Nonlinear Control

《非线性控制》

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Lyapunov Stability Theory

李雅普洛夫稳定性理论

Stability(稳定性)

- 一个自动控制系统要能正常工作,必须首先是一个稳定的系统.
  - 例如,电压自动调解系统中保持电机电压为恒定的能力;
    - ✓ 电机自动调速系统中保持电机转速为一定的能力以及火箭飞行中保持航向为一定的能力等。
    - ✓ 具有稳定性的系统称为稳定系统。

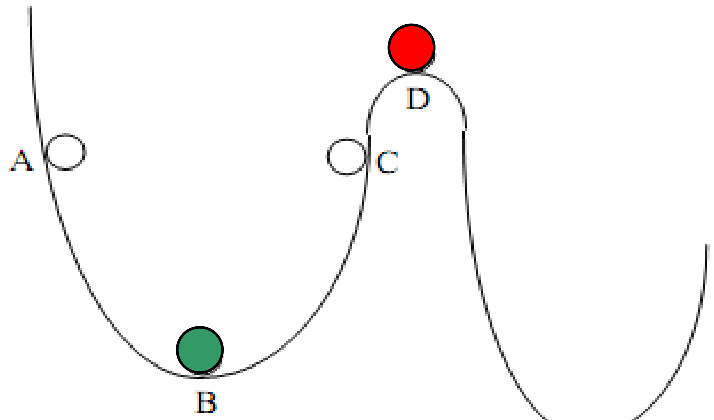
Stability(稳定性)

- 稳定性：

  - 表示系统在遭受外界扰动偏离原来的平衡状态，而在扰动消失后，系统本身仍有能力恢复到平衡状态的一种“顽性”，属于系统的基本结构特性，而与输入作用无关。

  - 当系统受到外界干扰后，显然它的平衡被破坏，但在外扰去掉以后，它仍有能力自动地恢复到平衡状态。

  - 如果一个系统不具有上述特性，则称为不稳定系统。



Stability(稳定性)

- 分析一个控制系统的稳定性,一直是控制理论中所关注的最重要问题.
  - 对于简单系统，常利用经典控制理论中线性定常系统的稳定性判据.
  - 在经典控制理论中,借助于常微分方程稳定性理论,产生了许多稳定性判据,如劳斯-赫尔维茨(Routh-Hurwitz)判据和奈奎斯特判据等,都给出了既实用又方便的判别系统稳定性的方法.
  - 但这些稳定性判别方法仅限于讨论SISO线性定常系统输入输出间动态关系，未研究系统的内部状态变化的稳定性，也不能推广到时变系统和非线性系统等复杂系统.

Lyapunov

- 1892年,俄国学者李亚普诺夫（Aleksandr Mikhailovich Lyapunov, 1857 -1918) 发表题为“运动稳定性一般问题”的博士论文,建立了关于运动稳定性研究的一般理论.
- 可是在相当长的一段时间里,李雅普诺夫第二法并没有引起研究动态系统稳定性的人们的重视,这是因为当时讨论系统输入输出间关系的经典控制理论占有绝对地位.
- 随着状态空间分析法的发展和现代控制理论的诞生，Lyapunov的工作才得到大量工程界人士的重视，成为近40年来研究系统稳定性的最主要方法.
- “稳定性理论在吸引着全世界数学家的注意，而且得到了工程师们的广泛赞赏，并快速成为美国培训控制工程师的标准内容之一”.
- 当今任何一本控制期刊都有李雅普诺夫的名字.

Dynamic Systems(动态系统)

考虑以下用常微分方程表示的系统:

$$\dot{x} = f(t, x), x(t_0) = x_0 \qquad (2.1)$$

其中 $t \in \mathbb{R}^+$ 表示时间, $x \in \mathbb{R}^n$ 表示状态, $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$。

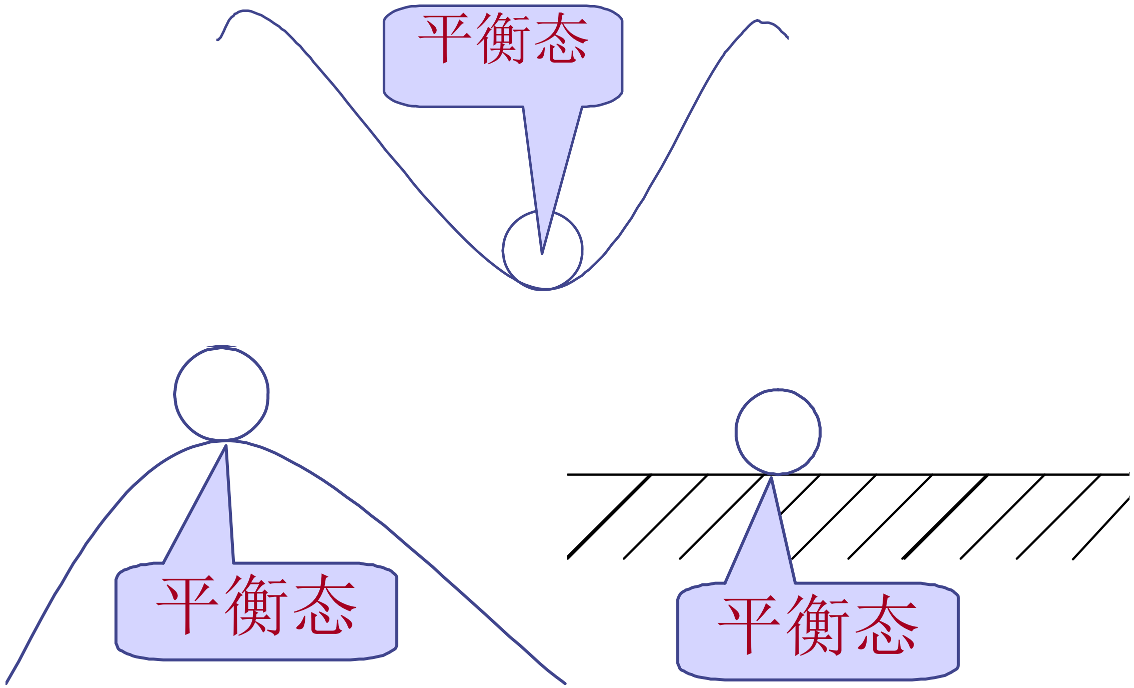
Equilibrium Point (平衡点)

