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Optimal Control

Static Optimization to Optimal Control

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Content

- Static optimization and dynamic optimization
- History and present of optimal control
- Calculus of variations

Example

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

- ❖ Optimize the quantities in one month. (multi-stage optimization)
- ❖ The quantities tomorrow are affected by the quantities today (dynamic optimization)

Static optimization to dynamic optimization

Static optimization

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) \qquad \text{s.t.} \quad \mathbf{x} \in X \subset \mathbb{R}^{n_x}.$$

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

Multi-stage optimization

$$\min_{\mathbf{x}_1,\ldots,\mathbf{x}_N} \quad \sum_{k=1}^{N} f(k, \mathbf{x}_k) \qquad \text{s.t.} \quad \mathbf{x}_k \in X_k \subset \mathbb{R}^{n_x}, \quad k = 1, \ldots, N.$$

$$\min_{\mathbf{x}(t)} \quad \int_{t_1}^{t_2} f(t, \mathbf{x}(t))\, dt \qquad \text{s.t.} \quad \mathbf{x}(t) \in X(t) \subset \mathbb{R}^{n_x}, \quad t_1 \le t \le t_2.$$

Dynamic optimization

$$\boxed{x_k, k = 1, 2, \ldots\ldots \qquad x(t), \quad t_1 \le t \le t_2}$$

$$\begin{array}{lll} \displaystyle\min_{\mathbf{x}_1,\ldots,\mathbf{x}_N} & \displaystyle\sum_{k=1}^{N} f(k, \mathbf{x}_k, \mathbf{x}_{k-1}) & \\ \text{s.t.} & \mathbf{x}_0 \text{ given} \quad x_k = g(x_{k-1}) & \\ & \mathbf{x}_k \in X_k \subset \mathbb{R}^{n_x}, \quad k = 1, \ldots, N. & \end{array}$$

$$\begin{array}{ll} \displaystyle\min_{\mathbf{x}(t)} & \displaystyle\int_{t_1}^{t_2} f(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \\ \text{s.t.} & \mathbf{x}(t) \in X(t) \subset \mathbb{R}^{n_x}, \quad t_1 \le t \le t_2. \\ & \dot{x}(t) = g(x(t)) \end{array}$$

Dynamic Optimization to Optimal Control

- Finding a control law for a given system such that a certain optimality criterion is achieved

Continuous time optimal control

min $$J = \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{S}_f\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\left(\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\right)\mathrm{d}t$$

s.t. $$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

Discrete time optimal control

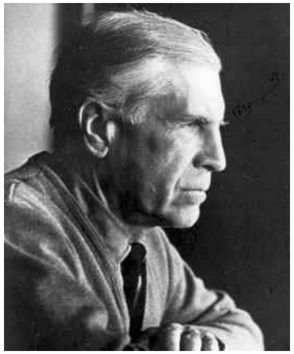
min $$\mathrm{J} = \frac{1}{2}x^T(N)S_N x(N) + \frac{1}{2}\sum_{n=1}^{N}[x^T(n)Q_n x(n) + u^T(n)R_n u(n)]$$

s.t. $$x(n+1) = A_n x(n) + B_n u(n), \qquad x(0) = x_0$$

Other constraints

History and Present

Calculus of variations（1600-1900）



Lev Pontryagin(1908-1988)

Maximum principle

Richard Ernest Bellman(1920-1984)

Dynamic programming

Approximate dynamic programming[1]

[3]

Reinforcement learning[2]

No model

[1] D. P. Bestsekas, Dynamic Programming and Optimal Control, Athena Scientific, 2011.

[2] R. S. Sutton and A. G. Barto, Reinforcement learning: an Introduction, MIT Press, 1998.

[3]P. Mehta and S. Meyn, Q-learning and Pontryagin's Minimum Principle, CDC 2009.

Main methods

- Calculus of variations
- Pontryagin maximum principle
- Bellman dynamic programming

Calculus of Variations

Calculus of variations

- **Main issue** – General control problem, the cost is a function of functions $\mathbf{x}(t)$ and $\mathbf{u}(t)$.

$$\min J = h(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

subject to

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ &\mathbf{x}(t_0), t_0 \text{ given} \\ &m(\mathbf{x}(t_f), t_f) = 0 \end{aligned}$$

– Call $J(\mathbf{x}(t), \mathbf{u}(t))$ a **functional**.

$$J(x(t), u(t))$$

Minimum of a Function

A function $f(\mathbf{x})$ has a local minimum at $\mathbf{x}^\star$ if

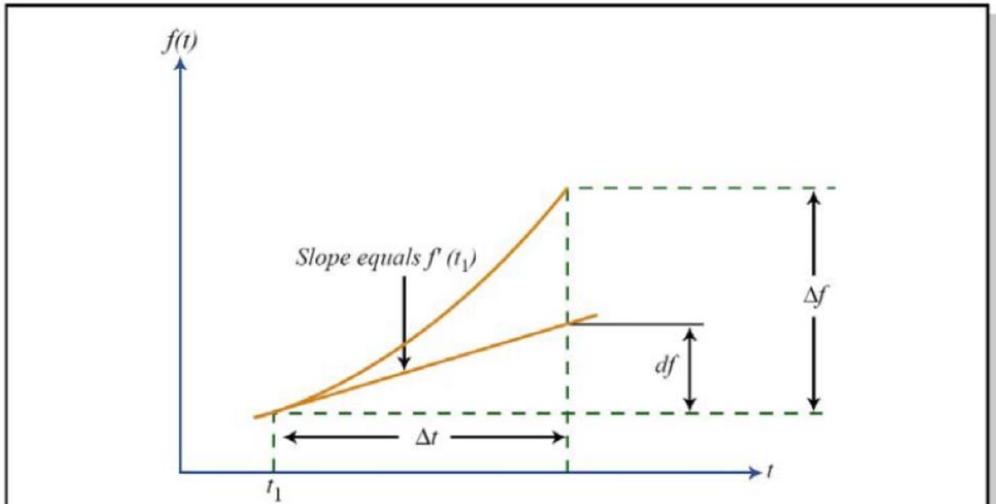
$$f(\mathbf{x}) \geq f(\mathbf{x}^\star)$$

for all admissible $\mathbf{x}$ in $\|\mathbf{x} - \mathbf{x}^\star\| \leq \epsilon$

$$\boxed{J(\mathrm{x}(t))} \quad \textbf{?}$$

Gradient $\dfrac{\partial f}{\partial \mathbf{x}} = 0$

Differential $df = \dfrac{\partial f}{\partial \mathbf{x}} d\mathbf{x} = 0$



Differential df versus increment Δf shown for a function, but the same difference holds for a functional.

