Quantum criticality of Fermi surfaces

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Physics 268br, Spring 2018

PHYSICS

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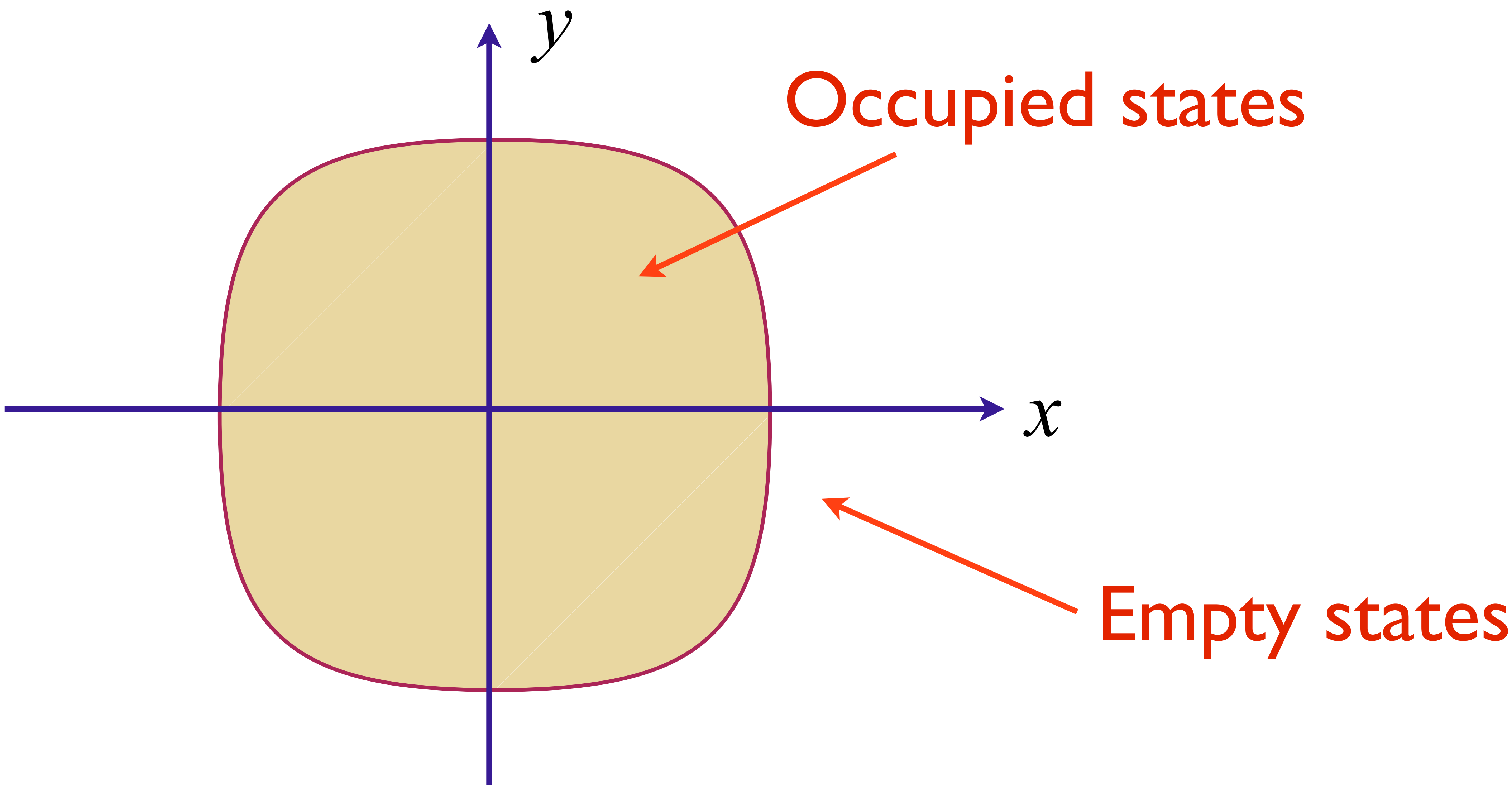
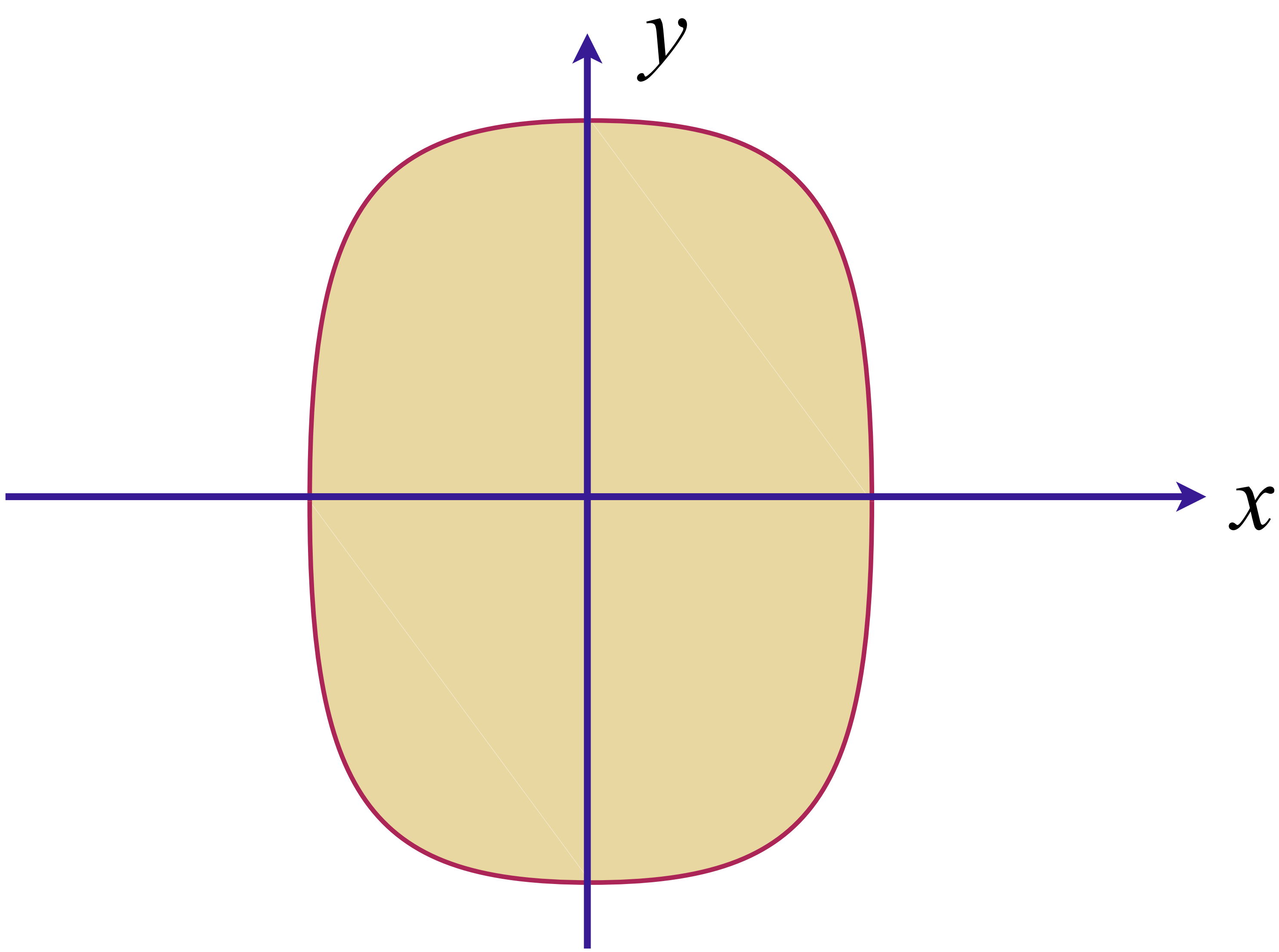
Quantum criticality of Ising-nematic ordering in a metal



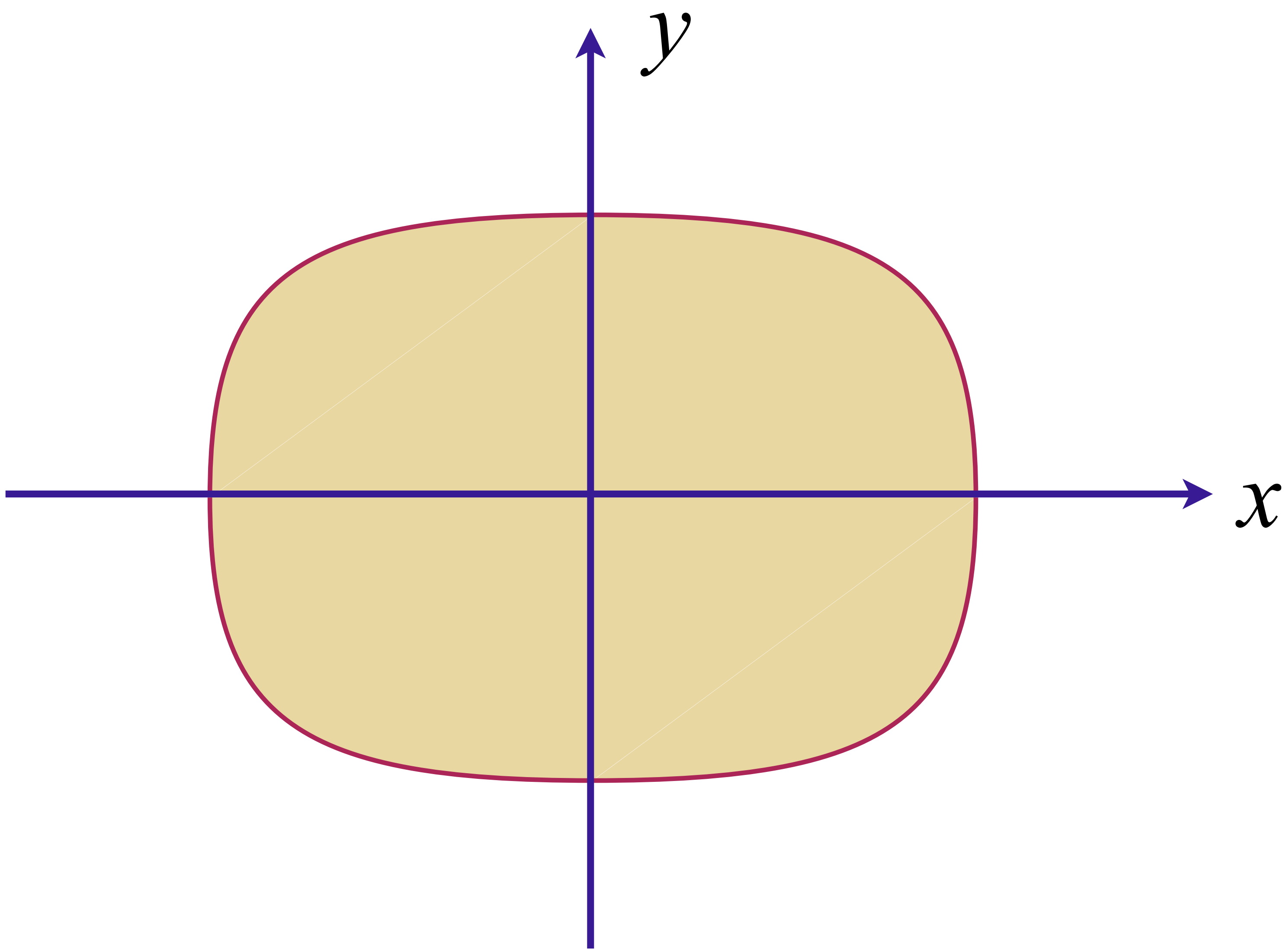
A metal with a Fermi surface
with full square lattice symmetry

Quantum criticality of Ising-nematic ordering in a metal



Spontaneous elongation along y direction:

Quantum criticality of Ising-nematic ordering in a metal



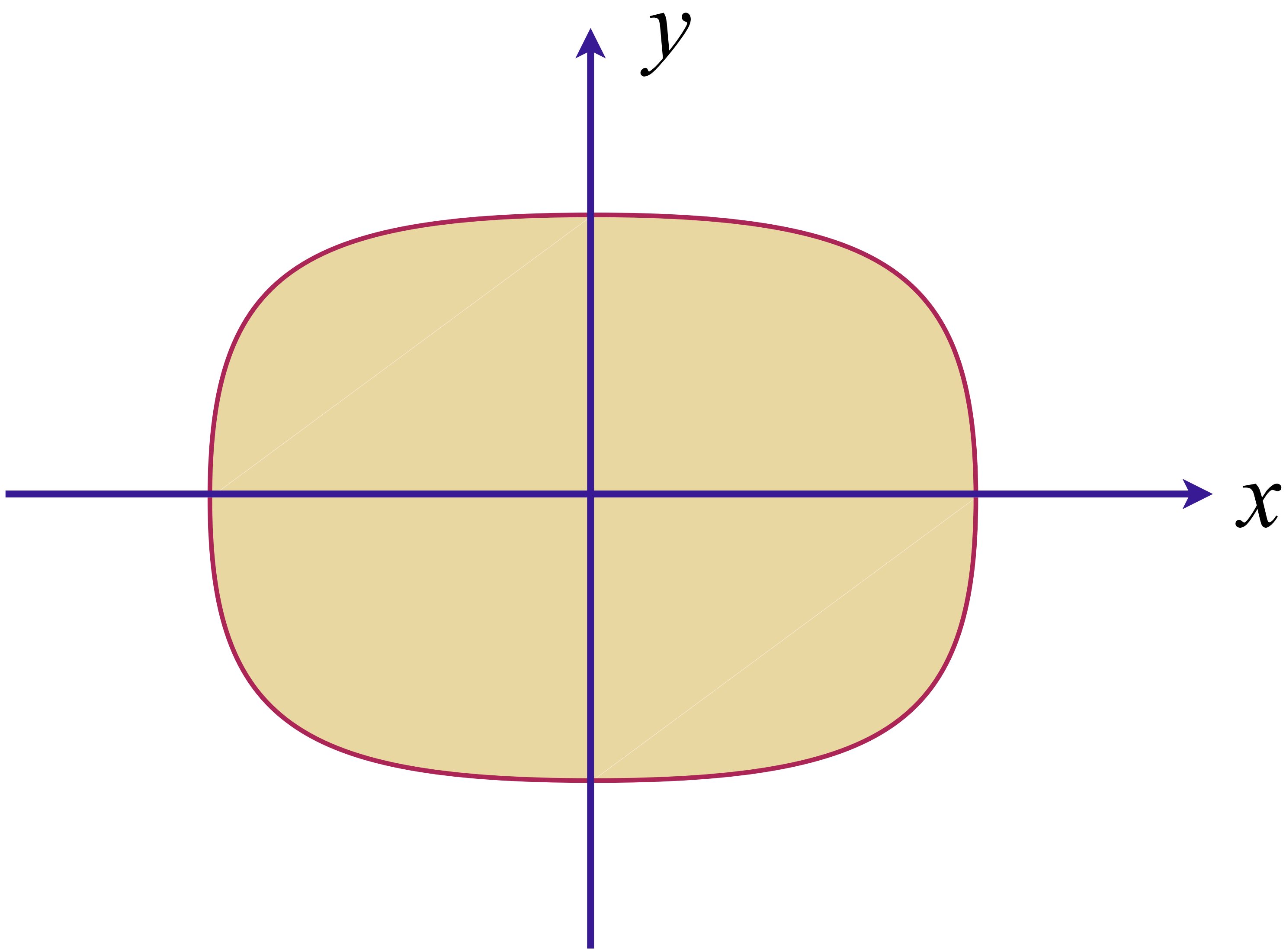
Spontaneous elongation along x direction:

Ising-nematic order parameter

$$\phi \sim \int d^2k \, (\cos k_x - \cos k_y) \, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$$

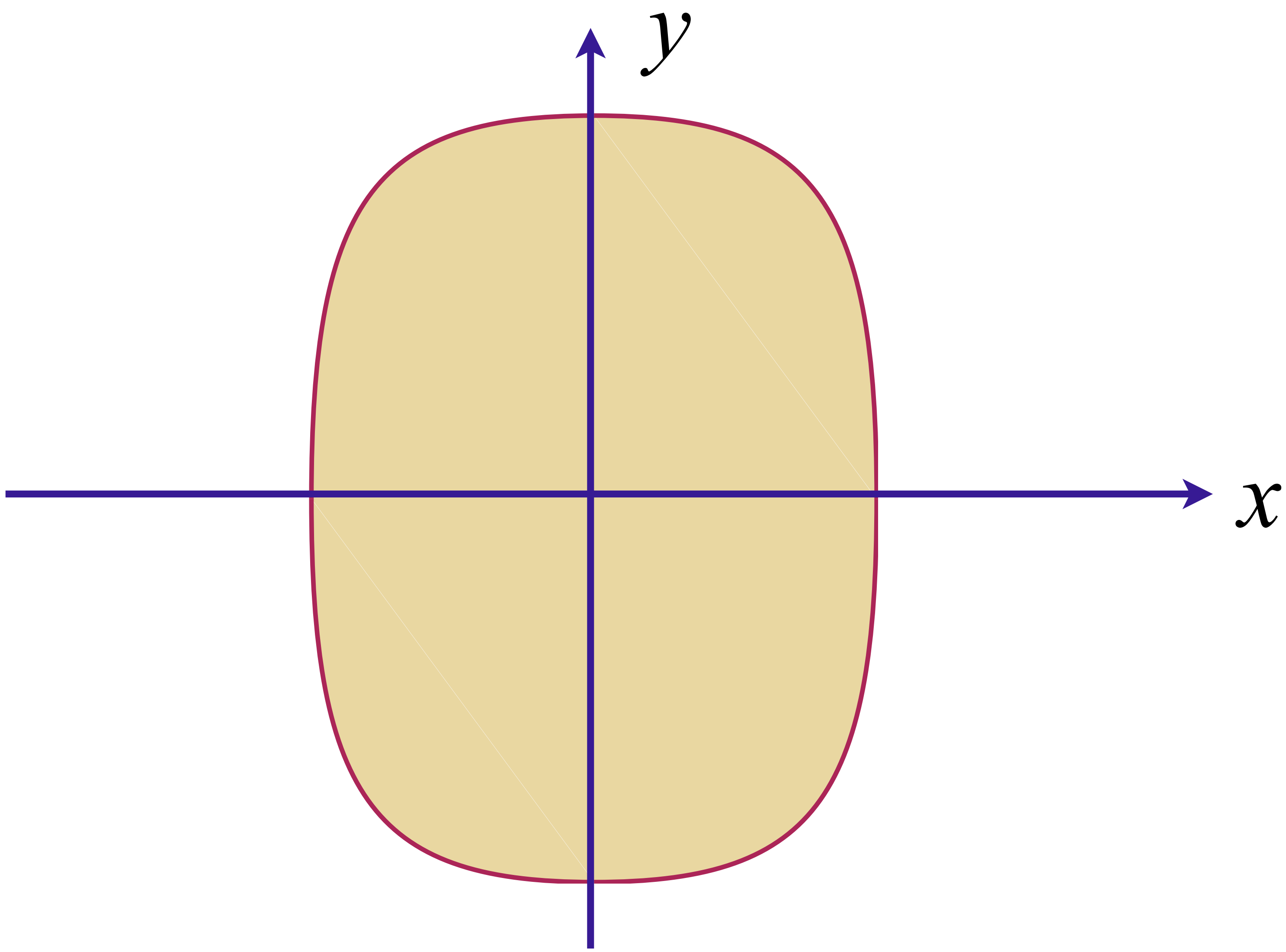
Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian

Quantum criticality of Ising-nematic ordering in a metal



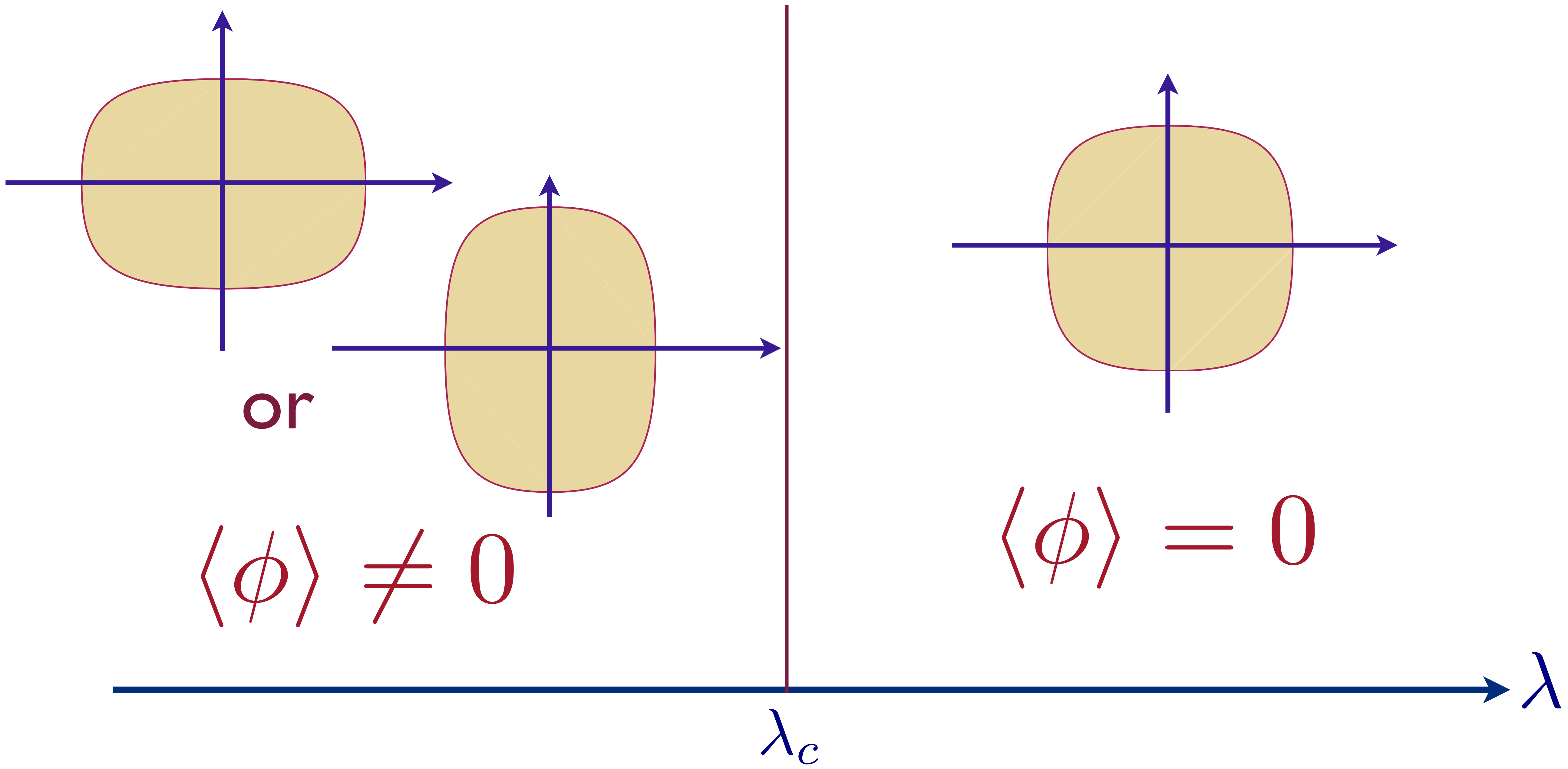
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Ising-nematic ordering in a metal



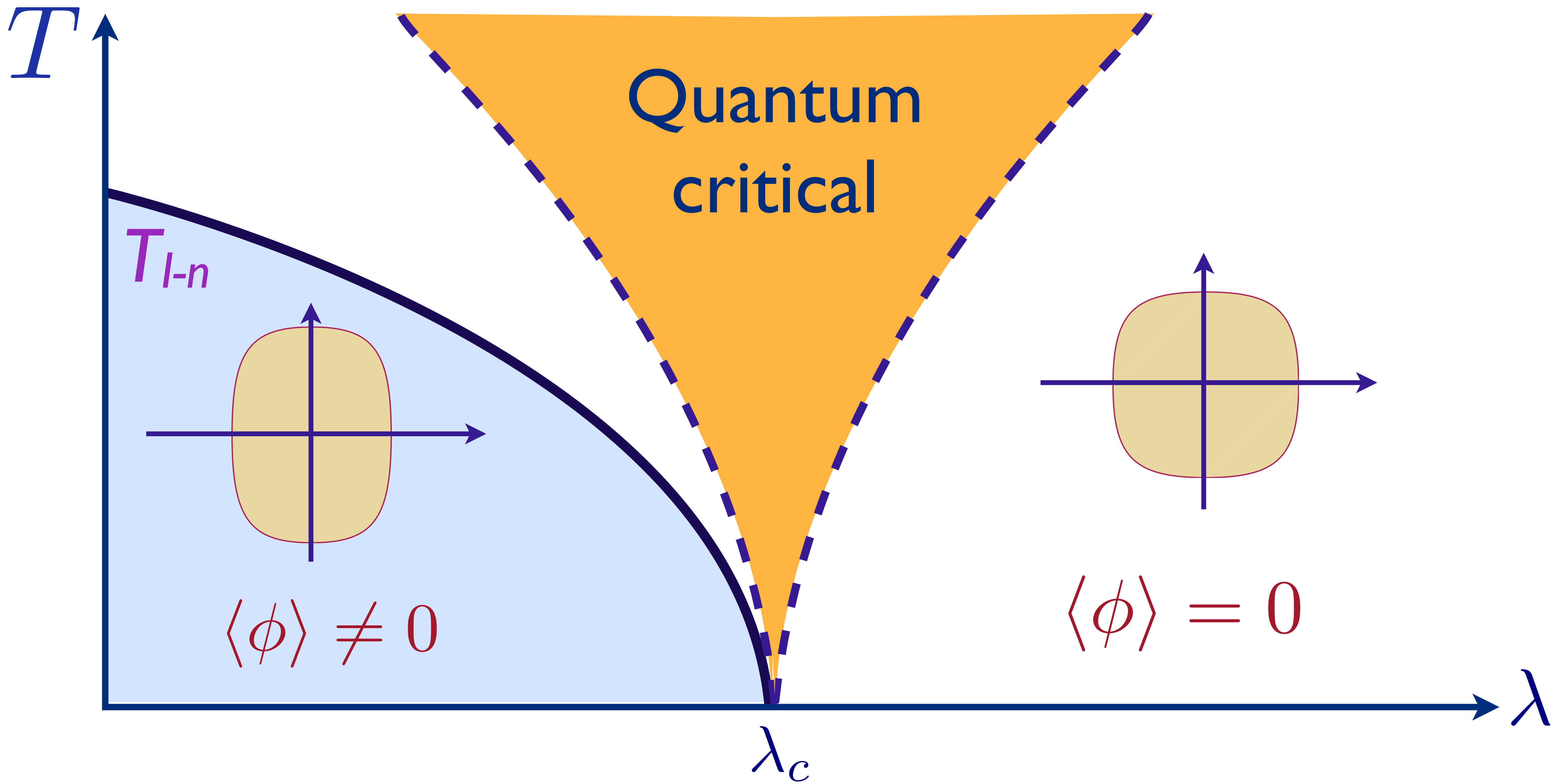
Spontaneous elongation along y direction:
Ising order parameter $\phi < 0$.

