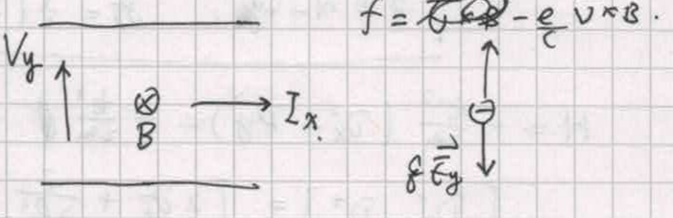
Lecture 12 量子霍尔效应

参考: 教材 Chap. 15.

E. Fradkin Field Theory of Condensed Matter Physics §13.1

1) 经典霍尔效应



* 电导的定义: $\vec J=\sigma\vec E$

Hall 电导: $j_x=\sigma_{xy}E_y=\sigma_H E_y$.

* Lorentz 力: $\vec f = -\frac{e}{c}\vec v\times\vec B = -(-e\vec E_y)$

$\therefore \quad v_x = -E_y c/B_z$.

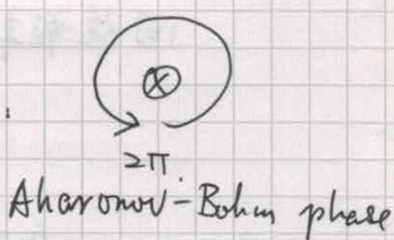
$j_x = -e\, n_e v_x = n_e \frac{ec}{B_z}E_y$

$\sigma_H = \frac{ec}{B} n_e$

n_e/B 与电子密度的关系:

定义 $\nu = $ 电子数/磁通量子数.

$\Phi = \frac{hc}{e}$.

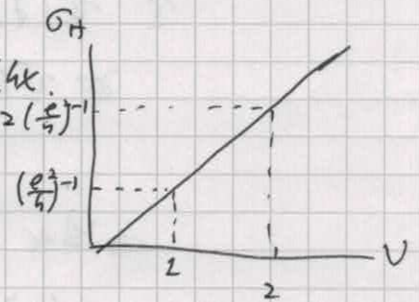


$= \frac{n_e\cdot A}{(B\cdot A/\frac{hc}{e})}$

$=\frac{n_e}{B}\cdot\frac{hc}{e}$

$\therefore \quad \sigma_H = \frac{h}{e^2}\nu = \frac{\nu}{\rho_0} \qquad \rho_0 = \frac{e^2}{h}$.

* $\sigma_H \propto \nu$: 在经典体系中, ν 连续变化 $\Rightarrow$ σ_H 连续变化.



2) 朗道能级 Landau Levels.

* 选择 symmetric gauge $\quad \vec B=\nabla\times\vec A \quad B_z=\partial_x A_y-\partial_y A_x$

取 $A_x=-\frac12 By$; $A_y=\frac12 Bx$. $\Rightarrow$ $B_z=B$.

* $H=\frac{1}{2m}\left(-i\hbar\nabla-\frac{e}{c}\vec A\right)^2 = -\frac{\hbar^2}{2m}\left(\nabla-\frac{ie}{\hbar c}\vec A\right)^2$.

$D_x = \partial_x - \frac{ie}{\hbar c}A_x = \partial_x - \frac{ie}{\hbar c}\left(-\frac12 By\right) = \partial_x+\frac12\frac{eB}{\hbar c}y = \partial_x + \frac{iy}{2l_0^2}$.

l_0: magnetic length $\quad \Phi_0=\frac{hc}{e}\sim B\pi l_0^2$. $\quad l_0=\sqrt{\frac{\hbar c}{eB}}$.

$D_y = \partial_y - \frac{ie}{\hbar c}A_y = \partial_y - \frac{ie}{\hbar c}\left(\frac12 Bx\right) = \partial_y - \frac{ix}{2l_0^2}$.

构造 $D^- = D_x + iD_y = \partial_x + i\partial_y + \frac{x+iy}{2l_0^2} = 2\partial_{\bar{z}} + \frac{z}{2l_0^2}$.

