Lecture 6 近藤(Kondo)效应

1) 从 Anderson 模型到 Kondo 模型

$\boxed{V1}\to$ * Single-impurity Anderson 模型

$$H=\sum_{k\sigma}\epsilon_k c^\dagger_{k\sigma}c_{k\sigma}+\sum_\sigma \epsilon_d c^\dagger_{d\sigma}c_{d\sigma}+\sum_{k\sigma}(V_{kd}c^\dagger_{k\sigma}c_{d\sigma}+h.c.)+Un_{d\uparrow}n_{d\downarrow}$$

考虑 $U\gg|\epsilon_d|\gg V$, $\epsilon_d<0$: 杂质形成一个局域磁矩. $\langle n_d\rangle\approx 1$

* 局域磁矩: 电荷自由度冻结 $\langle n_d\rangle\approx 1$; 自旋自由度 $\vec{S}=\frac{1}{2}c^\dagger_{d\alpha}\vec{\sigma}_{\alpha\beta}c_{d\beta}$

- 平均场: $\langle n_{d\uparrow}\rangle\neq\langle n_{d\downarrow}\rangle$: 存在磁矩, 但无法考虑磁矩的量子/热涨落.
- $V_{kd}=0$: 自由磁矩. $|\uparrow\rangle$ $|\downarrow\rangle$ 简并.
- $V_{kd}\ll|\epsilon_d|, U$: 微扰展开. 二阶微扰论 (简并).

$$H_0=\sum_{k\sigma}\epsilon_k c^\dagger_{k\sigma}c_{k\sigma}+\sum_\sigma\epsilon_d c^\dagger_{d\sigma}c_{d\sigma}+Un_{d\uparrow}n_{d\downarrow}$$

$$H_1=\sum_{k\sigma}(V_{kd}c^\dagger_{k\sigma}c_{d\sigma}+h.c.)$$

对称性:

$$H_{eff}=\sum_{kk'}J_{kk'}c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta}c_{k'\alpha}\cdot\vec{S}$$

自旋旋转对称性要求 $H\sim\vec{S}_{kk'}\cdot\vec{S}$.

$$\langle\uparrow|H_1|\uparrow\rangle=\langle\uparrow|H_1|\downarrow\rangle=\langle\downarrow|H_1|\uparrow\rangle=\langle\downarrow|H_1|\downarrow\rangle=0.$$

$\boxed{V2}\to$

$$\langle\alpha|H^{(2)}|\beta\rangle=\sum_n\langle\alpha|H_1|n\rangle\langle n|H_1|\beta\rangle\,\frac{1}{2}\left(\frac{1}{E_\alpha-E_n}+\frac{1}{E_\beta-E_n}\right).$$

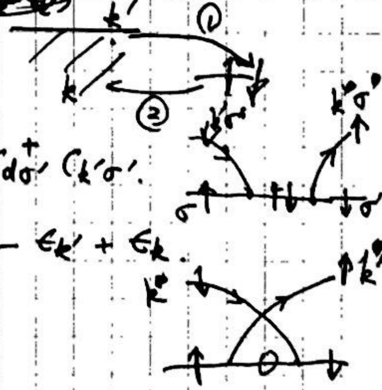
简并微扰: 令 $E_\alpha=E_\beta$. $\frac{1}{2}\left(\frac{1}{E_\alpha-E_n}+\frac{1}{E_\beta-E_n}\right)=\frac{1}{E_\alpha-E_n}$

$$H^{(2)}=\sum_{\alpha\beta}|\alpha\rangle\langle\beta|\,\frac{\langle\alpha|H_1|n\rangle\langle n|H_1|\beta\rangle}{E_\alpha-E_n}$$

$$=\sum_n H_1\frac{|n\rangle\langle n|}{E_\alpha-E_n}H_1=\hat{H}_1\frac{1}{E_\alpha-\hat{H}_0}\hat{H}_1.$$

这里: $H_1=\sum_{k\sigma}(V_{kd}c^\dagger_{k\sigma}c_{d\sigma}+h.c.)$.