Quantum criticality of Ising-nematic ordering in a metal



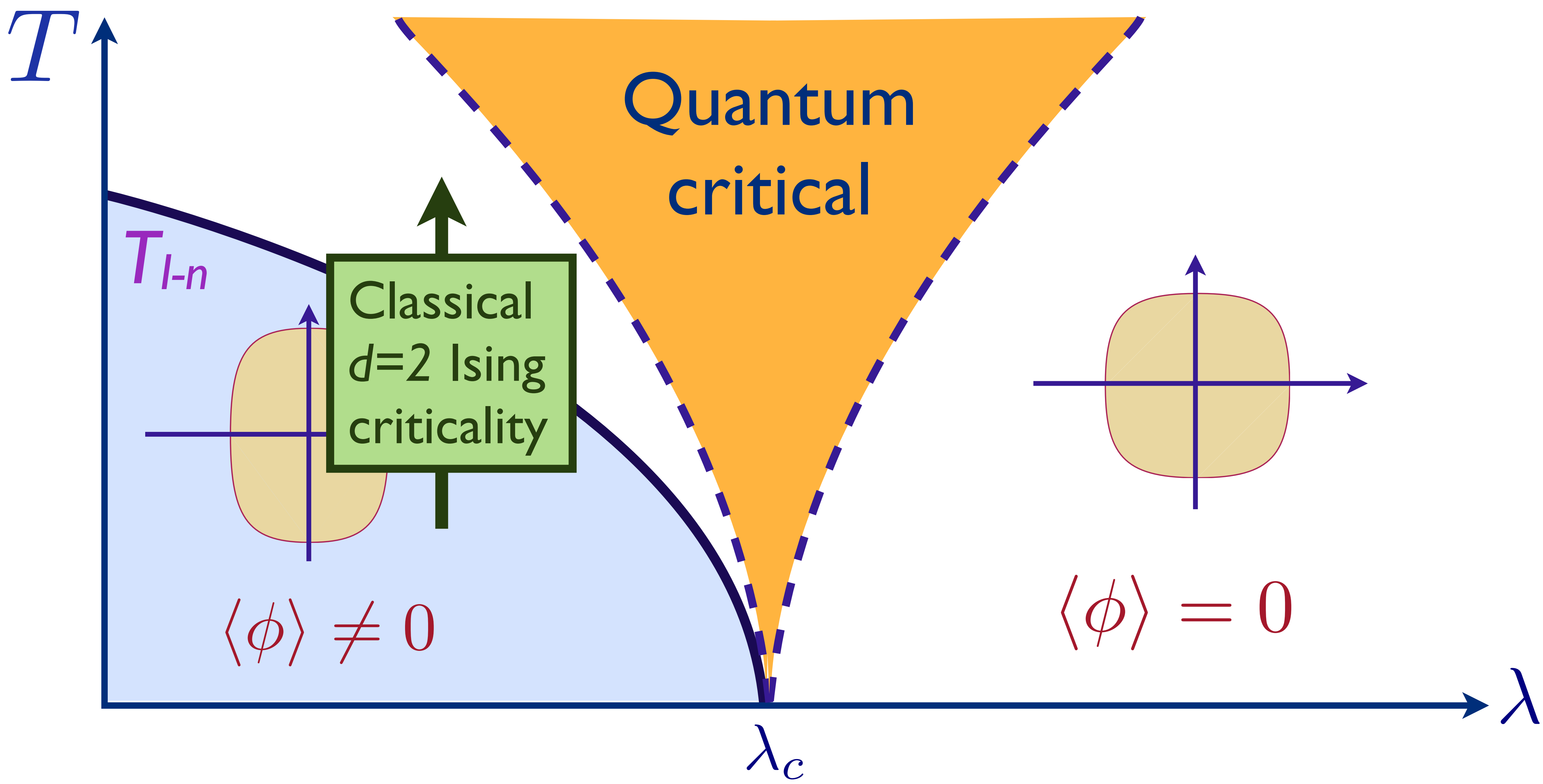
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ

Quantum criticality of Ising-nematic ordering in a metal



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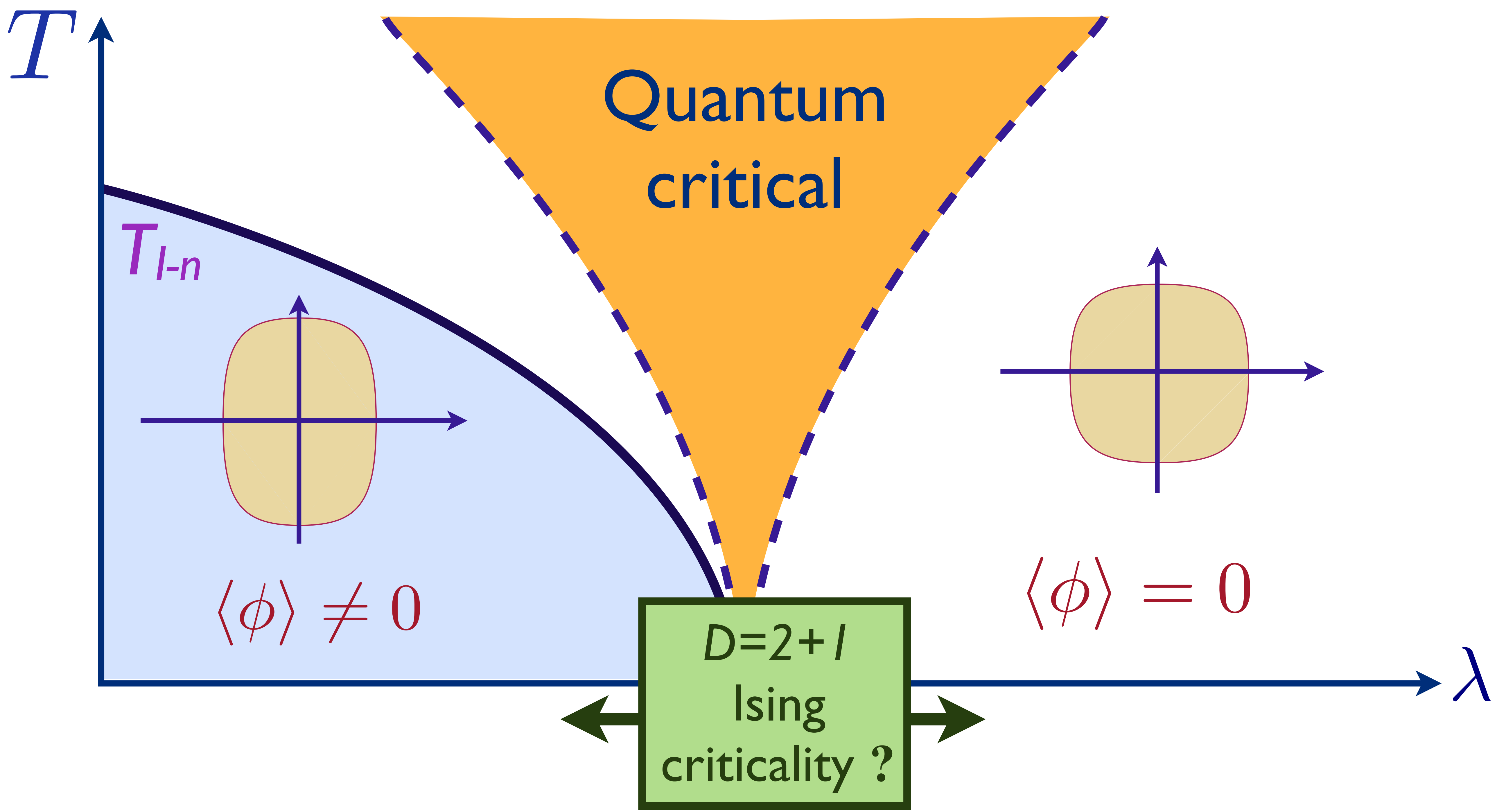

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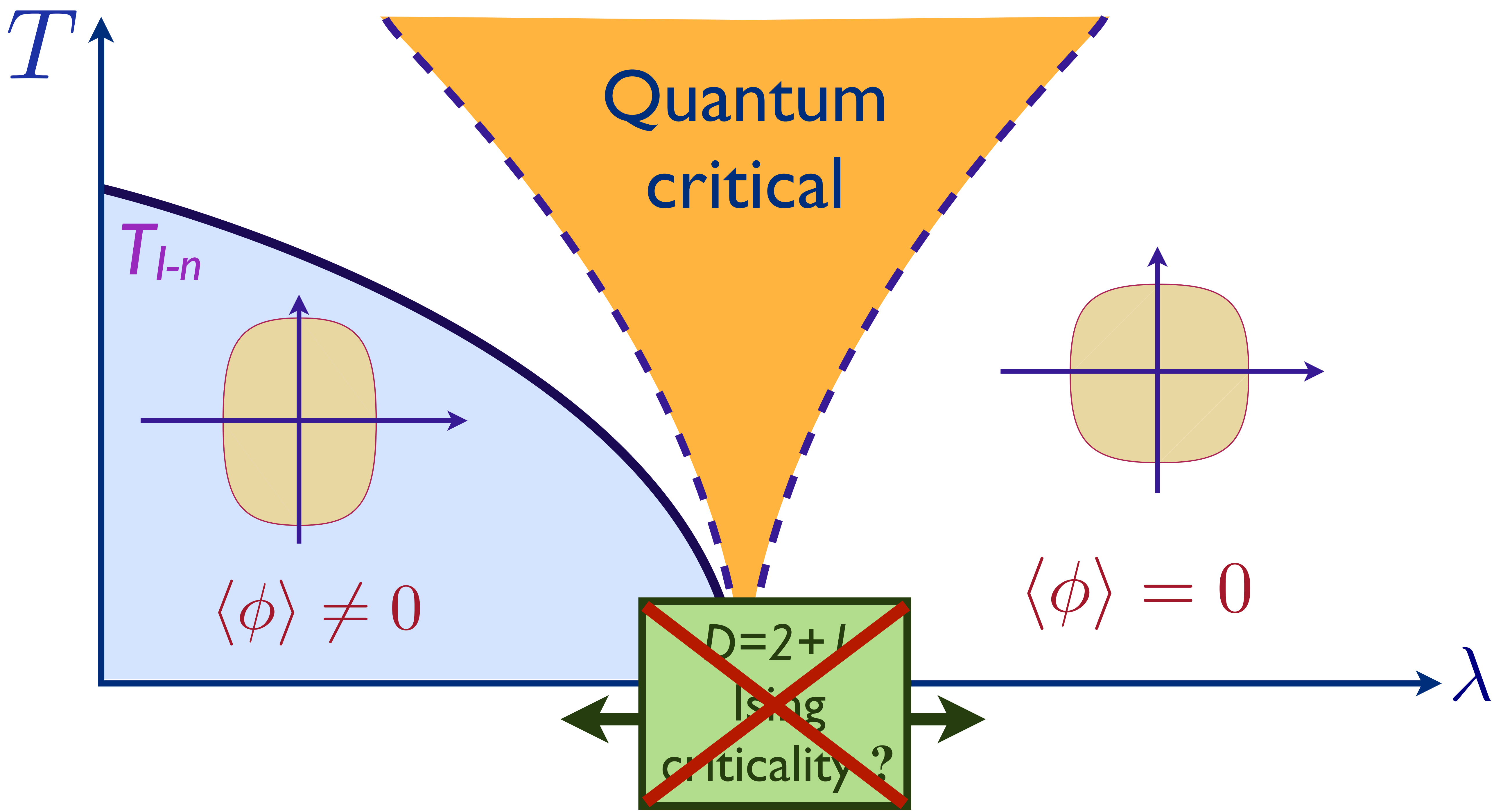

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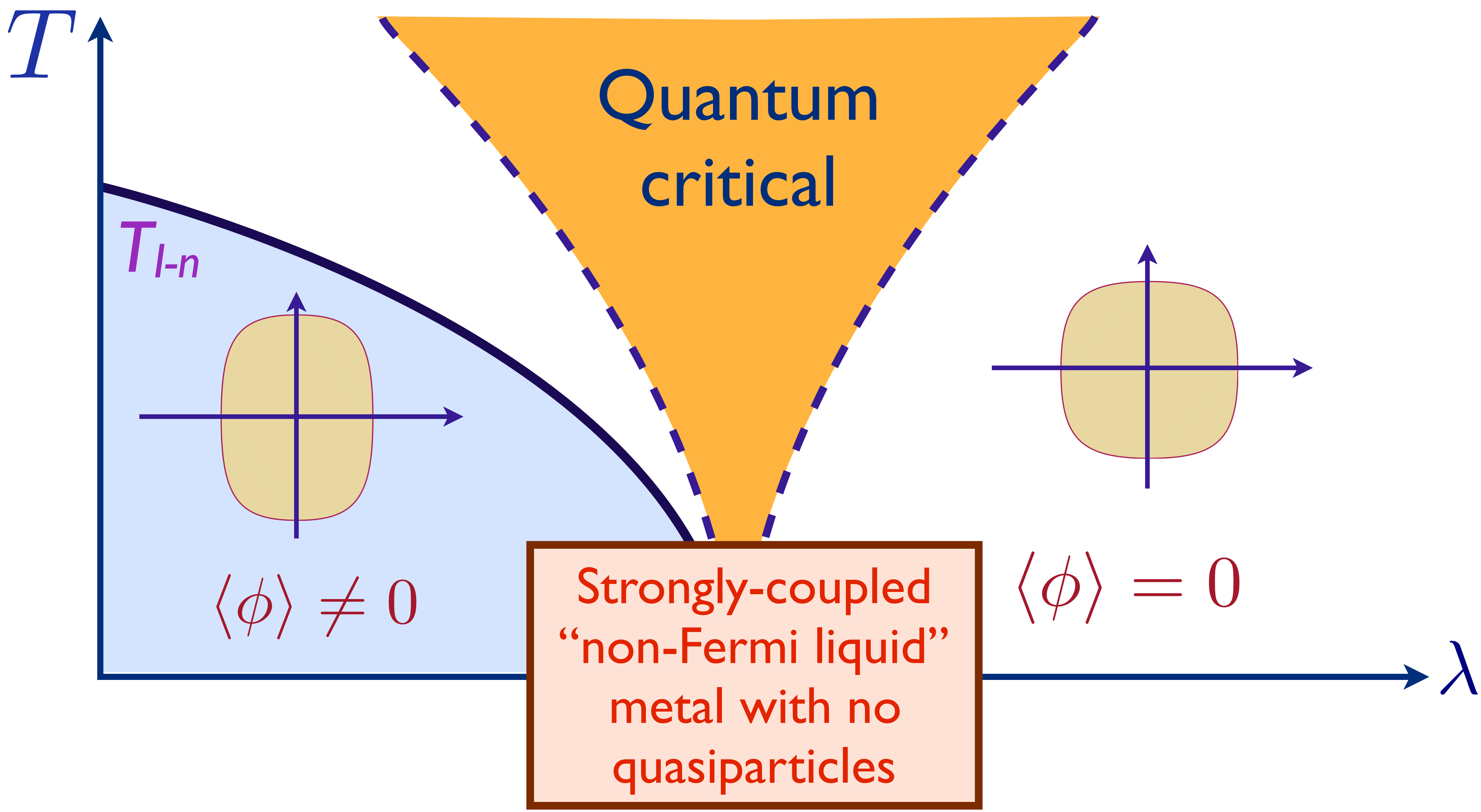

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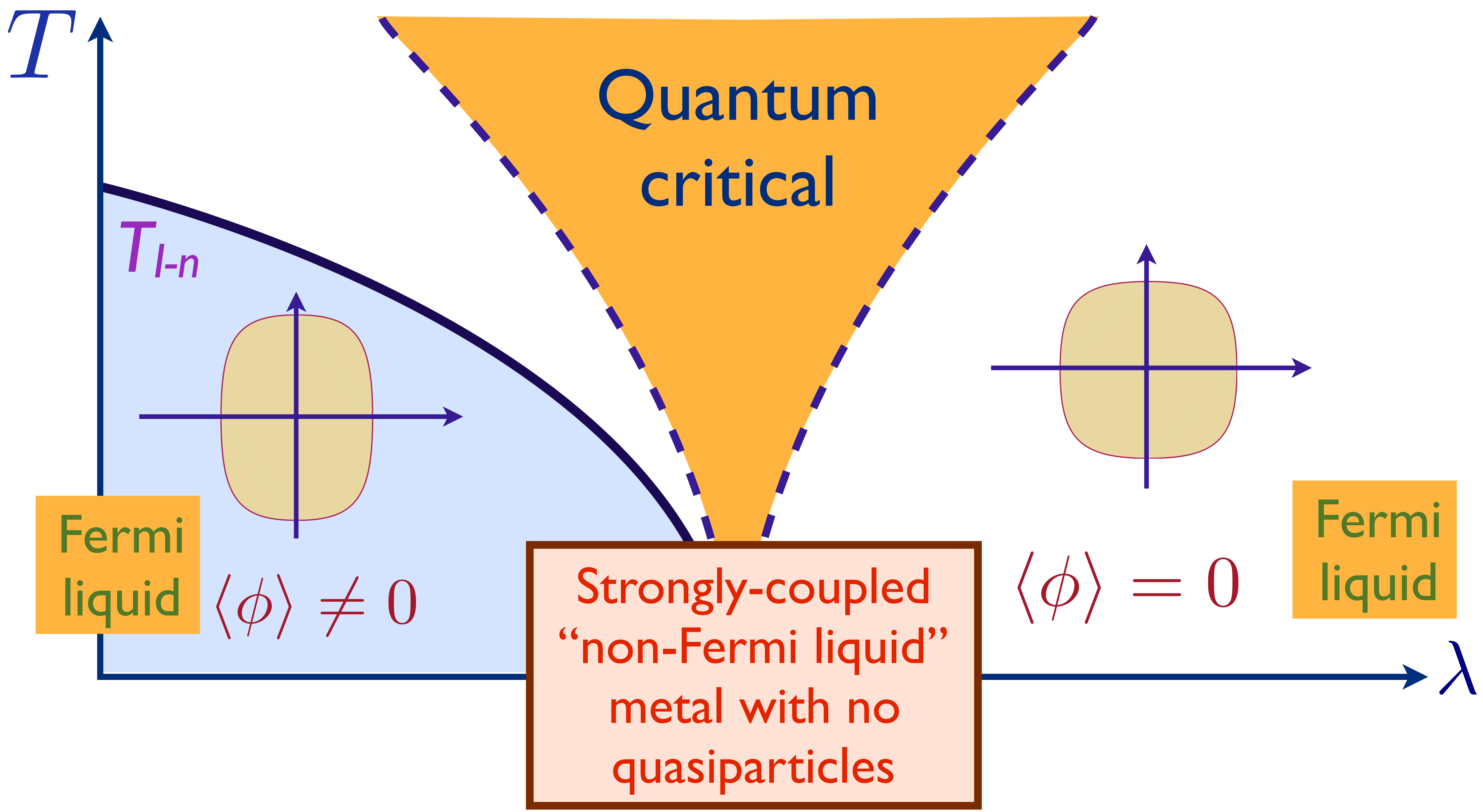

Phase diagram as a function of T and λ

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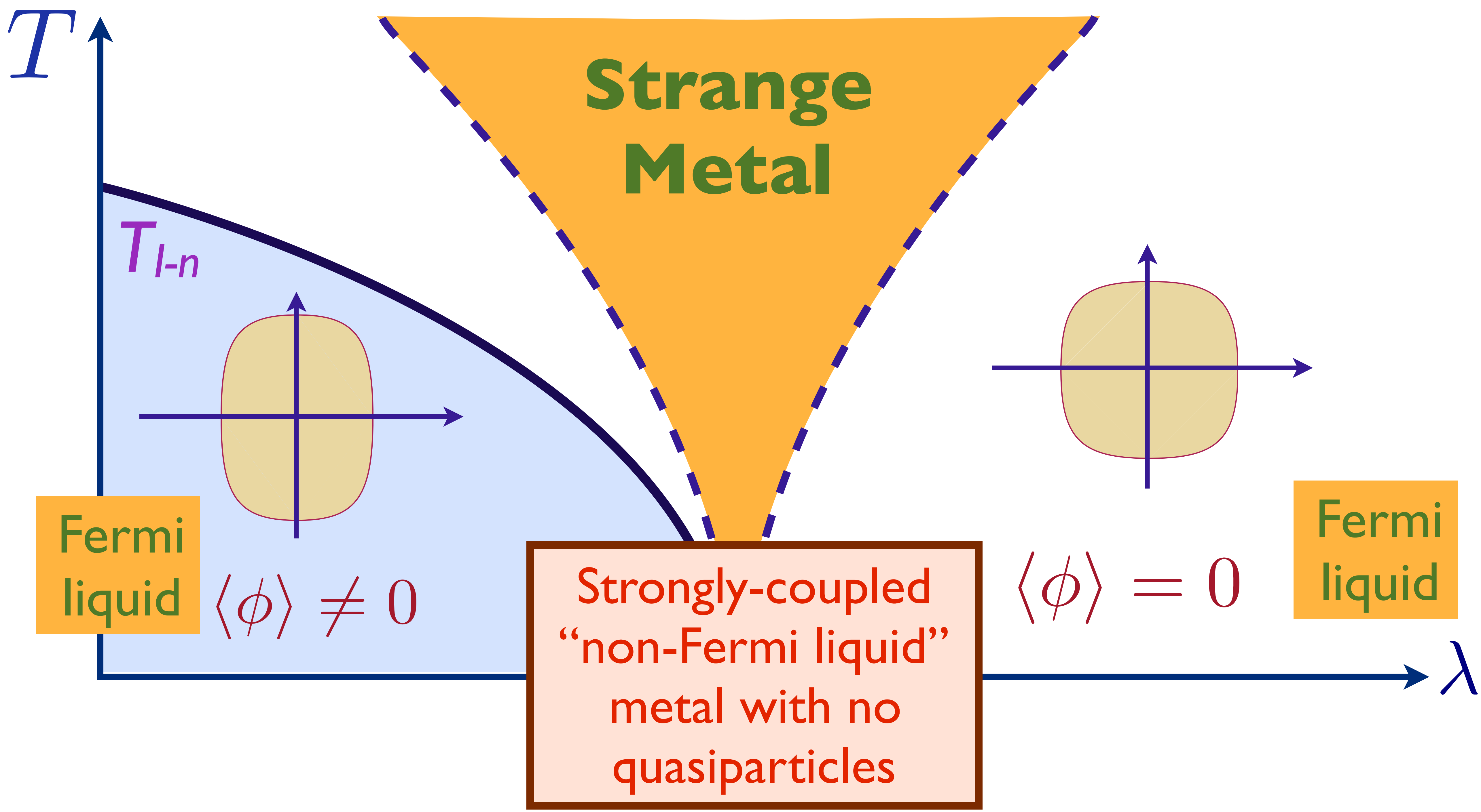
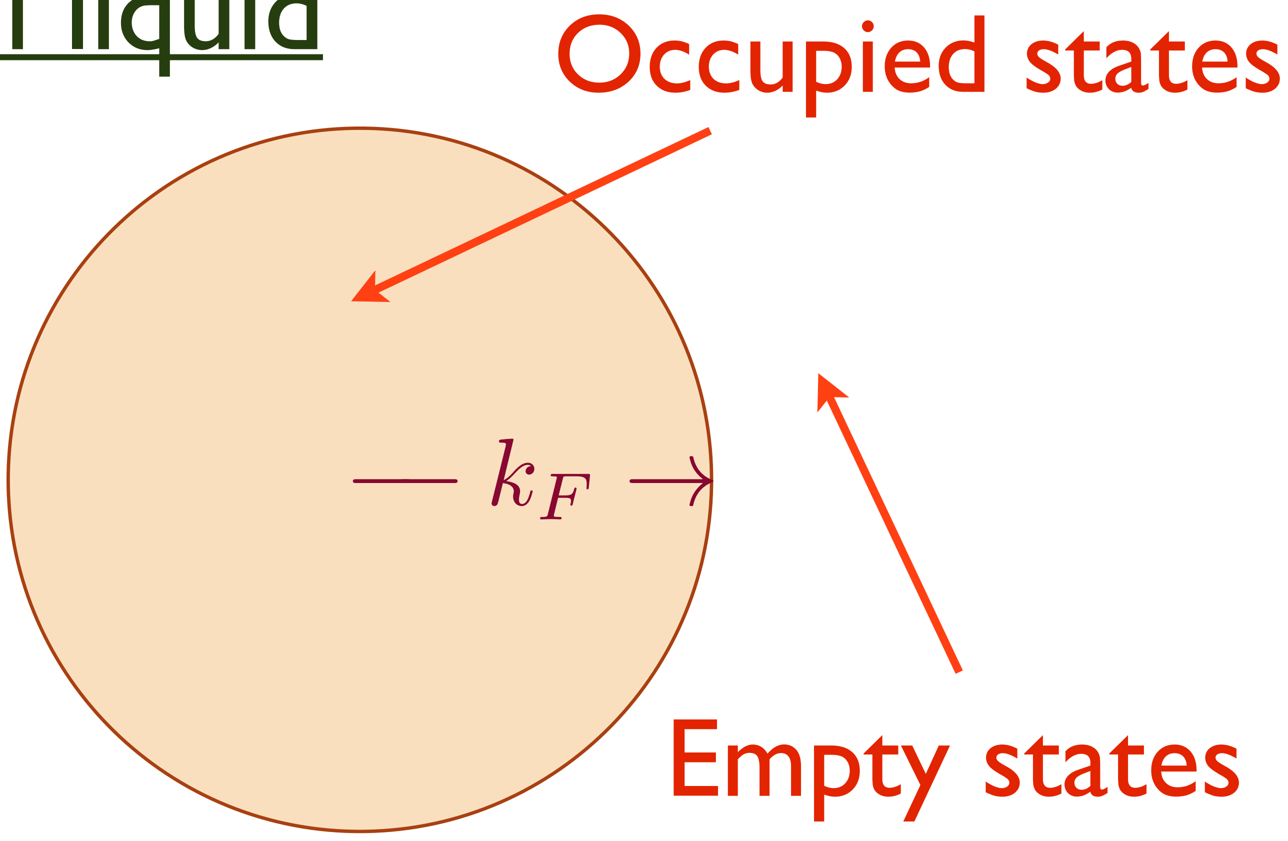


Phase diagram as a function of T and λ

The Fermi liquid

$$\mathcal{L} = f_\alpha^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f_\alpha + u\, f_\alpha^\dagger f_\beta^\dagger f_\beta f_\alpha$$



The Fermi liquid: RG

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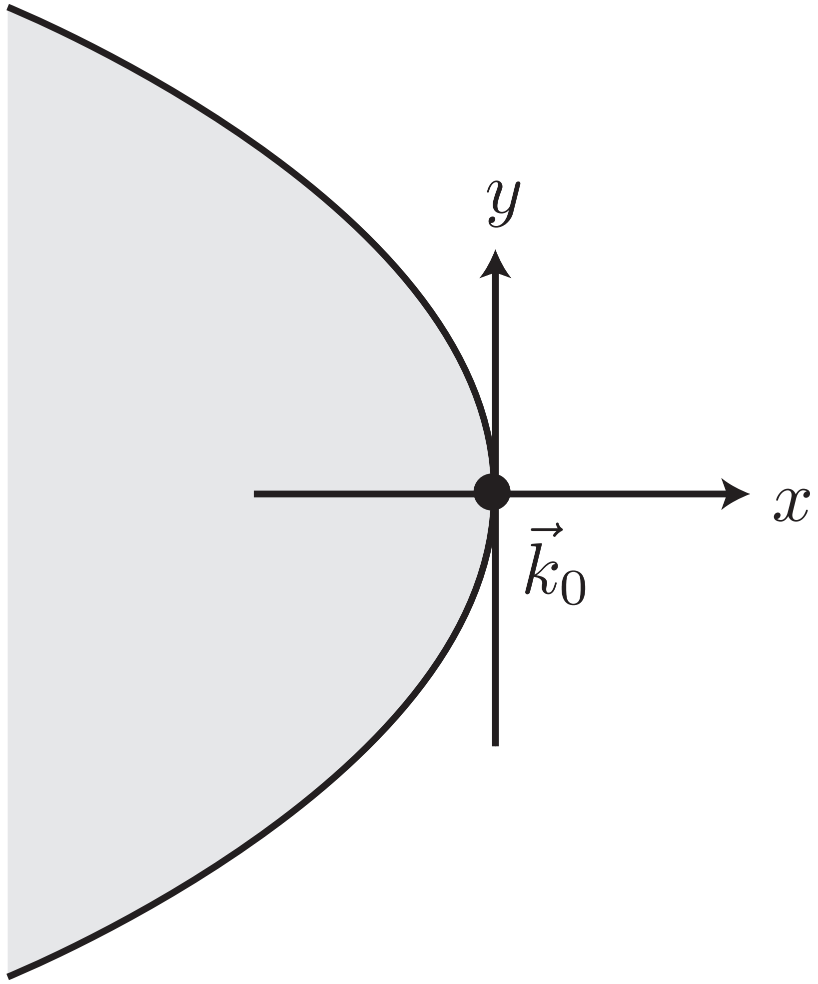