$D^+ = D_x - iD_y = \partial_x - i\partial_y - \frac{x-iy}{2l_0^2} = 2\partial_z - \frac{\bar{z}}{2l_0^2}$.

$$\boxed{\begin{array}{ll} z = x+iy, & \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \bar{z} = x-iy. & \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{array}}$$

$$H = -\frac{\hbar^2}{2m}(D_x^2 + D_y^2) = -\frac{\hbar^2}{2m}\cdot\frac{1}{2}(D^+D^- + D^-D^+).$$

$$[D^-, D^+] = [2\partial_{\bar{z}} + \frac{z}{2l_0^2}, 2\partial_z - \frac{\bar{z}}{2l_0^2}] = -\frac{2}{l_0^2}$$

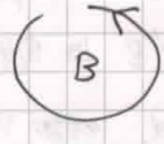
令 $a = i\frac{l_0}{\sqrt{2}}D^-$, $a^+ = i\frac{l_0}{\sqrt{2}}D^+$

$\therefore [a, a^+] = 1$.

$$H = \frac{\hbar^2}{ml_0^2}(a^+a + \frac{1}{2}). = \hbar\omega_c(a^+a + \frac{1}{2}).$$

* Landau 能级: $\hat{n} = \hat{a}^+\hat{a}$.

回旋频率 cyclotron frequency



$$m\omega_c^2 r = m\omega_c v = \frac{evB}{c}.$$

$$\therefore \omega_c = \frac{eB}{mc}.$$

*最低 Landau 能级: Lowest Landau Level (LLL).

$a|\psi\rangle = 0$, or $D^-|\psi\rangle = 0$.

$$(2\partial_{\bar{z}} + \frac{z}{2l_0^2})\Psi(z,\bar{z}) = 0.$$

令 $\Psi(z,\bar{z}) = f(z,\bar{z})\,e^{-|z|^2/4l_0^2}$.

$$\partial_{\bar{z}}\Psi(z,\bar{z}) = (\partial_{\bar{z}}f)e^{-|z|^2/4l_0^2} + f(-\frac{z}{4l_0^2})e^{-|z|^2/4l_0^2}$$

$$= (\partial_{\bar{z}}f)e^{-|z|^2/4l_0^2} - \frac{z}{4l_0^2}\Psi$$

因此 $2\partial_{\bar{z}}\Psi = (\partial_{\bar{z}}f)e^{-|z|^2/4l_0^2} - \frac{z}{2l_0^2}\Psi$

$$(2\partial_{\bar{z}} + \frac{z}{2l_0^2})\Psi = 0 \Rightarrow \partial_{\bar{z}}f(z,\bar{z}) = 0$$

$\therefore f(z,\bar{z}) = f(z)$ 为全纯函数.

$$\text{LLL} \Leftrightarrow \Psi(z,\bar{z}) = f(z)\,e^{-|z|^2/4l_0^2}.$$

3) Landau 能级简并度.

* $\Psi(z,\bar{z}) = f(z)\, e^{-|z|^2/4l_0^2}$

估计: Gaussian 波包, 大小 $\sim l_0 = \sqrt{\frac{\hbar c}{eB}}$.

# of states in LL $\sim A/\pi l_0^2 \sim A\cdot\frac{eB}{hc} = \frac{e\Phi}{hc} = \Phi/\Phi_0$.

$\therefore$ LL 填充数 $= \frac{n_e A}{\Phi/\Phi_0} = \frac{n_e}{B}\frac{hc}{e} = \nu$

$\nu=1$: 第一 Landau 能级填满.

* 稍微严格一点数态:

Basis: $f(z) = z^n$. (全纯函数 $f(z) = c_0 z^0 + c_1 z^1 + c_2 z^2 + \cdots$).

$$|\Psi|^2 = |z|^{2n} e^{-|z|^2/2l_0^2} = r^{2n} e^{-r^2/2l_0^2}$$

r_{max}: $0 = \frac{\partial|\Psi|^2}{\partial r} = \left(2n r^{2n-1} - r^{2n}\frac{r}{l_0^2}\right) e^{-r^2/2l_0^2}$