Figure by MIT OpenCourseWare.

$$\Delta f = f(x + dx) - f(x) = \frac{\partial f}{\partial x} dx + H.O.T$$

Differential df **is the linear part of increment** Δf

Minimal of a Functional

A functional $J(\mathbf{x}(t))$ has a local minimum at $\mathbf{x}^\star(t)$ if

$$J(\mathbf{x}(t)) \geq J(\mathbf{x}^\star(t))$$

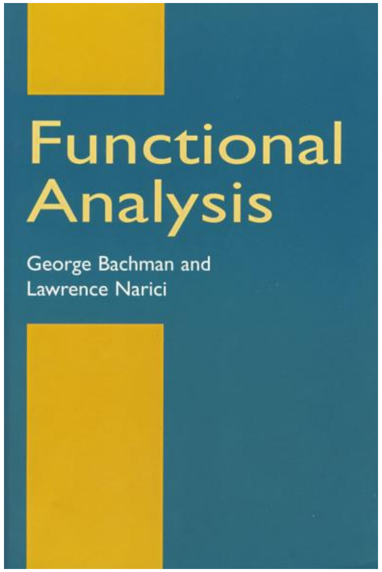
for all admissible $\mathbf{x}(t)$ in $\boxed{\|\mathbf{x}(t) - \mathbf{x}^\star(t)\|} \leq \epsilon$

Function Norm:

1. $\|\mathbf{x}(t)\| \geq 0$ for all $\mathbf{x}(t)$, and $\|\mathbf{x}(t)\| = 0$ only if $\mathbf{x}(t) = 0$ for all t in the interval of definition.
2. $\|a\mathbf{x}(t)\| = |a|\|\mathbf{x}(t)\|$ for all real scalars a.
3. $\|\mathbf{x}_1(t) + \mathbf{x}_2(t)\| \leq \|\mathbf{x}_1(t)\| + \|\mathbf{x}_2(t)\|$

Common function norm:

$$\|\mathbf{x}(t)\|_2 = \left(\int_{t_0}^{t_f} \mathbf{x}(t)^T \mathbf{x}(t) dt\right)^{1/2}$$



Minimum of a Functional

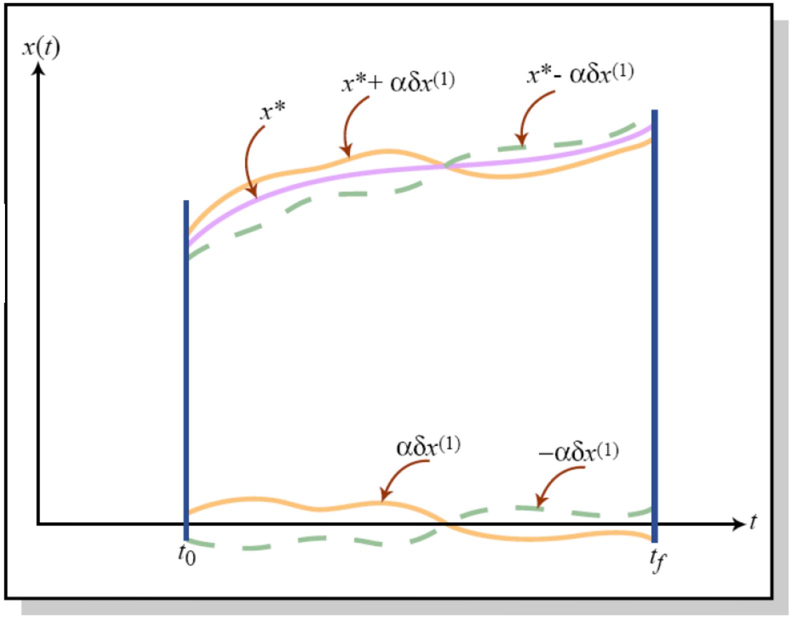
$\min\ f(x)$ ⟷ **Differential df is the linear part of increment Δf**

$\min\ J(x(t))$ ⟷ **Linear part of increment**

Increment: $\Delta J(\mathbf{x}(t), \delta\mathbf{x}(t)) = J(\mathbf{x}(t) + \delta\mathbf{x}(t)) - J(\mathbf{x}(t))$

Variation

$$\Delta J(\mathbf{x}(t), \delta\mathbf{x}(t)) = \delta J(\mathbf{x}(t), \delta\mathbf{x}(t)) + H.O.T.$$



Fundamental Theorem

Let $\mathbf{x}$ be a function of t in the class Ω, and $J(\mathbf{x})$ be a differentiable functional of $\mathbf{x}$. **Assume** that the functions in Ω are not constrained by any boundaries.

If $\mathbf{x}^\star$ is an extremal function, then the variation of J must vanish on $\mathbf{x}^\star$, i.e. for all admissible $\delta\mathbf{x}$,

$$\delta J(\mathbf{x}(t), \delta\mathbf{x}(t)) = 0$$

Proof: see [Kirk, P121].

Summary

❖ Optimal control is a functional optimization

$$\min \ f(x) \quad \Longleftrightarrow \quad J(x(t))$$

$$x = (x_1, \cdots, x_n)^T \in R^n \quad \Longleftrightarrow \quad x(t) = (x_1(t), \cdots, x_n(t))^T$$

$$df = 0 \quad \Longleftrightarrow \quad \delta J = 0$$

How to compute the variation?

How to compute the variation

- How compute the variation? If $J(\mathbf{x}(t)) = \int_{t_0}^{t_f} f(\mathbf{x}(t))dt$ where f has cts first and second derivatives with respect to $\mathbf{x}$, then

$$\begin{aligned}\delta J(\mathbf{x}(t), \delta\mathbf{x}) &= \int_{t_0}^{t_f} \left\{\frac{\partial f(\mathbf{x}(t))}{\partial \mathbf{x}(t)}\right\} \delta\mathbf{x}dt + f(\mathbf{x}(t_f))\delta t_f - f(\mathbf{x}(t_0))\delta t_0 \\ &= \int_{t_0}^{t_f} f_{\mathbf{x}}(\mathbf{x}(t))\delta\mathbf{x}dt + f(\mathbf{x}(t_f))\delta t_f - f(\mathbf{x}(t_0))\delta t_0\end{aligned}$$

The derivative of parametric variable integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{b(t)}^{a(t)} f(x,t)\mathrm{d}x = \int_{b(t)}^{a(t)} \frac{\partial f(x,t)}{\partial t}\mathrm{d}x + f[a(t),t]\frac{\mathrm{d}a(t)}{\mathrm{d}t} - f[b(t),t]\frac{\mathrm{d}b(t)}{\mathrm{d}t}$$

- For more general problems, first consider the cost evaluated on a scalar function $x(t)$ with t_0, t_f and the curve endpoints fixed.

$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt$$

$$\Rightarrow \quad \delta J(x(t), \delta x) = \int_{t_0}^{t_f} \left[\, g_x(x(t), \dot{x}(t), t)\delta x + g_{\dot{x}}(x(t), \dot{x}(t), t)\delta \dot{x}\right] dt$$

  – Note that

$$\delta \dot{x} = \frac{d}{dt}\delta x$$

  so δx and $\delta \dot{x}$ **are not independent**.

- Integrate by parts:

$$\int u dv \equiv uv - \int v du$$

with $u = g_{\dot{x}}$ and $dv = \delta \dot{x} dt$ to get:

$$\begin{aligned}
\delta J(x(t), \delta x) &= \int_{t_0}^{t_f} g_x(x(t), \dot{x}(t), t)\delta x dt + [g_{\dot{x}}(x(t), \dot{x}(t), t)\delta x]_{t_0}^{t_f} \\
&\quad - \int_{t_0}^{t_f} \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t)\delta x dt \\
&= \int_{t_0}^{t_f} \left[g_x(x(t), \dot{x}(t), t) - \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t) \right] \delta x(t) dt \\
&\quad + [g_{\dot{x}}(x(t), \dot{x}(t), t)\delta x]_{t_0}^{t_f}
\end{aligned}$$

- Since $x(t_0)$, $x(t_f)$ given, then $\delta x(t_0) = \delta x(t_f) = 0$, yielding

$$\delta J(x(t), \delta x) = \int_{t_0}^{t_f} \left[g_x(x(t), \dot{x}(t), t) - \frac{d}{dt} g_{\dot{x}}(x(t), \dot{x}(t), t) \right] \delta x(t) dt$$

- Recall need $\delta J = 0$ for all admissible $\delta x(t)$, which are arbitrary within (t_0, t_f) $\Rightarrow$ the (first order) necessary condition for a maximum or minimum is called **Euler Equation**:

$$\boxed{\frac{\partial g(x(t), \dot{x}(t), t)}{\partial x} - \frac{d}{dt}\left(\frac{\partial g(x(t), \dot{x}(t), t)}{\partial \dot{x}}\right) = 0}$$