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$, by writing $f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q})$

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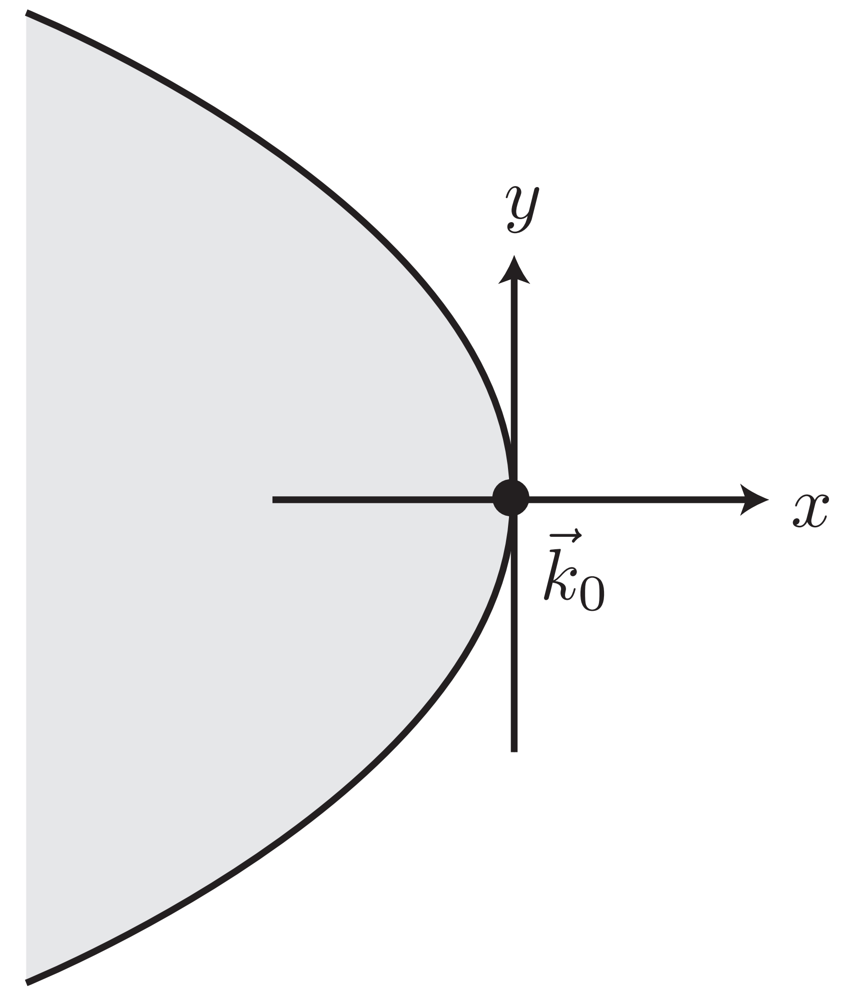

- Expand fermion kinetic energy at wavevectors about $\vec{k}_0$, by writing $f_\alpha(\vec{k}_0 + \vec{q}) = \psi_\alpha(\vec{q})$

$$\mathcal{L}[\psi_\alpha] = \psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2 \right) \psi_\alpha + u\, \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha$$

The Fermi liquid: RG

$$\mathcal{S}[\psi_\alpha] = \int d^{d-1}y\, dx\, d\tau \left[\psi_\alpha^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_\alpha + u\, \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha\right]$$

The Fermi liquid: RG

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The kinetic energy is invariant under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, provided $z = 1$ and

$$\psi \quad \to \quad \psi\, s^{(d+1)/4}.$$

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$$\psi \quad \to \quad \psi\, s^{(d+1)/4}.$$

Then we find $u \to u s^{(1-d)/2}$, and so we have the RG flow

$$\frac{du}{d\ell} \quad = \quad \frac{(1-d)}{2} u$$

Interactions are *irrelevant* in $d = 2$!

The Fermi liquid: RG

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The fermion Green's function to order u^2 has the form (upto logs)

$$G(\vec{q},\omega) = \frac{\mathcal{A}}{\omega - q_x - q_y^2 + ic\,\omega^2}$$

So the quasiparticle pole is sharp.

The Fermi liquid: RG

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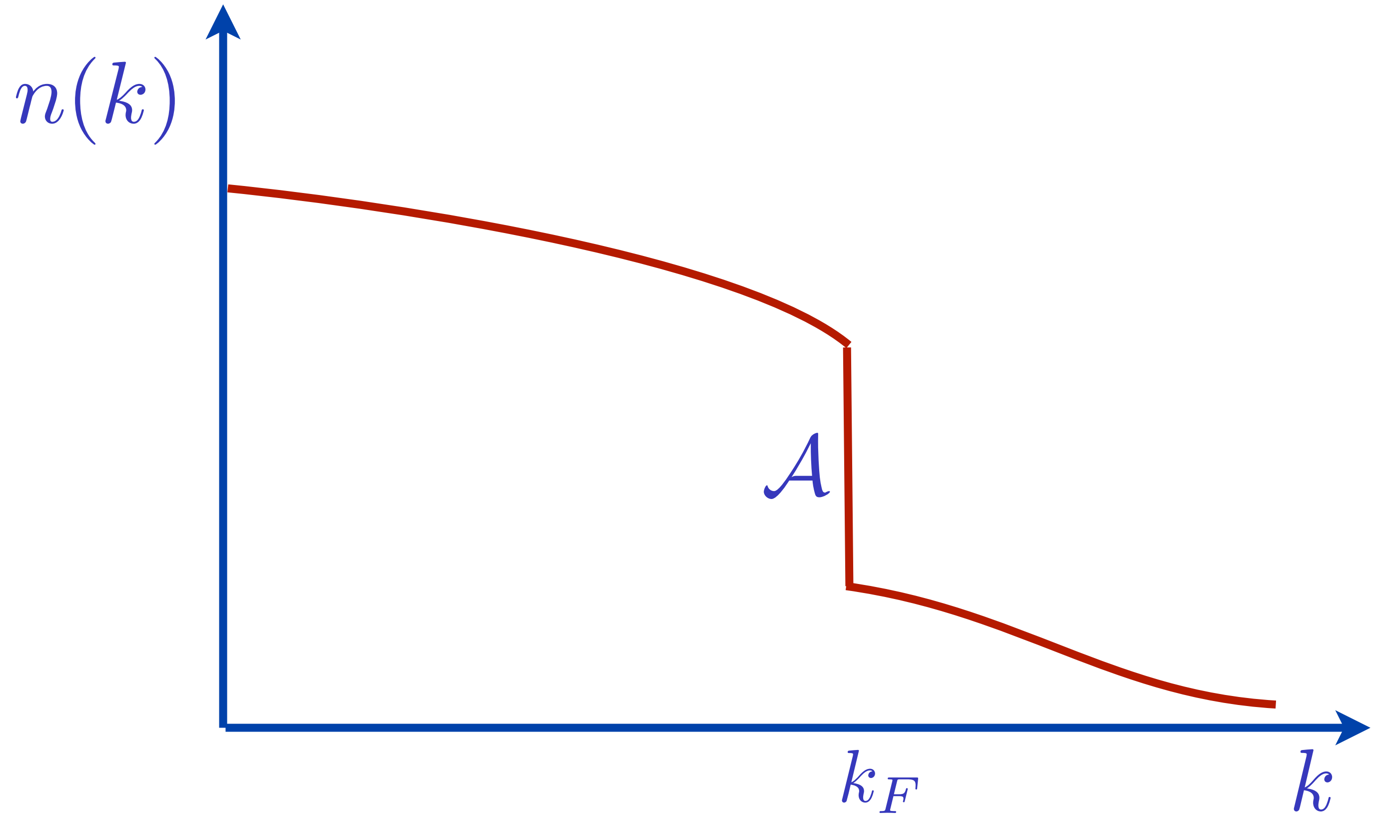
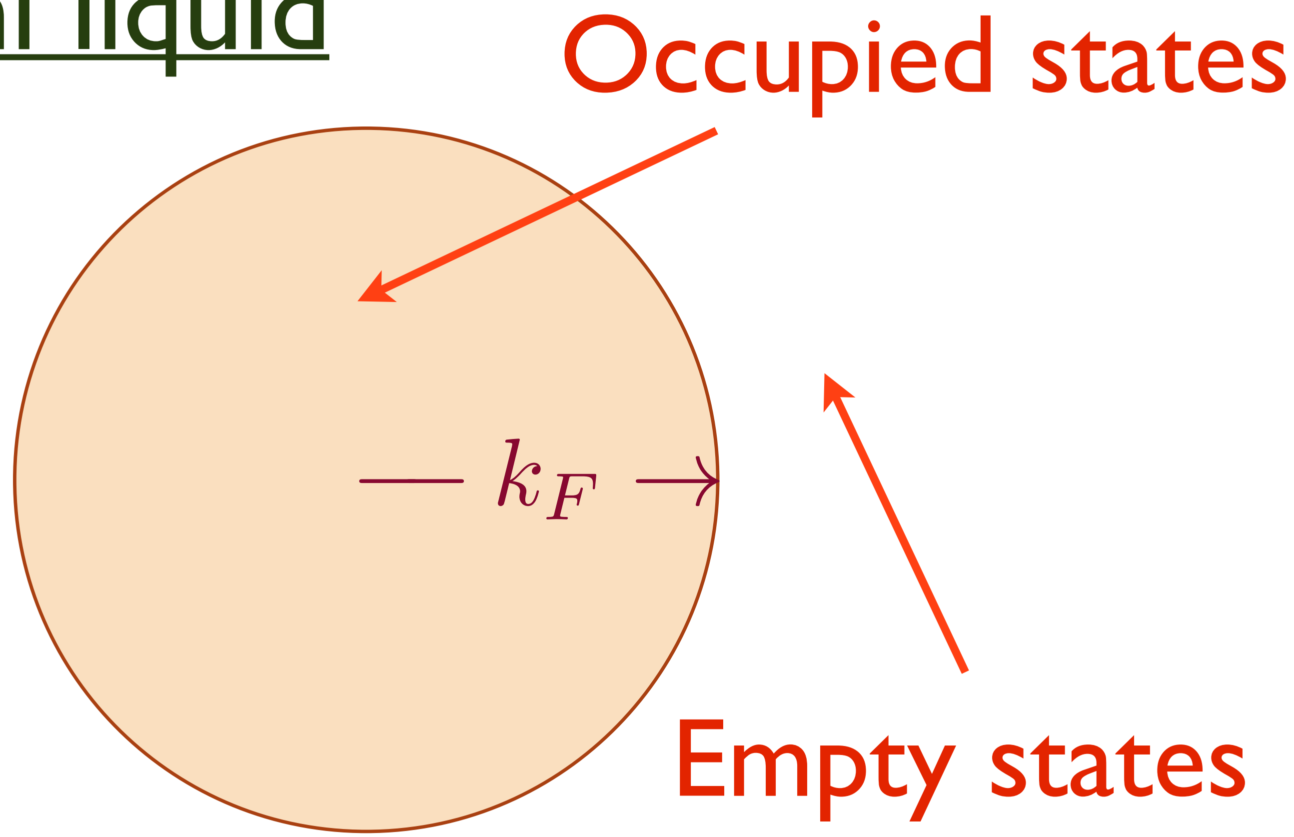
$$G(\vec{q}, \omega) = \frac{\mathcal{A}}{\omega - q_x - q_y^2 + i c\, \omega^2}$$

So the quasiparticle pole is sharp. And fermion momentum distribution function $n(\vec{k}) = \left\langle f_\alpha^\dagger(\vec{k}) f_\alpha(\vec{k}) \right\rangle$ had the following form:



The Fermi liquid

$$\mathcal{L} = f^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) f$$
$$+ 4 \text{ Fermi terms}$$



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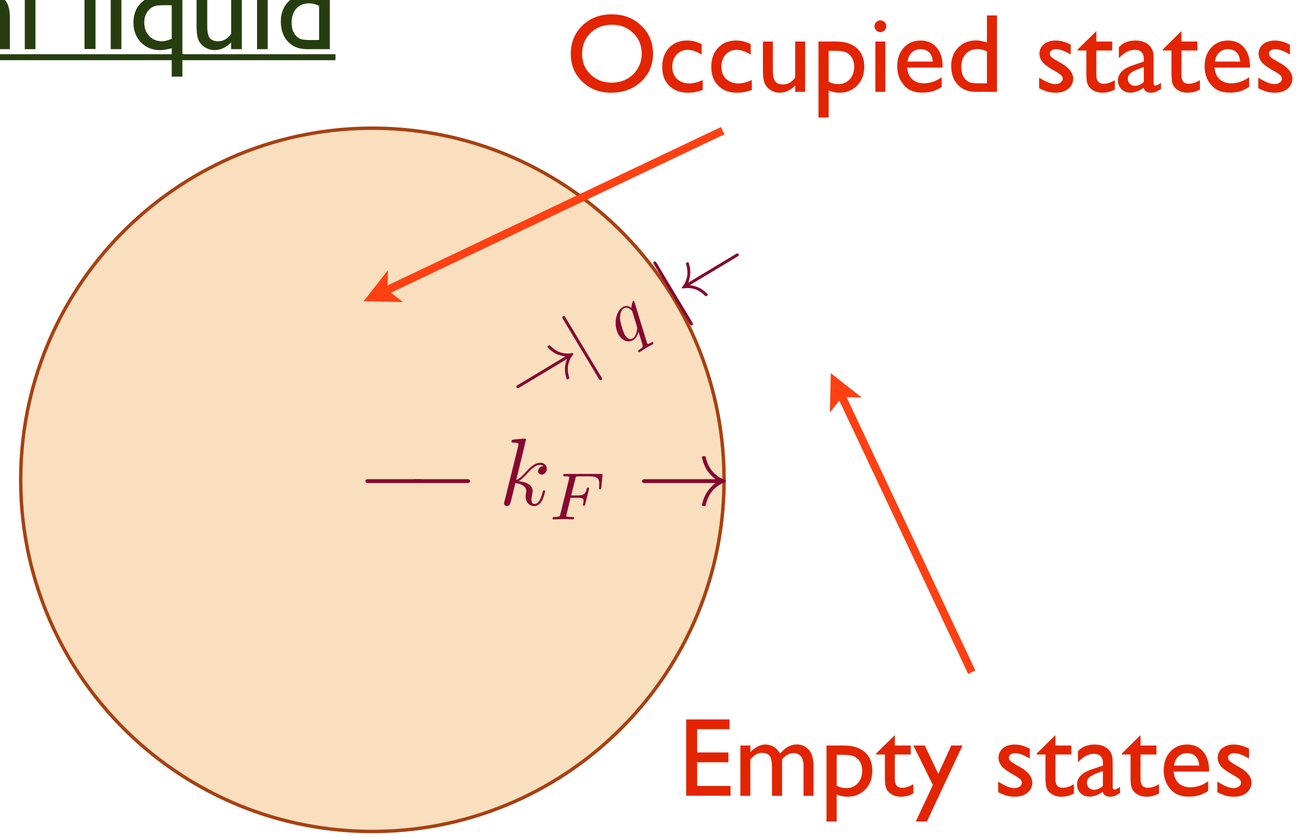

- Fermi wavevector obeys the Luttinger relation $k_F^d \sim \mathcal{Q}$, the fermion density

- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.

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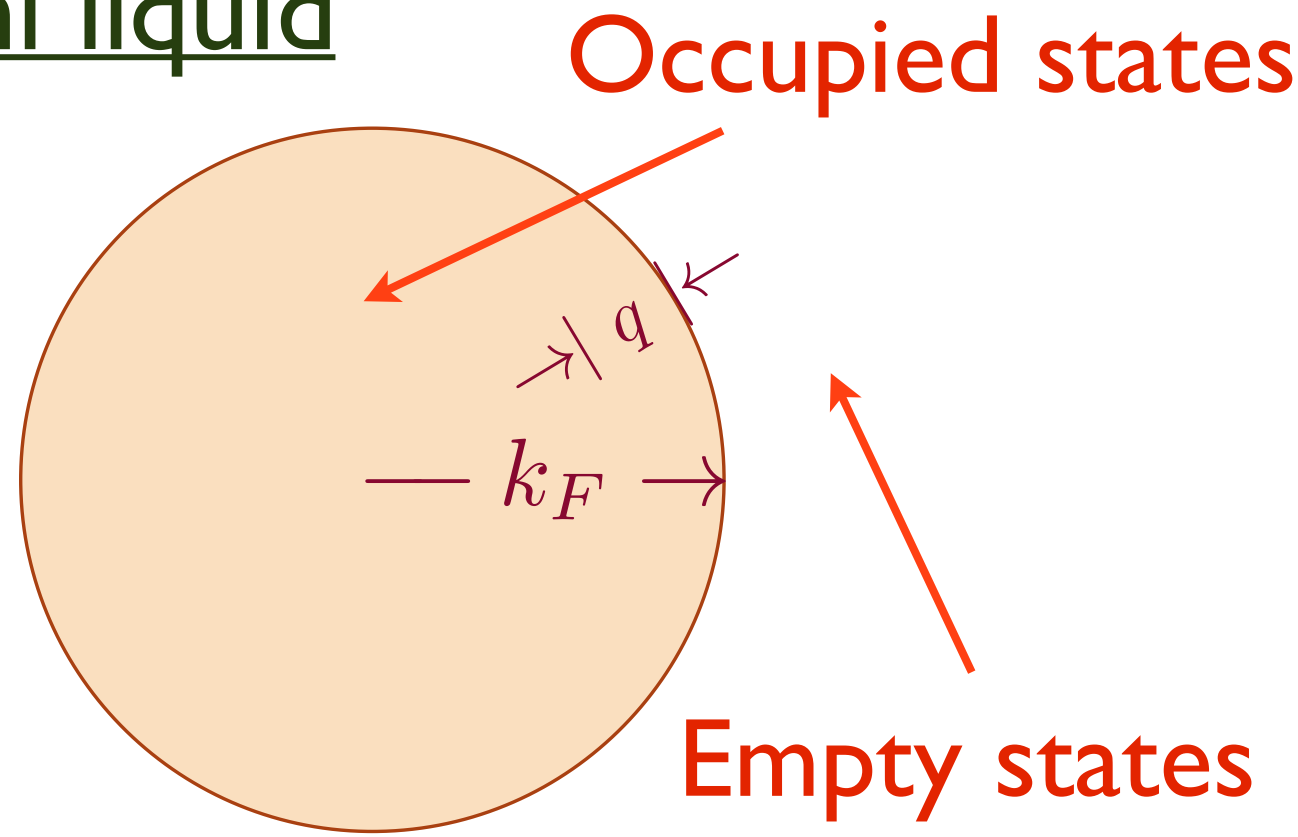

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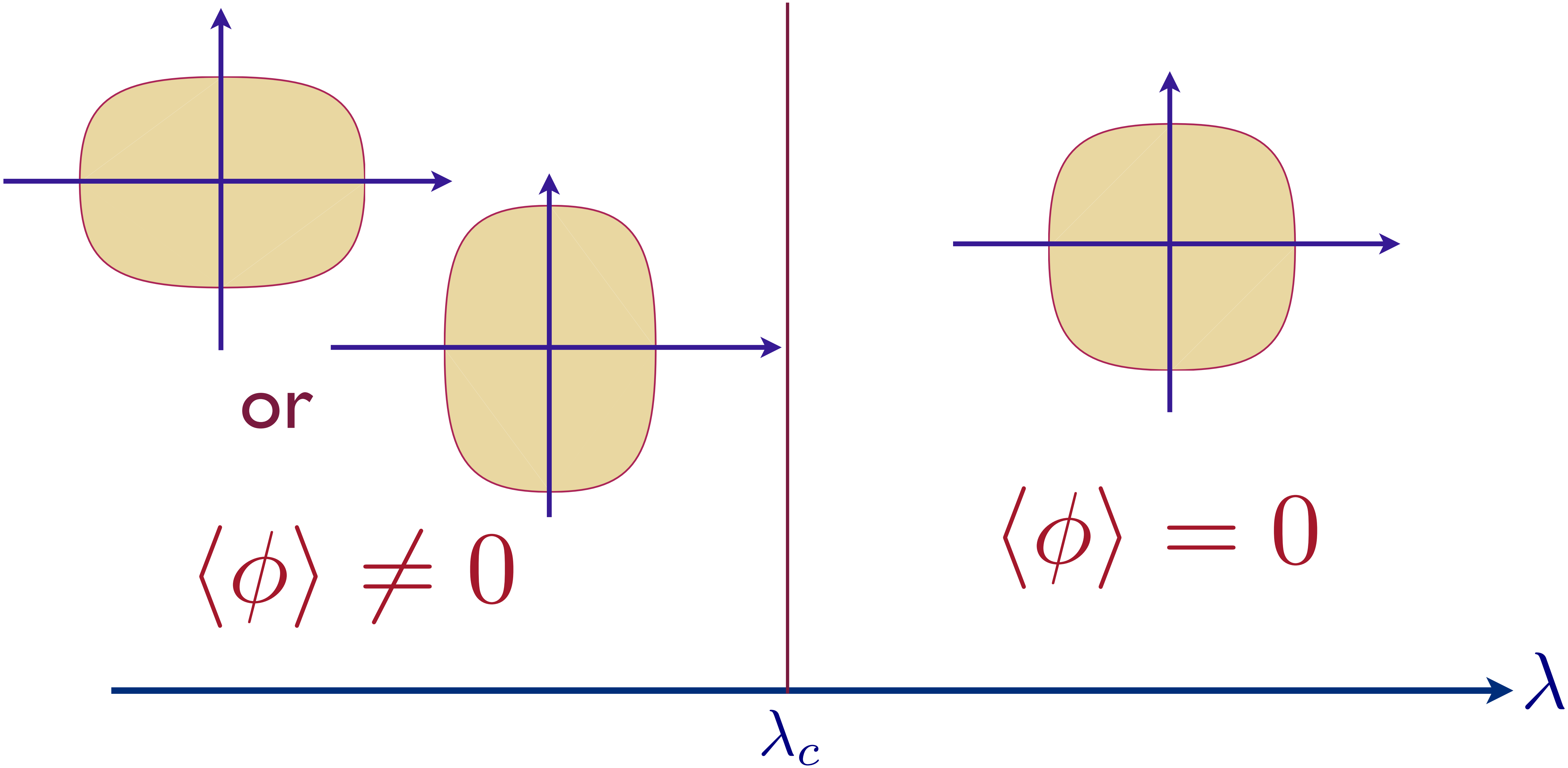
- Sharp particle and hole of excitations near the Fermi surface with energy $\omega \sim |q|^z$, with dynamic exponent $z = 1$.

- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T$. It is useful to write this is as $S \sim T^{(d-\theta)/z}$, with violation of hyperscaling exponent $\theta = d - 1$.