$$2n - r^2/l_0^2 = 0$$

$$r_{max} = \sqrt{2n}\, l_0.$$

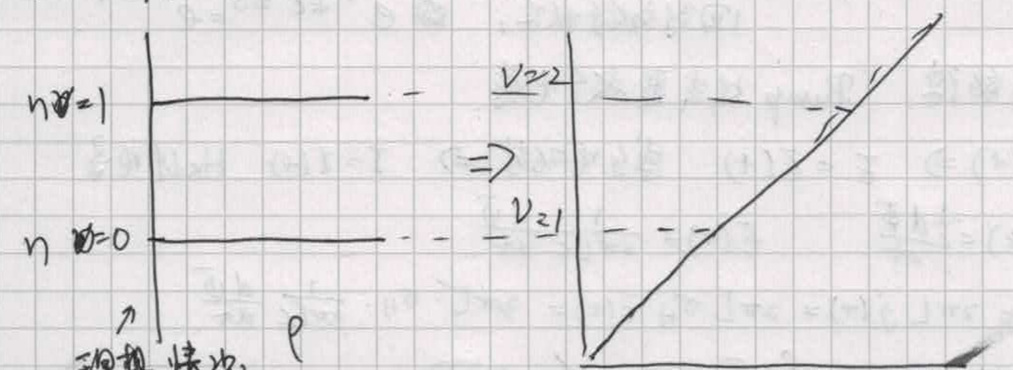
n^{th} state: 面积 $A = \pi\cdot(\sqrt{2n}\, l_0)^2$

$$= 2n\pi l_0^2 = n\cdot\frac{hc}{eB}.$$

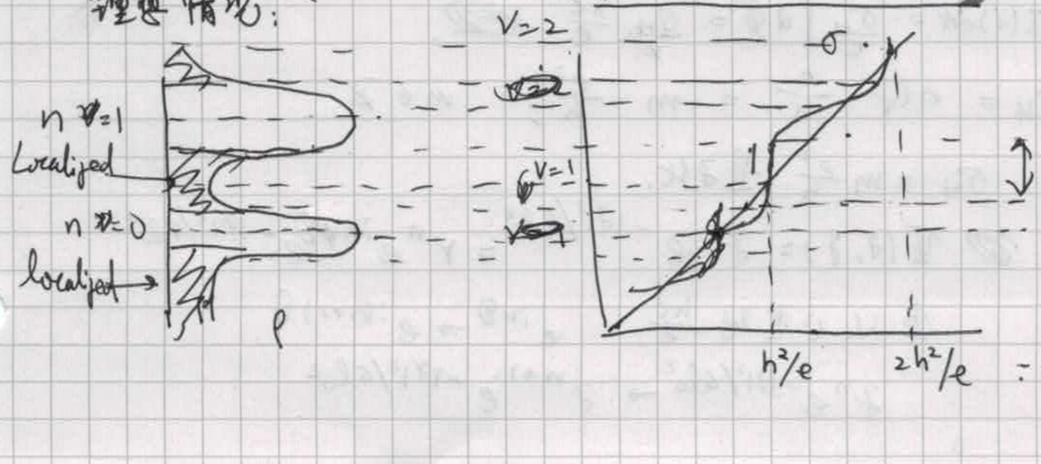
# of states in LL $= n = A\cdot\frac{eB}{hc}$.

4) Hall 电导/电阻量子化的原因:

杂质 $\Rightarrow$ 局域化态.



理想情况:



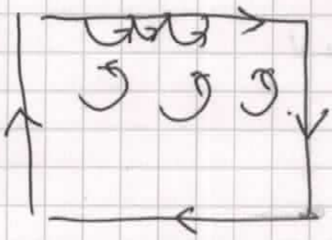
QH平台来自于无序!
无序越强, 平台越长.

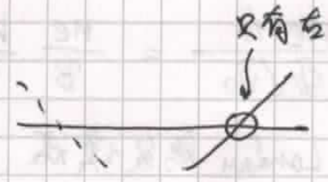
* 绝缘的体态与无耗散的边界态.

QHE 态：体态化学势落在两个朗道能级之间(的局域化态)

体态为有能隙的绝缘态. 能隙 $\approx \hbar\omega_c$.

边界态：手征 (chiral) 边界模式.



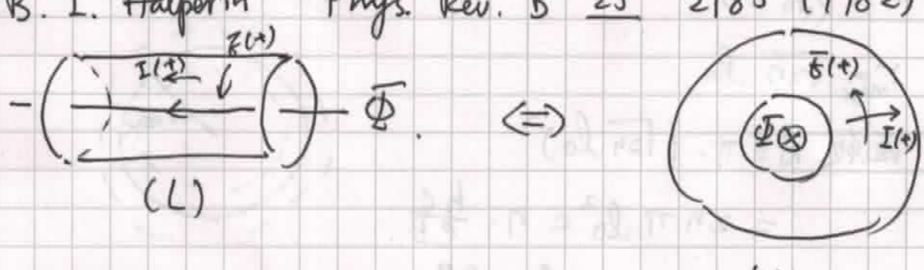


边界态无耗散：无背散射

5) Laughlin's argument.

