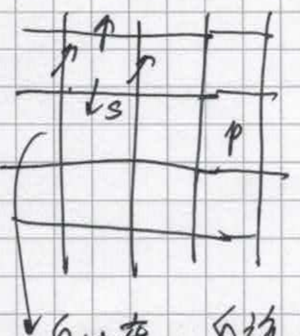
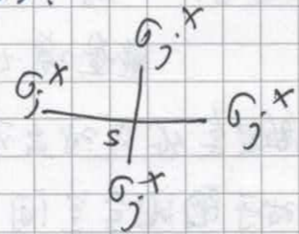
Lecture 13 拓扑序简介

1) Toric-code 模型.

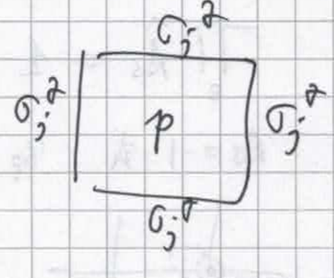
A. Kitaev Annals of Physics 303 2 (2003).



$$\hat{A}_s = \prod_{j\in s} \hat{\sigma}_j^x .$$



$$\hat{B}_p = \prod_{j\in \boxed{p}} \hat{\sigma}_j^z$$



自由度：自旋 $1/2$ in each bond.

$$\hat{H} = -\sum_s \hat{A}_s - \sum_p \hat{B}_p .$$

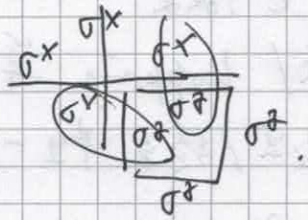
* 此模型严格可解： Commuting ~~projectors~~ stablizers

- Projector： $\hat{A}_s$ 和 $\hat{B}_p$ 均为 ~~投影算符~~ stablizer 算符.

$$\hat{A}_s^2 = 1 , \quad \hat{B}_p^2 = 1 .$$

$$\hat{O}^2 = 1 \Rightarrow \hat{O} \text{ 的本征值为 } \pm 1 .$$

- Commuting： $\hat{A}_s$ 和 $\hat{B}_p$ 对易.



$$(-1)^2 = +1 .$$

因此，可取 $\hat{A}_s$ 和 $\hat{B}_p$ 的共同本征态.

$$|\hat{A}_s = \pm 1 , \hat{B}_p = \pm 1 \rangle .$$

基态： $\hat{A}_s = +1$, $\hat{B}_p = +1$ $\forall$ s, p.

$\{\hat{A}_s, \hat{B}_p\}$ "基本上"确定量子态：

热力学极限下： N个格点 $\Rightarrow$ 2N个 bond $\Rightarrow$ 2N个自旋 $1/2$

Hilbert 空间维度 2^{2N}.

$N\times \hat{A}_s$; $N\times \hat{B}_p$: 2N个 (± 1) 自由度.

2) 激发态：任意子激发与弦算符.

∅基态：$A_s=+1$, $B_p=+1$.

* 激发态：$A_s=-1$ 或 $B_p=-1$

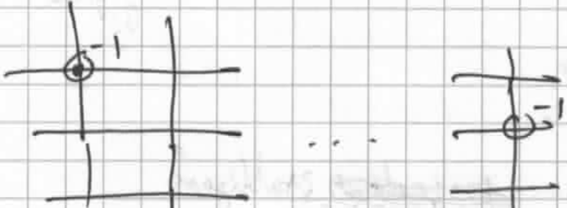
能量增加：$\Delta E = 2$

* 激发态必成对出现：

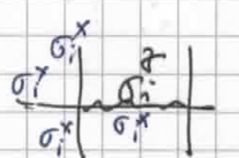
对于闭合空间，(如考虑 p.b.c.)

$$\prod_s \hat{A}_s = 1, \quad \prod_p \hat{B}_p = 1$$

∴ $A_s=-1$ 或 $B_p=-1$ 必须成对出现



*弦算符，

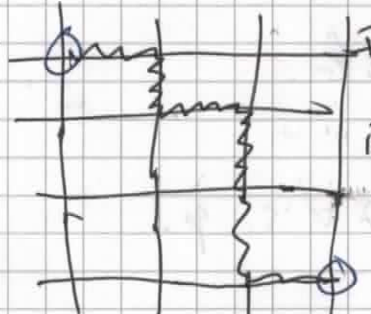


σ_i^z 与左右两侧的 $\hat{A}_s$ 反对易，

σ_i^z 产生一对 $A_s=-1$ 的 e 激发.