A:



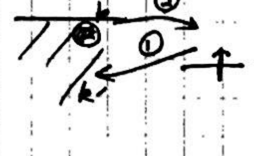
$$H^{(2)}_A=\sum_{kk',\sigma\sigma'}V_{kd}c^\dagger_{k\sigma}c_{d\sigma}\frac{1}{2}\left(\frac{1}{\epsilon_k-\epsilon_d-U}+\frac{1}{\epsilon_{k'}-\epsilon_d-U}\right)V^*_{k'd}c^\dagger_{d\sigma'}c_{k'\sigma'}.$$

初态 $E_i=\epsilon_d$. 中间 $E=2\epsilon_d+U-\epsilon_{k'}$ 末态 $E_f=\epsilon_d-\epsilon_{k'}+\epsilon_k$.

$$E_i-E=\epsilon_d-(2\epsilon_d+U-\epsilon_{k'})=\epsilon_{k'}-\epsilon_d-U$$

$$E_f-E=\epsilon_d-\epsilon_{k'}+\epsilon_k-(2\epsilon_d+U-\epsilon_{k'})=\epsilon_k-\epsilon_d+U.$$

$$H^{(2)}_B=\sum_{kk',\sigma\sigma'}V^*_{kd}c^\dagger_{d\sigma}c_{k\sigma}\frac{1}{2}\left(\frac{1}{\epsilon_d-\epsilon_{k'}}+\frac{1}{\epsilon_d-\epsilon_k}\right)V_{k'd}c^\dagger_{k'\sigma'}c_{d\sigma}.$$



$E_i=\epsilon_d$ 中间 $E=\epsilon_{k'}$ 末态 $E_f=\epsilon_d+\epsilon_{k'}-\epsilon_k$.

$\therefore H^{(2)}=H^{(2)}_A+H^{(2)}_B$.

$$H^{(2)}_A=\sum_{kk',\sigma\sigma'}V_{kd}V^*_{k'd}\,c^\dagger_{k\sigma}c_{k'\sigma'}c^\dagger_{d\sigma'}c_{d\sigma}\,\frac{1}{2}\left(\frac{1}{\epsilon_d+U-\epsilon_k}+\frac{1}{\epsilon_d+U-\epsilon_{k'}}\right).$$

$$H^{(2)}_B=\sum_{kk',\sigma\sigma'}V_{kd}V^*_{k'd}\,c^\dagger_{k\sigma}c_{k'\sigma'}c^\dagger_{d\sigma'}c_{d\sigma}\,\frac{1}{2}\left(\frac{1}{\epsilon_k-\epsilon_d}+\frac{1}{\epsilon_{k'}-\epsilon_d}\right).$$

$$H^{(2)} = \sum_{kk',\sigma\sigma'} V_{kd} V^*_{k'd}\, c^\dagger_{k\sigma} c_{k'\sigma'} c^\dagger_{d\sigma'} c_{d\sigma}\, \frac{1}{2}\left(\frac{1}{\epsilon_k-\epsilon_d}+\frac{1}{\epsilon_{k'}-\epsilon_d}+\frac{1}{U+\epsilon_d-\epsilon_k}+\frac{1}{U+\epsilon_d-\epsilon_{k'}}\right)$$

$\boxed{V3} \longrightarrow$ $$= \sum_{kk',\sigma\sigma'} \frac{J_{kk'}}{2}\, c^\dagger_{k\sigma} c_{k'\sigma'} c^\dagger_{d\sigma'} c_{d\sigma}$$

w/ $$J_{kk'} = V_{kd} V^*_{k'd}\left(\frac{1}{\epsilon_k-\epsilon_d}+\frac{1}{\epsilon_{k'}-\epsilon_d}+\frac{1}{U+\epsilon_d-\epsilon_k}+\frac{1}{U+\epsilon_d-\epsilon_{k'}}\right).$$