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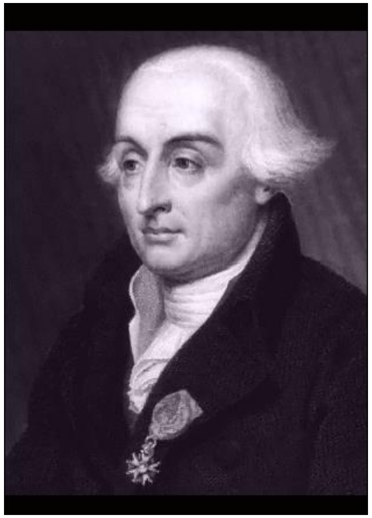
Euler



Lagrange

- **Example**: Find the curve that gives the shortest distance between 2 points in a plane (x_0, y_0) and (x_f, y_f).
  - Cost function – sum of differential arc lengths:

$$\begin{aligned} J &= \int_{x_0}^{x_f} ds = \int_{x_0}^{x_f} \sqrt{(dx)^2 + (dy)^2} \\ &= \int_{x_0}^{x_f} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

  - Take y as dependent variable, and x as independent one

$$\frac{dy}{dx} \to \dot{y}$$

  - New form of the cost:

$$J = \int_{x_0}^{x_f} \sqrt{1 + \dot{y}^2}\, dx \to \int_{x_0}^{x_f} g(\dot{y}) dx$$

  - Take partials: $\partial g / \partial y = 0$, and

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial g}{\partial \dot{y}}\right) &= \frac{d}{d\dot{y}}\left(\frac{\partial g}{\partial \dot{y}}\right)\frac{d\dot{y}}{dx} \\ &= \frac{d}{d\dot{y}}\left(\frac{\dot{y}}{(1+\dot{y}^2)^{1/2}}\right)\ddot{y} = \frac{\ddot{y}}{(1+\dot{y}^2)^{3/2}} = 0 \end{aligned}$$

    which implies that $\ddot{y} = 0$
  - Most general curve with $\ddot{y} = 0$ is a line $y = c_1 x + c_2$

Vector Functions

- Can generalize the problem by including several (N) functions $x_i(t)$ and possibly free endpoints

$$J(\mathbf{x}(t)) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

with t_0, t_f, $\mathbf{x}(t_0)$ fixed.

- Then (drop the arguments for brevity)

$$\delta J(\mathbf{x}(t), \delta\mathbf{x}) = \int_{t_0}^{t_f} \left[\, g_{\mathbf{x}}\delta\mathbf{x}(t) + g_{\dot{\mathbf{x}}}\delta\dot{\mathbf{x}}(t)\right] dt$$

  - Integrate by parts to get:

$$\delta J(\mathbf{x}(t), \delta\mathbf{x}) \;=\; \int_{t_0}^{t_f} \left[\; g_{\mathbf{x}} - \frac{d}{dt}g_{\dot{\mathbf{x}}}\right] \delta\mathbf{x}(t)dt + g_{\dot{\mathbf{x}}}(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f)\delta\mathbf{x}(t_f)$$

- The requirement then is that for $t \in (t_0, t_f)$, $\mathbf{x}(t)$ must satisfy

$$\frac{\partial g}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial g}{\partial \dot{\mathbf{x}}} = 0$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$ which are the given N boundary conditions, and the remaining N more BC follow from:

  – $\mathbf{x}(t_f) = \mathbf{x}_f$ if $\mathbf{x}_f$ is given as fixed,

  – If $\mathbf{x}(t_f)$ are free, then

$$\frac{\partial g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}(t_f)} = 0$$

- Note that we could also have a mixture, where parts of $\mathbf{x}(t_f)$ are given as fixed, and other parts are free – just use the rules above on each component of $x_i(t_f)$

Free Terminal Time

- Now consider a slight variation: the goal is to minimize

$$J(\mathbf{x}(t)) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

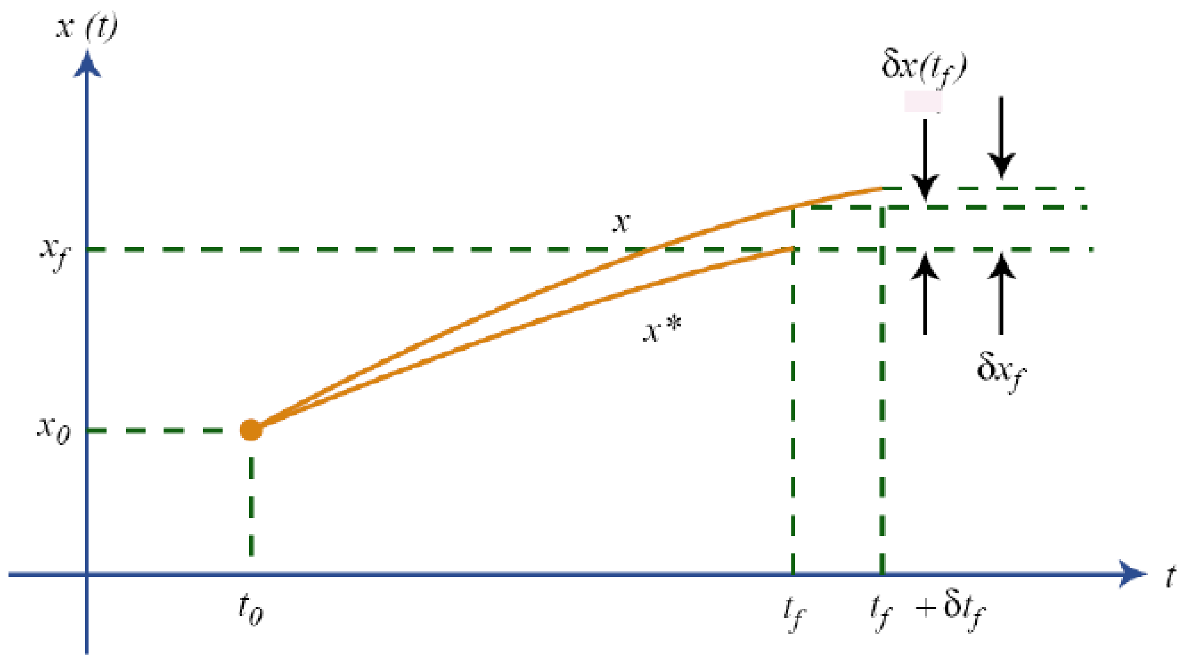
with t_0, $\mathbf{x}(t_0)$ fixed, t_f free, and various constraints on $\mathbf{x}(t_f)$

  - Compute variation of the functional considering 2 candidate solutions:
    - $\mathbf{x}(t)$, which we consider to be a perturbation of the optimal $\mathbf{x}^\star(t)$ (that we need to find)

$$\delta J(\mathbf{x}^\star(t), \delta\mathbf{x}) \;=\; \int_{t_0}^{t_f} [\, g_{\mathbf{x}}\delta\mathbf{x}(t) + g_{\dot{\mathbf{x}}}\delta\dot{\mathbf{x}}(t)]\, dt + g(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f)\delta t_f$$

    - Integrate by parts to get:

$$\begin{aligned}\delta J(\mathbf{x}^\star(t), \delta\mathbf{x}) \;=&\; \int_{t_0}^{t_f} \left[g_{\mathbf{x}} - \frac{d}{dt} g_{\dot{\mathbf{x}}} \right] \delta\mathbf{x}(t)dt \\ &+\; g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f)\delta\mathbf{x}(t_f) \\ &+\; g(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f)\delta t_f\end{aligned}$$



Comparison of possible changes to function at end time when t_f is free.

