Linear System Theory: HW4

2022/12/01

（Due: Dec. 8, 2022）

1. (20’+10’) Consider the linear system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{1}$$

   1) Show that the system (or the matrix A) is stable if and only if for any given positive definite matrix $Q \in \mathbb{R}^{n\times n}$, there exists a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that

$$A^T P + PA = -Q\,. \tag{2}$$

   2) Using the above theorem to show that the matrix $A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$ is stable.

2. (20’) If the matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz, then for any matrix $Q \in \mathbb{R}^{n\times n}$, there exists a unique solution of (2), and the solution is $P = \int_0^\infty e^{A^T t} Q e^{At} dt$.

3. (20’) Show that all eigenvalues of A have real parts less than $-\mu < 0$ if and only if, for any given positive definite symmetric matrix N, the equation

$$A^T M + MA + 2\mu M = -N \tag{3}$$

has a unique symmetric solution M and M is positive definite.

4. (30’) Consider the linear system

$$\dot{x} = Ax + Bu, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. Show that this system is controllable if and only if the matrix

$$W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T \tau} d\tau = \int_0^t e^{A(t-\tau)} BB^T e^{A^T (t-\tau)} d\tau \tag{5}$$

is positive definite for any $t > 0$.