R. B. Laughlin Phys. Rev. B 23 5632 (1981)

B. I. Halperin Phys. Rev. B 25 2185 (1982).



穿入磁通量 Φ： $\Phi = \Phi(t)$

$$\Phi(t=0) = 0 \ ; \qquad \Phi(t=t_0) = \frac{hc}{e} = \Phi_0$$

回到初始状态； $e^{i\frac{e}{\hbar c}\Phi_0} = e^{i2\pi}$.

- 假设 Bulk 有能隙：Pump 过去整数个电荷.

$$\Phi = \Phi(t) \Rightarrow E = E(t) \text{ 感生电动势} \Rightarrow I = I(t) \text{ Hall 电流.}$$

$$\oint d\vec{l}\cdot\vec{E} = 2\pi L\, E(t) = -\frac{1}{c}\frac{d\Phi}{dt}. \qquad E(t) = \frac{1}{2\pi L c}\frac{d\Phi}{dt}.$$

$$I(t) = 2\pi L\, j(t) = 2\pi L\, \sigma_H\, E(t) = \cancel{2\pi L}\cdot\sigma_H\cdot\frac{1}{\cancel{2\pi L}c}\frac{d\Phi}{dt}.$$

$$\Delta Q = \int I(t)\,dt = \frac{\sigma_H}{c}\int d\Phi = \frac{\sigma_H}{\cancel{c}}\frac{h\cancel{c}}{e}.$$

$$\sigma_H = \Delta Q\cdot\frac{e}{h} = m\cdot\frac{e^2}{h}; \qquad m\in\mathbb{Z}.$$

* Bulk 能隙 $\Rightarrow$ $\sigma_H = m\frac{e^2}{h}$ 量子化.

* 微观模型： $\Psi(z,\bar{z}) = z^n e^{-|z|^2/4l_0^2} = r^n e^{in\theta} e^{-|z|^2/4l_0^2}$.

Φ 从 0 变到 $\frac{hc}{e}$： $e^{in\theta} \to e^{i(n+1)\theta}$.

$$z^n e^{-|z|^2/4l_0^2} \to z^{n+1} e^{-|z|^2/4l_0^2}.$$

6) 分数量子霍尔效应; Laughlin 波函数.

例, $\nu = 1/3$. (一般地, $\nu = 1/m$).

Laughlin 波函数: 一次量子化形式; 直接猜出来的, 无变分参数.

$$\Psi(z_1, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_0^2}$$

— LLL: 全纯函数 $\times e^{-\sum_i |z_i|^2/4l_0^2}$.

— 库仑排斥相互作用: $(z_i - z_j)^m \to 0$ as $z_i - z_j \to 0$.

— 费米子: 反对称波函数 $\Psi(\cdots z_i, \cdots, z_j, \cdots) = -\Psi(\cdots z_j \cdots z_i \cdots)$.

m 为奇数.

— $\nu = 1/m$: 最高角动量为 z_i^{mN} 对应单粒子态 $\tilde{n} = mN$. 半径 $\tilde{L} = \sqrt{2\tilde{n}}\, l_0$

若 Ψ 充满整个系统, # of states in LLL $= \tilde{n} = mN$.

$$\nu = N/\tilde{n} = N/mN = \frac{1}{m}$$

7) 准粒子激发; 准粒子电荷.

* 空穴激发: 去掉电子 @ z_0.

$$\Psi(z_1, \cdots, z_N) = \prod_i (z_i - z_0)^m \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_0^2}$$

$$= \prod_i (z_i - z_0)^m \Psi_0(z_1, \cdots, z_N)$$

* 分数化 quasi-hole 激发:

$$\Psi(z_1, \cdots, z_N) = \prod_i (z_i - z_0)\, \Psi_0.$$

Quasihole @ z_0. 相当于 $1/m$ 个电子.

$$\text{“} a(z_0) = \text{”} \prod_i (z_i - z_0).$$

$$\text{“} \psi(z_0) = \text{”} \prod_i (z_i - z_0)^m = [a(z_0)]^m.$$

* 准粒子电荷: Plasma analogy.

