Lecture 11 量子相变.

参考资料: 教材 Chap 14; S. Sachdev Quantum Phase Transitions (2nd Ed). Chap 18

1) 经典相变回顾.

*连续相变(二级相变): 对称性自发破缺.

对称性 $\longrightarrow$ 序参量 $\longrightarrow$ Ginzburg-Landau 理论

例: Ising model $H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j$.

对称性: $\sigma_i^z \to -\sigma_i^z \Rightarrow$ 序参量 $\phi = \langle \sigma_i^z \rangle$. $\phi \to -\phi$.

例: 超导体: 对称性 $C_i \to C_i e^{i\theta}$.

序参量: $\Phi = \Delta = \langle C_{k\uparrow} C_{-k\downarrow} \rangle$. $\Phi \to \Phi e^{i2\theta}$

Landau: $F = r|\Phi|^2 + u|\Phi|^4$.

F 必须满足对称性: $\cancel{\Phi^3}$ $\cancel{\Phi^2}$ etc.

Ginzburg + Landau: $\Phi = \Phi(\vec{r})$

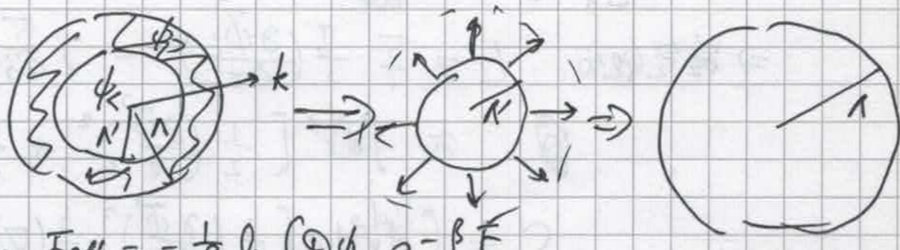
超导: $F = \int d\vec{r}\left[|\nabla\Phi|^2 + r|\Phi|^2 + u|\Phi|^4\right]$

$\nabla - i\frac{2e}{c}\vec{A}$ ...

Ising: $F = \int d\vec{r}\left[(\nabla\phi)^2 + r\phi^2 + u\phi^4\right]$.

*重整化群.

$\phi = \phi_< + \phi_>$



① 积掉 $\phi_>$:

$e^{-\beta F_{eff}} = \int \mathcal{D}\phi_> e^{-\beta F}$ or $F_{eff} = -\frac{1}{\beta}\ln\int\mathcal{D}\phi_> e^{-\beta F}$.

② Rescale: $\Lambda' \to \Lambda$.

自由理论: ① $F_{eff} = F$.

② 量纲分析: $k \to k\frac{\Lambda}{\Lambda'}$ $[k] = 1$. $[x] = -1$.

$[F] = 0 \Rightarrow [\phi] = \frac{D-2}{2}$. $[r] = 2$ $[u] = 4-D$.

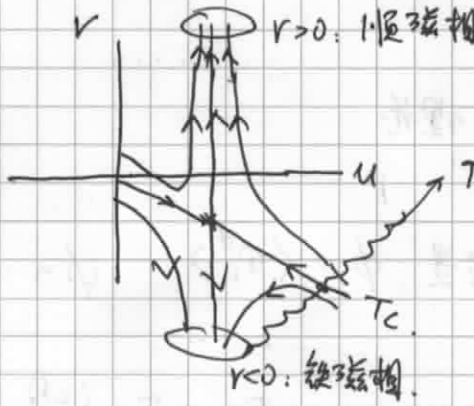
$[\lambda] > 0$: relevant
< 0: irrelevant
$= 0$: marginal.
} perturbation.

为什么不放高阶项, 如 $\phi\nabla^4\phi$, ϕ^6 etc? 因为更加 irrelevant.

*重整化群；临界指数.

例：Wilson - Fisher fixed pt.

相变点：有且仅有一个 relevant perturbation.



临界指数：2个独立指数 $[r] = 1/\nu$，$[\phi] = \frac{D-2+\eta}{2}$.

2) 经典 ⟺ 量子相变.