Figure by MIT OpenCourseWare.

Comparison of possible changes to function at end time when t_f is free.

- By definition, $\delta \mathbf{x}(t_f)$ is the difference between two admissible functions at time t_f (in this case the optimal solution $\mathbf{x}^\star$ and another candidate $\mathbf{x}$).
  - But in this case, must also account for possible changes to δt_f.
  - Define $\delta \mathbf{x}_f$ as being the difference between the ends of the two possible functions – **total possible change** in the final state:

$$\delta \mathbf{x}_f \approx \delta \mathbf{x}(t_f) + \dot{\mathbf{x}}^\star(t_f)\delta t_f$$

so $\delta \mathbf{x}(t_f) \neq \delta \mathbf{x}_f$ in general.

- Substitute to get

$$\begin{aligned}\delta J(\mathbf{x}^\star(t), \delta \mathbf{x}) \;&=\; \int_{t_0}^{t_f} \left[g_{\mathbf{x}} - \frac{d}{dt} g_{\dot{\mathbf{x}}} \right] \delta \mathbf{x}(t) dt + g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f)\delta \mathbf{x}_f \\ &+\; \left[g(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) - g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f)\dot{\mathbf{x}}^\star(t_f) \right] \delta t_f\end{aligned}$$

- Independent of the terminal constraint, the conditions on the solution $\mathbf{x}^\star(t)$ to be an extremal for this case are that it satisfy the Euler equations

$$\boxed{g_{\mathbf{x}}(\mathbf{x}^\star(t), \dot{\mathbf{x}}^\star(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t), \dot{\mathbf{x}}^\star(t), t) \ = \ 0}$$

  – Now consider the additional constraints on the individual elements of $\mathbf{x}^\star(t_f)$ and t_f to find the other boundary conditions

- Type of terminal constraints determines how we treat $\delta\mathbf{x}_f$ and δt_f
  1. Unrelated
  2. Related by a simple function $\mathbf{x}(t_f) = \Theta(t_f)$
  3. Specified by a more complex constraint $\mathbf{m}(\mathbf{x}(t_f), t_f) = 0$

- **Type 1:** If t_f and $\mathbf{x}(t_f)$ are free but unrelated, then $\delta\mathbf{x}_f$ and δt_f are independent and arbitrary $\Rightarrow$ their coefficients must both be zero.

$$\boxed{\begin{aligned} g_{\mathbf{x}}(\mathbf{x}^\star(t),\dot{\mathbf{x}}^\star(t),t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t),\dot{\mathbf{x}}^\star(t),t) &= 0 \\ g(\mathbf{x}^\star(t_f),\dot{\mathbf{x}}^\star(t_f),t_f) - g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f),\dot{\mathbf{x}}^\star(t_f),t_f)\dot{\mathbf{x}}^\star(t_f) &= 0 \\ g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f),\dot{\mathbf{x}}^\star(t_f),t_f) &= 0 \end{aligned}}$$

  - Which makes it clear that this is a **two-point boundary value problem**, as we now have conditions at both t_0 and t_f

- **Type 2:** If t_f and $\mathbf{x}(t_f)$ are free but related as $\mathbf{x}(t_f) = \Theta(t_f)$, then

$$\delta \mathbf{x}_f = \frac{d\Theta}{dt}(t_f)\delta t_f$$

  – Substitute and collect terms gives

$$\begin{aligned} \delta J &= \int_{t_0}^{t_f} \left[g_{\mathbf{x}} - \frac{d}{dt} g_{\dot{\mathbf{x}}} \right] \delta \mathbf{x} dt + \left[g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) \frac{d\Theta}{dt}(t_f) \right. \\ &+ \left. g(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) - g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) \dot{\mathbf{x}}^\star(t_f) \right] \delta t_f \end{aligned}$$

  – Set coefficient of δt_f to zero (it is arbitrary) $\Rightarrow$ full conditions

$$\boxed{\begin{aligned} g_{\mathbf{x}}(\mathbf{x}^\star(t), \dot{\mathbf{x}}^\star(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t), \dot{\mathbf{x}}^\star(t), t) &= 0 \\ g_{\dot{\mathbf{x}}}(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) \left[\frac{d\Theta}{dt}(t_f) - \dot{\mathbf{x}}^\star(t_f) \right] + g(\mathbf{x}^\star(t_f), \dot{\mathbf{x}}^\star(t_f), t_f) &= 0 \end{aligned}}$$

• Type 3:

with t_f free and $\mathbf{x}(t_f)$ given by a condition:

$$\mathbf{m}(\mathbf{x}(t_f), t_f) = 0$$

- Constrained optimization, so as before, augment the cost functional

$$J(\mathbf{x}(t), t) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

with the constraint using Lagrange multipliers:

$$J_a(\mathbf{x}(t), \boldsymbol{\nu}, t) = h(\mathbf{x}(t_f), t_f) + \boldsymbol{\nu}^T \mathbf{m}(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

- Considering changes to $\mathbf{x}(t)$, t_f, $\mathbf{x}(t_f)$ and $\boldsymbol{\nu}$, the variation for J_a is

$$\begin{aligned}
\delta J_a &= h_{\mathbf{x}}(t_f)\delta\mathbf{x}_f + h_{t_f}\delta t_f + \mathbf{m}^T(t_f)\delta\boldsymbol{\nu} + \boldsymbol{\nu}^T\left(\mathbf{m}_{\mathbf{x}}(t_f)\delta\mathbf{x}_f + \mathbf{m}_{t_f}(t_f)\delta t_f\right) \\
&\quad + \int_{t_0}^{t_f}\left[g_{\mathbf{x}}\delta\mathbf{x} + g_{\dot{\mathbf{x}}}\delta\dot{\mathbf{x}}\right]dt + g(t_f)\delta t_f \\
&= \left[h_{\mathbf{x}}(t_f) + \boldsymbol{\nu}^T\mathbf{m}_{\mathbf{x}}(t_f)\right]\delta\mathbf{x}_f + \left[h_{t_f} + \boldsymbol{\nu}^T\mathbf{m}_{t_f}(t_f) + g(t_f)\right]\delta t_f \\
&\quad + \mathbf{m}^T(t_f)\delta\boldsymbol{\nu} + \int_{t_0}^{t_f}\left[g_{\mathbf{x}} - \frac{d}{dt}g_{\dot{\mathbf{x}}}\right]\delta\mathbf{x}dt + g_{\dot{\mathbf{x}}}(t_f)\delta\mathbf{x}(t_f)
\end{aligned}$$

  - Now use that $\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) + \dot{\mathbf{x}}(t_f)\delta t_f$ as before to get

$$\begin{aligned}
\delta J_a &= \left[h_{\mathbf{x}}(t_f) + \boldsymbol{\nu}^T\mathbf{m}_{\mathbf{x}}(t_f) + g_{\dot{\mathbf{x}}}(t_f)\right]\delta\mathbf{x}_f \\
&\quad + \left[h_{t_f} + \boldsymbol{\nu}^T\mathbf{m}_{t_f}(t_f) + g(t_f) - g_{\dot{\mathbf{x}}}(t_f)\dot{\mathbf{x}}(t_f)\right]\delta t_f + \mathbf{m}^T(t_f)\delta\boldsymbol{\nu} \\
&\quad + \int_{t_0}^{t_f}\left[g_{\mathbf{x}} - \frac{d}{dt}g_{\dot{\mathbf{x}}}\right]\delta\mathbf{x}dt
\end{aligned}$$