$\sum_{\sigma\sigma'} c^\dagger_{k\sigma} c_{k'\sigma'} c^\dagger_{d\sigma'} c_{d\sigma}$ 与 $\underbrace{c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\beta}}_{\vec{s}_{kk'}} \cdot \vec{S}$ 的关系:

$$\vec{s}_{kk'} = \frac{1}{2}\sum_{\alpha\beta} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\beta}, \quad \vec{S} = \frac{1}{2}\sum_{\alpha\beta} c^\dagger_{d\alpha}\vec{\sigma}_{\alpha\beta} c_{d\beta}.$$

$$\vec{s}_{kk'}\cdot\vec{S} = \frac{1}{4} c^\dagger_{k\alpha} c_{k'\beta} c^\dagger_{d\alpha'} c_{d\beta'}\, \vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\alpha'\beta'}.$$

$$\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\alpha'\beta'} = 2\delta_{\alpha\beta'}\delta_{\beta\alpha'} - \delta_{\alpha\beta}\delta_{\alpha'\beta'}.$$

$$\therefore\ \vec{s}_{kk'}\cdot\vec{S} = \frac{1}{4} c^\dagger_{k\alpha} c_{k'\beta} c^\dagger_{d\alpha'} c_{d\beta'}\left(2\delta_{\alpha\beta'}\delta_{\beta\alpha'} - \delta_{\alpha\beta}\delta_{\alpha'\beta'}\right)$$

$$= \frac{1}{2} c^\dagger_{k\alpha} c_{k'\beta} c^\dagger_{d\beta} c_{d\alpha} - \frac{1}{4} c^\dagger_{k\alpha} c_{k'\alpha} c^\dagger_{d\beta} c_{d\beta}$$

$$\therefore\ \sum_{\sigma\sigma'} c^\dagger_{k\sigma} c_{k'\sigma'} c^\dagger_{d\sigma'} c_{d\sigma} = 2\,\vec{s}_{kk'}\cdot\vec{S} + \underline{4\, c^\dagger_{k\alpha} c_{k'\alpha}\, n_d}$$

忽略这项 相当于对 ϵ_d 修正.

$$H^{(2)} = \sum_{kk',\alpha\beta} J_{kk'}\frac{1}{2} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\beta}\cdot\vec{S}.$$

* 近藤 (Kondo) 模型

$$H = \sum_{k\sigma}\epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk',\alpha\beta} J_{kk'}\, c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\beta}\cdot\vec{S}.$$

近似: 低能有效模型: $\epsilon_k \approx \epsilon_{k'} \approx 0$. $\epsilon_d < 0$, $U+\epsilon_d > 0$.

$$J_{kk'} = V_{kd} V^*_{k'd}\left(\frac{2}{-\epsilon_d} + \frac{2}{U+\epsilon_d}\right).$$

$$J_{kk'} \approx 2|V_d|^2 \frac{U}{|\epsilon_d|(U-|\epsilon_d|)}$$

对于 $\epsilon_d = -U/2$: $J = \frac{4|V_d|^2}{U}$

注意和书上的记号的区别: $H = \sum_{kk'} J_{kk'}\vec{s}_{kk'}\cdot\vec{S}$

反铁磁: $J > 0$, 这里的 J 和书上差了一个负号!

归一化: $J = \frac{J_0}{V}$ ← 系统体积.

$$H = \sum_{k\sigma}\epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{J_0}{2V}\sum_{kk',\alpha\beta} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\beta}\cdot\vec{S}.$$

2) ~~Kondo 模型的杂质散射率~~：磁性杂质的~~电阻率~~.