例：~~Heisenberg 模型~~ 横场 Ising 模型：

$$H = -J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - h\sum_i \hat{\sigma}_i^x.$$

$$\langle \hat{\sigma}_i^z \rangle = \phi.$$

例：Heisenberg 模型：

$$H = -J\sum_{\langle ij\rangle} \vec{\Phi}_i \cdot \vec{\Phi}_j + \sum_i \frac{\vec{L}_i^2}{2I} \qquad \vec{L}_i：\vec{\Phi}_i \text{ 的角动量.}$$

$$\vec{L}_i \sim \frac{d\vec{\Phi}_i}{dt}.$$

⇒ 路径积分：$L = \sum_i \frac{I}{2}\left(\frac{\partial \vec{\Phi}_i}{\partial \tau}\right)^2 - J\sum_{\langle ij\rangle} \vec{\Phi}_i \cdot \vec{\Phi}_j.$

$$\approx \int d^d\vec{r}\left[\frac{1}{2}\left(\frac{\partial \vec{\Phi}}{\partial \tau}\right)^2 + \frac{1}{2}(\nabla\vec{\phi})^2 + r\vec{\phi}^2 + u(\vec{\phi}^2)^2\right]$$

$$S = \int d^d\vec{r}\, d\tau \left[\frac{1}{2}\left(\frac{\partial \vec{\Phi}}{\partial \tau}\right)^2 + \frac{1}{2}(\nabla\vec{\phi})^2 + r\vec{\phi}^2 + u(\vec{\phi}^2)^2 + \cdots\right]$$

τ 与 $\vec{r}$ 同样 scale：$[\tau] = [\vec{r}] = -1$

$[\omega] = [k] = 1.$

相当于 $D = d+1$ 维经典体系.

*Dynamical exponent：$[\omega] = [k^z]$.

$d \Leftrightarrow D = d+z$.

这里 $z = 1$.

3) 费米液体中的密度波长程序.

自旋密度波，电荷密度波，…

(SDW)　　(CDW)

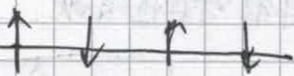
密度波：序参量在实空间周期性变化. $\Delta(\vec{r}) = \Delta_0 e^{i\vec{Q}\cdot\vec{r}}$

或 $\Delta_0 \cos(\vec{Q}\cdot\vec{r})$.

例：SDW 2维正方晶格.

$$H = \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} + J\sum_{\langle ij\rangle}\vec{S}_i\cdot\vec{S}_j$$

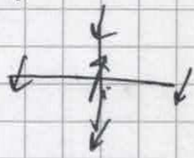
w/ $\vec{S}_i = c^\dagger_{i\alpha}\vec{\sigma}_{\alpha\beta}c_{i\beta}$;　$J>0$：反铁磁.



序参量：$\vec{\phi} \sim \langle (-1)^i \vec{S}_i\rangle$.　$\langle\vec{S}_i\rangle = (-)^i\vec{\phi}$.

不失一般性，假设 $\vec{\phi} = \phi\, e^{\hat{z}}$.

平均场分解：
$$H = \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} + J\sum_{\langle ij\rangle}\left(\langle\vec{S}_i\rangle\cdot\vec{S}_j + \vec{S}_i\cdot\langle\vec{S}_j\rangle - \langle\vec{S}_i\rangle\cdot\langle\vec{S}_j\rangle\right)$$
$$= \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} + J\sum_{i}\sum_{\langle j,i\rangle}\langle\vec{S}_j\rangle\cdot\vec{S}_i$$
$$= \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} - \sum_i(-)^i 4J\phi\, S_i^z \qquad \left(\langle\vec{S}_j\rangle = -(-)^i\phi\, e^{\hat{z}}\right)$$



令 $\Delta = 4J\phi$

$$H = \sum_{k\alpha}(\epsilon_k-\mu)c^\dagger_{k\alpha}c_{k\alpha} - \sum_i(-)^i\Delta S_i^z.$$

$(-)^i = e^{i\vec{Q}\cdot\vec{r}_i}$，$\vec{Q} = (\pi,\pi)$ ← 反铁磁动量.

$$H = \sum_{k\alpha}(\epsilon_k-\mu)(c^\dagger_{k\uparrow}c_{k\uparrow} + c^\dagger_{k\downarrow}c_{k\downarrow}) - \Delta\sum_i \underbrace{e^{i\vec{Q}\cdot\vec{r}_i}(c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow})}$$

