alpha}-e^{-\beta E_\beta})\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle\delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right)$$

$$\delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right) \Rightarrow \hbar\omega = E_\beta-E_\alpha. \quad E_\alpha = E_\beta+\hbar\omega.$$

$$\therefore e^{-\beta E_\alpha}-e^{-\beta E_\beta} = e^{-\beta E_\alpha}(1-e^{-\beta\hbar\omega}).$$

$$\operatorname{Im}\chi^{ret}_{BA}(\omega) = \frac{\pi(1-e^{-\beta\hbar\omega})}{\hbar}\frac{1}{Z}\sum_{\alpha\beta}e^{-\beta E_\alpha}\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle\delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right).$$

类似地, $S_{AB}(\omega) = \int_{-\infty}^{\infty}dt\,\langle A(t)B(0)\rangle e^{i\omega t}$

$$= \int_{-\infty}^{\infty}dt\,tr\left\{\frac{1}{Z}e^{-\beta H}e^{iHt/\hbar}Ae^{-iHt/\hbar}B\right\}e^{i\omega t}$$

$$= \frac{1}{Z}\int_{-\infty}^{\infty}dt\sum_{\alpha\beta}\langle\alpha|e^{-\beta H}e^{iHt/\hbar}Ae^{-iHt/\hbar}|\beta\rangle\langle\beta|B|\alpha\rangle e^{i\omega t}$$

$$= \frac{1}{Z}\int_{-\infty}^{\infty}dt\sum_{\alpha\beta}\langle\alpha|e^{-\beta E_\alpha}e^{iE_\alpha t/\hbar}Ae^{-iE_\beta t/\hbar}|\beta\rangle\langle\beta|B|\alpha\rangle e^{i\omega t}$$

$$= \frac{1}{Z}\sum_{\alpha\beta}\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle e^{-\beta E_\alpha}\int_{-\infty}^{\infty}dt\,e^{i\frac{E_\alpha-E_\beta}{\hbar}t}e^{i\omega t}$$

$$= \frac{2\pi}{Z}\sum_{\alpha\beta}e^{-\beta E_\alpha}\langle\alpha|A|\beta\rangle\langle\beta|B|\alpha\rangle\delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right).$$

$$\therefore \operatorname{Im}\chi^{ret}_{BA}(\omega) = \frac{1-e^{-\beta\hbar\omega}}{2\hbar}S_{BA}(\omega).$$

5) 定义谱函数: $A_{BA}(\omega) = \frac{1}{\pi}\operatorname{Im}\chi^{ret}_{BA}(\omega)$

$$= \frac{1-e^{-\beta\hbar\omega}}{\hbar}\frac{1}{Z}\sum_{\alpha\beta} e^{-\beta E_\alpha}\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle \delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right).$$

$$= (1-e^{-\beta\hbar\omega})\frac{1}{Z}\sum_{\alpha\beta} e^{-\beta E_\alpha}\langle\alpha|B|\beta\rangle\langle\beta|A|\alpha\rangle\delta(\hbar\omega+E_\alpha-E_\beta)$$

$$\operatorname{Im}\chi^{ret}_{BA}(\omega) = \pi A_{BA}(\omega) = \int_{-\infty}^{\infty} d\omega' A_{BA}(\omega')\pi\delta(\omega'-\omega).$$

$$\operatorname{Re}\chi^{ret}_{BA}(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\operatorname{Im}\chi^{ret}_{BA}(\omega')}{\omega'-\omega}\, d\omega'.$$

$$= \mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{A_{BA}(\omega')}{\omega'-\omega}\, d\omega'.$$

$$\therefore\ \chi^{ret}_{BA}(\omega) = \int_{-\infty}^{\infty} d\omega' A_{BA}(\omega')\left\{\mathcal{P}\frac{1}{\omega'-\omega} + i\pi\delta(\omega'-\omega)\right\}.$$

$$= \int_{-\infty}^{\infty} d\omega' \frac{A_{BA}(\omega')}{\omega'-\omega-i0^+}$$

奇报点 ⓐ $\omega=\omega'-i0^+$ 确实在下半平面 ☺.

7) 费米子(电子)格林函数.

类比玻色子格林函数, 我们定义:

$$G^R_{\sigma\sigma'}(\vec r,t;\vec r',t') = -\frac{i}{\hbar}\theta(t-t')\langle\{\psi_\sigma(\vec r,t),\psi^\dagger_{\sigma'}(\vec r',t')\}\rangle.$$

对于自旋旋转对称的体系, 只有 $G^R_{\sigma\sigma}\neq 0$, 可以省去 σ 的角标.