$\boxed{4}\to 2)$ Kondo 效应：用微扰展开计算杂质散射率 $\Rightarrow$ 电阻率

$$H = \sum_k \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{J_0}{2V}\sum_{kk'} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\alpha}\cdot \vec{S}_d$$

假设 $\vec{S}$ 为 自旋-S： $\vec{S}_d^2 = S(S+1)$.

$H = H_0 + H_1$; $H_0 = \sum_k \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}$; $H_1 = \frac{J_0}{2V}\sum_{kk'} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta} c_{k'\alpha}\cdot\vec{S}_d$

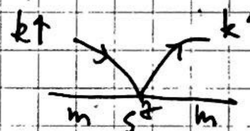
令 $S^\pm = S^x \pm i S^y$; $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$

$$\therefore H_1 = \frac{J_0}{2V}\sum_{kk'}\left[S_d^z\left(c^\dagger_{k\uparrow}c_{k'\uparrow} - c^\dagger_{k\downarrow}c_{k'\downarrow}\right) + S_d^+ c^\dagger_{k\downarrow}c_{k'\uparrow} + S_d^- c^\dagger_{k\uparrow}c_{k'\downarrow}\right]$$

① J_0 一阶微扰：

* $T^{(0)}_{k\uparrow\to k'\uparrow;m}$ 来自 $\frac{J_0}{2V}S_d^z c^\dagger_{k'\uparrow}c_{k\uparrow}$.



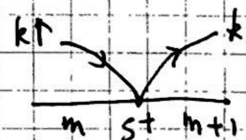
$$T^{(0)}_{k\uparrow\to k'\uparrow;m} = \frac{J_0}{2V} m$$

$$\Gamma^{(0)}_{k\uparrow\to k'\uparrow}(m) = 2\pi\sum_{k'}\left|T^{(0)}_{k\uparrow\to k'\uparrow;m}\right|^2\delta(\epsilon_k-\epsilon_{k'}) \leftarrow \text{Fermi 黄金定则}$$

$$= 2\pi\frac{J_0^2 m^2}{4V^2}\sum_{k'}\delta(\epsilon_{k'}-\epsilon_k)$$

$$= \frac{\pi}{2V}J_0^2 m^2 N(0) \longrightarrow \text{non flip}$$

* $T^{(0)}_{k\uparrow\to k'\downarrow;m}$ 来自 ~~...~~ $\frac{J_0}{2V}S_d^+ c^\dagger_{k'\downarrow}c_{k\uparrow}$.



$$\langle m+1|S_d^+|m\rangle = \sqrt{S(S+1)-m(m+1)}$$

$$\therefore T^{(0)}_{k\uparrow\to k'\downarrow;m} = \frac{J_0}{2V}\sqrt{S(S+1)-m(m+1)}$$

$$\Gamma^{0}_{k\uparrow\to k'\downarrow}(m) = 2\pi\sum_{k'}\frac{J_0^2}{4V^2}\left[S(S+1)-m(m+1)\right]$$

$$= \frac{\pi}{2V}J_0^2\left[S(S+1)-m(m+1)\right]N(0) \longrightarrow \text{flip}$$

$$\therefore \Gamma^0(m) = (\text{nonflip}) + (\text{flip})$$

$$= \frac{\pi}{2V}J_0^2\left[S(S+1)-m\right]N(0)$$

Average over m:

$$\Gamma^0 = \frac{1}{2S+1}\sum_{m=-S}^{S}\Gamma^0(m) = \frac{\pi}{2V}J_0^2 S(S+1)N(0)$$

为什么 $\propto \frac{1}{V}$：这是一个自旋杂质的 Γ^0，不同杂质的贡献为 Γ^0 的叠加（假设杂质间独立的）

$$\therefore \Gamma^0_{total} = (\#\text{ of impurity})\times\Gamma^0$$

$$= \frac{(\#\text{ of impurity})}{V}\frac{\pi}{2}J_0^2 S(S+1)N(0)$$

$$= n_{imp}\cdot\frac{\pi}{2}J_0^2 S(S+1)N(0)$$

$T\to 0$ 时：有良好行为（不发散）的贡献. $\Delta\rho \propto \Gamma^0$.