Quantum criticality of Ising-nematic ordering in a metal



Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal

Effective action for Ising order parameter

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

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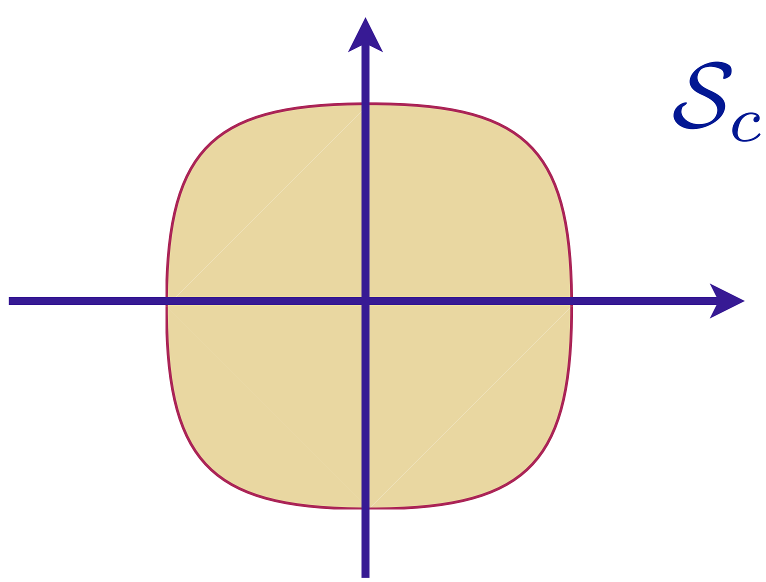
Effective action for electrons:



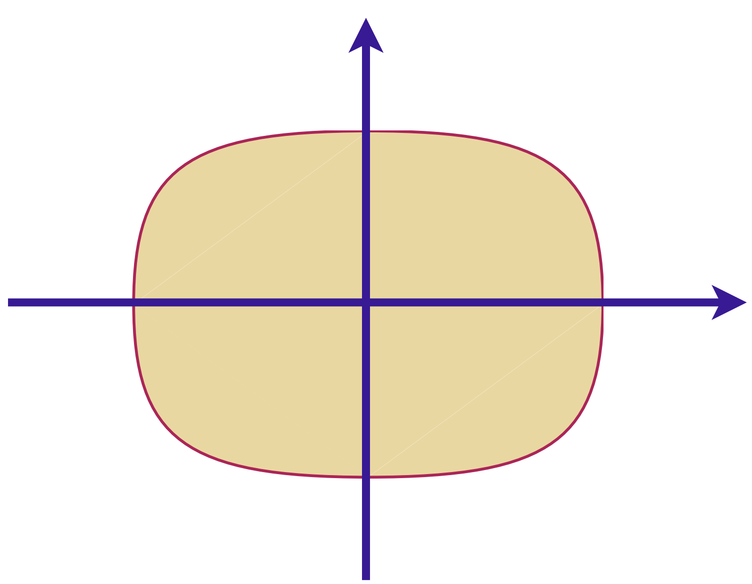
$$\begin{aligned} \mathcal{S}_c &= \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \\ &\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \end{aligned}$$

Quantum criticality of Ising-nematic ordering in a metal

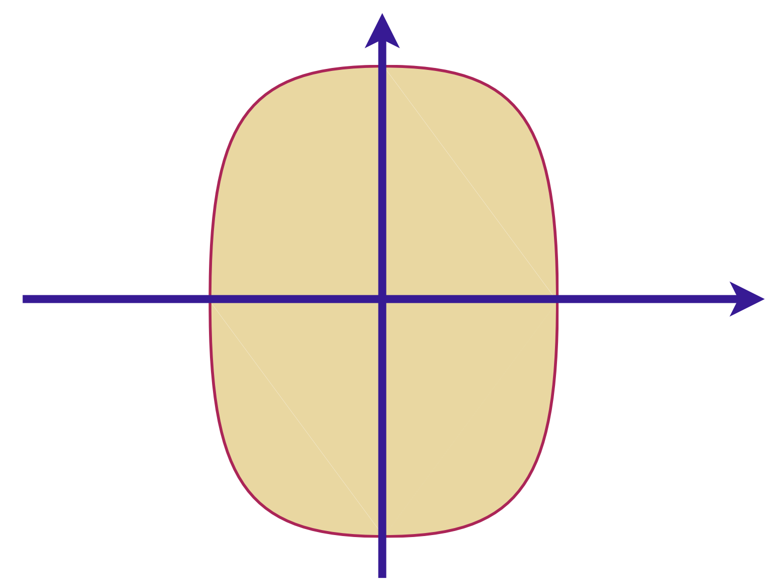
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\,g \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y\right) c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

for spatially dependent ϕ



$\langle \phi \rangle > 0$



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Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{S}_\phi = \int d^2 r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

$$\mathcal{S}_c = \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c^\dagger_{\mathbf{k}\alpha} \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

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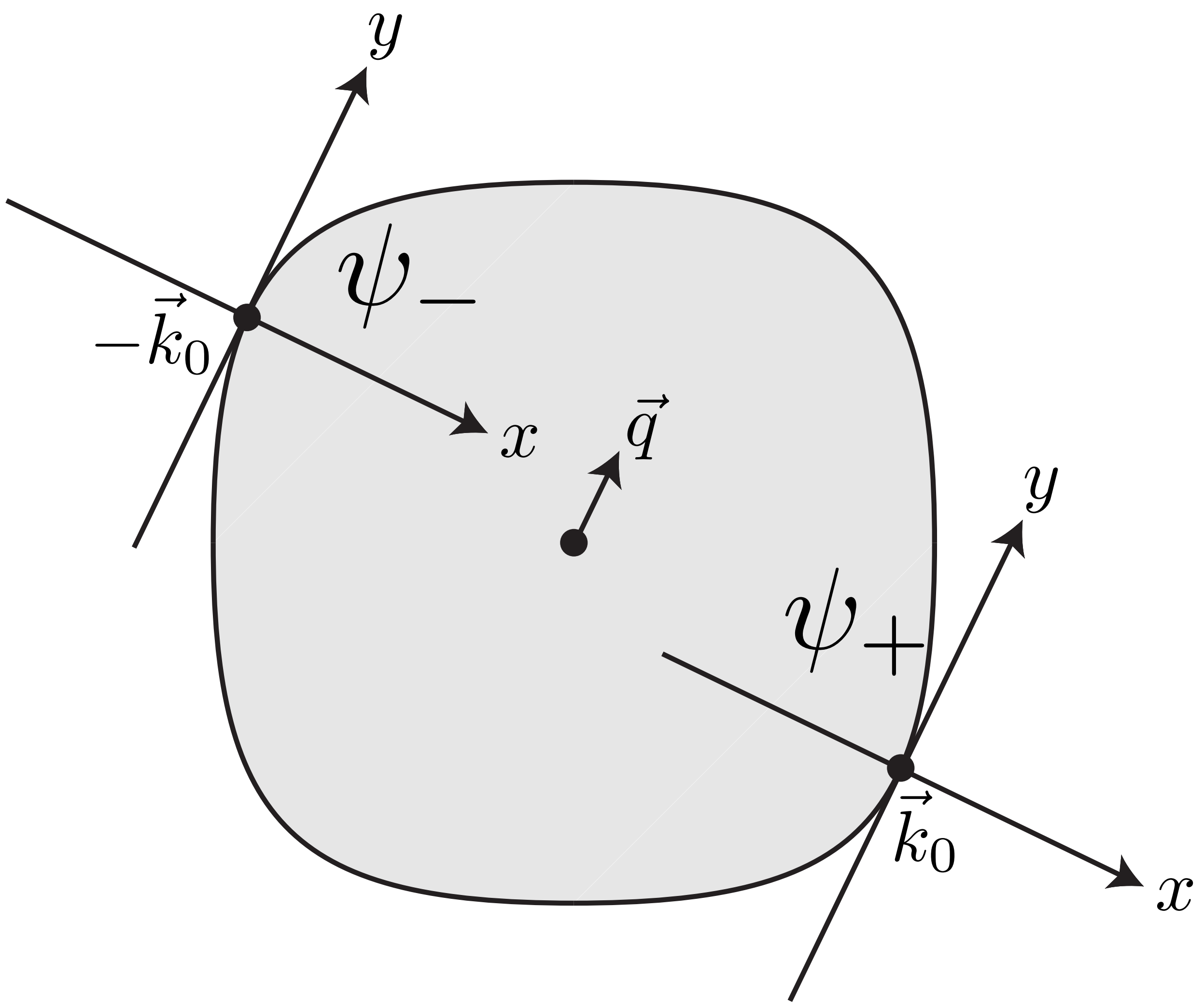
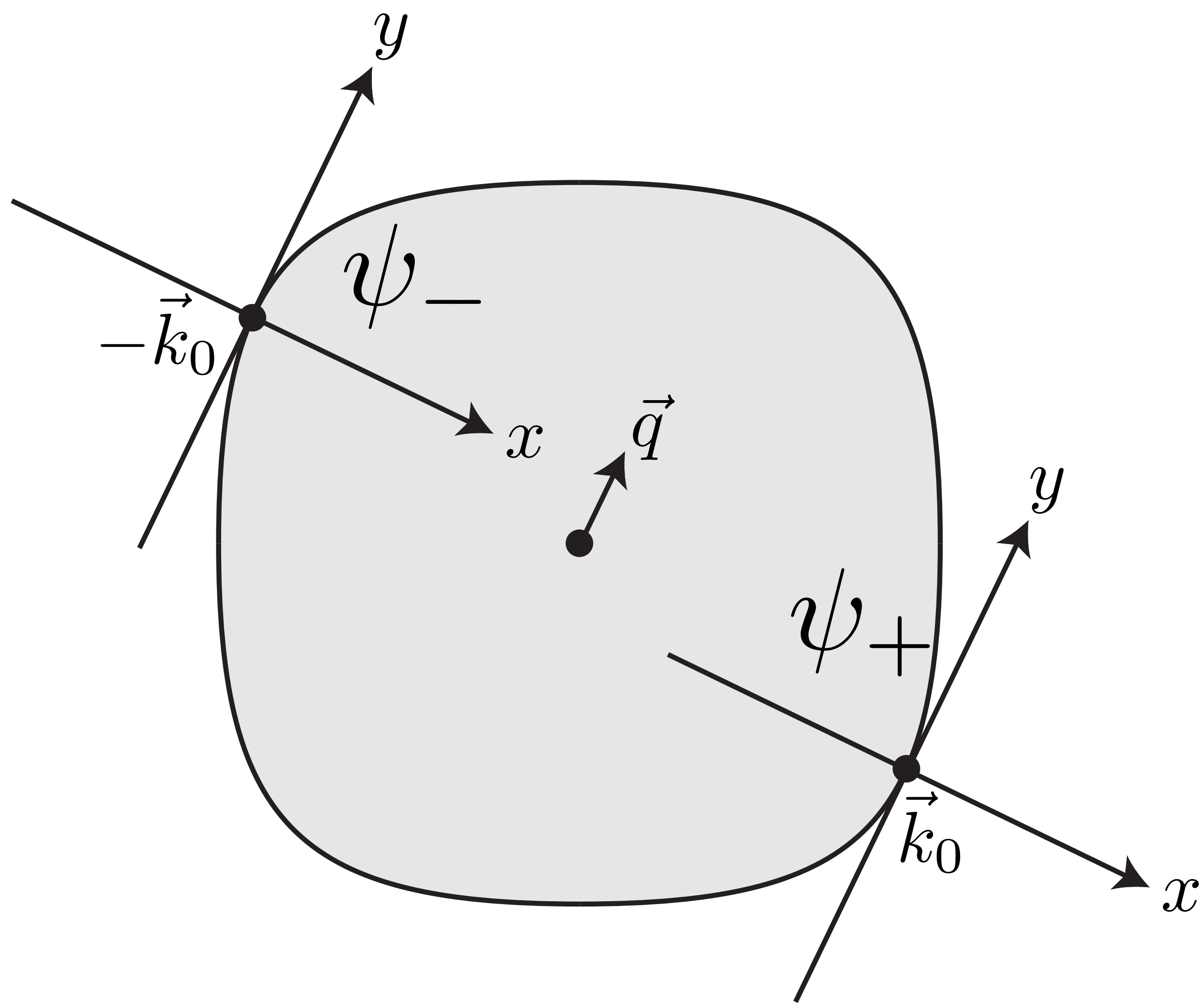


- ϕ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

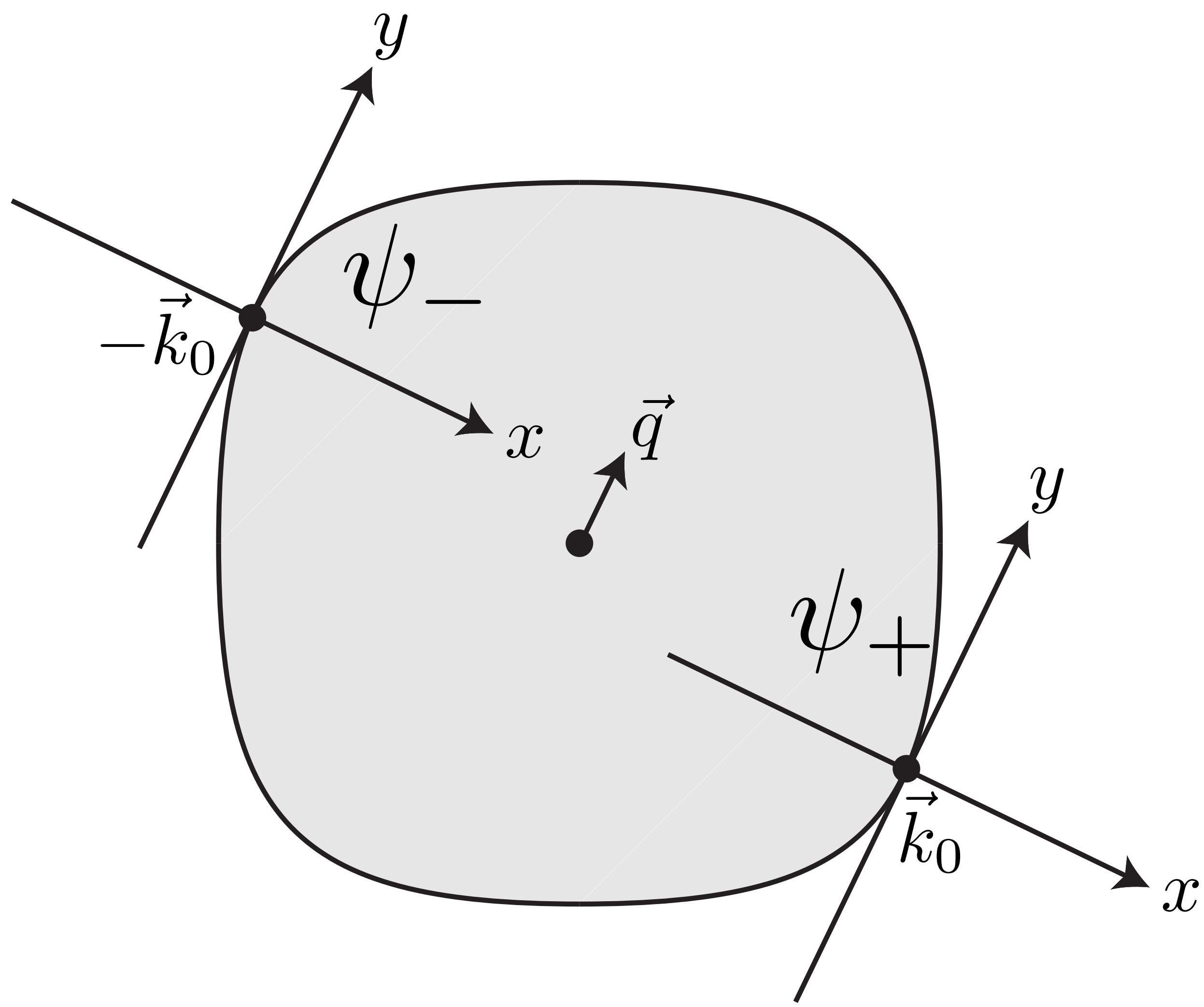
Quantum criticality of Ising-nematic ordering in a metal



- ϕ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

- Expand fermion kinetic energy at wavevectors about $\pm\vec{k}_0$ and boson (ϕ) kinetic energy about $\vec{q} = 0$.

Quantum criticality of Ising-nematic ordering in a metal



$$\mathcal{L}[\psi_\pm, \phi] =$$

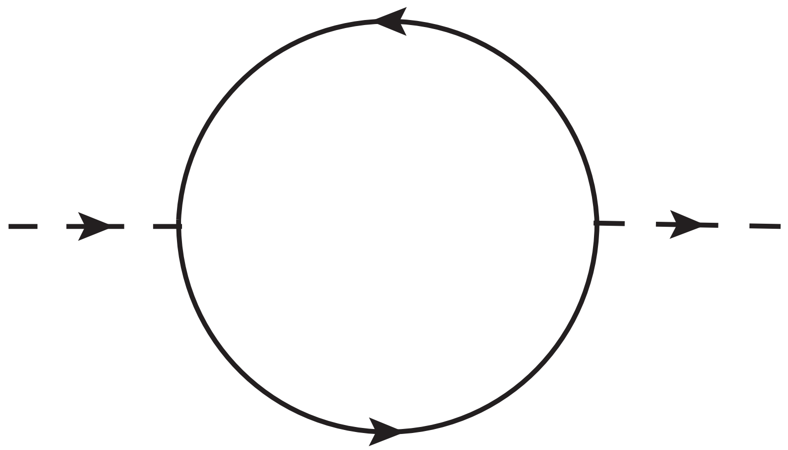
$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-\phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger\left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger\left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$



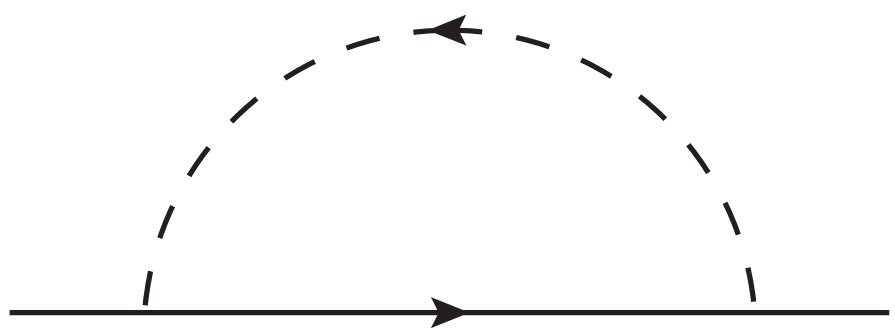
One loop ϕ self-energy with N_f fermion flavors:

$$\begin{aligned}\Sigma_\phi(\vec{q},\omega) &= N_f \int \frac{d^2k}{4\pi^2}\frac{d\Omega}{2\pi}\frac{1}{\left[-i(\Omega+\omega)+k_x+q_x+(k_y+q_y)^2\right]\left[-i\Omega-k_x+k_y^2\right]} \\ &= \frac{N_f}{4\pi}\frac{|\omega|}{|q_y|}\end{aligned}$$

Landau-damping

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2$$



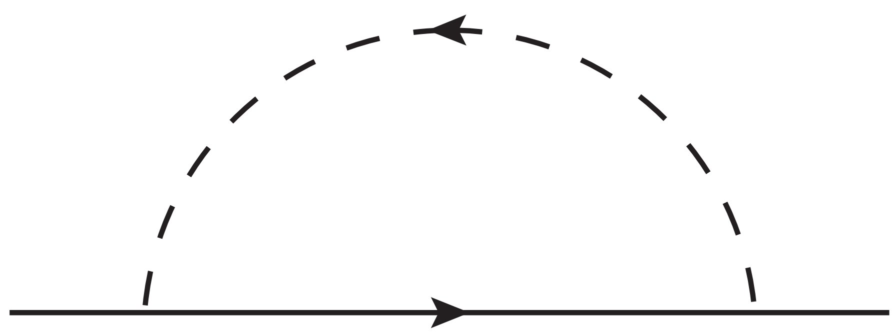
Electron self-energy at order $1/N_f$:

$$\Sigma(\vec{k},\Omega) = -\frac{1}{N_f}\int \frac{d^2q}{4\pi^2}\frac{d\omega}{2\pi}\frac{1}{\left[-i(\omega+\Omega)+k_x+q_x+(k_y+q_y)^2\right]\left[\dfrac{q_y^2}{g^2}+\dfrac{|\omega|}{|q_y|}\right]}$$

$$= -i\frac{2}{\sqrt{3}N_f}\left(\frac{g^2}{4\pi}\right)^{2/3}\operatorname{sgn}(\Omega)|\Omega|^{2/3}$$

Quantum criticality of Ising-nematic ordering in a metal

$$\boxed{\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}}$$



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$\boxed{\sim |\Omega|^{d/3} \text{ in dimension } d.}$

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_- - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

Schematic form of ϕ and fermion Green's functions in d dimensions

$$D(\vec{q}, \omega) = \frac{1/N_f}{q_\perp^2 + \dfrac{|\omega|}{|q_\perp|}} \quad , \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_\perp^2 - i\mathrm{sgn}(\omega)|\omega|^{d/3}/N_f}$$

In the boson case, $q_\perp^2 \sim \omega^{1/z_b}$ with $z_b = 3/2$.
In the fermion case, $q_x \sim q_\perp^2 \sim \omega^{1/z_f}$ with $z_f = 3/d$.

Note $z_f < z_b$ for $d > 2 \Rightarrow$ Fermions have *higher* energy than bosons, and perturbation theory in g is OK.
Strongly-coupled theory in $d = 2$.

Quantum criticality of Ising-nematic ordering in a metal

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Schematic form of ϕ and fermion Green's functions in $d = 2$

$$D(\vec{q},\omega) = \frac{1/N_f}{q_y^2 + \dfrac{|\omega|}{|q_y|}} \quad , \quad G_f(\vec{q},\omega) = \frac{1}{q_x + q_y^2 - i\,\mathrm{sgn}(\omega)|\omega|^{2/3}/N_f}$$

In *both* cases $q_x \sim q_y^2 \sim \omega^{1/z}$, with $z = 3/2$. Note that the bare term $\sim \omega$ in G_f^{-1} is irrelevant.

Strongly-coupled theory without quasiparticles.

Quantum criticality of Ising-nematic ordering in a metal

$$\begin{aligned}\mathcal{L} &= \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- \\ &\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2\end{aligned}$$

Simple scaling argument for $z = 3/2$.

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2$$

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$$
\begin{aligned}
\mathcal{L} \;&=\; \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2\right)\psi_- \\
&\quad - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y \phi\right)^2
\end{aligned}
$$

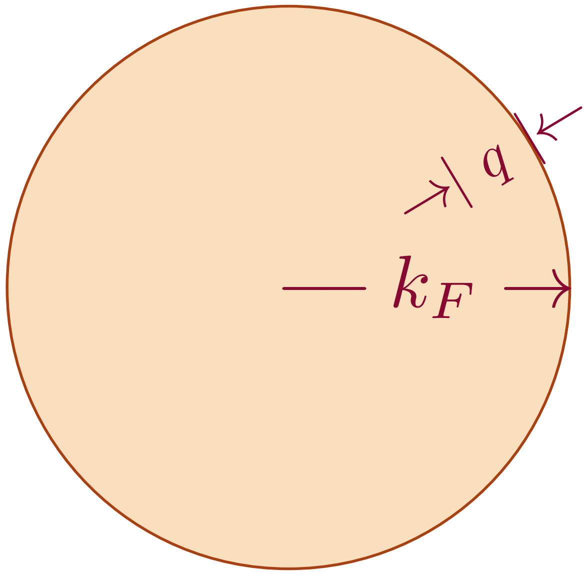
Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$
\begin{aligned}
\phi &\to \phi\, s \\
\psi &\to \psi\, s^{(2z+1)/4} \\
g &\to g\, s^{(3-2z)/4}
\end{aligned}
$$

So the action is invariant provided $z = 3/2$.

<u>FL</u>
<u>Fermi</u>
<u>liquid</u>

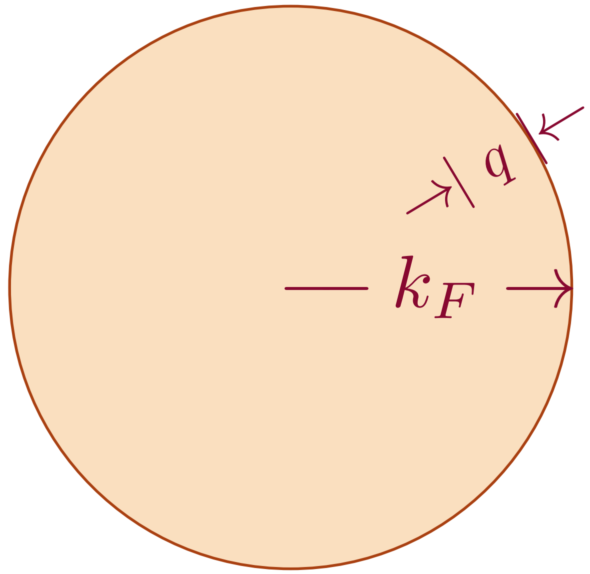


- $k_F^d \sim \mathcal{Q}$, the fermion density

- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.

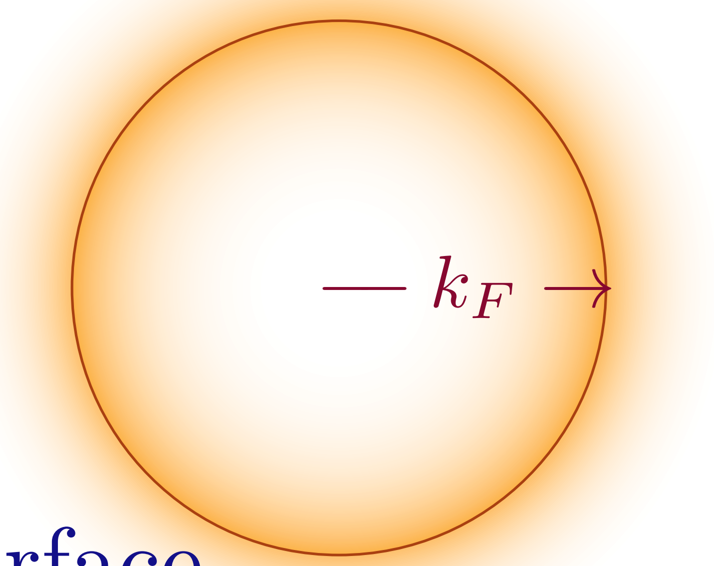
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

FL Fermi liquid

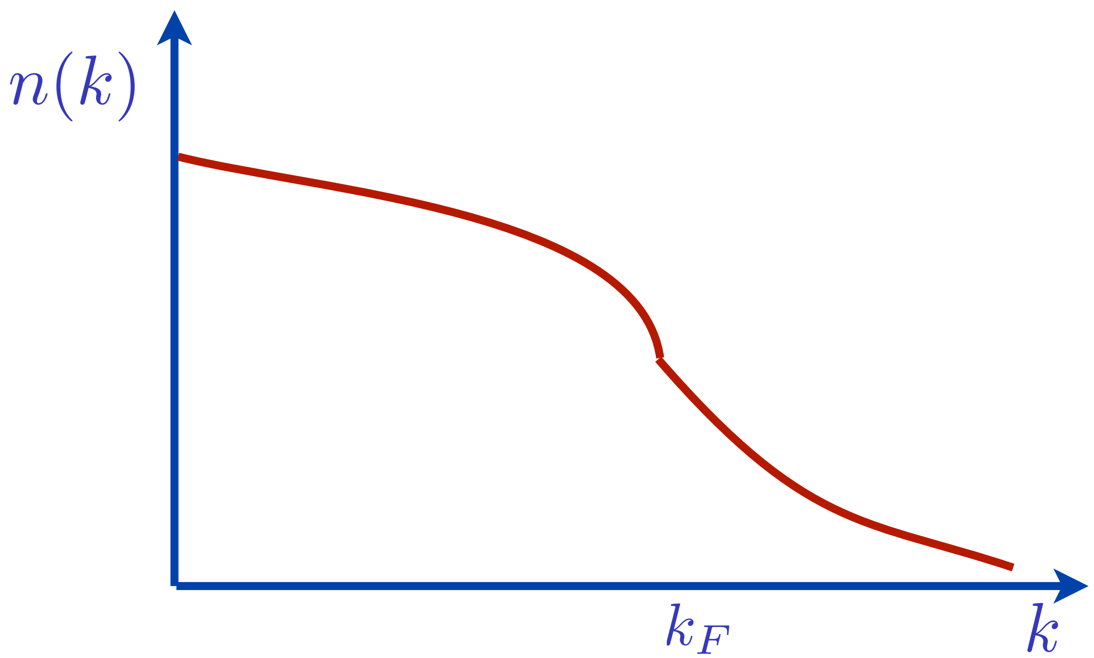


- $k_F^d \sim \mathcal{Q}$, the fermion density
- Sharp fermionic excitations near Fermi surface with $\omega \sim |q|^z$, and $z = 1$.
- Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.

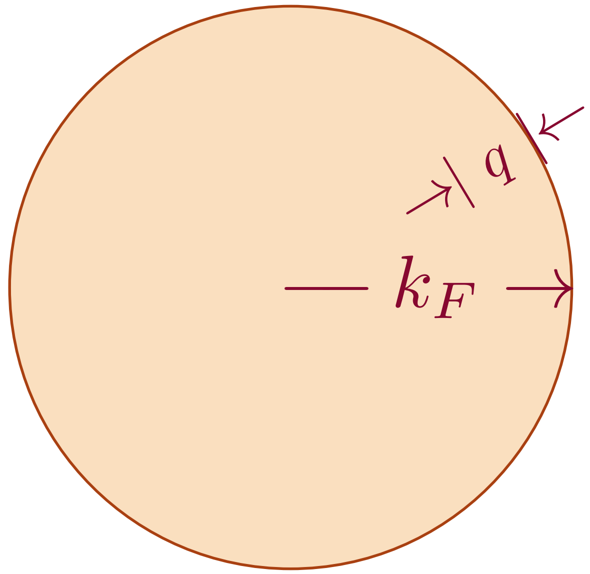
NFL Nematic QCP



- Fermi surface with $k_F^d \sim \mathcal{Q}$.