- Looks like a bit of a mess, but we can clean it up a bit using

$$w(\mathbf{x}(t_f), \boldsymbol{\nu}, t_f) = h(\mathbf{x}(t_f), t_f) + \boldsymbol{\nu}^T \mathbf{m}(\mathbf{x}(t_f), t_f)$$

to get

$$\begin{aligned}
\delta J_a \quad &= \left[w_{\mathbf{x}}(t_f) + g_{\dot{\mathbf{x}}}(t_f)\right] \delta \mathbf{x}_f \\
&+ \left[w_{t_f} + g(t_f) - g_{\dot{\mathbf{x}}}(t_f)\dot{\mathbf{x}}(t_f)\right] \delta t_f + \mathbf{m}^T(t_f)\delta\boldsymbol{\nu} \\
&+ \int_{t_0}^{t_f} \left[g_{\mathbf{x}} - \frac{d}{dt} g_{\dot{\mathbf{x}}}\right] \delta \mathbf{x} dt
\end{aligned}$$

  - Given the extra degrees of freedom in the multipliers, can treat all of the variations as independent $\Rightarrow$ all coefficients must be zero to achieve $\delta J_a = 0$

- So the necessary conditions are

$$\boxed{\begin{aligned} g_{\mathbf{x}} - \frac{d}{dt} g_{\dot{\mathbf{x}}} &= 0 \quad (\texttt{dim}\ n) \\ w_{\mathbf{x}}(t_f) + g_{\dot{\mathbf{x}}}(t_f) &= 0 \quad (\texttt{dim}\ n) \\ w_{t_f} + g(t_f) - g_{\dot{\mathbf{x}}}(t_f)\dot{\mathbf{x}}(t_f) &= 0 \quad (\texttt{dim}\ 1) \end{aligned}}$$

  – With $\mathbf{x}(t_0) = \mathbf{x}_0$ ($\texttt{dim}\ n$) and $\mathbf{m}(\mathbf{x}(t_f), t_f) = 0$ ($\texttt{dim}\ m$) combined with last 2 conditions $\Rightarrow 2n + m + 1$ constraints

  – Solution of Eulers equation has $2n$ constants of integration for $x(t)$, and must find $\boldsymbol{\nu}$ ($\texttt{dim}\ m$) and $t_f \Rightarrow 2n + m + 1$ unknowns

Optimal Control Problems

- Are now ready to tackle the optimal control problem
  - Start with simple terminal constraints

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

    with the system dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

  - t_0, $\mathbf{x}(t_0)$ fixed
  - t_f free
  - $\mathbf{x}(t_f)$ are fixed or free by element

- Note that this looks a bit different because we have $\mathbf{u}(t)$ in the integrand, but consider that with a simple substitution, we get

$$\tilde{g}(\mathbf{x}, \dot{\mathbf{x}}, t) \overset{\dot{\mathbf{x}}=\mathbf{a}(\mathbf{x},\mathbf{u},t)}{\longrightarrow} \hat{g}(\mathbf{x}, \mathbf{u}, t)$$

- Note that the differential equation of the dynamics acts as a constraint that we must adjoin using a Lagrange multiplier, as before:

$$J_a = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \left[g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T \{ \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}} \} \right] dt$$

- Find the variation:

$$\begin{aligned} \delta J_a \; = \; & h_{\mathbf{x}} \delta \mathbf{x}_f + h_{t_f} \delta t_f + \int_{t_0}^{t_f} \left[g_{\mathbf{x}} \delta \mathbf{x} + g_{\mathbf{u}} \delta \mathbf{u} + (\mathbf{a} - \dot{\mathbf{x}})^T \delta \mathbf{p}(t) \right. \\ & \left. + \mathbf{p}^T(t) \{ \mathbf{a}_{\mathbf{x}} \delta \mathbf{x} + \mathbf{a}_{\mathbf{u}} \delta \mathbf{u} - \delta \dot{\mathbf{x}} \} \right] dt + \left[g + \mathbf{p}^T (\mathbf{a} - \dot{\mathbf{x}}) \right] (t_f) \delta t_f \end{aligned}$$

- Clean this up by defining the **Hamiltonian**:

$$H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t) \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- Then

$$\delta J_a = h_{\mathbf{x}}\delta\mathbf{x}_f + \left[h_{t_f} + g + \mathbf{p}^T(\mathbf{a} - \dot{\mathbf{x}})\right](t_f)\delta t_f + \int_{t_0}^{t_f}\left[H_{\mathbf{x}}\delta\mathbf{x} + H_{\mathbf{u}}\delta\mathbf{u} + (\mathbf{a} - \dot{\mathbf{x}})^T\delta\mathbf{p}(t) - \mathbf{p}^T(t)\delta\dot{\mathbf{x}}\right]dt$$

- To proceed, note that by integrating by parts – we get:

$$\begin{aligned}
-\int_{t_0}^{t_f}\mathbf{p}^T(t)\delta\dot{\mathbf{x}}dt &= -\int_{t_0}^{t_f}\mathbf{p}^T(t)d\delta\mathbf{x} \\
&= -\mathbf{p}^T\delta\mathbf{x}\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f}\left(\frac{d\mathbf{p}(t)}{dt}\right)^T\delta\mathbf{x}dt \\
&= -\mathbf{p}^T(t_f)\delta\mathbf{x}(t_f) + \int_{t_0}^{t_f}\dot{\mathbf{p}}^T(t)\delta\mathbf{x}dt \\
&= -\mathbf{p}^T(t_f)\left(\delta\mathbf{x}_f - \dot{\mathbf{x}}(t_f)\delta t_f\right) + \int_{t_0}^{t_f}\dot{\mathbf{p}}^T(t)\delta\mathbf{x}dt
\end{aligned}$$

- So now can rewrite the variation as:

$$\begin{aligned}
\delta J_a &= h_{\mathbf{x}} \delta \mathbf{x}_f + \left[h_{t_f} + g + \mathbf{p}^T(\mathbf{a} - \dot{\mathbf{x}}) \right] (t_f) \delta t_f \\
&\quad + \int_{t_0}^{t_f} \left[H_{\mathbf{x}} \delta \mathbf{x} + H_{\mathbf{u}} \delta \mathbf{u} + (\mathbf{a} - \dot{\mathbf{x}})^T \delta \mathbf{p}(t) \right] dt - \int_{t_0}^{t_f} \mathbf{p}^T(t) \delta \dot{\mathbf{x}} dt \\
&= \left(h_{\mathbf{x}} - \mathbf{p}^T(t_f) \right) \delta \mathbf{x}_f + \left[h_{t_f} + g + \mathbf{p}^T(\mathbf{a} - \dot{\mathbf{x}}) + \mathbf{p}^T \dot{\mathbf{x}} \right] (t_f) \delta t_f \\
&\quad + \int_{t_0}^{t_f} \left[\left(H_{\mathbf{x}} + \dot{\mathbf{p}}^T \right) \delta \mathbf{x} + H_{\mathbf{u}} \delta \mathbf{u} + (\mathbf{a} - \dot{\mathbf{x}})^T \delta \mathbf{p}(t) \right] dt
\end{aligned}$$

- So necessary conditions for $\delta J_a = 0$ are that for $t \in [t_0, t_f]$

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}, \mathbf{u}, t) && (\texttt{dim } n) \\ \dot{\mathbf{p}} &= -H_{\mathbf{x}}^T && (\texttt{dim } n) \\ H_{\mathbf{u}} &= 0 && (\texttt{dim } m) \end{aligned}}$$

  – With the boundary condition (lost if t_f is fixed) that

$$h_{t_f} + g + \mathbf{p}^T\mathbf{a} = h_{t_f} + H(t_f) = 0$$

  – Add the boundary constraints that $\mathbf{x}(t_0) = \mathbf{x}_0$ (dim n)