（非磁性杂质：$\Gamma^0 = \frac{1}{3}\cdot\frac{\pi}{2}J_0^2 N(0)\, n_{imp}$ 类似的行为）.

我们下面会看到：考虑二阶修正之后，自旋翻转的散射率当 $T\to 0$ 时"发散"。

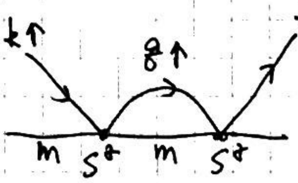
$\boxed{V5}$ → 二阶微扰论：[严格的推导需要用到完整的有限温微扰计算，如 Feynman 图等方法，本课程不讨]。

$$T^0_{\alpha\to\beta} = \langle\beta|H_1|\alpha\rangle$$
$$T^1_{\alpha\to\beta} = \sum_\gamma \langle\beta|H_1|\gamma\rangle\frac{1}{E_\alpha-E_\gamma}\langle\gamma|H_1|\alpha\rangle$$

有限温修正，多体性为基态

① $T^1_{k\uparrow\to k'\uparrow;\,m}$:

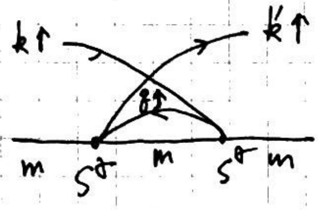
(1a)



$$T^1_{1a} = \sum_q \frac{J_0 m}{2V}\,\frac{(1-f_q)}{\epsilon_k-\epsilon_q}\,\frac{J_0 m}{2V}$$
$$= \frac{J_0^2 m^2}{4V^2}\sum_q\frac{1-f_q}{\epsilon_k-\epsilon_q} = \frac{J_0^2 m^2}{4V}\alpha$$

w/ $\alpha = \frac{1}{V}\sum_q \frac{1-f_q}{\epsilon_k-\epsilon_q}$.

(1b)



$$|GS\rangle \to \frac{J_0}{2V}c^\dagger_{k\uparrow}c_{q\uparrow}S^z|GS\rangle \to \left(\frac{J_0}{2V}\right)^2 c^\dagger_{q\uparrow}c_{k\uparrow}S^z c^\dagger_{k'\uparrow}c_{q\uparrow}S^z|GS\rangle$$

中间态 $\xleftarrow{q\uparrow}$ 相当于一个空穴态，能量为 $-\epsilon_q$.

$$T^1_{1b} = \sum_q \frac{J_0 m}{2V}\left(-\frac{f_q}{\epsilon_q-\epsilon_{k'}}\right)\frac{J_0 m}{2V}$$

$\epsilon_{k'}=\epsilon_k$

负号来自
$c^\dagger_{q\uparrow}c_{k\uparrow}c^\dagger_{k'\uparrow}c_{q\uparrow}$
$= -c^\dagger_{k'\uparrow}c_{k\uparrow}c^\dagger_{q\uparrow}c_{q\uparrow}$

$$= \frac{J_0^2 m^2}{4V}\frac{1}{V}\sum_q\frac{f_q}{\epsilon_k-\epsilon_q} = \frac{J_0^2 m^2}{4V}\gamma$$

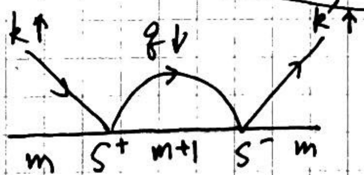
令 $\gamma = \frac{1}{V}\sum_q\frac{f_q}{\epsilon_k-\epsilon_q}$.

$$T^1_{1a} + T^1_{1b} = \frac{J_0^2 m^2}{4V}(\alpha+\gamma)$$

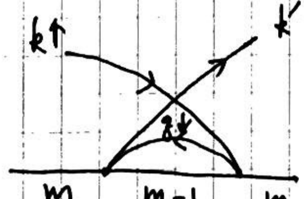
$$\alpha+\gamma = \frac{1}{V}\sum_q\frac{1}{\epsilon_k-\epsilon_q}$$ 与温度无关.