FL Fermi liquid



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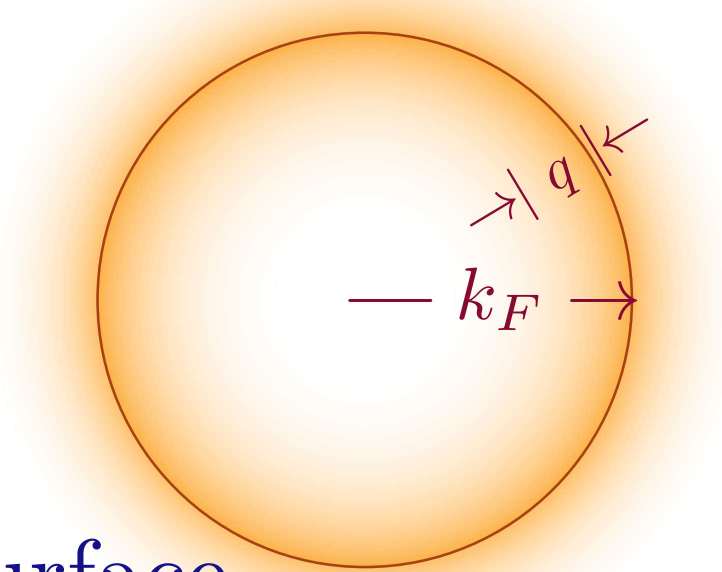
NFL Nematic QCP

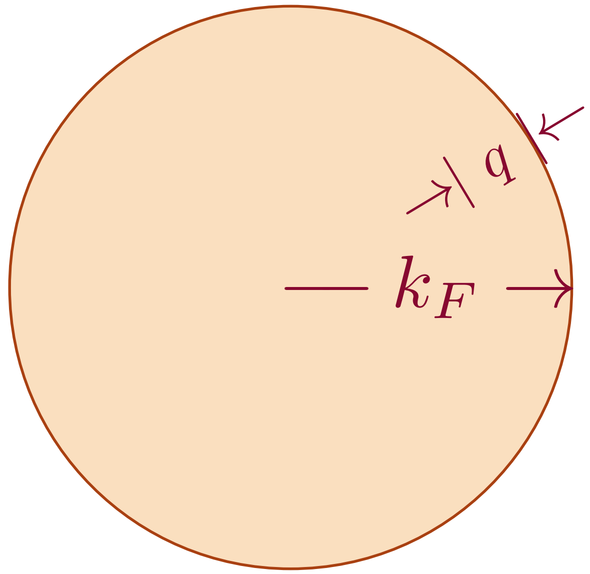


- Fermi surface with $k_F^d \sim \mathcal{Q}$.

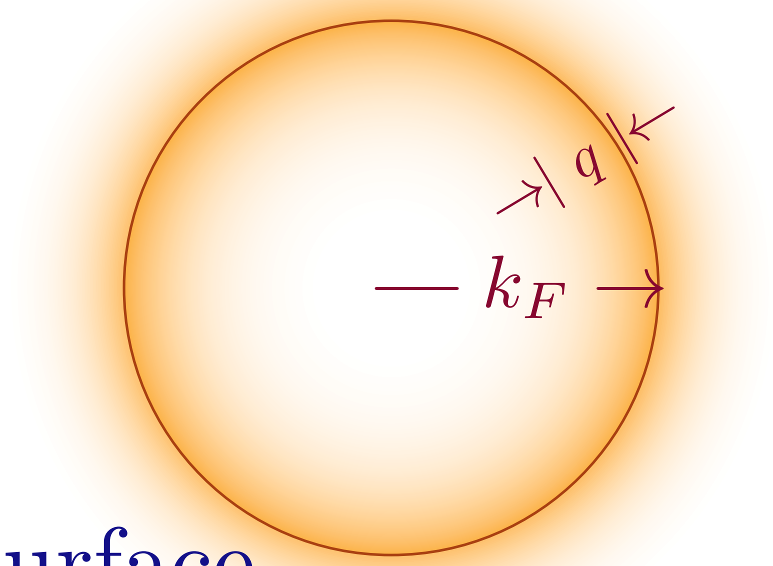
- Diffuse fermionic excitations with $z = 3/2$ to three loops.

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

FL Fermi liquid

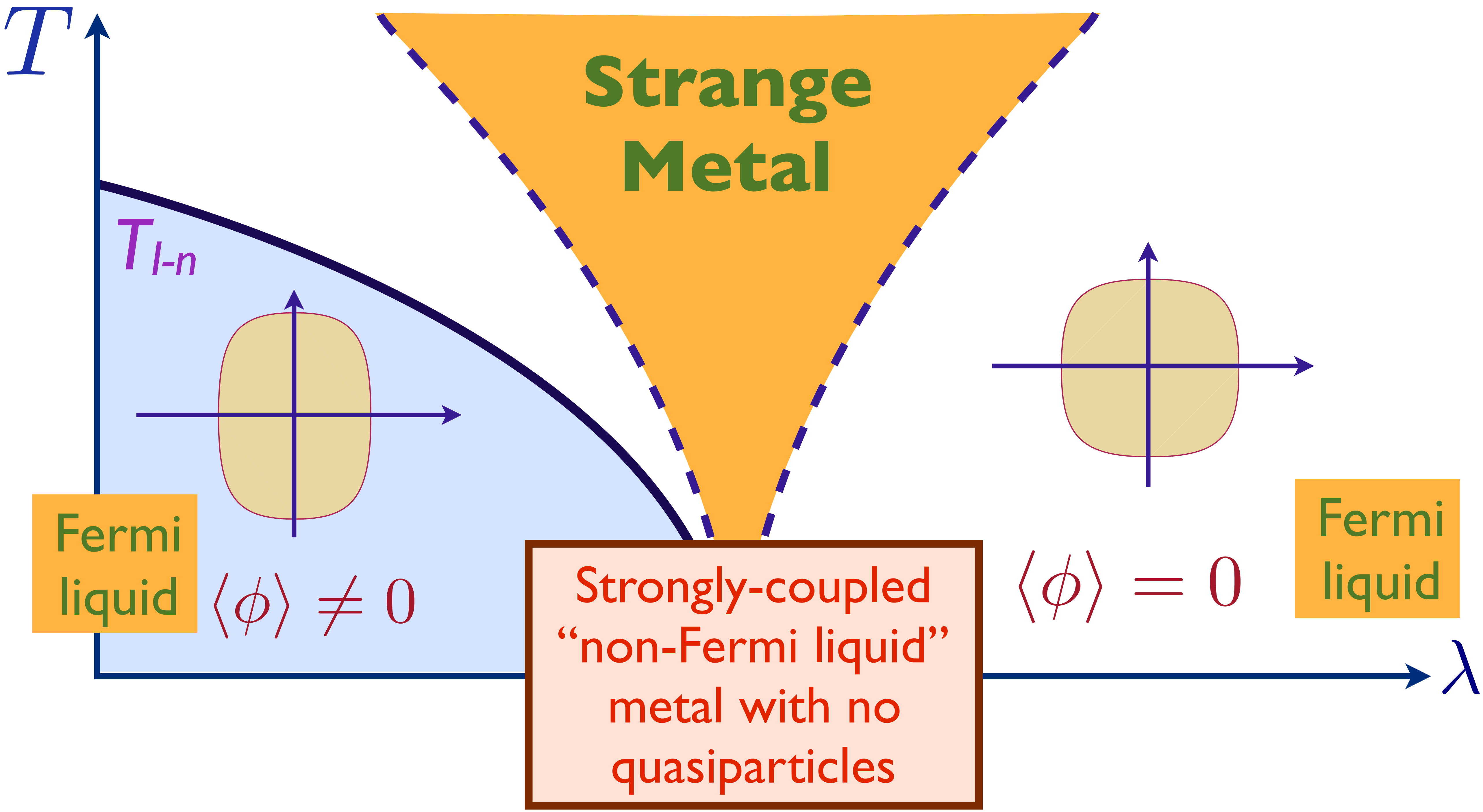


NFL Nematic QCP



FL Fermi liquid	NFL Nematic QCP
• $k_F^d \sim \mathcal{Q}$, the fermion density	• Fermi surface with $k_F^d \sim \mathcal{Q}$.
• Sharp fermionic excitations near Fermi surface with $\omega \sim \lvert q \rvert^z$, and $z = 1$.	• Diffuse fermionic excitations with $z = 3/2$ to three loops.
• Entropy density $S \sim T^{(d-\theta)/z}$ with violation of hyperscaling exponent $\theta = d - 1$.	• $S \sim T^{(d-\theta)/z}$ with $\theta = d - 1$.

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ

The Hubbard Model

$$H = -\sum_{i<j} t_{ij} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^{\dagger} c_{i\alpha}$$

$t_{ij} \rightarrow$ "hopping". $U \rightarrow$ local repulsion, $\mu \rightarrow$ chemical potential

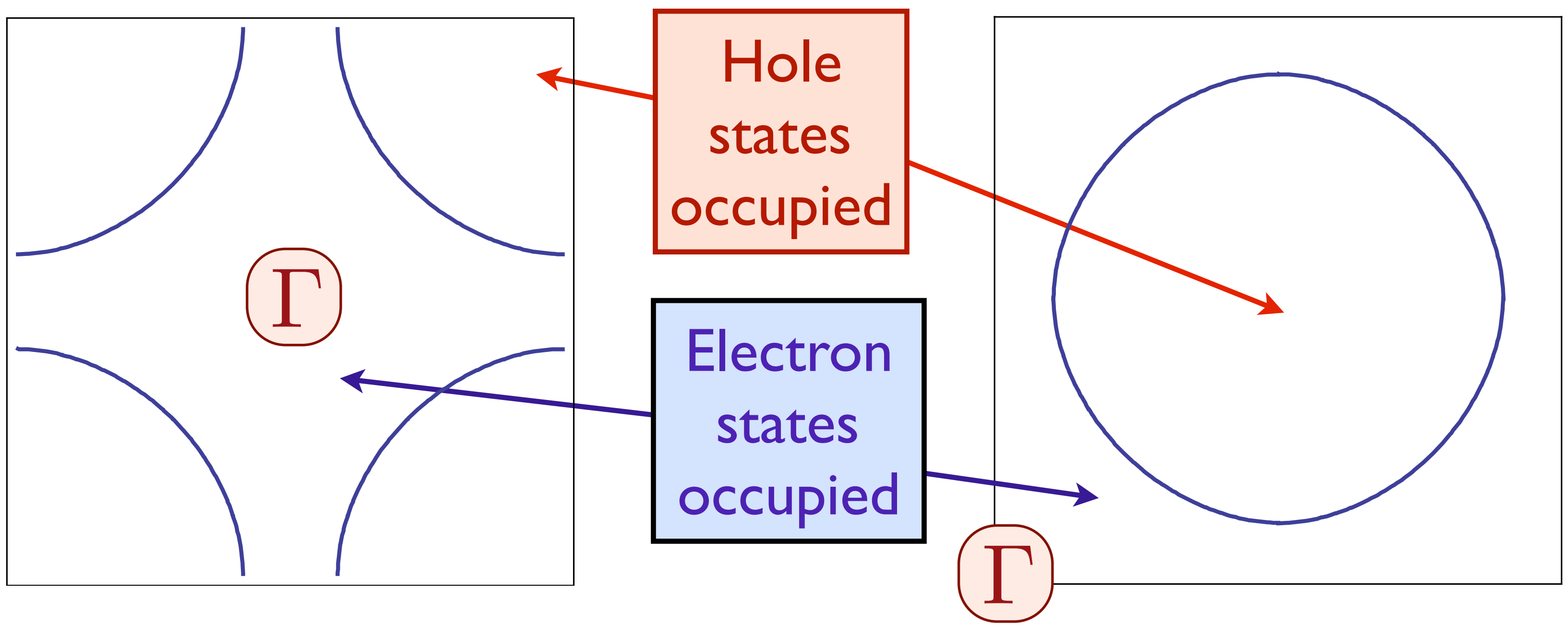
Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c_{i\alpha}^{\dagger} c_{i\alpha}$$

$$\begin{gathered} c_{i\alpha}^{\dagger} c_{j\beta} + c_{j\beta} c_{i\alpha}^{\dagger} = \delta_{ij} \delta_{\alpha\beta} \\ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0 \end{gathered}$$

Will study on the square lattice

Fermi surfaces in electron- and hole-doped cuprates



Effective Hamiltonian for quasiparticles:

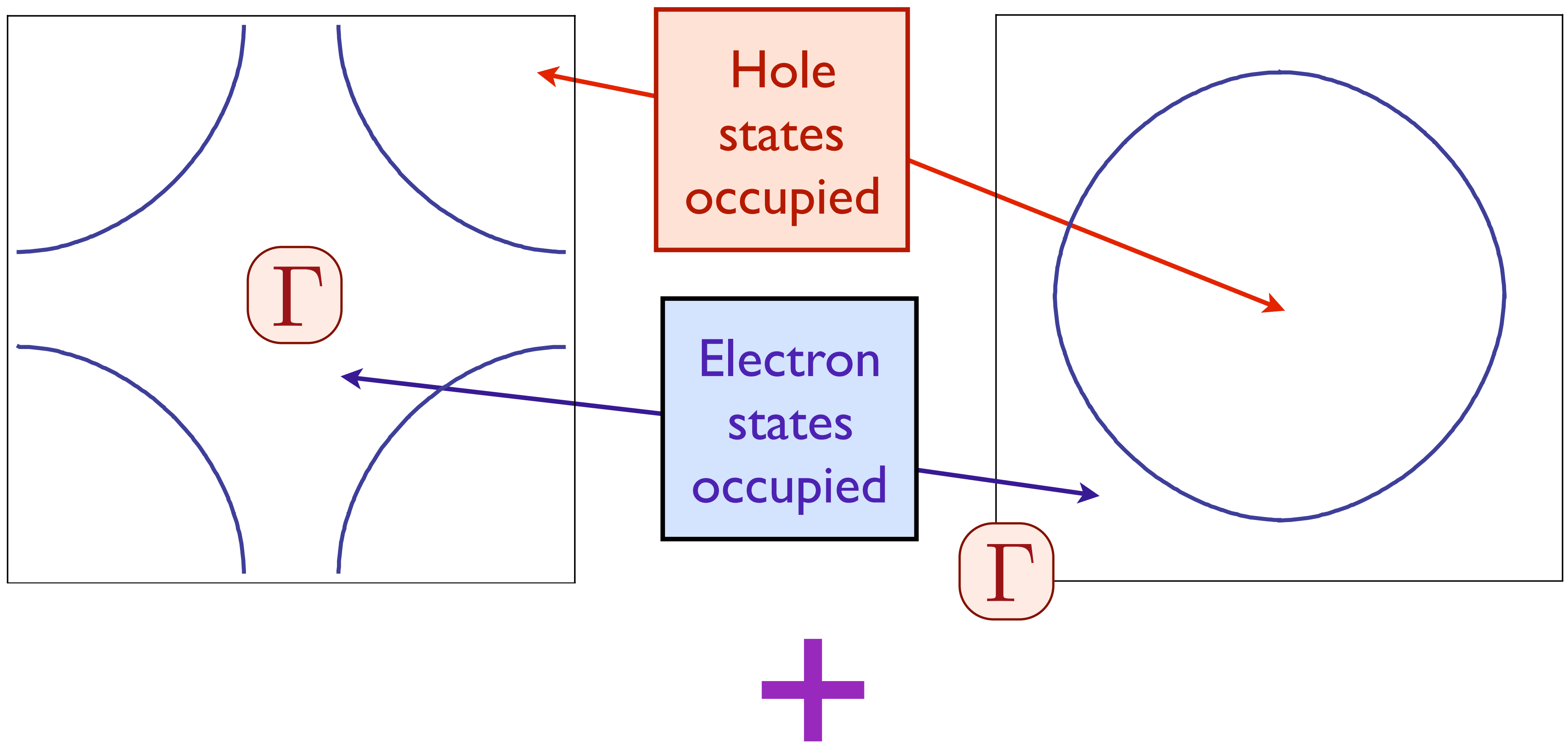
$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}$$

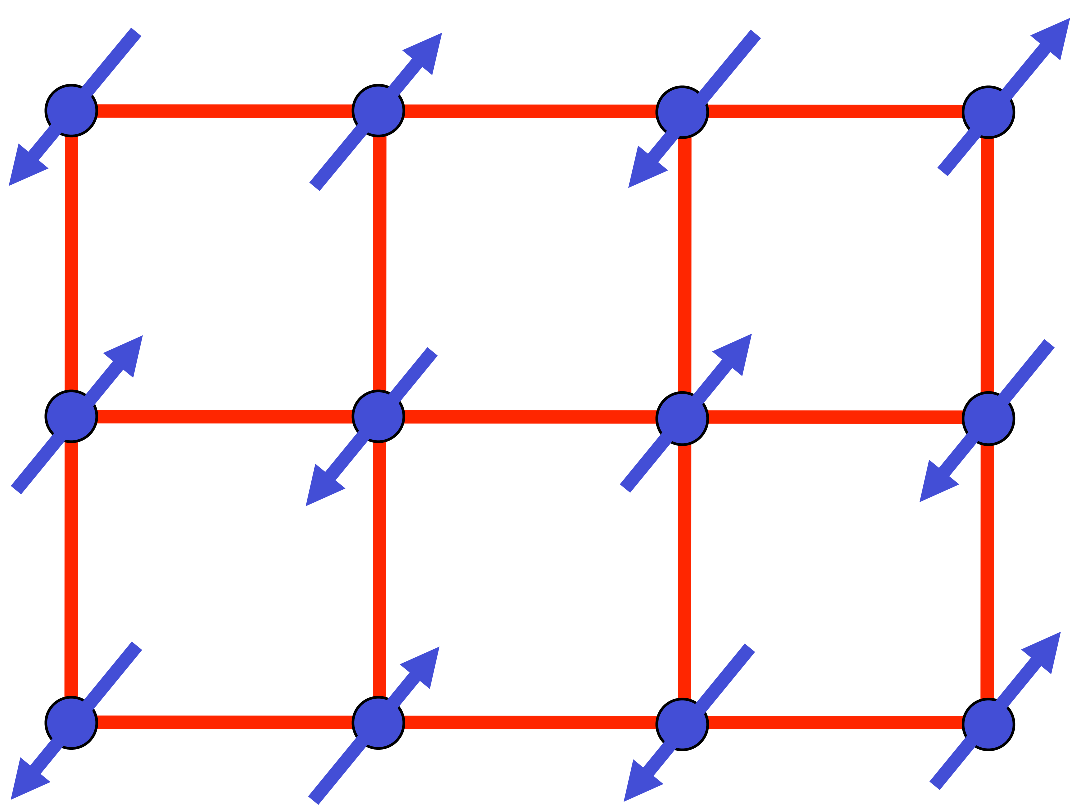
with t_{ij} non-zero for first, second and third neighbor, leads to satisfactory agreement with experiments. The area of the occupied electron states, $\mathcal{A}_e$, from Luttinger's theory is

$$\mathcal{A}_e = \begin{cases} 2\pi^2(1-x) & \text{for hole-doping } x \\ 2\pi^2(1+p) & \text{for electron-doping } p \end{cases}$$

The area of the occupied hole states, $\mathcal{A}_h$, which form a closed Fermi surface and so appear in quantum oscillation experiments is $\mathcal{A}_h = 4\pi^2 - \mathcal{A}_e$.

Fermi surface+antiferromagnetism





The electron spin polarization obeys

$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau) e^{i\mathbf{K}\cdot\mathbf{r}}$$

where $\mathbf{K}$ is the ordering wavevector.

Fermi surface+antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3}\vec{S}^2 + \frac{U}{4} \tag{1}$$

Then we decouple the interaction via

$$\exp\left(\frac{2U}{3}\sum_i \int d\tau \vec{S}_i^2\right) = \int \mathcal{D}\vec{J}_i(\tau)\exp\left(-\sum_i \int d\tau \left[\frac{3}{8U}\vec{J}_i^2 - \vec{J}_i\vec{S}_i\right]\right) \tag{2}$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for $\vec{J}_i$. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field $\vec{\varphi}_i$ where

$$\vec{J}_i = \vec{\varphi}_i\, e^{i\mathbf{K}\cdot\mathbf{r}_i} \tag{3}$$

Fermi surface+antiferromagnetism

In this manner, we obtain the "spin-fermion" model

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right)$$

$$\mathcal{S} = \int d\tau \sum_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger \left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

$$- \lambda \int d\tau \sum_i c_{i\alpha}^\dagger \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i}$$

$$+ \int d\tau d^2 r \left[\frac{1}{2} \left(\nabla_r \vec{\varphi} \right)^2 + \frac{1}{2} \left(\partial_\tau \vec{\varphi} \right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]$$

Fermi surface+antiferromagnetism

In the Hamiltonian form (ignoring, for now, the time dependence of $\vec{\varphi}$), the coupling between $\vec{\varphi}$ and the electrons takes the form

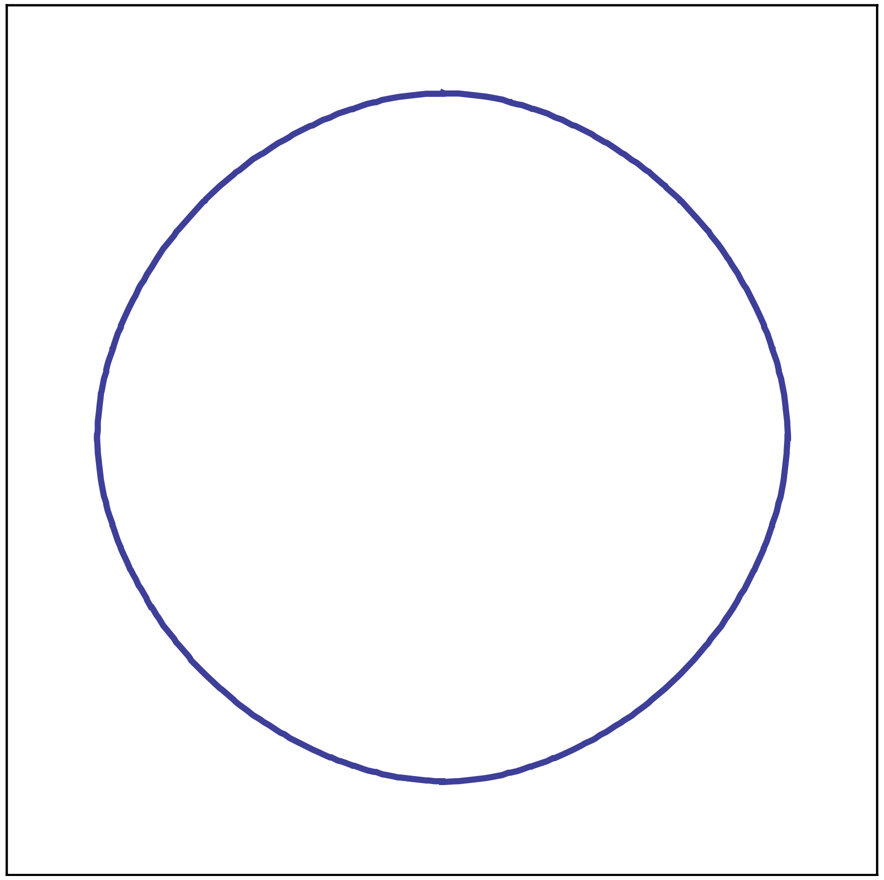
$$H_{\mathrm{sdw}} = \lambda \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} \vec{\varphi}_{\mathbf{q}} \cdot c^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}$$

where $\vec{\sigma}$ are the Pauli matrices, the boson momentum $\mathbf{q}$ is small, while the fermion momenum $\mathbf{k}$ extends over the entire Brillouin zone. In the antiferromagnetically ordered state, we may take $\vec{\varphi} \propto (0, 0, 1)$, and the electron dispersions obtained by diagonalizing $H_0 + H_{\mathrm{sdw}}$ are

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right)^2 + \lambda^2 |\vec{\varphi}|^2}$$

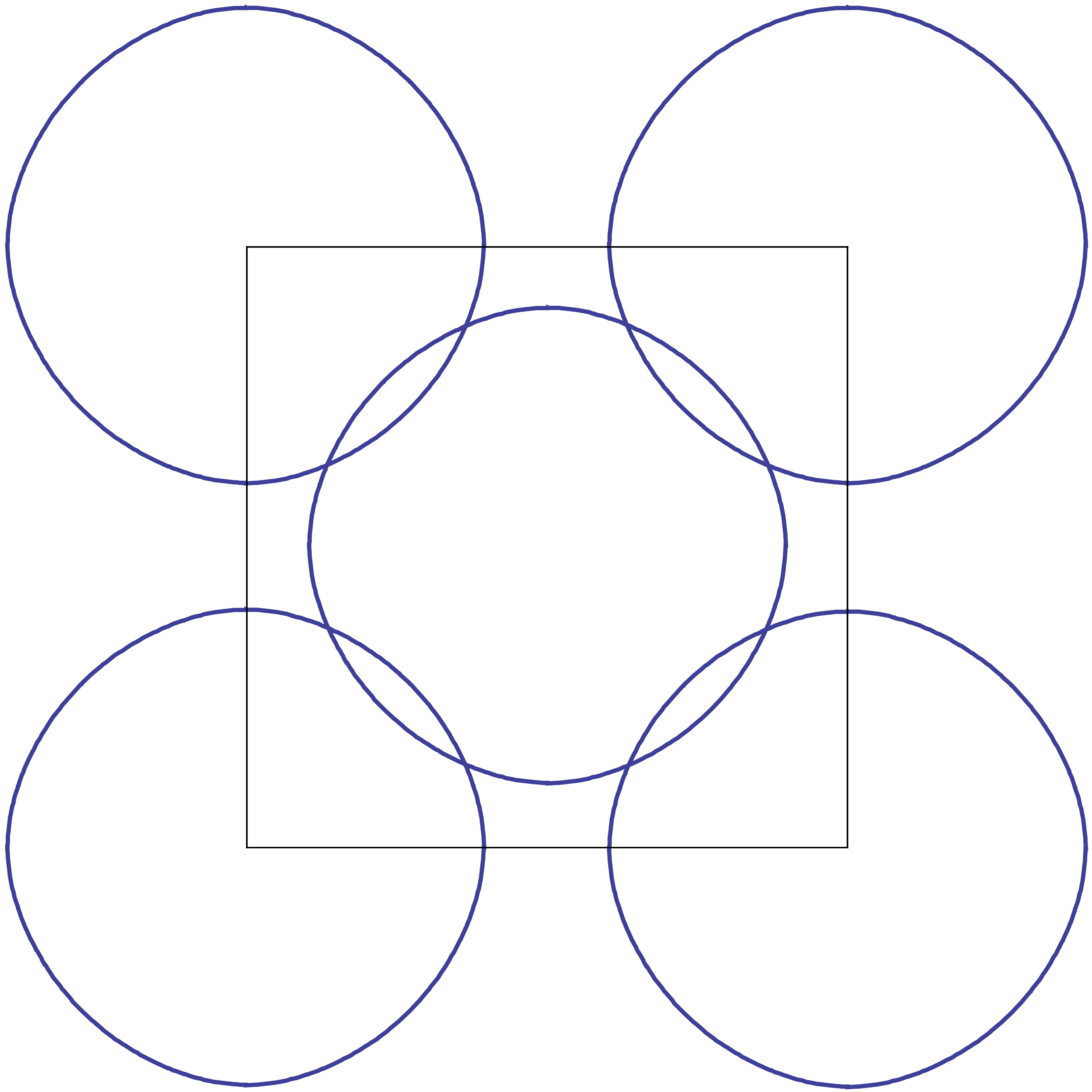
This leads to the Fermi surfaces shown in the following slides as a function of increasing $|\vec{\varphi}|$.