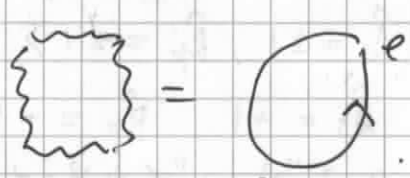
σ_i^z：e-e 对产生/对湮灭/hopping 算符.

— σ_i^z 弦算符：在弦的起点/终点产生一对 e 激发.



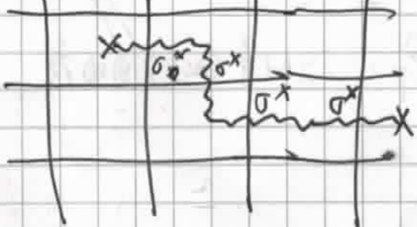
开弦：$\prod \sigma_i^z$ e-e 对产生/湮灭/移动算符.

闭弦：

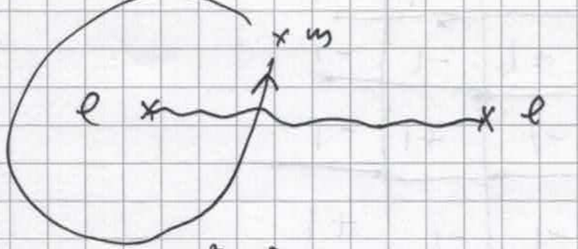


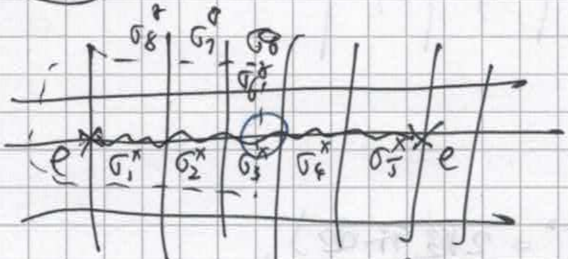
$\hat{O}|0\rangle = 0$.

类似地，σ_i^x 算符产生一对 m 激发.



* 统计: mutual braiding stat. b/w e & m = −1.





$$\hat{O}_1 = \hat{\sigma}_1^x \hat{\sigma}_2^x \hat{\sigma}_3^x \hat{\sigma}_4^x \sigma_5^x.$$

$$\hat{O}_2 = \cdots \sigma_3^z \sigma_6^z \sigma_7^z \sigma_8^z \cdots$$

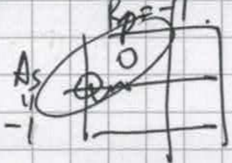
$$\hat{O}_1 \hat{O}_2 = -\hat{O}_2 \hat{O}_1 \quad \text{b/c} \quad \hat{\sigma}_3^x \hat{\sigma}_3^z = -\hat{\sigma}_3^z \sigma_3^x.$$

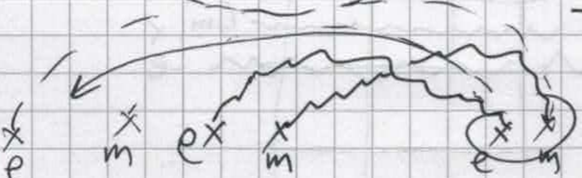
$$\hat{O}_2 \hat{O}_1 |0\rangle = -\hat{O}_1 \hat{O}_2 |0\rangle = -\hat{O}_1 |0\rangle.$$

3) 任意子激发:

* Bound state $\epsilon = e \otimes m$.