$\boxed{V6}$

(1c)



$$T^1_{1c} = \sum_q \frac{J_0}{2V}\langle s,m|S_d^-|s,m+1\rangle\frac{1-f_q}{\epsilon_k-\epsilon_q}\frac{J_0}{2V}\langle s,m+1|S_d^+|s,m\rangle$$
$$= \frac{J_0^2}{4V}\alpha\cdot|\langle s,m+1|S_d^+|s,m\rangle|^2$$
$$= \frac{J_0^2}{4V}[s(s+1)-m(m+1)]\alpha$$



$$T^1_{1d} = \sum_q\left(\frac{J_0}{2V}\right)^2|\langle s,m-1|S_d^-|s,m\rangle|^2\frac{-f_q}{\epsilon_q-\epsilon_k}$$
$$= \frac{J_0^2}{4V}[s(s+1)-m(m-1)]\gamma$$

$$T^1_{1c}+T^1_{1d} = \frac{J_0^2}{4V}\left\{\overset{T\text{-无关}}{\cancel{[s(s+1)-m(m+1)](\alpha+\gamma)}} + 2m\gamma\right\}$$
$$\approx \frac{J_0^2}{4V}\cdot 2m\gamma$$

$\boxed{\cancel{V7}}$ → 这里 $\gamma = \frac{1}{V}\sum_q\frac{f_q}{\epsilon_k-\epsilon_q} = \int d\epsilon\, N(\epsilon)\frac{f(\epsilon)}{\epsilon_k-\epsilon}$.

$T\to 0$ 极限: $f(\epsilon) = \begin{cases}0, & \epsilon>0\\ 1, & \epsilon<0\end{cases}$

$\epsilon_k\approx 0$. $\gamma = \int_{-\infty}^{0} d\epsilon\, N(\epsilon)\frac{1}{-\epsilon} \sim N(0)\int_{-\infty}^{0}\frac{d\epsilon}{-\epsilon}$ 对数发散.

有限温度：对数发散被 T 截断.

$$\gamma \approx N(0)\int_{-\epsilon_F}^{-k_BT}\frac{d\epsilon}{-\epsilon} \approx N(0)\ln\frac{T_F}{T}.$$

$$\therefore\quad T^{(1)} \approx \frac{J_0^2}{4V}\cdot 2m\,N(0)\ln\frac{T_F}{T}$$

$$= \frac{J_0}{2V}m\cdot J_0 N(0)\ln\frac{T_F}{T}.$$

Recall $T^{(0)}_{k\uparrow\to k\uparrow,m} = \frac{J_0}{2V}m.$

$\therefore$ 到第一阶：$T_{k\uparrow\to k\uparrow,m} = \frac{J_0}{2V}m\left(1+J_0N(0)\ln\frac{T_F}{T}+\cdots\right).$

有效 $J_0^{eff} = J_0\left(1+J_0N(0)\ln\frac{T_F}{T}+\cdots\right).$

$\Gamma_{k\uparrow\to k\uparrow,m} \propto J_0^2 \propto \left(1+2J_0N(0)\ln\frac{T_F}{T}+\cdots\right).$

$\Gamma = \Gamma^0\left(1+2J_0N(0)\ln\frac{T_F}{T}+\cdots\right).$

随 $T\to 0$，$\Gamma \sim \ln\frac{T_F}{T}$ 对数增大.

对于自旋翻转的情形：

- $\Gamma_{total} > \Gamma_{k\uparrow\to k\uparrow}$. 因此 Γ_{total} 至少也会出现 $\ln\frac{T_F}{T}$ 的增长.
- 自详细计算表明，$\Gamma_{k\uparrow\to k\downarrow}$ 也有 $\Gamma=\Gamma^0\left(1+2J_0N(0)\ln\frac{T_F}{T}\right)$ 的修正.