$$G^R(\vec r,t;\vec r',t') = G^R(\vec r-\vec r',t-t').$$

动量空间: $G^R(\vec k,\omega) = \int d\vec r\int_{-\infty}^{\infty} dt\, G^R(\vec r,t)e^{i\omega t-i\vec k\cdot\vec r}$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\int d\vec r\left[-\frac{i}{\hbar}\theta(t-t')\right]\langle\{\psi(\vec r,t),\psi^\dagger(0,0)\}\rangle e^{-i\vec k\cdot\vec r}$$

$$= \frac{1}{V}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\int d\vec r\, d\vec r'\left[-\frac{i}{\hbar}\theta(t-t')\right]\langle\{\psi(\vec r,t),\psi^\dagger(\vec r',0)\}\rangle e^{-i\vec k\cdot(\vec r-\vec r')}$$

$$= -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \overset{\theta(t-t')}{\cdot}\, e^{i\omega t}\left\langle\left\{\frac{1}{\sqrt V}\int d\vec r\,\psi(\vec r,t)e^{-i\vec k\cdot\vec r},\ \frac{1}{\sqrt V}\int d\vec r'\,\psi^\dagger(\vec r',0)e^{i\vec k\cdot\vec r'}\right\}\right\rangle$$

$$= -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \overset{\theta(t-t')}{\cdot}\, e^{i\omega t}\langle\{C_k(t), C^\dagger_k(0)\}\rangle.$$

定义谱函数 $A(\vec k,\omega) = -\frac{1}{\pi}\operatorname{Im}G^R(\vec k,\omega)$.

可以推导类似玻色子格林函数的关系: $G^R(\vec k,\omega) = \int d\omega' \frac{A(\vec k,\omega')}{\omega-\omega'+i0^+}$.

~~证明方式: 谱表示~~ 原因: $G^R(\vec k,t) \propto \theta(t-t')$ 满足 K-K 关系

8) 费米子的谱展开*:

考虑多体 $\hat{H}$ 的本征态. $\hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle$.

$$A(\vec{k},\omega) = -\frac{1}{\pi}\mathrm{Im}\,G^R(\vec{k},\omega)$$

$$= -\frac{1}{\pi}\frac{1}{2\pi}\left(-\frac{i}{\hbar}\right)\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle\{c_k(t), c_k^\dagger(0)\}\rangle$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\frac{1}{Z}\sum_{\alpha\beta}e^{-\beta E_\alpha}\Big[\langle\alpha|e^{iHt/\hbar}c_k e^{-iHt/\hbar}|\beta\rangle\langle\beta|c_k^\dagger|\alpha\rangle$$
$$+ \langle\alpha|c_k^\dagger|\beta\rangle\langle\beta|e^{iHt/\hbar}c_k e^{-iHt/\hbar}|\alpha\rangle\Big]$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\frac{1}{Z}\sum_{\alpha\beta}\left(e^{-\beta E_\alpha}+e^{-\beta E_\beta}\right)e^{i(E_\alpha-E_\beta)t/\hbar}$$
$$\langle\alpha|c_k|\beta\rangle\langle\beta|c_k^\dagger|\alpha\rangle$$

$$= \frac{1}{\hbar}\frac{1}{Z}\sum_{\alpha\beta}\left(e^{-\beta E_\alpha}\underset{\uparrow}{+}e^{-\beta E_\alpha}\right)\langle\alpha|c_k|\beta\rangle\langle\beta|c_k^\dagger|\alpha\rangle\,\delta\left(\omega+\frac{E_\alpha-E_\beta}{\hbar}\right)$$

与玻色子的区别.

9) 自由费米子的格林函数

自由费米子 $\hat{H} = \sum_k \xi_k c_k^\dagger c_k$, $\xi_k = \epsilon_k - \mu$. 令 $\hbar = 1$.

$$A(\vec{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle\{c_k(t), c_k^\dagger(0)\}\rangle$$

* $c_k(t) = e^{-i\xi_k t}c_k$ 类似 $|k(t)\rangle = e^{i\xi_k t}|k\rangle$

证明. $c_k(t) = e^{iHt}c_k e^{-iHt}$

考虑 $[e^{iHt}, c_k] = [e^{i\xi_k c_k^\dagger c_k t}, c_k] = [e^{i\xi_k n_k t}, c_k]$

$= [\sum_{p=0}^{\infty}\frac{1}{p!}(i\xi_k t n_k)^p, c_k]$

费米子: $n_k^p = n_k^2 = \dots = n_k$. $[n_k^p, c_k] = 0$; $[n_k, c_k] = -c_k$

∴ $[e^{iHt}, c_k] = \sum_{p=1}^{\infty}\frac{1}{p!}(i\xi_k t)^p[n_k, c_k] = -c_k\sum_{p=1}^{\infty}\frac{1}{p!}(i\xi_k t)^p$

$= -c_k(e^{i\xi_k t} - 1)$

证明. 本征态 $|\{n_k\}\rangle$ $\quad \hat{H}|\{n_k\}\rangle = \sum_k n_k\xi_k$

$$c_k|\{n_k\}\rangle = \begin{cases}|\dots, 0, \dots\rangle, & n_k = 1\\ 0, & n_k = 0.\end{cases}$$

$$\therefore\ e^{iHt}c_k e^{-iHt}|\{n_k\}\rangle = e^{iHt}c_k e^{-i\sum n_k\xi_k t}|\{n_k\}\rangle$$
$$= c_k e^{-i\xi_k t}|\{n_k\}\rangle$$

$$\therefore\ A(\vec{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{i\omega t}e^{-i\xi_k t}\overbrace{\langle\{c_k, c_k^\dagger\}\rangle}^{=1}$$
$$= \delta(\omega - \xi_k)$$

$$\therefore\ G(\vec{k},\omega) = \int d\omega'\frac{\delta(\omega'-\xi_k)}{\omega-\omega'+i0^+} = \frac{1}{\omega-\xi_k+i0^+}.$$