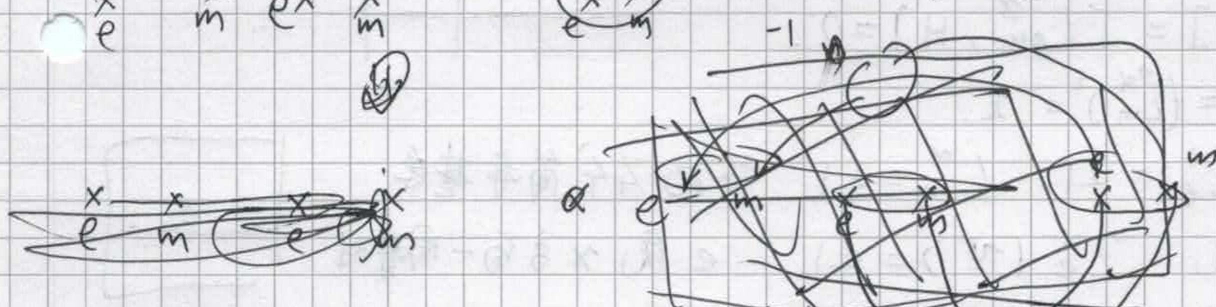
(e x x m) = ε.





$= -1 \implies e \otimes m = \epsilon$ 是费米子.

−1



* 4 anyons: $\mathbb{1}$, e, m, $\epsilon = e \otimes m$.

e 和 m 为 boson, ε 为 fermion.

e (m) = −1.

Fusion rule: $e \otimes e = \mathbb{1}$, $m \otimes m = \mathbb{1}$, ...

⊗	1	e	m	ε
1	1	e	m	ε
e	e	1	ε	m
m	m	ε	1	e
ε	ε	m	e	1

Mutual braiding:

M_{ab}	$\mathbb{1}$	e	m	ϵ
$\mathbb{1}$	+1	+1	+1	+1
e	+1	+1	−1	−1
m	+1	−1	+1	−1
ϵ	+1	−1	−1	+1

~~4) 基态~~

4) 基态简并度与流形拓扑.

* 数一下自由度: 考虑 p.b.c. (空间 = T^2 = 2维环面).

N个格点: 2N个 $s=\frac{1}{2}$.

2N个本征值 $\{A_s, B_p\}$

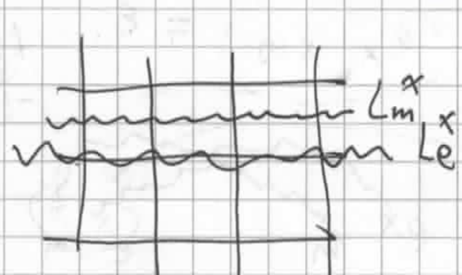
−2个约束条件: $\prod_s A_s = \prod_p B_p = +1$.

∴ 还剩 $2N-(2N-2)=2$ 个自由度.

事实上, 基态是 $2^2=4$ 重简并的

— 这个4重简并在 toric code 中被用于存储 logic qubit (量子信息)

* 弦算符:



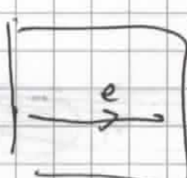
定义 $L_e^x = \prod \sigma_i^z$

$L_m^x = \prod \sigma_i^x$

$[L_e^x, H] = [L_m^x, H] = 0$.

$(\hat{L}_e^x)^2 = (\hat{L}_m^x)^2 = \mathbb{1}$.

∴ $L_e^x = \pm 1$ $L_m^x = \pm 1$ 标记4个简并基态.



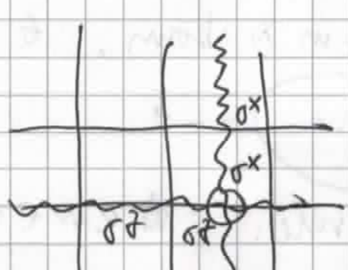
物理意义: $L_e^x|\Psi\rangle = \pm 1$: e 线 x 方向一周看到的 flux 为 ±1.

* 不同的标记方式: 用类似的方式可以定义 L_e^y, L_m^y: 沿 y 方向绕一周的算符.

反对易: $L_e^x L_m^y = -L_m^y L_e^x$.

令 $|0\rangle = |L_e^y = 1, L_m^y = 1\rangle$.

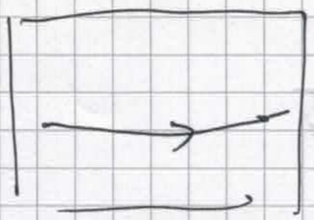
$|e\rangle = \hat{L}_e^x|0\rangle$: e 线 x 方向一周.