  – If $\mathbf{x}_i(t_f)$ is fixed, then $\mathbf{x}_i(t_f) = x_{i_f}$

  – If $\mathbf{x}_i(t_f)$ is free, then $\mathbf{p}_i(t_f) = \dfrac{\partial h}{\partial x_i}(t_f)$ for a total (dim n)

Transversal Condition 横截条件

- Note the symmetry in the differential equations:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}, \mathbf{u}, t) = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T \\
\dot{\mathbf{p}} &= -\left(\frac{\partial H}{\partial \mathbf{x}}\right)^T = -\frac{\partial (g + \mathbf{p}^T \mathbf{a})}{\partial \mathbf{x}}^T \\
&= -\left(\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right)^T \mathbf{p} - \left(\frac{\partial g}{\partial \mathbf{x}}\right)^T
\end{aligned}
$$

Example

- Simple double integrator system starting at $y(0) = 10$, $\dot{y}(0) = 0$, must drive to origin $y(t_f) = \dot{y}(t_f) = 0$ to minimize the cost ($b > 0$)

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} bu^2(t)dt$$

- Define the dynamics with $x_1 = y$, $x_2 = \dot{y}$ so that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- With $\mathbf{p}(t) = [p_1(t) \;\; p_2(t)]^T$, define the Hamiltonian

$$H = g + \mathbf{p}^T(t)\mathbf{a} = \frac{1}{2}bu^2 + \mathbf{p}^T(t)\left(A\mathbf{x}(t) + Bu(t)\right)$$

- The necessary conditions are then that:

$$\dot{\mathbf{p}} = -H_{\mathbf{x}}^T, \quad \rightarrow \quad \begin{aligned} \dot{p}_1 &= -\tfrac{\partial H}{\partial x_1} = 0 \rightarrow p_1(t) = c_1 \\ \dot{p}_2 &= -\tfrac{\partial H}{\partial x_2} = -p_1 \rightarrow p_2(t) = -c_1 t + c_2 \end{aligned}$$

$$H_u = bu + p_2 = 0 \quad \rightarrow \quad u = -\frac{p_2}{b} = -\frac{c_2}{b} + \frac{c_1}{b}t$$

- Now go back to the state equations:

$$\dot{x}_2(t) = -\frac{c_2}{b} + \frac{c_1}{b}t \quad \longrightarrow \quad x_2(t) = c_3 - \frac{c_2}{b}t + \frac{c_1}{2b}t^2$$

and since $x_2(0) = 0$, $c_3 = 0$, and

$$\dot{x}_1(t) = x_2(t) \quad \longrightarrow \quad x_1(t) = c_4 - \frac{c_2}{2b}t^2 + \frac{c_1}{6b}t^3$$

and since $x_1(0) = 10$, $c_4 = 10$

- Now note that

$$\begin{aligned}
x_2(t_f) &= -\frac{c_2}{b}t_f + \frac{c_1}{2b}t_f^2 = 0 \\
x_1(t_f) &= 10 - \frac{c_2}{2b}t_f^2 + \frac{c_1}{6b}t_f^3 = 0 \\
&= 10 - \frac{c_2}{6b}t_f^2 = 0 \quad \longrightarrow \quad c_2 = \frac{60b}{t_f^2}, \qquad c_1 = \frac{120b}{t_f^3}
\end{aligned}$$

- Now impose the boundary conditions:

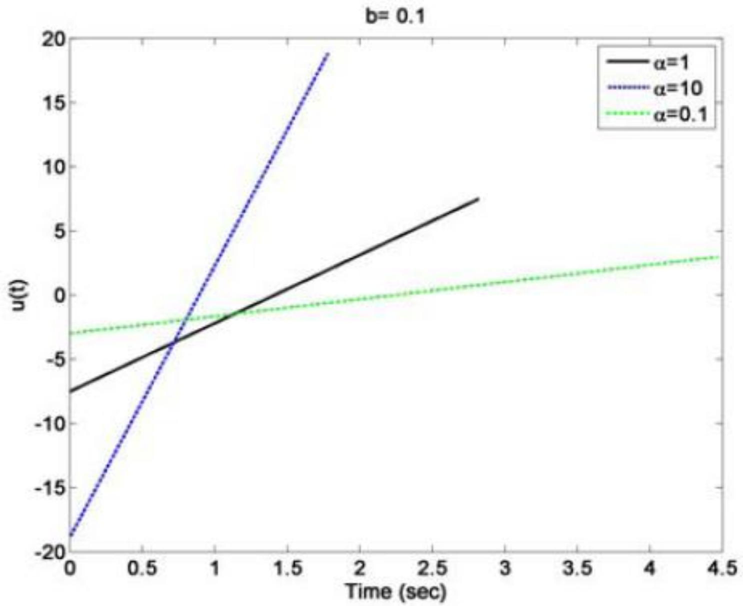
$$
\begin{aligned}
H(t_f) + h_t(t_f) &= \frac{1}{2}bu^2(t_f) + p_1(t_f)x_2(t_f) + p_2(t_f)u(t_f) + \alpha t_f = 0 \\
&= \frac{1}{2}bu^2(t_f) + (-bu(t_f))u(t_f) + \alpha t_f \\
&= -\frac{1}{2}bu^2(t_f) + \alpha t_f = 0 \;\rightarrow\; t_f = \frac{1}{2b\alpha}\left(-c_2 + c_1 t_f\right)^2
\end{aligned}
$$

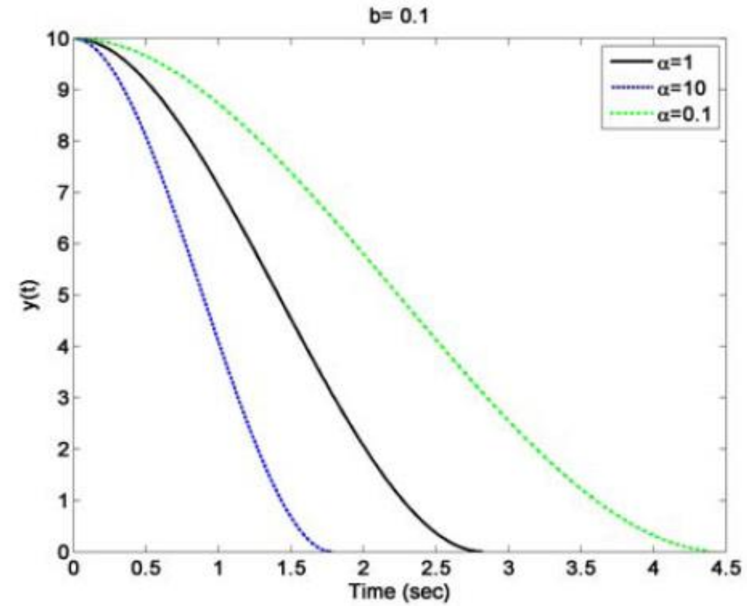
– But that gives us:

$$t_f = \frac{1}{2b\alpha}\left(-\frac{60b}{t_f^2} + \frac{120b}{t_f^3}t_f\right)^2 = \frac{(60b)^2}{2b\alpha t_f^4}$$

so that $t_f^5 = 1800b/\alpha$ or $t_f \approx 4.48(b/\alpha)^{1/5}$, which makes sense because t_f goes down as α goes up.