Fermi surface+antiferromagnetism



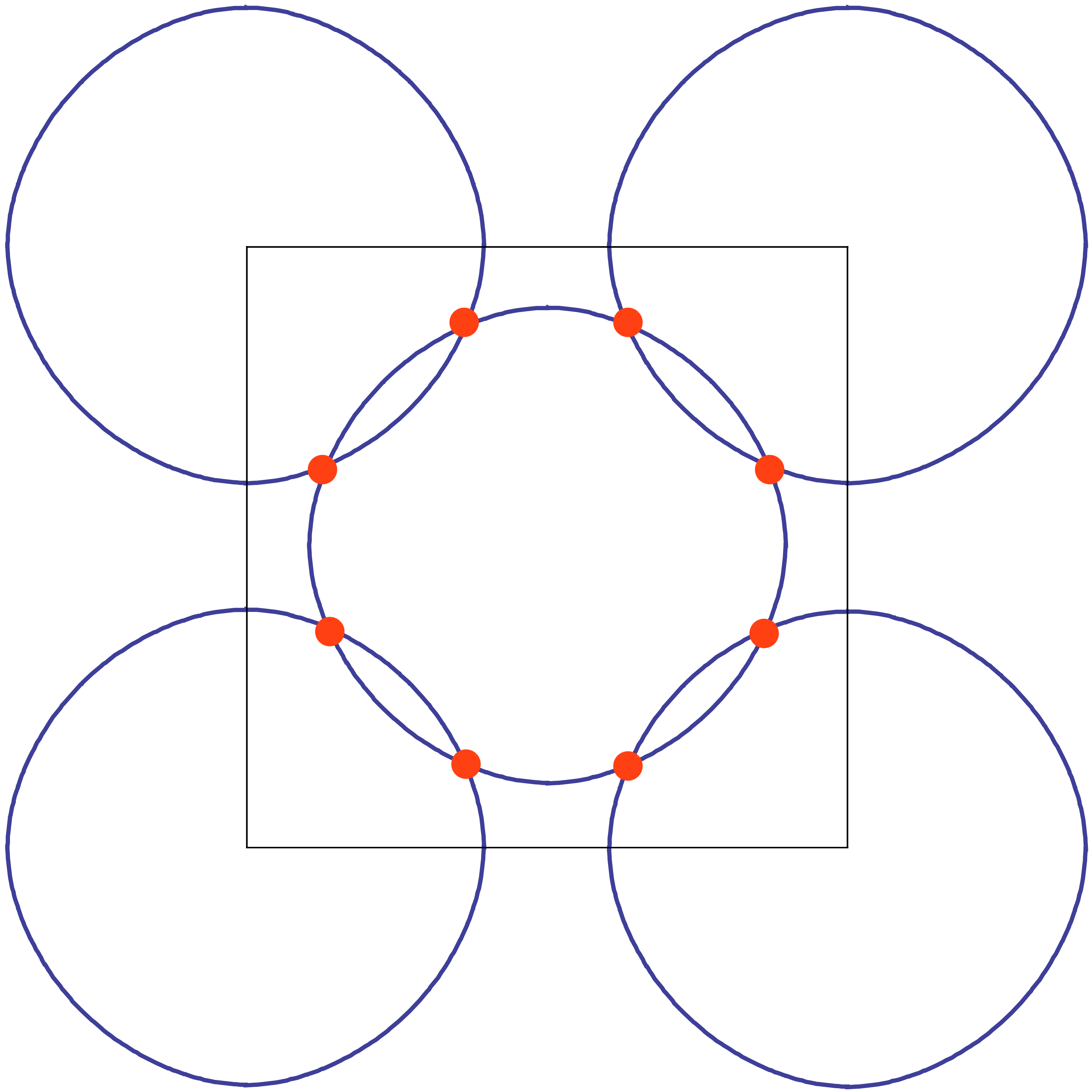
Metal with “large” Fermi surface

Fermi surface+antiferromagnetism



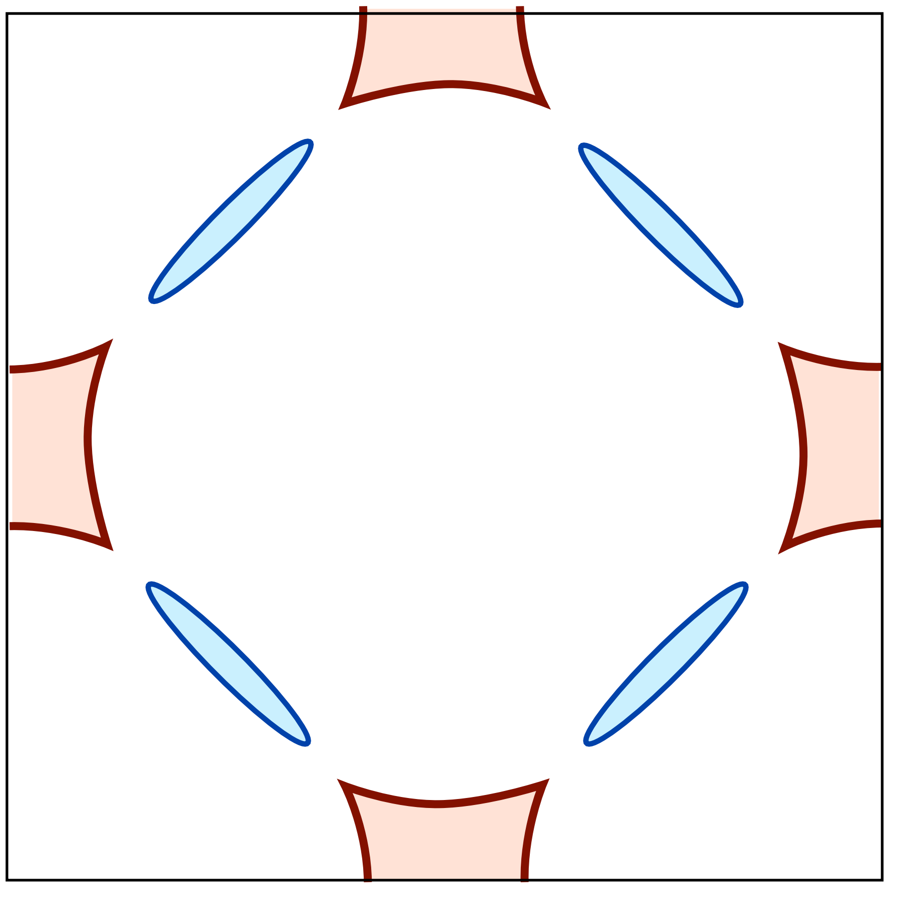
Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$.

Fermi surface+antiferromagnetism



"Hot" spots

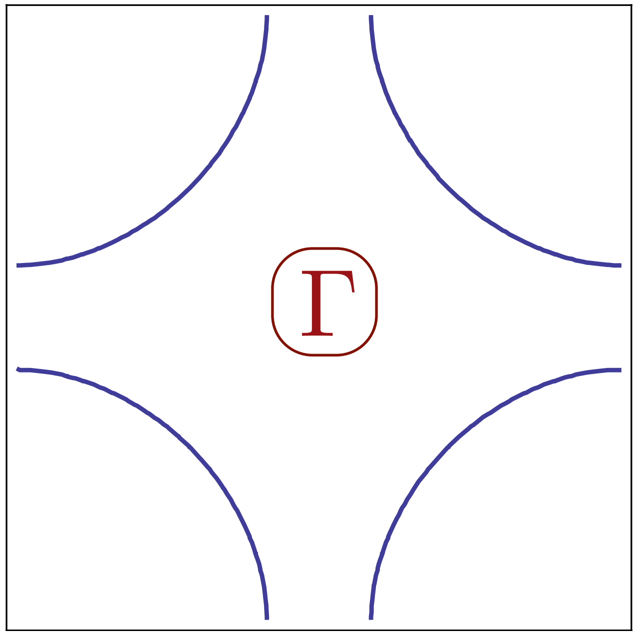
Fermi surface+antiferromagnetism



Electron and hole pockets in antiferromagnetic phase with $\langle \vec{\varphi} \rangle \neq 0$

Square lattice Hubbard model with hole doping

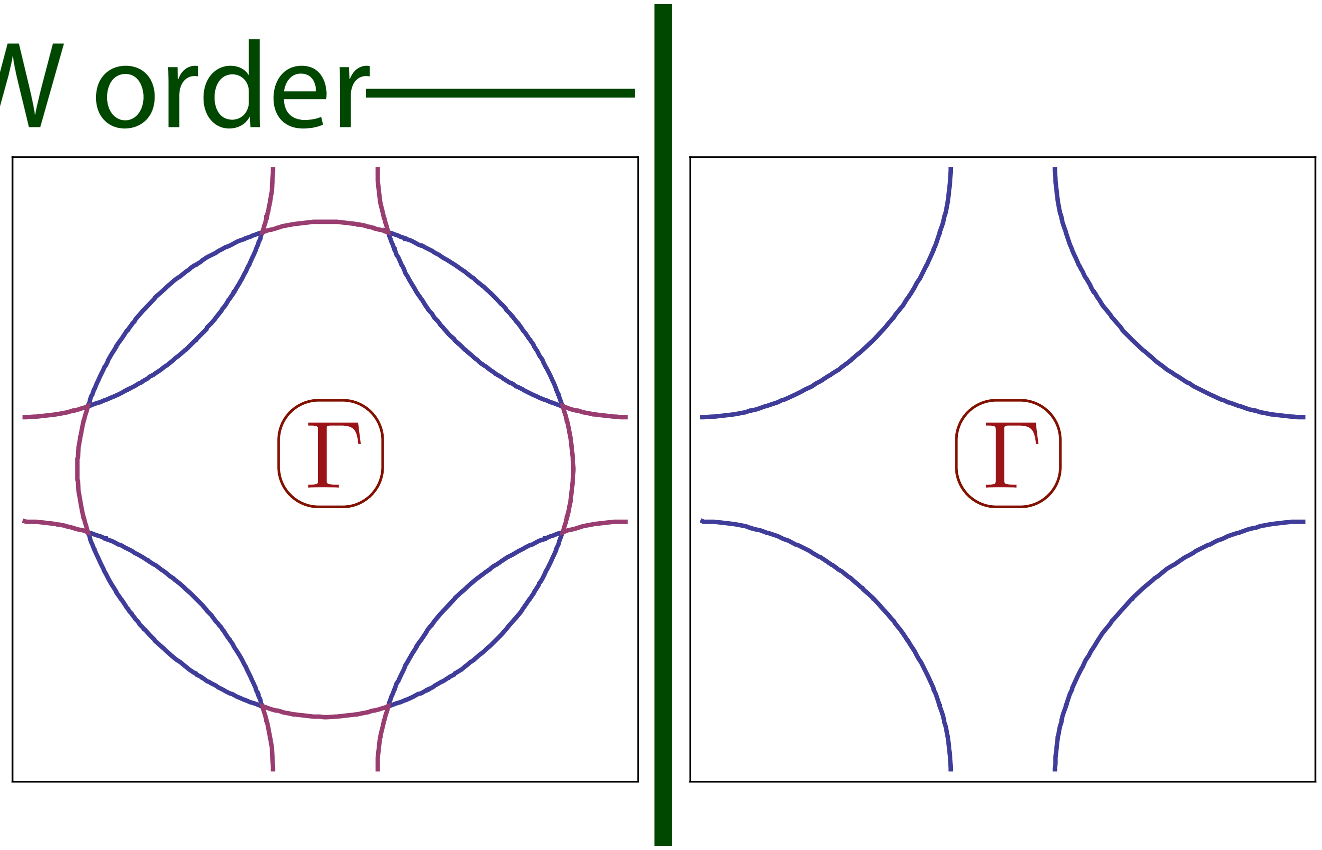
⟵ Increasing SDW order ——



S. Sachdev, A.V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A.V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Square lattice Hubbard model with hole doping

Increasing SDW order



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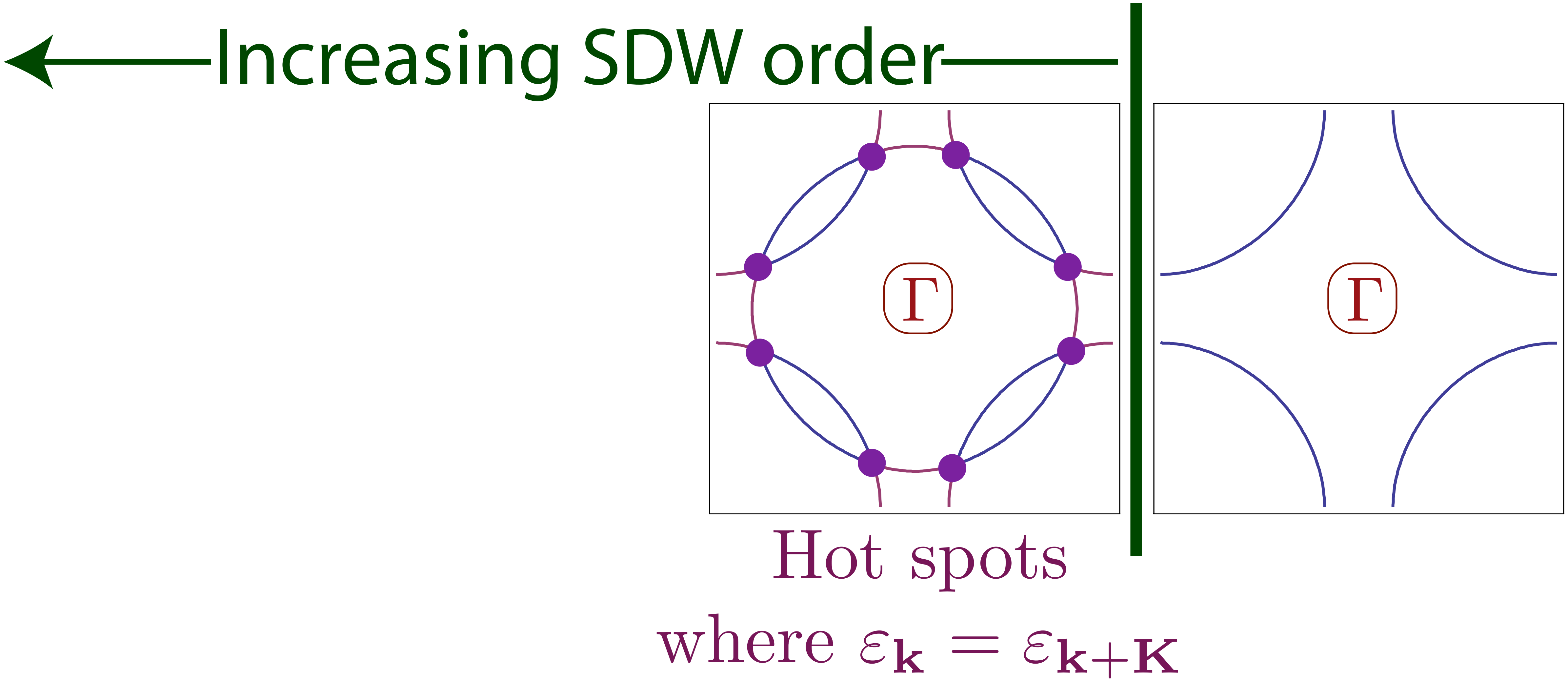
Square lattice Hubbard model with hole doping



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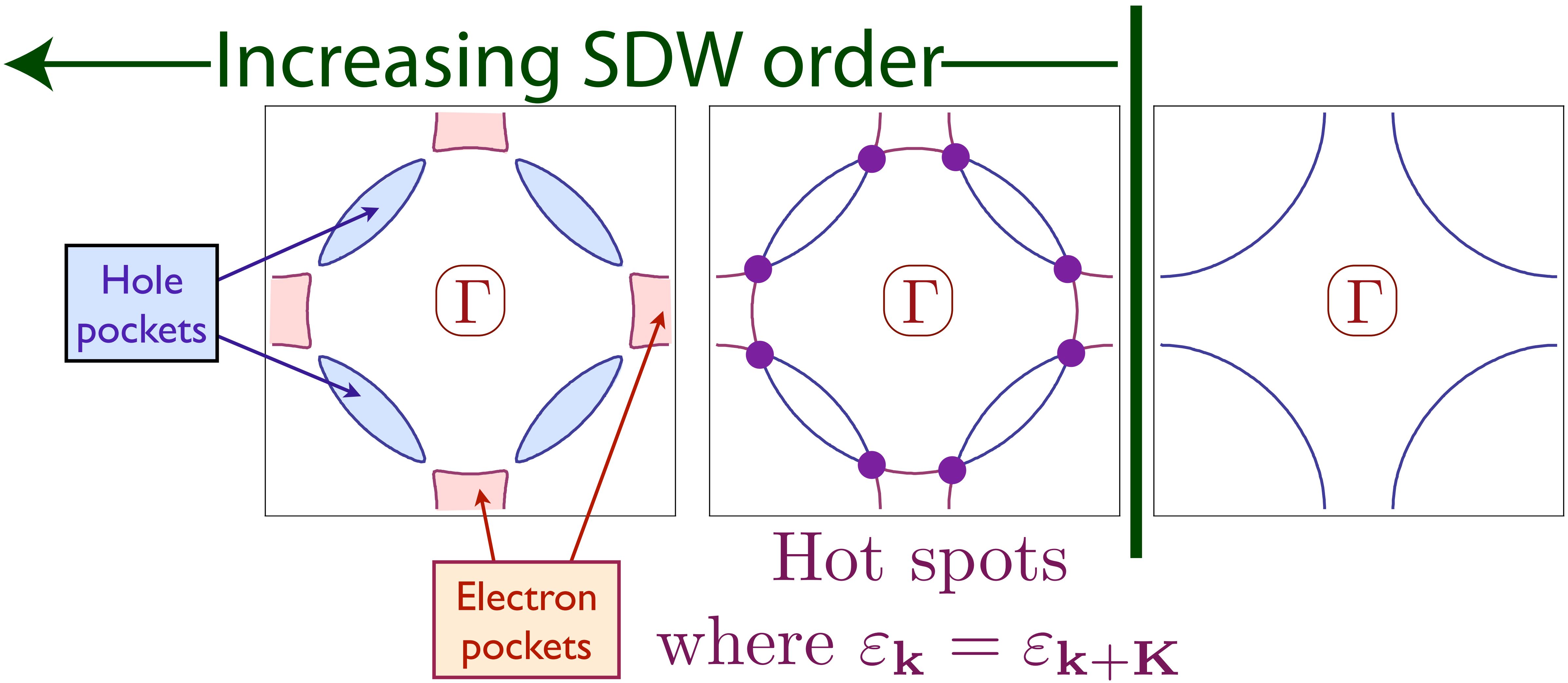
Square lattice Hubbard model with hole doping



Fermi surface breaks up at hot spots into electron and hole "pockets"

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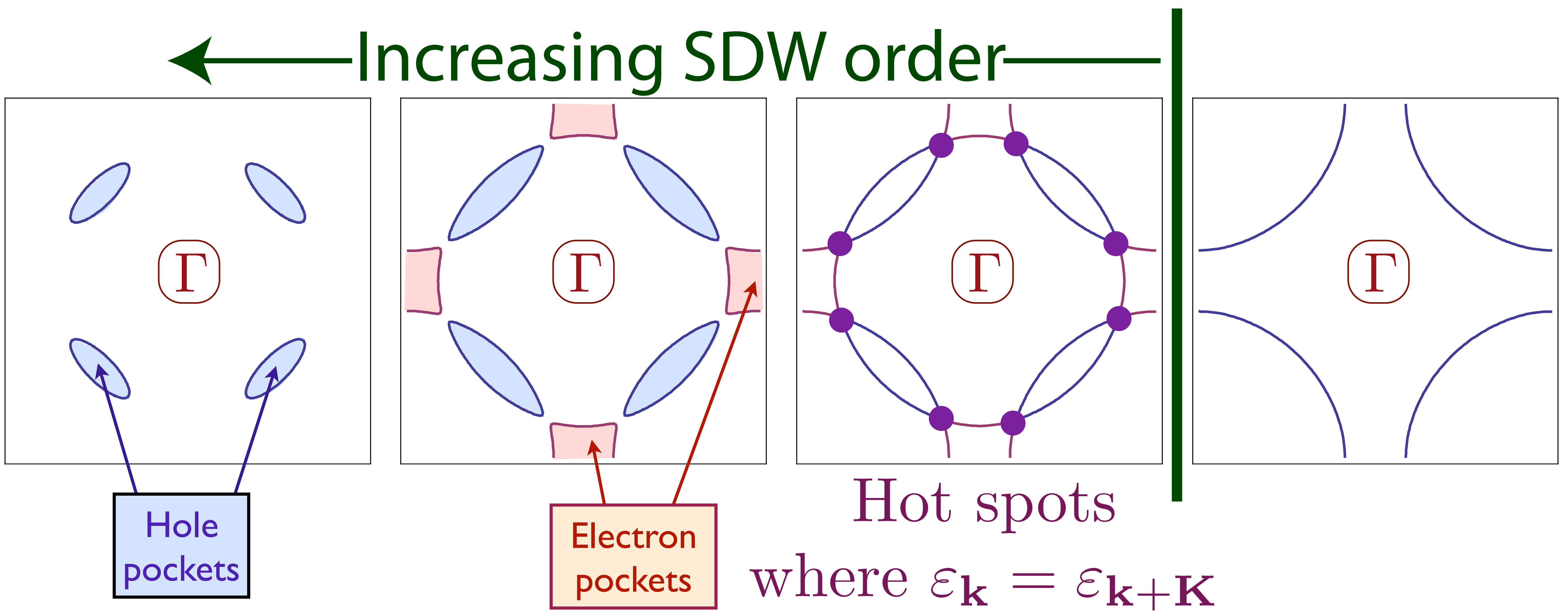

Fermi surface breaks up at hot spots into electron and hole "pockets"

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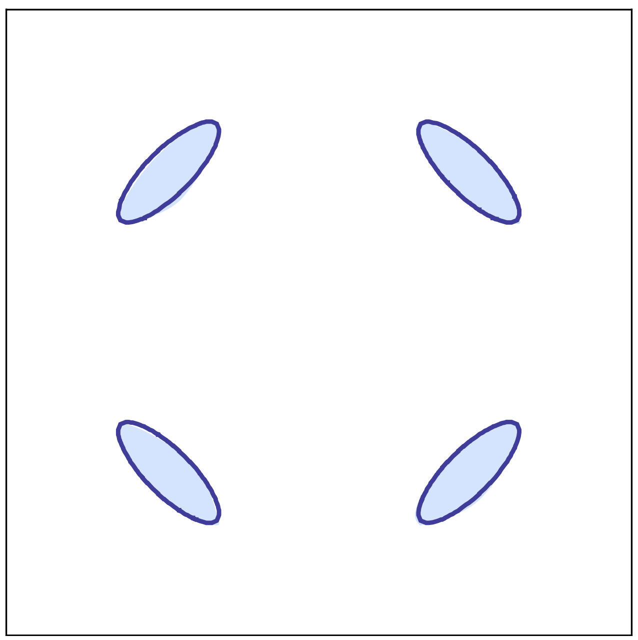
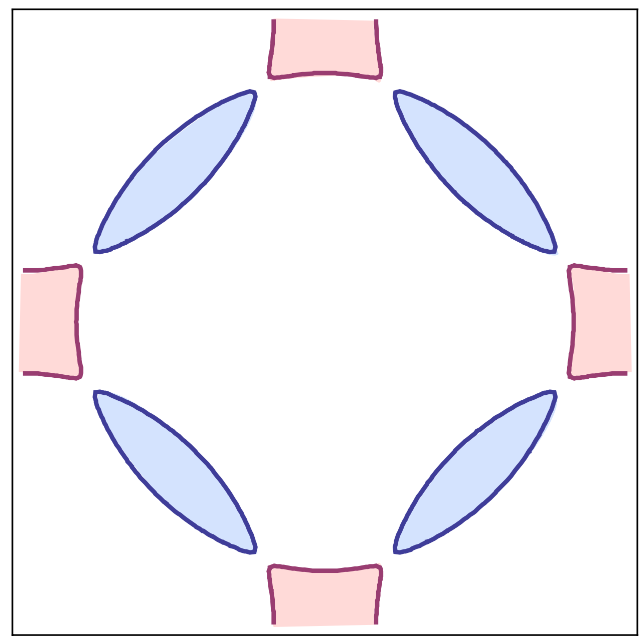
Square lattice Hubbard model with hole doping

$\langle \vec{\varphi} \rangle \neq 0$ and large



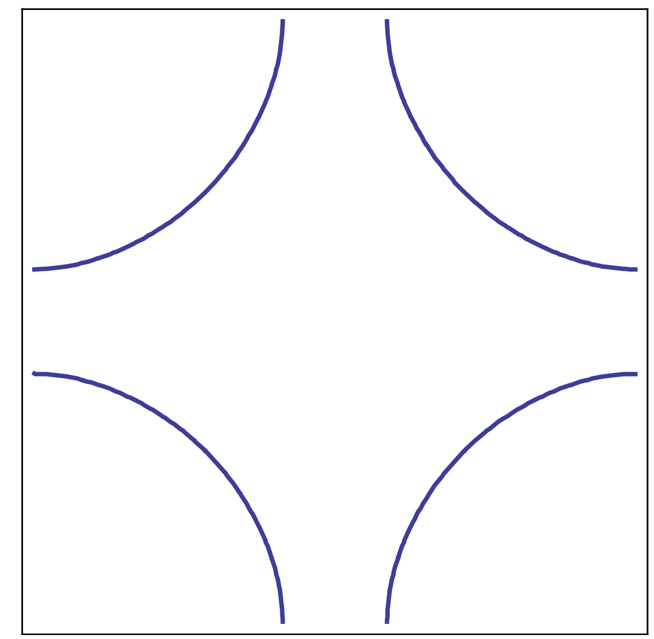
Metal with hole pockets

$\langle \vec{\varphi} \rangle \neq 0$ and small



Metal with electron and hole pockets

$\langle \vec{\varphi} \rangle = 0$

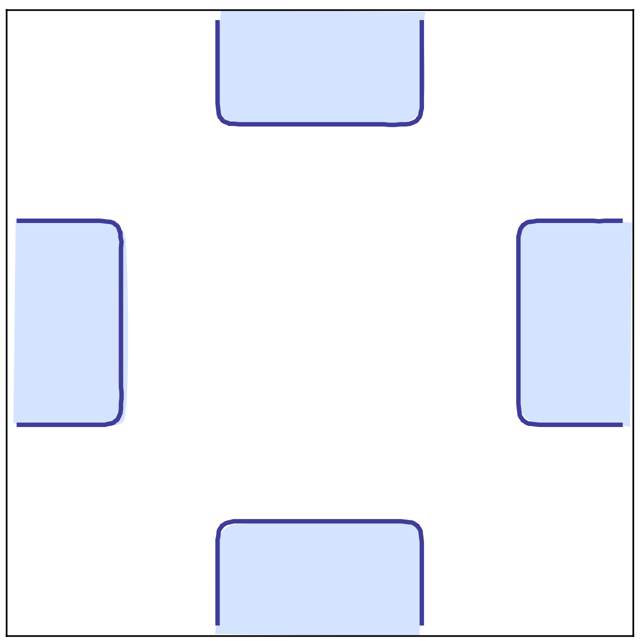


Metal with "large" Fermi surface

s

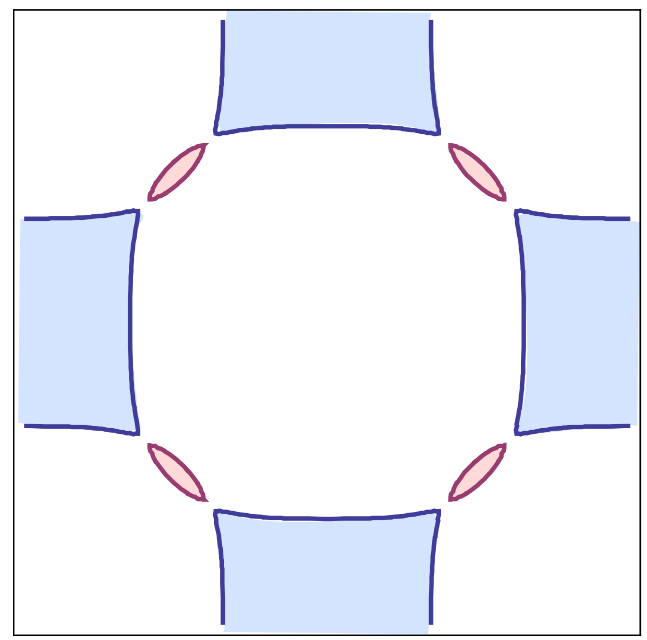
Square lattice Hubbard model with electron doping