~~与自旋翻转~~

$\Rightarrow$ 电阻 $\rho = \rho^0\left(1+2J_0N(0)\ln\frac{T_F}{T}+\cdots\right)$ $T\to 0$时 对数发散.

3) Unitarity 极限；微扰论失效. $\boxed{V7}$

$\Gamma \le 1$

微扰论算出来 $\Gamma > 1$ 说明微扰论错了…

$$J^{eff} = J_0\left(1+J_0N(0)\ln\frac{T_F}{T}\right)$$

$T\to 0$，J^{eff} 变大；微扰展开不再适用

$$J_0N(0)\ln\frac{T_F}{T} \sim 1.$$

$$\Rightarrow \ln\frac{T}{T_F} = -\frac{1}{J_0N(0)}$$

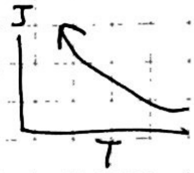
$$T = T_F\,e^{-1/J_0N(0)}.$$

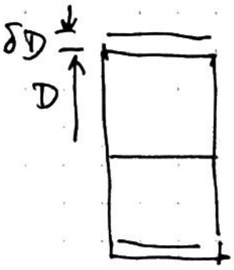
定义 Kondo 温度 $T_K = T_F\,e^{-\frac{4}{3J_0N(0)}}.$

系因子：超越微扰论的修正.

4) ~~Poor~~ Poor Man's Renormalization Group 低配版重整化群

r8 P. W. Anderson $\beta(J) = -\frac{dJ}{d\ln T} = J^2 N(0)$.



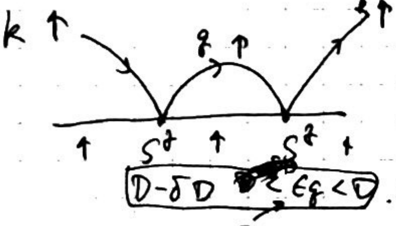


截断 $D \to D - \delta D$

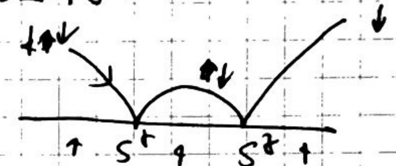
"积掉" $D-\delta D < \epsilon < D$ 的自由度，得到有效Hamiltonian

$S=\frac{1}{2}$，自旋旋转不变：只需计算 $J_z S_d^z (c_{k\uparrow}^\dagger c_{k\uparrow} - c_{k\downarrow}^\dagger c_{k\downarrow})$

的修正即可.



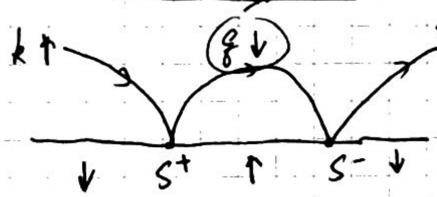
vs



∴ 二者相等，不会修正J.

(给出的是 $c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow}$ 的修正)

a:



$\epsilon_k \approx \epsilon_{k'} \approx 0,\ \epsilon_q \approx D$

$$H_a^{(2)} = \left\langle \sum_q \frac{J}{2V} S^- c_{k'\uparrow}^\dagger c_{q\downarrow} \frac{J}{2V} S^+ c_{q\downarrow}^\dagger c_{k\uparrow} \right\rangle \frac{1}{2}\left(\frac{1}{\epsilon_k - \epsilon_q} + \frac{1}{\epsilon_{k'} - \epsilon_q}\right)$$

$\sum_q$: $D-\delta D < \epsilon_q < D$. $\langle\ \rangle$: "积掉"能量为 $D-\delta D < \epsilon < D$ 的模式.