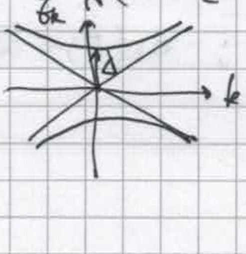
⇓ Fourier 变换 $\sum_k c^\dagger_{k\uparrow}c_{k+Q\uparrow}$.

$$H = \sum_{k}\left[(\epsilon_k-\mu)(c^\dagger_{k\uparrow}c_{k\uparrow} + c^\dagger_{k\downarrow}c_{k\downarrow}) - \Delta c^\dagger_{k\uparrow}c_{k+Q\uparrow} - \Delta c^\dagger_{k\downarrow}c_{k+Q\downarrow}\right].$$

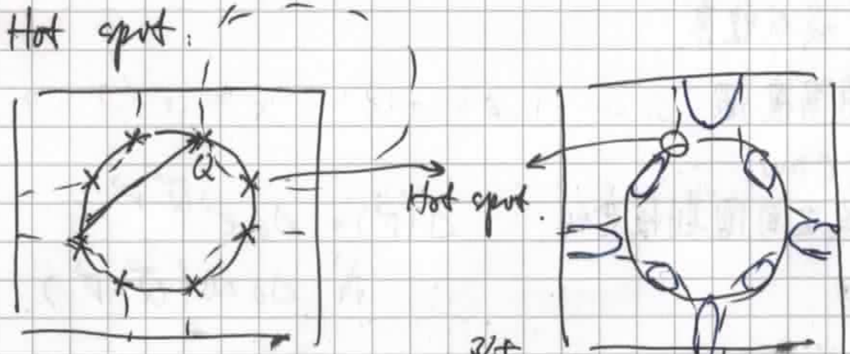
$$= \sum_k \begin{pmatrix} c^\dagger_{k\alpha} & c^\dagger_{k+Q,\alpha}\end{pmatrix}\begin{pmatrix}\epsilon_k-\mu & -\alpha\Delta \\ -\alpha\Delta & \epsilon_{k+Q}-\mu\end{pmatrix}\begin{pmatrix}c_{k\alpha} \\ c_{k+Q,\alpha}\end{pmatrix}.$$

$$E_k = \frac{\epsilon_k+\epsilon_{k+Q}}{2} - \mu \pm \sqrt{\left(\frac{\epsilon_{k+Q}-\epsilon_k}{2}\right)^2 + \Delta^2}$$

$\mu = \epsilon_{k+Q} = \epsilon_k$：打开能隙

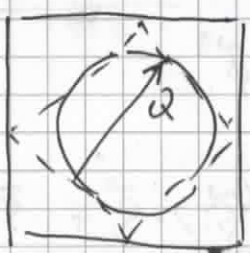


4) Hot spot:



费米面上在"hot spot"附近打开能隙.

平移对称性破缺：晶胞×2 ⇒ 布里渊区 $\times\frac{1}{2}$



$\vec{Q}$ 变成新的倒格矢.

费米面和新的布里渊区相交的地方打开能隙.

*低能自由度：$\phi(\vec{q})$：当 $\vec{q}\approx\vec{Q}$ 时能量低.

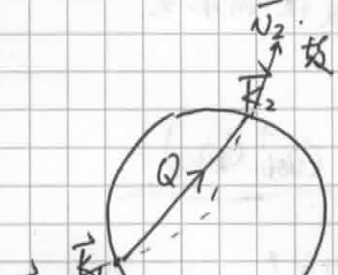
C_k：$\epsilon_k=\mu$ 时能量低.

$$\vec{\phi}(\vec{q})\ C^{\dagger}_{k+q,\alpha}\ \vec{\sigma}_{\alpha\beta}\ C_{k,\beta}.$$

当 $k\in$ hot spot 时：同时满足

$\epsilon_k\approx\mu$； $\epsilon_{k+q}\approx\mu$， $q\approx Q$

故三者均为低能激发.



*低能有效理论：

$\vec{\phi}(\vec{q})$：从 $\vec{Q}$ 量起 $\vec{q}_{ph}=\vec{Q}+\vec{q}$

$\psi_1(\vec{k})$： $\vec{k}_{ph}=\vec{K}_1+\vec{k}$.