– Finally, $c_2 = 2.99b^{3/5}\alpha^{2/5}$ and $c_1 = 1.33b^{2/5}\alpha^{3/5}$





例3-3

设系统状态方程为

$$\dot{x} = -x(t) + u(t)$$

$x(t)$ 的边界条件为 $x(0) = 1, x(t_f) = 0$ 。求最优控制 $u(t)$，使下列性能指标

$$J = \frac{1}{2}\int_0^{t_f} \left(x^2 + u^2\right) dt$$

为最小。

解：这里$x(0)$ 、$x(t_f)$ 均给定，故不需要横截条件。作哈密顿函数

$$H=\frac{1}{2}(x^2+u^2)+\lambda(-x+u)$$

则协态方程和控制方程为

$$\dot{\lambda}=-\frac{\partial H}{\partial x}=-x+\lambda$$

$$\frac{\partial H}{\partial u}=u+\lambda=0$$

即

$$u=-\lambda$$

故可得正则方程

$$\dot{x}(t) = -x(t) - \lambda(t)$$

$$\dot{\lambda}(t) = -x(t) + \lambda(t)$$

对正则方程进行拉氏变换，可得

$$sX(s) - x(0) = -X(s) - \lambda(s) \qquad \textbf{（3-25）}$$

$$s\lambda(s) - \lambda(0) = -X(s) + \lambda(s) \qquad \textbf{（3-26）}$$

由（3-25）式可求得

$$X(s) = \frac{x(0) - \lambda(s)}{s+1} \qquad \textbf{（3-27）}$$

代入（**3-26**），即得

$$(s^2-2)\lambda(s)=(s+1)\lambda(0)-x(0)$$

于是，解出$\lambda(s)$为

$$\lambda(s)=\frac{(s+1)\lambda(0)-x(0)}{s^2-2}=\frac{s+1}{(s+\sqrt{2})(s-\sqrt{2})}\lambda(0)-\frac{1}{(s+\sqrt{2})(s-\sqrt{2})}x(0) \quad \text{（3-28）}$$

反变换可求得

$$\lambda(t)=\frac{1}{2\sqrt{2}}(e^{-\sqrt{2}\,t}-e^{\sqrt{2}\,t})x(0)+$$

$$+\frac{1}{2\sqrt{2}}\left[(\sqrt{2}-1)e^{-\sqrt{2}\,t}+(\sqrt{2}+1)e^{\sqrt{2}\,t}\right]\lambda(0) \qquad \text{（3-29）}$$

将（3-28）代入（3-26）可得

$$X(s)=\frac{s-1}{(s+\sqrt{2})(s-\sqrt{2})}x(0)-\frac{1}{(s+\sqrt{2})(s-\sqrt{2})}\lambda(0)$$

故

$$x(t)=\frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)e^{-\sqrt{2}\,t}+(\sqrt{2}-1)e^{\sqrt{2}\,t}\right]x(0)+$$

$$\frac{1}{2\sqrt{2}}(e^{-\sqrt{2}t}-e^{\sqrt{2}t})\lambda(0)$$

由 $x(0)=1,\ x(t_f)=0$ 从上式可得

$$\lambda(0)=\frac{(\sqrt{2}+1)e^{-\sqrt{2}\,t_f}+(\sqrt{2}-1)e^{\sqrt{2}\,t_f}}{e^{\sqrt{2}\,t_f}-e^{-\sqrt{2}\,t_f}}$$

把 $\lambda(0)$代入（3-29），可得$\lambda(t)$，而最优控制为

$$u^*(t)=-\lambda(t)=-\frac{1}{2\sqrt{2}}\left\{e^{-\sqrt{2}t}-e^{\sqrt{2}t}+\frac{(\sqrt{2}+1)e^{-\sqrt{2}t_f}+(\sqrt{2}-1)e^{\sqrt{2}t_f}}{e^{\sqrt{2}t_f}-e^{-\sqrt{2}t_f}}\left[(\sqrt{2}-1)e^{-\sqrt{2}t}+(\sqrt{2}+1)e^{\sqrt{2}t}\right]\right\}$$

代入 $H(t_f)+h_{t_f}=0$ 求得 t_f

例3-4　　设系统的状态方程为

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = u(t)$$

初始条件为

$$x_1(0) = 1 \qquad x_2(0) = 1$$

终端条件为

$$x_1(1) = 0 \qquad x_2(1)$$ 自由

要求确定最优控制 $u^*(t)$，使指标泛函

$$J(u) = \frac{1}{2}\int_0^1 u^2(t)dt$$

取极小值

这里$x_2(1)$是自由的，所以要用到横截条件，因终端指标

$$h(x(t_f),t_f)=0$$

所以

$$p_2(1)=\frac{\partial h}{\partial X_2(1)}=0 \qquad \textbf{（3-30）}$$

定义哈密顿函数

$$H=\frac{1}{2}u^2+p_1x_2+p_2u \qquad \textbf{（3-31）}$$

由必要条件可求得

$$\dot{p}_1=-\frac{\partial H}{\partial x_1}=0$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1$$

$$\frac{\partial H}{\partial u} = 0$$

即

$$u + p_2 = 0$$

得

$$u^*(t) = -p_2(t) \qquad \textbf{（3-32）}$$

将 $u^*(t)$ 代入状态方程，可得

正则方程：

$$\dot{x}_1 = x_2(t) \qquad (3\text{-}33)$$

$$\dot{x}_2 = -p_2(t) \qquad (3\text{-}34)$$

$$\dot{p}_1 = 0 \qquad (3\text{-}35)$$

$$\dot{p}_2 = -p_1(t) \qquad (3\text{-}36)$$

边界条件为

$$x_1(0) = 1 \qquad x_2(0) = 1$$

$$x_1(1) = 0 \qquad p_2(1) = 0 \qquad (3\text{-}37)$$

可见这是两点边值问题，对正则方程（**3-33**）~（**3-36**）进行拉氏变换，可得

$$sX_1(s)-x_1(0)=X_2(s)$$ （**3-38**）

$$sX_2(s)-x_2(0)=-p_2(s)$$ （**3-39**）

$$sp_1(s)-p_1(0)=0$$ （**3-40**）

$$sp_2(s)-p_2(0)=-p_1(s)$$ （**3-41**）

由（**3-38**）~（**3-41**）可解出

$$s^4 X_1(s) = s^3 x_1(0) + s^2 x_2(0) - s p_2(0) + p_1(0)$$

代入初始条件 $x_1(0)=1$ ，$x_2(0)=1$ ， 可得

$$X_1(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} p_2(0) + \frac{1}{s^4} p_1(0)$$