$$\approx \frac{J^2}{4V^2}\sum_q S^- S^+ c_{k'\uparrow}^\dagger c_{k\uparrow} \langle c_{q\downarrow} c_{q\downarrow}^\dagger \rangle \frac{1}{-D} \qquad \langle c_{q\downarrow} c_{q\downarrow}^\dagger \rangle \to 1$$

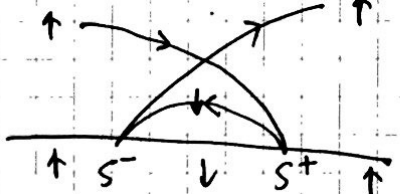
$$= -\frac{J^2}{4V^2} S^- S^+ c_{k'\uparrow}^\dagger c_{k\uparrow} \frac{1}{D} V N(0)\delta D \qquad \sum_q = V N(0)\delta D.$$

$$= -\frac{J^2}{4V} S^- S^+ c_{k'\uparrow}^\dagger c_{k\uparrow} N(0)\frac{\delta D}{D}.$$

$S^- S^+ = \frac{1}{2} - S^z$.

$$\therefore H_a^{(2)} \approx \frac{J^2}{4V} N(0)\frac{\delta D}{D} S^z c_{k'\uparrow}^\dagger c_{k\uparrow}$$

b:



$$H_b^{(2)} = \left\langle \sum_q \frac{J}{2V} S^+ c_{q\downarrow}^\dagger c_{k\uparrow} \frac{J}{2V} S^- c_{k'\uparrow}^\dagger c_{q\downarrow} \right\rangle \frac{1}{2}\left(\frac{1}{\epsilon_q - \epsilon_k} + \frac{1}{\epsilon_q - \epsilon_{k'}}\right)$$

q: 空穴，$\epsilon_q = -D$. $\epsilon_k \approx 0$. $\frac{1}{2}\left(\frac{1}{\epsilon_q - \epsilon_k} + \frac{1}{\epsilon_q - \epsilon_{k'}}\right) \approx \frac{1}{-D}$.

$$H_b^{(2)} \approx -\sum_q \frac{J^2}{4V^2} S^+ S^- c_{k'\uparrow}^\dagger c_{k\uparrow} \langle c_{q\downarrow}^\dagger c_{q\downarrow} \rangle \frac{1}{-D}$$

对换 $c_{k\uparrow}$ 与 $c_{k'\uparrow}^\dagger$. $\langle c_{q\downarrow}^\dagger c_{q\downarrow} \rangle = 1$

$$H_b^{(2)} = \frac{J^2}{4V^2} S^+ S^- c_{k'\uparrow}^\dagger c_{k\uparrow} N(0)\frac{\delta D}{D} \qquad w/\ S^+ S^- = \frac{1}{2} + S^z.$$

$\therefore\ H_b^{(2)} \approx \frac{J^2}{4V} N(0)\frac{\delta D}{D} S^z C_{k'\uparrow}^{+} C_{k\uparrow}.$

$H^{(2)} = \frac{J^2}{2V} N(0)\frac{\delta D}{D} S^z C_{k'\uparrow}^{+} C_{k\uparrow}$

与 $H_1 = \frac{J}{2V} S^z (C_{k'\uparrow}^{+} C_{k\uparrow} - C_{k'\downarrow}^{+} C_{k\downarrow})$ 比较，得到

$$\delta J = J^2 N(0)\frac{\delta D}{D} = -J^2 N(0)\,\delta(\ln D).$$

$D \to D - \delta D$

$\delta(\ln D) = -\frac{D}{|\delta D|}.$

$$\therefore\quad \frac{\delta J}{\delta \ln D} = -J^2 N(0)$$

有限温：
$$\frac{dJ}{d\ln D/T} = -J^2 N(0)$$

求解此微分方程，得到

$$J(D) = \frac{J(D_0)}{1 + J(D_0)N(0)\ln(D/D_0)}.$$