$\psi_2(\vec{k})$： $\vec{k}_{ph}=\vec{K}_2+\vec{k}$

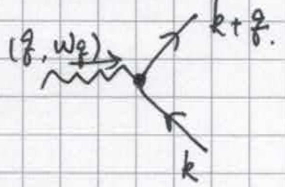
$$\vec{K}_2=\vec{K}_1+\vec{Q}.$$

$$H=H[\vec{\phi}]+\sum_{k}\left[\epsilon_1(\vec{k})\psi_1^{\dagger}(\vec{k})\psi_1(\vec{k})+\epsilon_2(\vec{k})\psi_2^{\dagger}(\vec{k})\psi_2(\vec{k})\right]$$
$$+\lambda\sum_{k,q}\left(\vec{\phi}(\vec{q})\cdot\psi^{\dagger}_{1\alpha}(\vec{k}+\vec{q})\,\vec{\sigma}_{\alpha\beta}\,\psi_{2\beta}(\vec{k})+h.c.\right)$$

$\epsilon_1(\vec{k})=\vec{v}_1\cdot\vec{k}$ ； $\epsilon_2(\vec{k})=\vec{v}_2\cdot\vec{k}$.

5) Landau damping.

物理过程：一个自旋波激发(magnon)衰变为一对费米子 粒子/空穴对.



$$\phi_q(\vec q)\,\psi_1^\dagger(\vec k+\vec q)\,\psi_2(\vec k)+h.c.$$

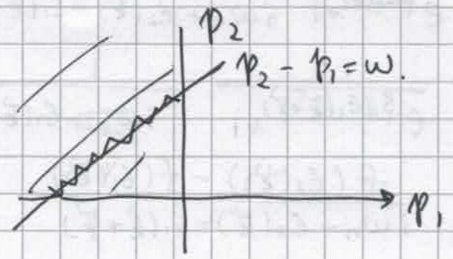
Fermi Golden Rule: $T=0$

$$\frac{1}{\tau}\propto\lambda^2\int\frac{d^2k}{(2\pi)^2}\,\delta(\omega+\epsilon_1(\vec k)-\epsilon_2(\vec k+\vec q))\,\theta(-\epsilon_1(\vec k))\,\theta(\epsilon_2(\vec k))$$

引入斜坐标系: $p_1=\vec v_1\cdot\vec k=\epsilon_1(\vec k)$ $\quad p_2=\vec v_2\cdot(\vec k+\vec q)=\epsilon_2(\vec k+\vec q)$.

$$\frac{1}{\tau}\propto\lambda^2\int\frac{dp_1\,dp_2}{(2\pi)^2\underbrace{|\vec v_1\times\vec v_2|}_{\text{Jacobian}}}\,\delta(\omega+p_1-p_2)\,\theta(-p_1)\,\theta(p_2)$$

$$\propto\frac{\lambda^2}{4\pi^2}\frac{1}{|\vec v_1\times\vec v_2|}\int_{-\infty}^{0}dp_1\int_0^{\infty}dp_2\,\delta(\omega+p_1-p_2)$$

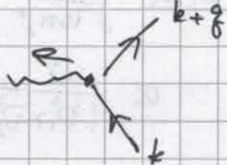


$$\propto\frac{\lambda^2}{4\pi^2}\frac{1}{|\vec v_1\times\vec v_2|}\sqrt{2}\,\omega$$

$$=\gamma\cdot\omega$$

$$\gamma\propto\frac{\lambda^2}{|\vec v_1\times\vec v_2|}.$$

当 $\omega<0$ 时，有类似的反过程.



$$\frac{1}{\tau}=\gamma\cdot|\omega|.$$

ϕ的格林函数:

$$G_\phi^{-1}=i\gamma|\omega|=i\gamma\omega\,\mathrm{sgn}(\omega)$$

解析延拓: $G_\phi^{-1}(\omega_n)=\gamma|\omega_n|$

对应 $S_{eff}=\sum_{q,\omega_n}\vec\phi(-\vec q,-\omega_n)\left(\gamma|\omega_n|+c^2q^2+\cdots\right)\vec\phi(\vec q,\omega_n)$.