故

$$x_1(t) = 1 + t - \frac{1}{2} p_2(0) t^2 + \frac{1}{6} p_1(0) t^3$$

同样可解得

$$p_2(s)=\frac{1}{s}p_2(0)-\frac{1}{s^2}p_1(0) \quad \text{（3-42）}$$

$$p_2(t)=p_2(0)-p_1(0)\,t \quad \text{（3-43）}$$

利用终端条件 $x_1(1)=0$ ， $p_2(1)=0$ ， **可得**

$$2-\frac{1}{2}p_2(0)+\frac{1}{6}p_1(0)=0$$

$$p_2(0)-p_1(0)=0$$

由上二式可解出

$$p_1(0) = 6 \qquad p_2(0) = 6$$

由（3-42）式可得最优状态轨迹

$$x_1^*(t) = 1 + t - 3t^2 + t^3$$

由（3-43）式可得最优协态

$$p_2^*(t) = 6(1-t)$$

由（3-32）式可得最优控制

$$u^*(t) = 6(t-1)$$

同理还可求出

$$x_2^*(t) = 1 - 6t + 3t^2$$

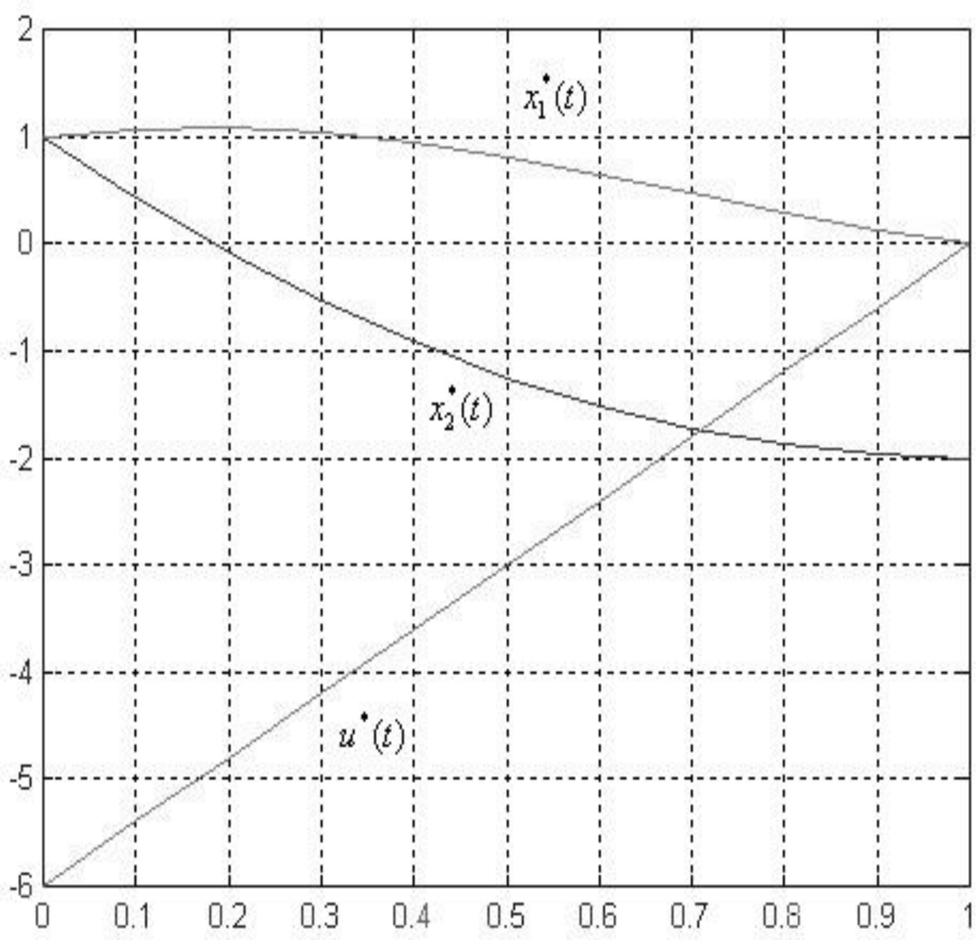


图3-2 最优控制和最优状态轨迹解

图3-2是最优解的轨迹曲线。

注意，这个系统是线性定常系统，这种线性两点边值问题的解可以通过寻找缺少的边界条件，并且进行一次积分而求得其解。

对非线性两点边值问题，则要借助于迭代方法产生一个序列，来多次修正缺少的初始条件的试探值，直到满足两点边值的条件。

Pontryagin maximum principle

- Two Assumptions:
  - δu is free, no constraint
  - H_u exists

- In practice, the control is limited

- Hamilton function may be not differential w.r.t control **U**

  Optimal fuel control problem

$$J = \int_0^{t_f} |u(t)| dt$$

❖ Pontryagin maximum principle

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{a}(\mathbf{x}, \mathbf{u}, t) \\ \dot{\mathbf{p}} &= -H_{\mathbf{x}}^T \\ H_{\mathbf{u}} &= 0\end{aligned}}$$

$$\min_{u \in \Omega} H(x^*, u, p^*, t) = H(x^*, u^*, p^*, t)$$

For the system

$$\dot{x}(t) = f\left(x,u,t\right)$$

$$J = \int_{t_0}^{t_f} L\left(x,u,t\right)dt$$

$$-1 \le u \le 2$$

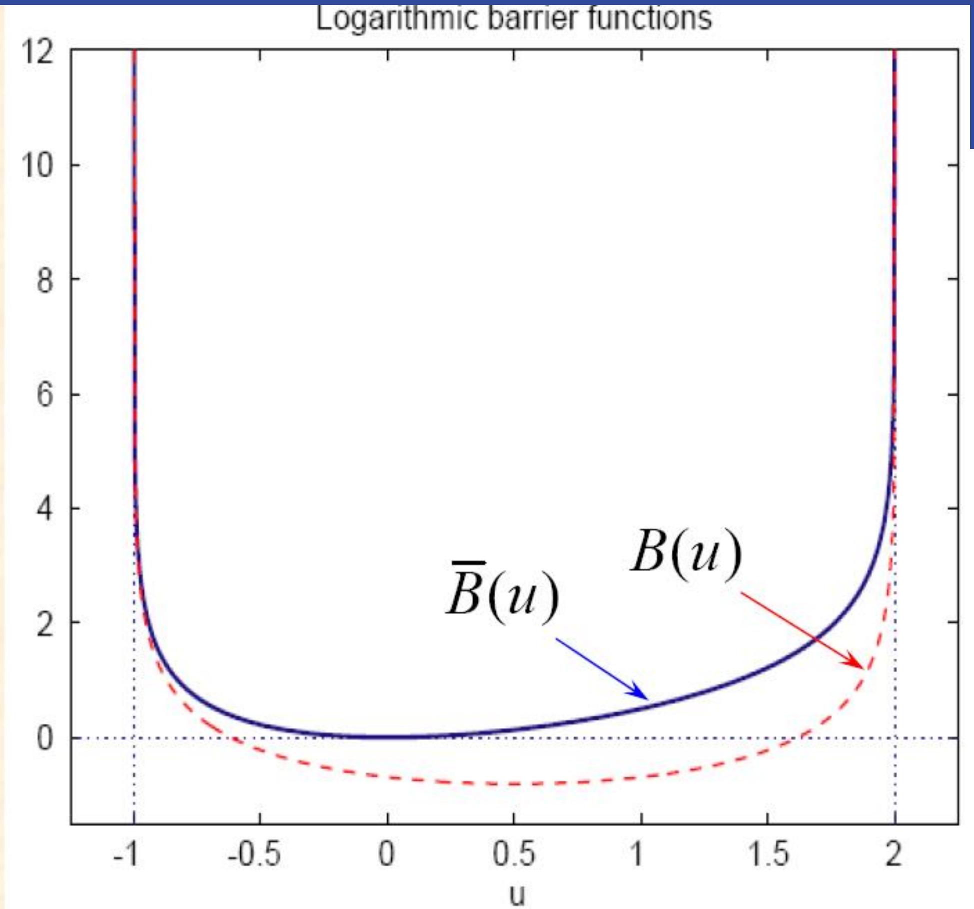
Introduce a barrier function

$$B(u) = -\ln(2-u) - \ln(1+u)$$

its gradient recentred barrier function

$$\overline{B}(u) = B(u) - B(0) - \left[\nabla B(0)\right]^T u$$

$$= \ln(2) - \ln(2-u) - \ln(1+u) + \frac{1}{2}u$$



Build a new objective functional without constraint

$$\tilde{J} = \int_{t_0}^{t_f} \left\{L\left(x,u,t\right) + \lambda(t)^T \left[f\left[x(t),u(t),t\right] - \dot{x}(t)\right] + \mu\overline{B}(u)\right\}dt$$

Then, $\min\limits_u J \quad \Leftrightarrow \quad \min\limits_u \tilde{J}$

s.t. $-1 \le u \le 2$