计算波函数平均值:

$$|\Psi|^2 = e^{-\beta U(z_1, \cdots, z_N)}$$ → 看成一个2维经典统计的问题.

$$\langle \rho(z) \rangle = \frac{\int d^2z_1 \cdots d^2z_N\, \rho(z)\, |\Psi(z_1, \cdots, z_N)|^2}{\int d^2z_1 \cdots d^2z_N\, |\Psi(z_1, \cdots, z_N)|^2}$$ w/ $\rho(z) = \sum_i e\,\delta(z - z_i)$

$$= \frac{\int \mathcal{D}z\, \rho(z)\, e^{-\beta U(z_1, \cdots, z_N)}}{\int \mathcal{D}z\, e^{-\beta U(z_1, \cdots, z_N)}}$$: 经典统计平均.

w/ $U = -\frac{1}{\beta} \ln |\Psi|^2$.

* 基态波函数:

$$U_0 = -\frac{1}{\beta}\ln|\Psi_0(z_1,\cdots,z_N)|^2 = -\frac{1}{\beta}\ln\left[\prod_{i<j}|z_i-z_j|^{2m} e^{-\sum_i \frac{|z_i|^2}{2l_0^2}}\right]$$

$$= -\frac{1}{\beta}\sum_{i<j}\ln|z_i-z_j|^{2m} + \frac{1}{\beta}\sum_i \frac{|z_i|^2}{2l_0^2}.$$

令 $l_0=1$, 取 $\beta=m$.

$$U_0(z_1,\cdots,z_N) = \underbrace{-2\sum_{i<j}\ln|z_i-z_j|}_{\text{2维电荷相互作用 库仑}} + \underbrace{\frac{1}{2m}\sum_i|z_i|^2}_{\text{背景电荷, } \rho_0=\frac{1}{2\pi m}}.$$

m 不太大: (如 $m=1,3,5,\cdots$): 高温, 库仑屏蔽态.

$$\rho(z)=\rho_0=\frac{1}{2\pi m}.$$

* 准空穴激发态:

$$U = -\frac{1}{m}\ln|\Psi(z_1,\cdots,z_N)|^2$$

$$= -\frac{1}{m}\ln\prod_i|z_i-z_0|^2 - \frac{1}{m}\ln|\Psi_0|^2$$

$$= -\frac{2}{m}\sum_i\ln|z_i-z_0| - 2\sum_{i<j}\ln|z_i-z_j| + \frac{1}{2m}\sum_i|z_i|^2.$$

外电荷 $\frac{1}{m}$ @ z_0.

屏蔽之后: $\int_{z\sim z_0} d^2z\,\Delta\rho = -\frac{1}{m}$. $\quad q=\frac{e}{m}$.

8) 准粒子激发的分数统计.

* 两个准粒子: @ u 和 w.

$$\Psi(z_1,\cdots,z_N) = N(u,w)\prod_{j=1}^{N}(z_j-u)(z_j-w)\Psi_0(z_1,\cdots,z_N).$$

希望有平移不变性: Ψ 为 $(u-w)$的函数.

$$U = -\frac{1}{m}\ln|\Psi|^2$$

$$= -\frac{1}{m}\ln|N(u,w)|^2 - \frac{2}{m}\sum_i\ln|z_i-u| - \frac{2}{m}\sum_i\ln|z_i-w| - 2\sum_{i<j}\ln|z_i-z_j| + \frac{1}{2m}\sum_i|z_i|^2$$

令 $\ln|N(u,w)| = \frac{1}{m}\ln|u-w|$

$$U = -\frac{2}{m^2}\ln|u-w| - \frac{2}{m}\sum_i\left(\ln|z_i-u| + \ln|z_i-w|\right)$$

$$-2\sum_{i<j}\ln|z_i-z_j| + \frac{1}{2m}\sum_i|z_i|^2.$$

静止电荷: $\frac{1}{m}$ @ u $\quad \frac{1}{m}$ @ w. $\quad$ 可动电荷: 1 @ z_i.

$$\therefore\quad \Psi = (u-w)^{\frac{1}{m}}\prod_{j=1}^{N}(z_j-u)(z_j-w)\Psi_0(z_1,\cdots,z_N).$$

交换 $u\leftrightarrow w$: $\frac{\pi}{m}$ phase $\Rightarrow$ 分数统计.