$\langle \vec{\varphi} \rangle \neq 0$ and large



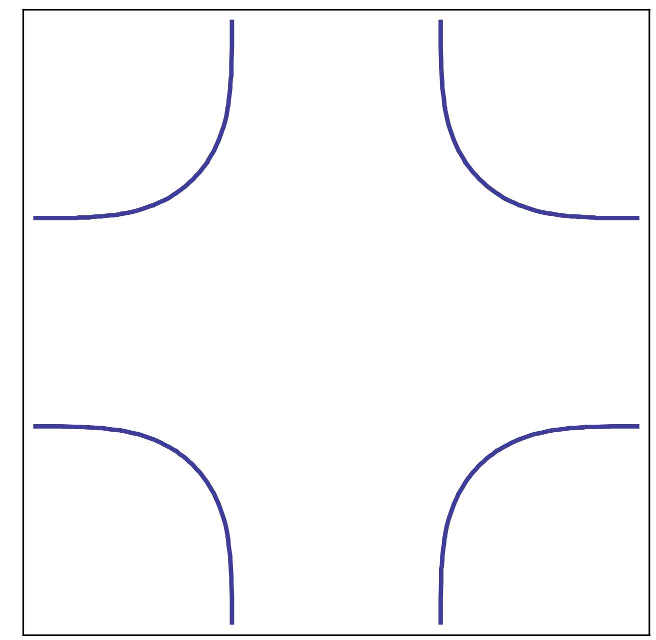
Metal with electron pockets

$\langle \vec{\varphi} \rangle \neq 0$ and small



Metal with electron and hole pockets

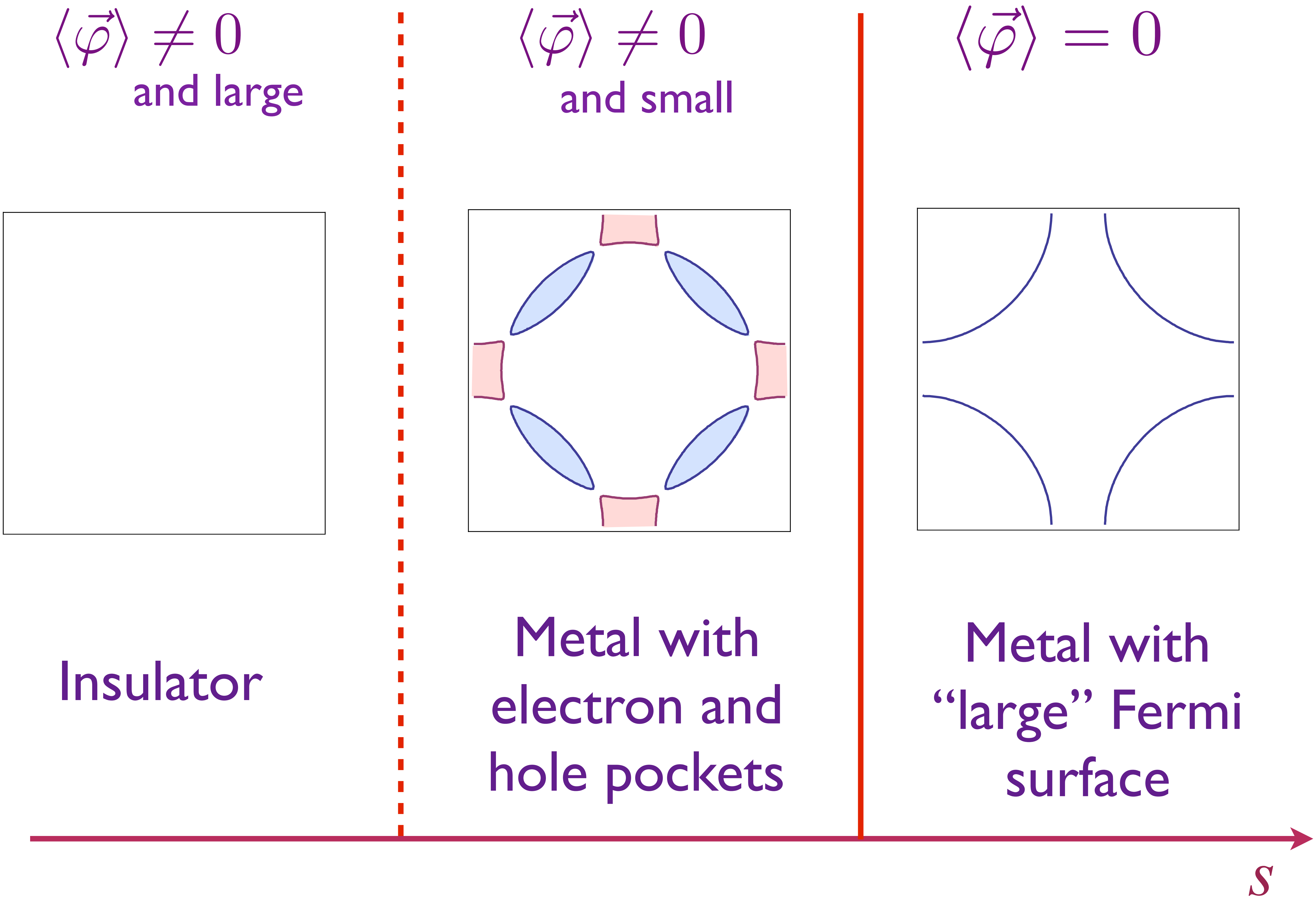
$\langle \vec{\varphi} \rangle = 0$



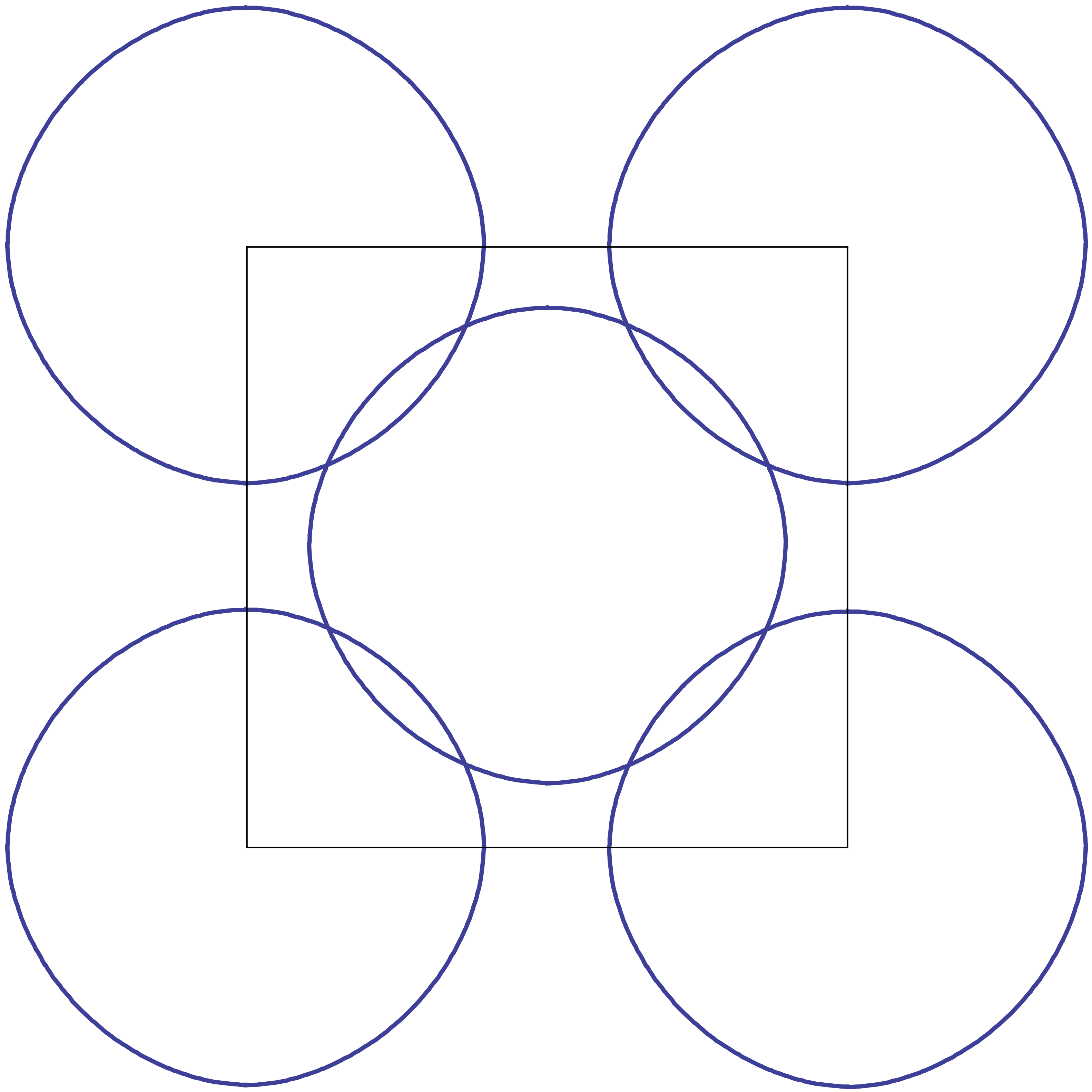
Metal with "large" Fermi surface

s

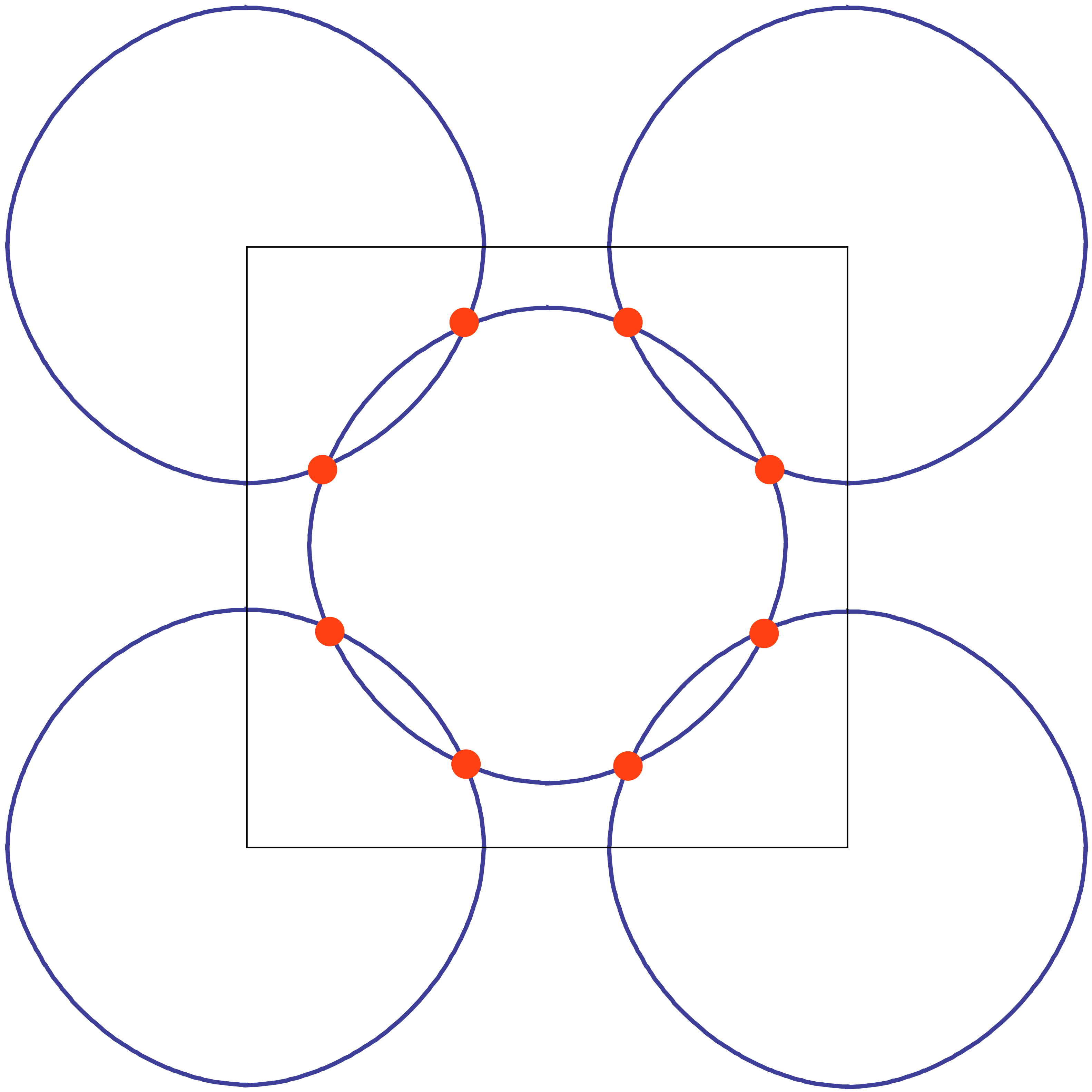
Square lattice Hubbard model with no doping



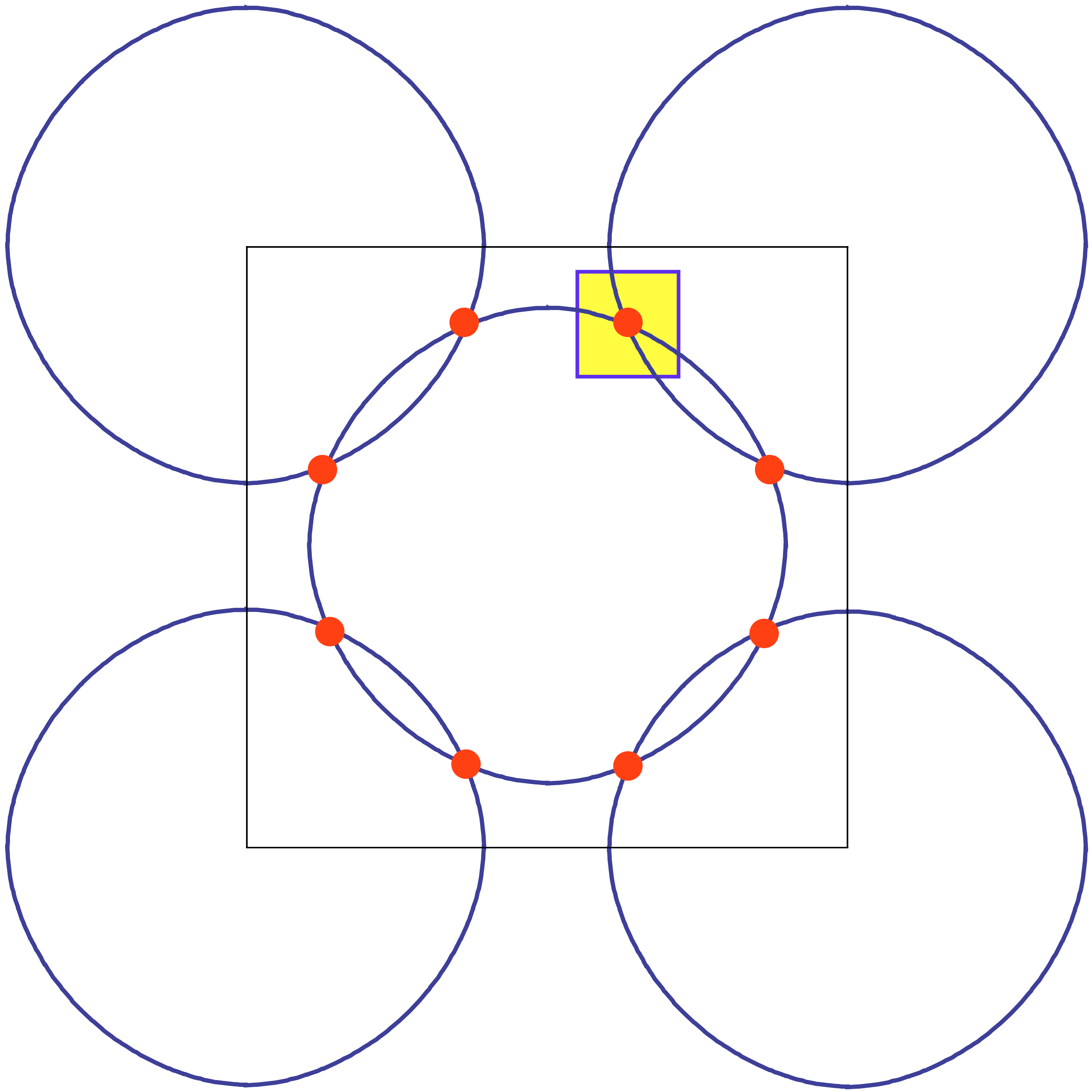
Fermi surface+antiferromagnetism



Fermi surfaces translated by $\mathbf{K} = (\pi, \pi)$.

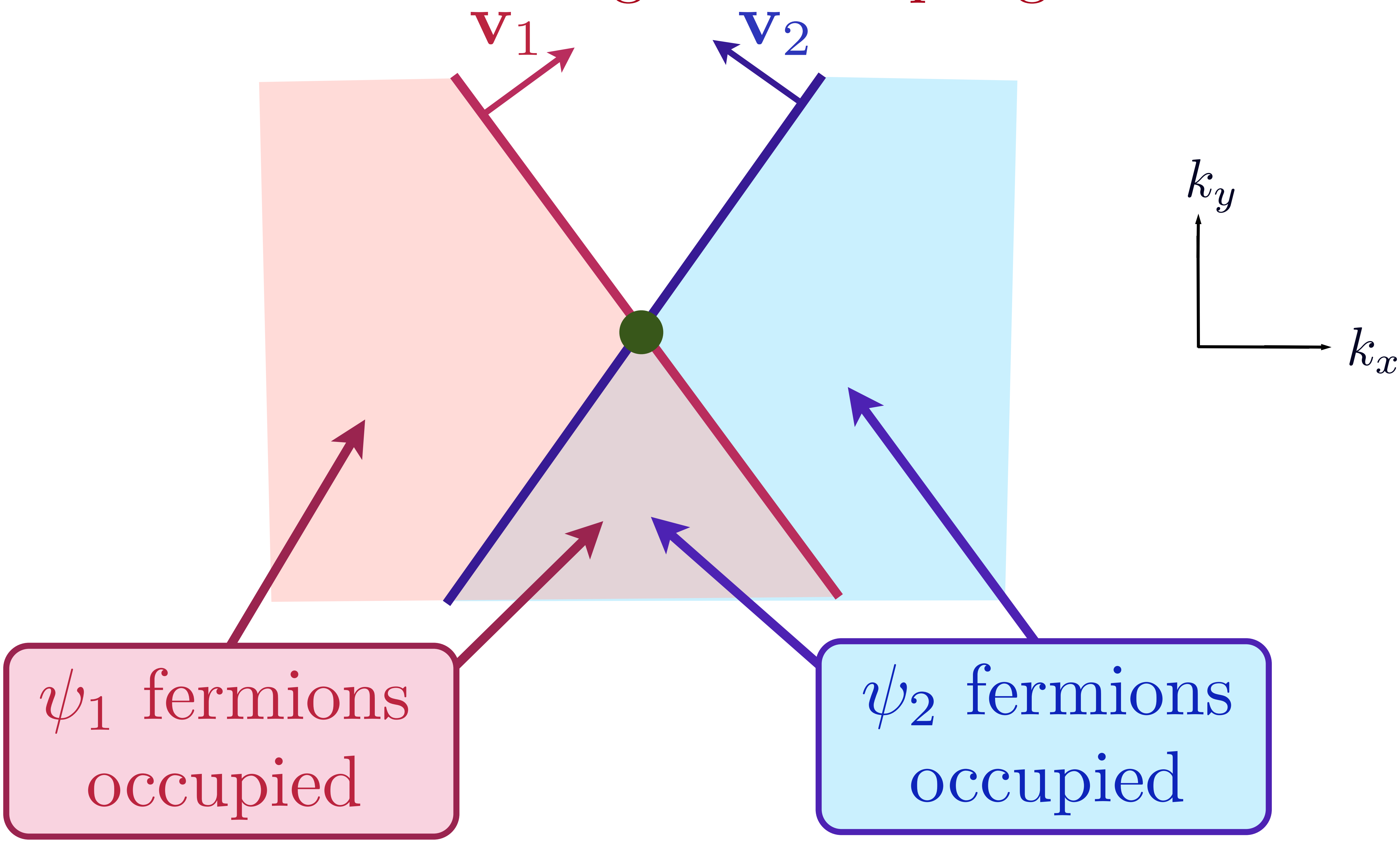


“Hot” spots



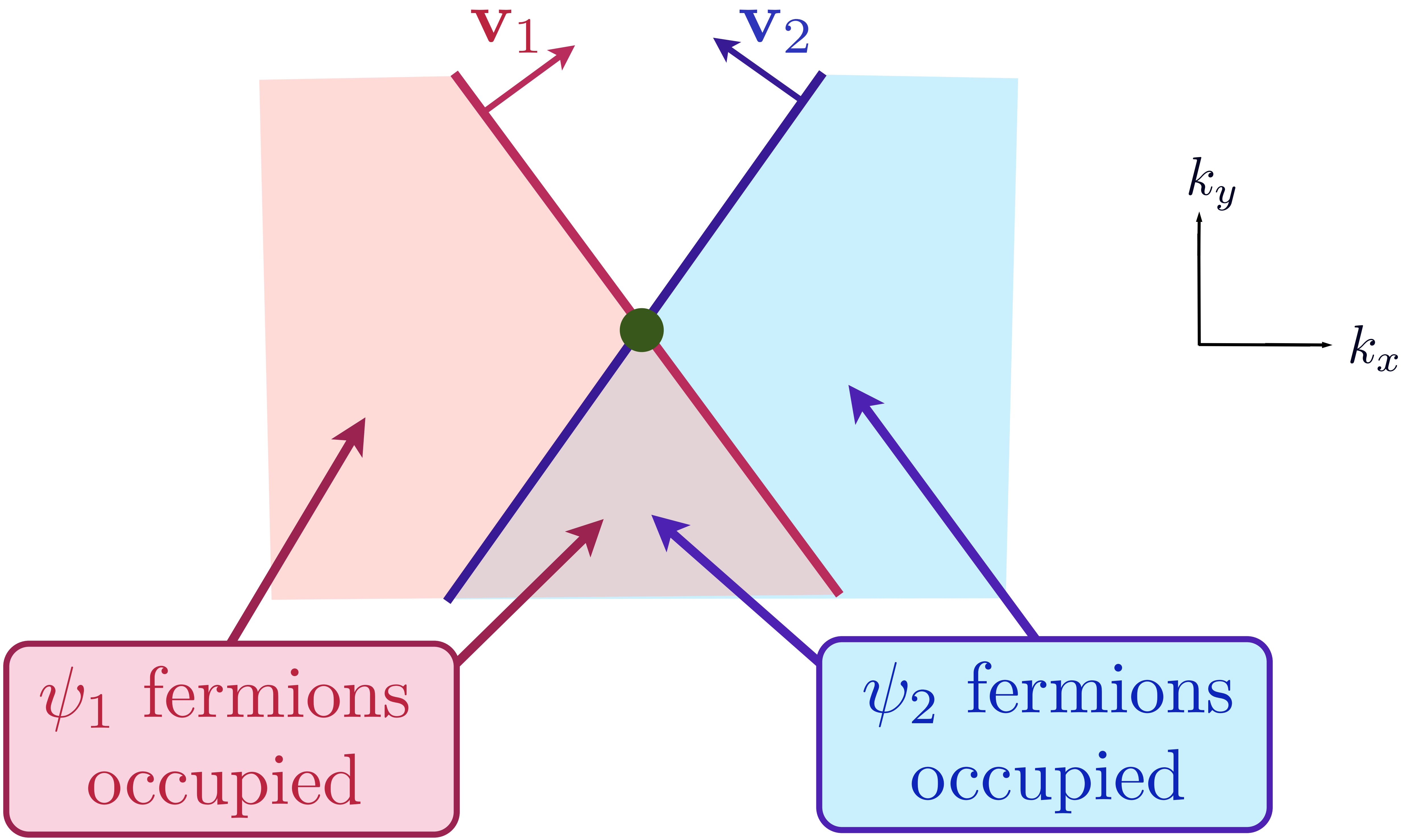
Low energy theory for critical point near hot spots

Theory has fermions $\psi_{1,2}$ (with Fermi velocities $\mathbf{v}_{1,2}$) and boson order parameter $\vec{\varphi}$, interacting with coupling λ



Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004).

$$\mathcal{L}_f = \psi^\dagger_{1\alpha}\left(\partial_\tau - i\mathbf{v}_1 \cdot \boldsymbol{\nabla}_r\right)\psi_{1\alpha} + \psi^\dagger_{2\alpha}\left(\partial_\tau - i\mathbf{v}_2 \cdot \boldsymbol{\nabla}_r\right)\psi_{2\alpha}$$



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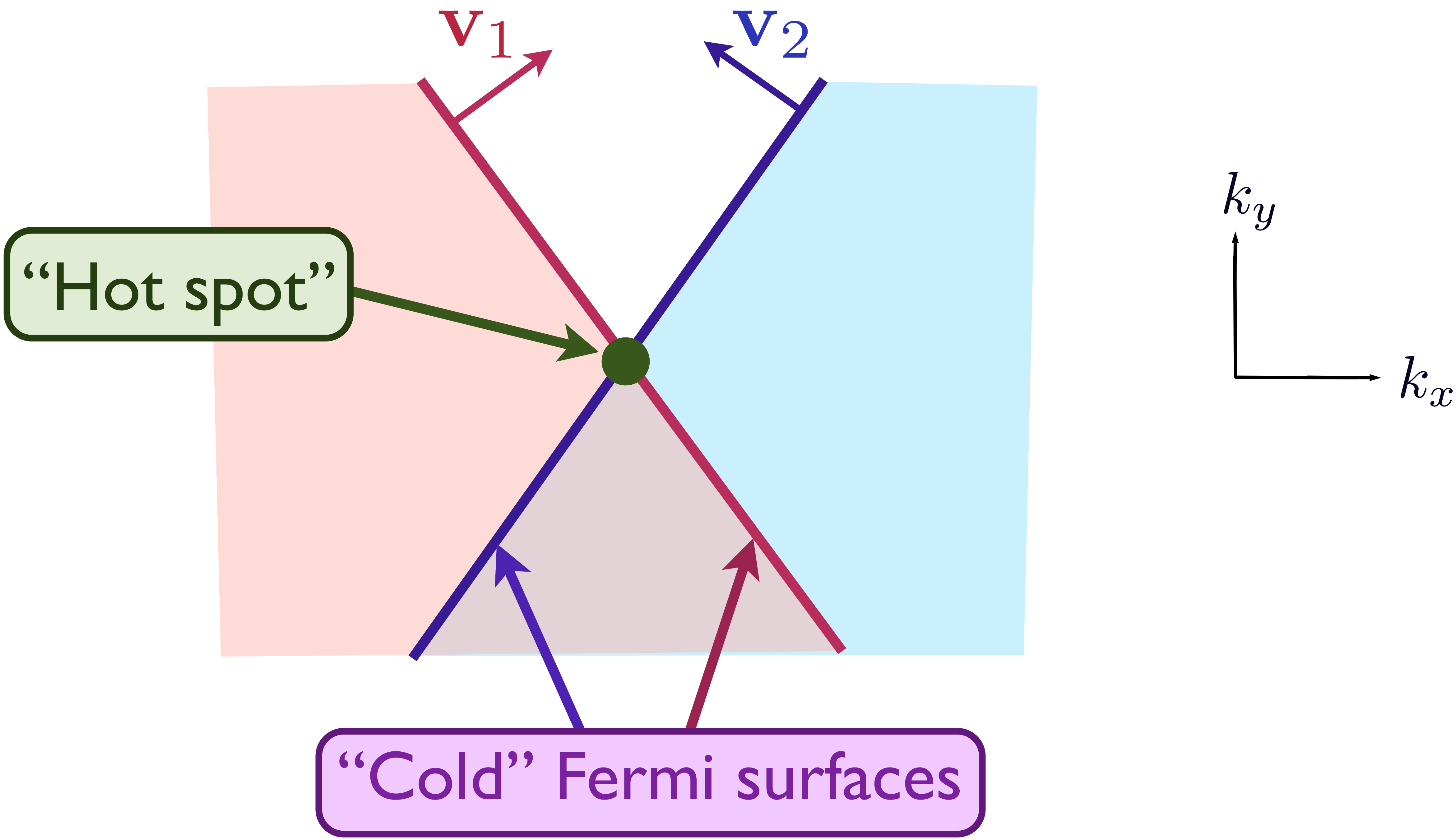

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Order parameter: $\mathcal{L}_\varphi = \frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{1}{2} \left(\partial_\tau \vec{\varphi}\right)^2 + \frac{s}{2} \vec{\varphi}^{\,2} + \frac{u}{4} \vec{\varphi}^{\,4}$

Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004).