$\hat{L}_e^y|e\rangle = |e\rangle$

$\hat{L}_m^y|e\rangle = \hat{L}_m^y \hat{L}_e^x|0\rangle = -\hat{L}_e^x \hat{L}_m^y|0\rangle = -|e\rangle$.

$\therefore\ |e\rangle = \hat{L}_e^x |0\rangle = |L_e^y = 1,\ L_m^y = -1\rangle$

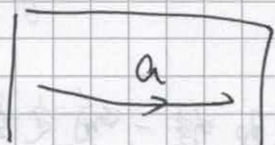


e绕x方向一周导致 $|0\rangle \to |e\rangle$.

类似地，$|m\rangle = \hat{L}_m^x |0\rangle = |L_e^y = -1,\ L_m^y = +1\rangle$.

$|\varepsilon\rangle = \hat{L}_m^x \hat{L}_e^x |0\rangle = |L_e^y = -1,\ L_m^y = -1\rangle$.

4× anyon ⟹ 4× 简并基态.

$|a\rangle =$   $= \hat{L}_a^x |0\rangle$.

一般地，简并度 $= (\#\text{ of anyons})^g$. ← 拓扑序.

$g=0$ $g=1$ $g=2$

* 在任意二维曲面上数自由度：

自由度：自旋 1/2 数量 $= E$ (Edge 数目).

$\hat{A}_s$ 算符个数 $= V$ (Vertex 个数).

$\hat{B}_p$ ···· $= F$ (Face 个数).

2×约束：$\prod_s A_s = \prod_p B_p = 1$.

剩余自由度 $= E - (V + F - 2) = 2g$

(Eular 公式：$V - E + F = \chi = 2 - 2g$).

$\therefore$ 简并度 $= 2^{2g} = 4^g$.

5) 分数量子霍尔效应：

FQHE 也是拓扑序.

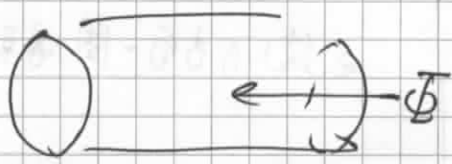
* 整米型拓扑序：$\mathbb{1}$ 包括电子激发.

三种 anyon：$\mathbb{1}$, a, $a^2 = a \otimes a$.

$a \otimes a \otimes a = \mathbb{1}$

3个准粒子 = 电子

基态3重简并：



$\Delta\Phi = \Phi_0 = \frac{hc}{e}$：

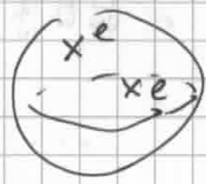
穿过一个准粒子 $\Delta Q = \frac{e}{3}$.

Bulk 变成另一个基态.

因此和IQH子的 Laughlin's argument 并不矛盾.

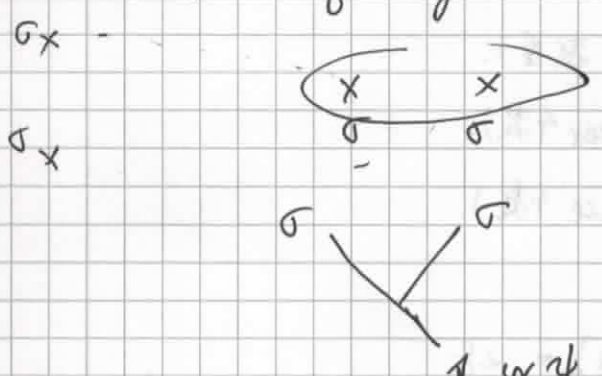
6) 非阿贝尔任意子与非阿贝尔统计.

Toric code：固定Anyon位置及种类即唯一确定量子态.



Non-Abelian Anyon：固定Anyon位置及种类，仍有拓扑简并度.

例：Ising anyon.

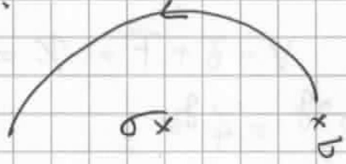


$\sigma \otimes \sigma = 1 \oplus \psi$.

二重简并.

1 or ψ.

Non-Abelian 统计：



2×2 矩阵.