Homework 5

Due: Friday, March 18 at noon

Instructions: Submit a pdf of your solutions to the HW 5 assignment on Gradescope.

0. If you would like any of these problems to be graded for proficiency with the core skills, list the skill and the corresponding problem.

1. Solve each of the following simultaneous systems of congruences or explain why no solution exists.

   (a) $\begin{cases} x \equiv 5 \pmod{13} \\ x \equiv 2 \pmod{7} \\ x \equiv 4 \pmod{11} \end{cases}$

   (b) $\begin{cases} x \equiv 3 \pmod{9} \\ x \equiv 2 \pmod{6} \\ x \equiv 1 \pmod{5} \end{cases}$

   (Note: this problem is worth 8 points–4 for each part)

2. Find three distinct (modulo 105) solutions to the equation

$$x^2 \equiv 4 \pmod{105}.$$

3. This exercise will help us work towards understanding square roots.

   (a) Let p be a prime number, let $e \geq 1$ be an integer, and let b be an integer such that $x^2 \equiv b \pmod{p}$ has a solution. Show that, if a_1 and a_2 are two such solutions, then $a_1 \equiv a_2 \pmod{p}$ or $a_1 \equiv -a_2 \pmod{p}$.

   (b) Provide a counterexample to show that the above statement is not necessarily true if p is not prime. In other words, provide integers m, b, a_1, and a_2 such that $a_1^2 \equiv b \pmod{m}$, $a_2^2 \equiv b \pmod{m}$, but $a_1 \not\equiv \pm a_2 \pmod{m}$. Make sure you explain why your counterexample is, in fact, a counterexample.

   (Note: This problem is worth 7 points–4 for part a, and 3 for part b.)

4. Prove or disprove:
   For any positive integers m and n, and any integers a and b, the system of equations

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

   will have at least one solution.