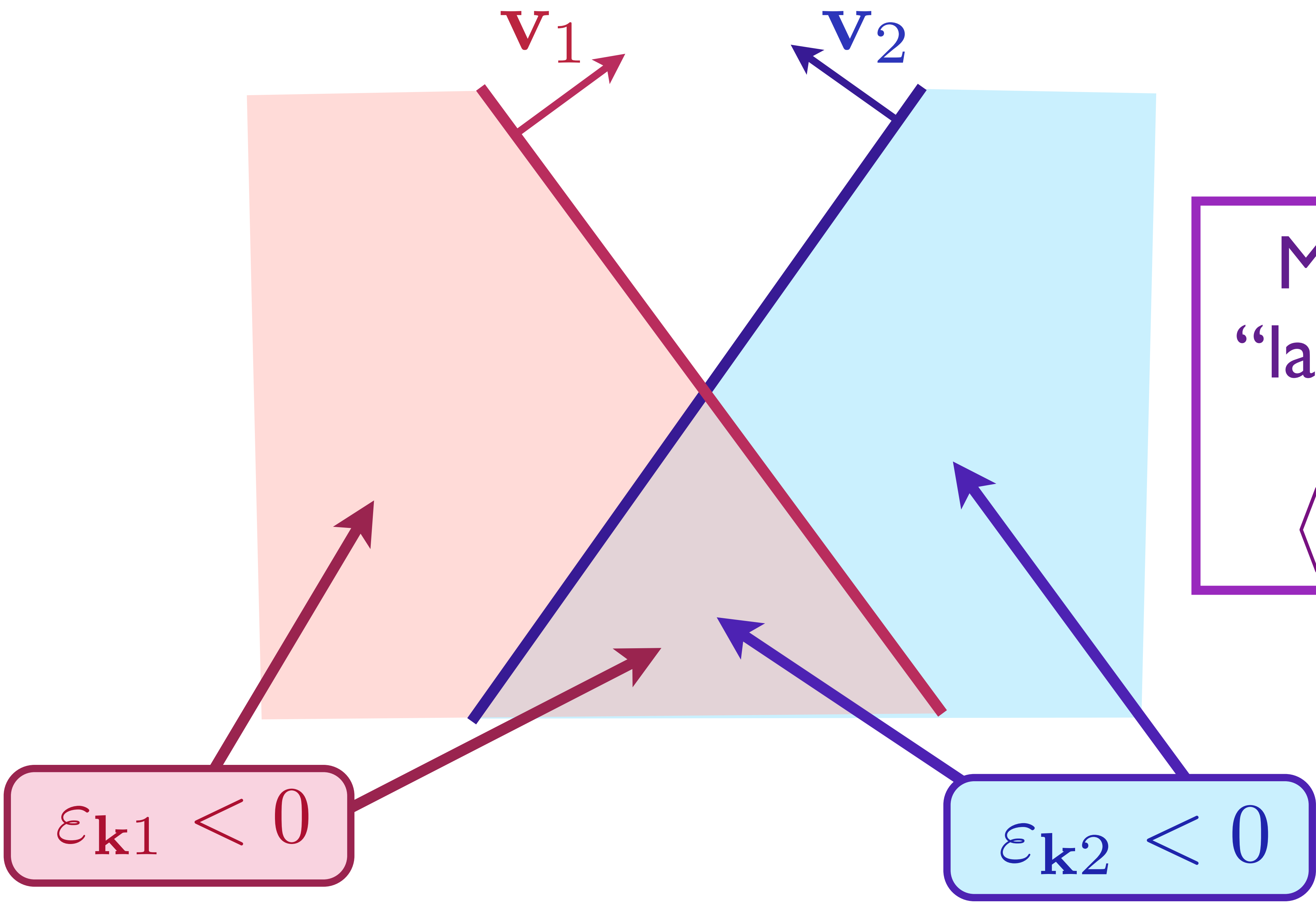
$$\mathcal{L}_f = \psi_{1\alpha}^\dagger \left(\partial_\tau - i\mathbf{v}_1 \cdot \boldsymbol{\nabla}_r\right) \psi_{1\alpha} + \psi_{2\alpha}^\dagger \left(\partial_\tau - i\mathbf{v}_2 \cdot \boldsymbol{\nabla}_r\right) \psi_{2\alpha}$$

Order parameter: $$\mathcal{L}_\varphi = \frac{1}{2}\left(\boldsymbol{\nabla}_r \vec{\varphi}\right)^2 + \frac{1}{2}\left(\partial_\tau \vec{\varphi}\right)^2 + \frac{s}{2}\vec{\varphi}^2 + \frac{u}{4}\vec{\varphi}^4$$

"Yukawa" coupling: $$\mathcal{L}_c = -\lambda \vec{\varphi} \cdot \left(\psi_{1\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \psi_{2\beta} + \psi_{2\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \psi_{1\beta}\right)$$

Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004).

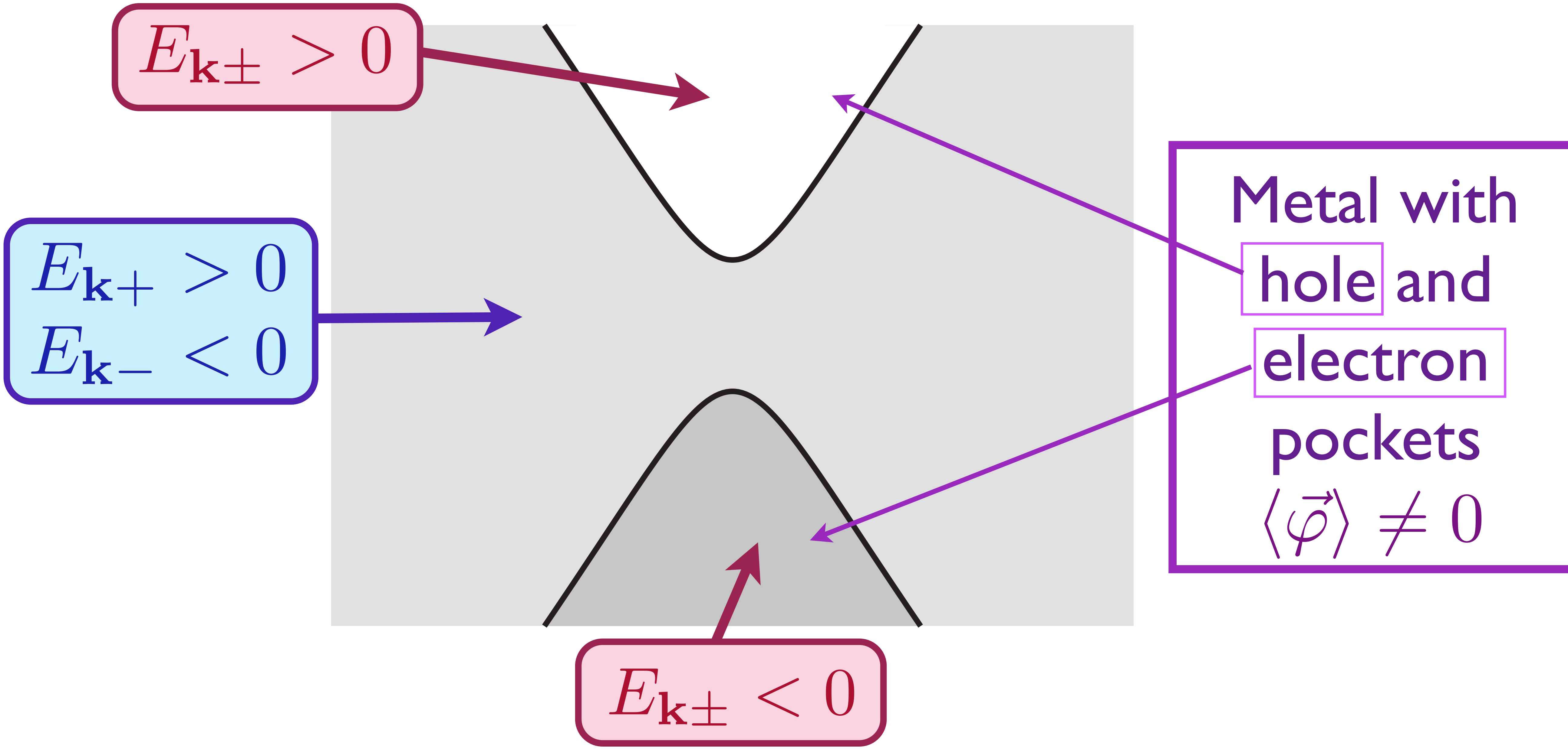
$$\mathcal{L}_f = \psi^\dagger_{1\alpha}\left(\partial_\tau - i\mathbf{v}_1\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha} + \psi^\dagger_{2\alpha}\left(\partial_\tau - i\mathbf{v}_2\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha}$$



Metal with "large" Fermi surface
$\langle\vec{\varphi}\rangle = 0$

Fermion dispersions: $\varepsilon_{\mathbf{k}1} = \mathbf{v}_1 \cdot \mathbf{k}$ and $\varepsilon_{\mathbf{k}2} = \mathbf{v}_2 \cdot \mathbf{k}$

$$\begin{aligned}\mathcal{L}_f &= \psi_{1\alpha}^\dagger\left(\partial_\tau - i\mathbf{v}_1\cdot\boldsymbol{\nabla}_r\right)\psi_{1\alpha} + \psi_{2\alpha}^\dagger\left(\partial_\tau - i\mathbf{v}_2\cdot\boldsymbol{\nabla}_r\right)\psi_{2\alpha} \\ & -\lambda\vec{\varphi}\cdot\left(\psi_{1\alpha}^\dagger\vec{\sigma}_{\alpha\beta}\psi_{2\beta} + \psi_{2\alpha}^\dagger\vec{\sigma}_{\alpha\beta}\psi_{1\beta}\right)\end{aligned}$$

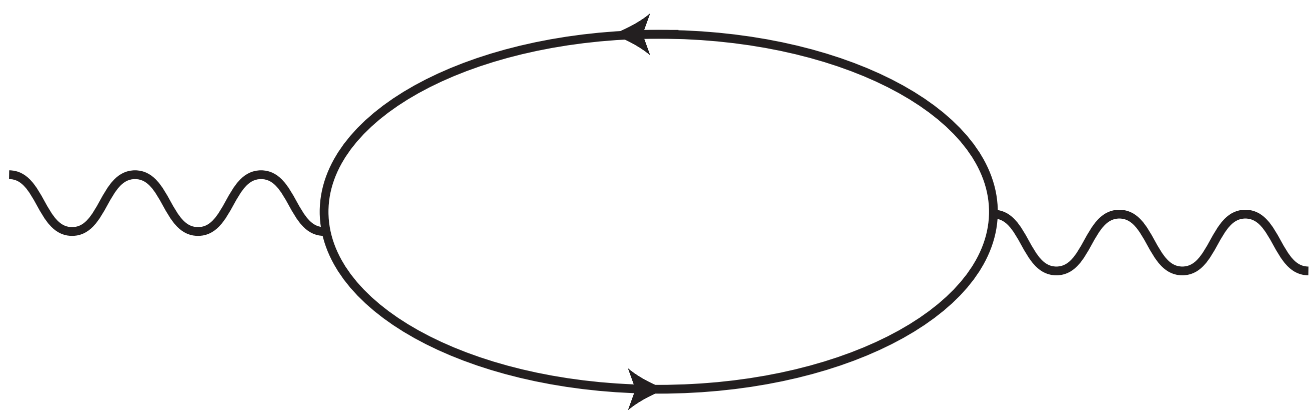


Fermion dispersions:

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}1} + \varepsilon_{\mathbf{k}2}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}1} - \varepsilon_{\mathbf{k}2}}{2}\right)^2 + \lambda^2|\vec{\varphi}|^2}$$

Hertz action.

Upon integrating the fermions out, the leading term in the $\vec{\varphi}$ effective action is $-\Pi(q,\omega_n)|\vec{\varphi}(q,\omega_n)|^2$, where $\Pi(q,\omega_n)$ is the fermion polarizability. This is given by a simple fermion loop diagram



$$\Pi(q,\omega_n) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{[-i(\epsilon_n+\omega_n)+\mathbf{v}_1\cdot(\mathbf{k}+\mathbf{q})][-i\epsilon_n+\mathbf{v}_2\cdot\mathbf{k}]}.$$

We define oblique co-ordinates $p_1 = \mathbf{v}_1 \cdot \mathbf{k}$ and $p_2 = \mathbf{v}_2 \cdot \mathbf{k}$. It is then clear that the integrand is independent of the $(d-2)$ transverse momenta, whose integral yields an overall factor Λ^{d-2} (in $d=2$ this factor is precisely 1).

Also, by shifting the integral over k_1 we note that the integral is independent of q. So we have

$$\Pi(q,\omega_n) = \frac{\Lambda^{d-2}}{|\mathbf{v}_1 \times \mathbf{v}_2|} \int \frac{dp_1 dp_2 d\epsilon_n}{8\pi^3} \frac{1}{[-i(\epsilon_n + \omega_n) + p_1][-i\epsilon_n + p_2]}.$$

Next, we evaluate the frequency integral to obtain

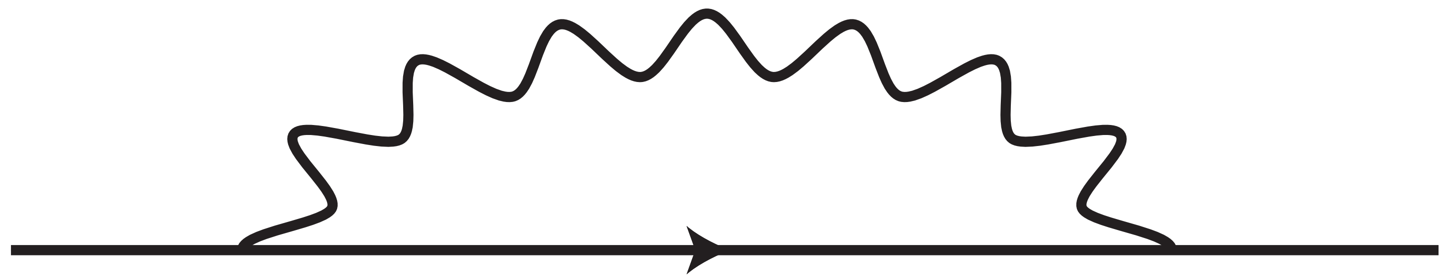
$$\begin{aligned} \Pi(q,\omega_n) &= \frac{\Lambda^{d-2}}{\zeta|\mathbf{v}_1 \times \mathbf{v}_2|} \int \frac{dp_1 dp_2}{4\pi^2} \frac{[\mathrm{sgn}(p_2) - \mathrm{sgn}(p_1)]}{-i\zeta\omega_n + p_1 - p_2} \\ &= -\frac{|\omega_n|\Lambda^{d-2}}{4\pi|\mathbf{v}_1 \times \mathbf{v}_2|}. \end{aligned}$$

In the last step, we have dropped a frequency-independent, cutoff-dependent constant which can absorbed into a redefinition of r. Inserting this fermion polarizability in the effective action for $\vec{\varphi}$, we obtain the Hertz action for the SDW transition:

$$\begin{aligned} \mathcal{S}_H = \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{2} \left[k^2 + \gamma|\omega_n| + s\right] |\vec{\varphi}(k,\omega_n)|^2 \\ + \frac{u}{4} \int d^d x d\tau \left(\vec{\varphi}^2(x,\tau)\right)^2. \end{aligned}$$

Fate of the fermions.

Let us, for now, assume the validity of the Hertz Gaussian action, and compute the leading correction to the electronic Green's function. This is given by the following Feynman graph for the electron self energy, Σ. At zero momentum for the ψ_1 fermion we have



$$\Sigma_1(0,\omega_n) = \lambda^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d\epsilon_n}{2\pi} \frac{1}{\left[q^2 + \gamma|\epsilon_n|\right]\left[-i(\epsilon_n+\omega_n) + \mathbf{v}_2 \cdot \mathbf{q}\,\right]}.$$

We first perform the integral over the $\mathbf{q}$ direction parallel to $\mathbf{v}_2$, while ignoring the subdominant dependence on this momentum in the boson propagator. Then we have

$$\begin{aligned}\Sigma_1(0,\omega_n) &= i\,\frac{\lambda^2}{|v_2|} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \int \frac{d\epsilon_n}{2\pi} \frac{\mathrm{sgn}(\epsilon_n+\omega_n)}{|q|^2 + \gamma|\epsilon_n|} \\ &= i\,\frac{\lambda^2}{\pi|v_2|\gamma} \mathrm{sgn}(\omega_n) \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \ln\left(\frac{|q|^2+\gamma|\omega_n|}{|q|^2}\right).\end{aligned}$$

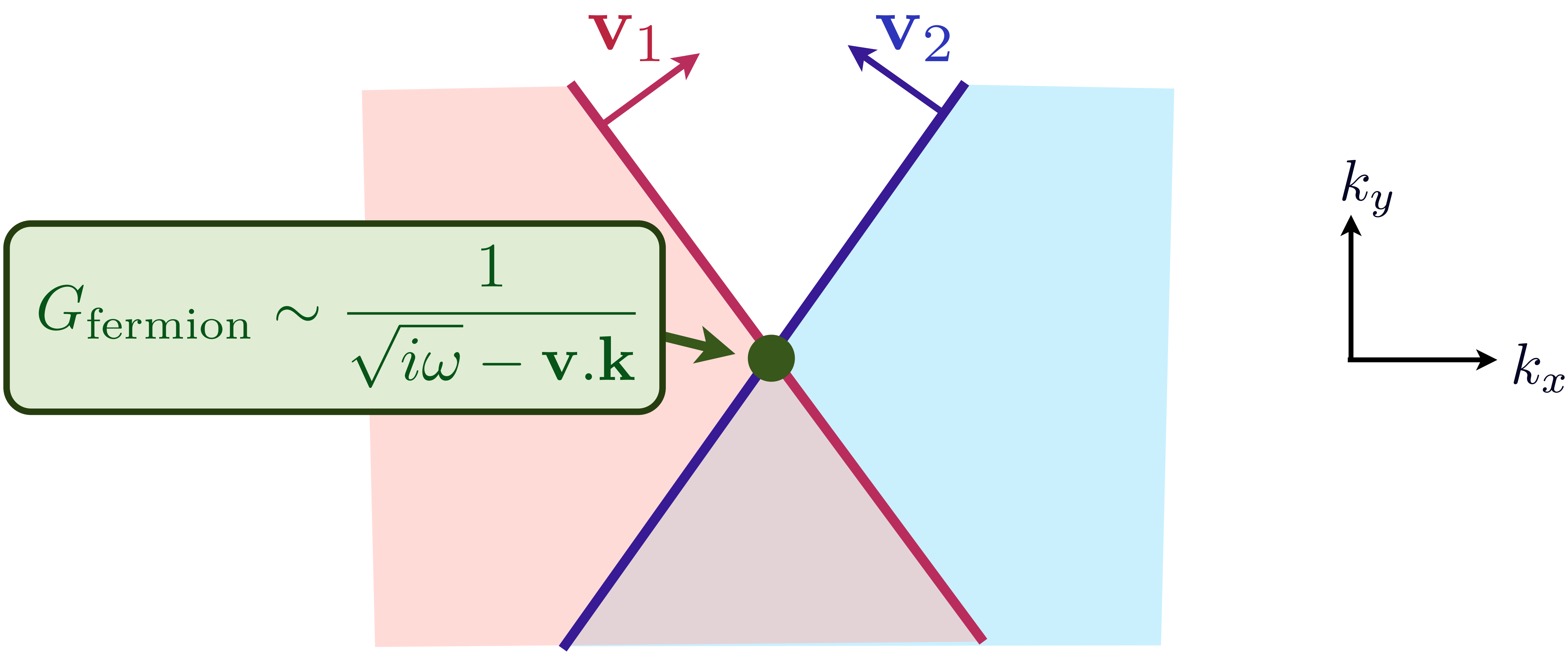
Evaluation of the q integral shows that

$$\Sigma_1(0, \omega_n) \sim |\omega_n|^{(d-1)/2}$$

The most important case is $d = 2$, where we have

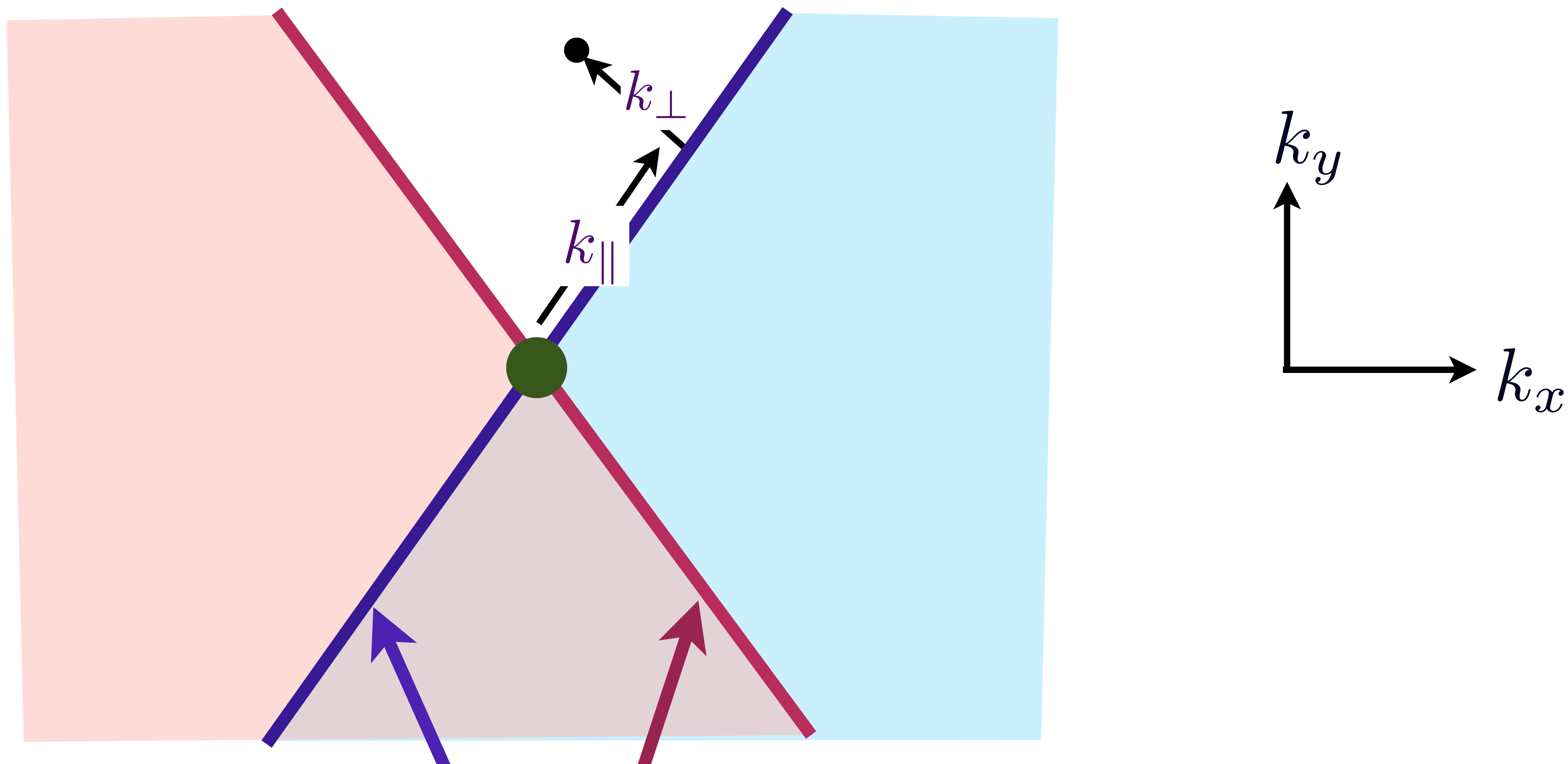
$$\Sigma_1(0, \omega_n) = i \frac{\lambda^2}{\pi |v_2| \sqrt{\gamma}} \mathrm{sgn}(\omega_n) \sqrt{|\omega_n|} \quad , \quad d = 2.$$

Critical point theory is strongly coupled in $d = 2$



A. J. Millis, *Phys. Rev. B* **45**, 13047 (1992)
Ar. Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004)

Critical point theory is strongly coupled in $d = 2$



$$G_{\text{fermion}} = \frac{Z(k_\parallel)}{\omega - v_F(k_\parallel) k_\perp}, \quad Z(k_\parallel) \sim v_F(k_\parallel) \sim k_\parallel$$

M. A. Metlitski and S. Sachdev, *Phys. Rev. B* **85**, 075127 (2010)

Strong coupling physics in $d = 2$

The theory so far has the boson propagator

$$\sim \frac{1}{q^2 + \gamma|\omega|}$$

which scales with dynamic exponent $z_b = 2$, and now a fermion propagator

$$\sim \frac{1}{-i\omega + c_1|\omega|^{(d-1)/2} + \mathbf{v}\cdot\mathbf{q}}.$$

First note that for $d < 3$, the bare $-i\omega$ term is less important than the contribution from the self energy at low frequencies. Ignoring the $i\omega$ term, we see that the fermion propagator scales with dynamic exponent $z_f = 2/(d-1)$. For $d > 2$, $z_f < z_b$, and so at small momenta the boson fluctuations have lower energy than the fermion fluctuations. Thus it seems reasonable to assume that the fermion fluctuations are not as singular, and we can focus on an effective theory of the SDW order parameter $\vec{\varphi}$ alone. In other words, the Hertz assumptions appear valid for $d > 2$.

However, in $d = 2$, we have $z_f = z_b = 2$. Thus fermionic and bosonic fluctuations are equally important, and it is not appropriate to integrate the fermions out at an initial stage. We have to return to the original theory of coupled bosons and fermions. This turns out to be strongly coupled, and exhibits complex critical behavior. For more details, see

M. A. Metlitski and S. Sachdev, arXiv:1005.1288 (Physical Review B **82**, 075127 (2010)).