6) 用格林函数计算 $\vec{\phi}$ 的低能有效模型.

低能有效模型: $Z = \int \mathcal{D}\vec{\phi} \exp\left[-\sum_{q,\omega_n} \vec{\phi}^*(\vec{q},\omega_n)\cdot G_\phi^{-1}(\vec{q},\omega_n)\,\vec{\phi}(\vec{q},\omega_n)\right]$.

$$G_\phi^{-1}(\vec{q},\omega_n) = \frac{\partial^2(-\ln Z)}{\partial\phi(\vec{q},\omega_n)\,\partial\phi^*(\vec{q},\omega_n)}.$$

$$= \frac{\partial^2 F}{\partial\phi\,\partial\phi^*}$$

$$= \chi_0(\vec{q},\omega_n)$$

$$= \int d\tau \sum_{k,k'} e^{i\omega_n\tau}\, T_\tau \langle \psi_1^+(\vec{k}+\vec{q},\tau)\,\psi_2(\vec{k},\tau)\,\psi_2^+(\vec{k}'-\vec{q},0)\,\psi_1(\vec{k}',0)\rangle$$

$$\propto \sum_k \sum_{\varepsilon_n} \frac{1}{i\varepsilon_n + i\omega_n - \epsilon_1(\vec{k}+\vec{q})}\,\frac{1}{i\varepsilon_n - \epsilon_2(\vec{k})}$$

$$= \int \frac{d^2k}{(2\pi)^2} \oint \frac{dz}{2\pi i}\, \underbrace{f(z)}_{=\frac{1}{e^{\beta z}+1}}\, \frac{1}{z + i\omega_n - \epsilon_1(\vec{k}+\vec{q})}\,\frac{1}{z - \epsilon_2(\vec{k})}$$

$$= \int \frac{d^2k}{(2\pi)^2}\left[f(\epsilon_2(\vec{k}))\,\frac{1}{\epsilon_2(\vec{k}) + i\omega_n - \epsilon_1(\vec{k}+\vec{q})} + f(\epsilon_1(\vec{k}))\,\frac{1}{\epsilon_1(\vec{k}+\vec{q}) - i\omega_n - \epsilon_2(\vec{k})}\right]$$

$$= \int \frac{d^2k}{(2\pi)^2}\, \frac{f(\epsilon_2(\vec{k})) - f(\epsilon_1(\vec{k}))}{i\omega_n + \epsilon_2(\vec{k}) - \epsilon_1(\vec{k}+\vec{q})}$$

解析延拓: $i\omega_n \to \omega + i0^+$.

$$G_\phi^{-1}(\vec{q},\omega) = \int \frac{d^2k}{(2\pi)^2}\, \frac{f(\epsilon_2(\vec{k})) - f(\epsilon_1(\vec{k}))}{\omega - \epsilon_1(\vec{k}+\vec{q}) + \epsilon_2(\vec{k}) + i0^+}$$

$$\mathrm{Im}\, G_\phi^{-1}(\vec{q},\omega) = \int \frac{d^2k}{(2\pi)^2}\left[f(\epsilon_2(\vec{k})) - f(\epsilon_1(\vec{k}))\right]\delta(\omega - \epsilon_1(\vec{k}+\vec{q}) + \epsilon_2(\vec{k})).$$

$$\propto \frac{\omega}{|\vec{v}_1\times\vec{v}_2|}. \quad (对于\ \omega>0).$$

对应 $G_\phi^{-1}(\vec{q},\omega_n) \sim C + \frac{|\omega_n|}{|\vec{v}_1\times\vec{v}_2|}$.

$$S_{eff} = \sum_{q,\omega_n} \phi^*(\vec{q},\omega_n)\left(\gamma|\omega_n| + \omega_n^2 + c^2q^2 + \cdots\right)\phi(\vec{q},\omega_n).$$

7) Hertz 理论: 当 $\omega_n\to 0$ 时, $\gamma|\omega_n| \gg \omega_n^2$ 忽略后者.

$$\gamma|\omega_n| + c^2q^2 \qquad |\omega_n| \sim q^2.$$

$$\therefore\ [\omega_n] = 2 \Rightarrow z = 2.$$