$$\dot{x} = f(t,x), x(t_0) = x_0 \qquad (2.1)$$

若对所有t，总存在 $\boxed{\dot{x}^* = f(t,x^*) \equiv 0}$，则称 x^*为系统的平衡状态或平衡点。

从上述可知,平衡点即指状态空间中状态变量的导数向量为零向量的点.

由于导数表示状态的运动变化方向,因此平衡点即指能够保持平衡、维持现状不运动的状态,如上图所示.



Equilibrium Point (平衡点)

- 显然, 对于线性定常系统

$$\dot{x} = Ax$$

的平衡点x^*是满足下述方程的解:

$$Ax^* = 0$$

- 当矩阵A 为非奇异时, 线性系统只有一个孤立的平衡态 $x^* = 0$;
  - 而当A为奇异时, 则存在无限多个平衡态, 且这些平衡态不为孤立平衡态,而构成状态空间中的一个子空间.
  - 对于非线性系统, 通常可有一个或几个孤立平衡态, 它们分别为对应于式 $f(t,x)=0$ 的常值解.

Equilibrium Point (平衡点)

- 例如，对于非线性系统

$$\boxed{\begin{aligned}\dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_1 + x_2 - x_2^3\end{aligned}}$$

其平衡态为下列代数方程组：

$$\begin{aligned}x_1 &= 0 \\ x_1 + x_2 - x_2^3 &= 0\end{aligned}$$

的解,即下述状态空间中的三个状态为其孤立平衡态.

$$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad x_3^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Equilibrium Point (平衡点)

- 线性系统在平衡点稳定，则系统稳定；

  而非线性系统在平衡点稳定，则只是在该点稳定，而不是整个系统稳定----可见，稳定性问题是相对于平衡状态而言的。

- 线性系统的稳定性只取决于系统的结构和参数，而与系统的初始条件及外界扰动的大小无关；

  但非线性系统的稳定性除了与系统的结构和参数有关外，还与初始条件及外界扰动的大小有关。

Equilibrium Point (平衡点)

- 孤立的平衡状态：

  在某一平衡状态的充分小的领域内不存别的平衡状态，即称为孤立的平衡状态。

  对于孤立的平衡状态，总可以经过适当的坐标变换，把它变换到状态空间的原点。

Equilibrium Point (平衡点)

Suppose we define a nonzero vector x^* to be an equilibrium point of (2.1):

$$f(t, x^*) = 0, \quad \forall t \geq t_0$$

We can redefine time $\tau = t - t_0$, introduce the new state

$$z(\tau) = x(\tau + t_0) - x^*$$

and arrive at the transformed system dynamics

$$\boxed{\frac{d\,z(\tau)}{d\,\tau}} = \frac{d\,x(\tau + t_0)}{d\,t} = f(\tau + t_0,\, z(\tau) + x^*) = \boxed{g(\tau,\, z(\tau))}$$

with $g(0, 0) = f(t_0, x^*) = 0$. Thus, we have shifted the equilibrium point to the origin and the initial time to zero.

Equilibrium Point (平衡点)

- 孤立的平衡状态：

  在某一平衡状态的充分小的领域内不存别的平衡状态，即称为孤立的平衡状态。

  对于孤立的平衡状态，总可以经过适当的坐标变换，把它变换到状态空间的原点。

  因此，仅仅需要讨论系统在 $x^* = 0$ 这个平衡状态处的稳定性即可。

- “原点稳定性问题”极大简化了研究，又不失一般性，是Lyapunov的重要贡献。

稳定性定义

$$\dot{x} = f(t, x), x(t_0) = x_0 \qquad (2.1)$$

$\dot{x} = f(x)$	$f : \mathbb{R}^n \to \mathbb{R}^n$	time-invariant (autonomous)
$\dot{x} = f(t, x)$	$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$	time-varying (non-autonomous)

稳定性定义

For autonomous systems $\dot{x} = f(x), \qquad t \geq 0$

Let $x(t)$ be its solution satisfying $x(0)=x_0$ and let $y(t)=x(t-t_0)$. Then $y(t_0)= x(0)=x_0$ and

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t-t_0)}{\mathrm{d}t} = f(x(t-t_0)) = f(y(t))$$

That is, $x(t-t_0)$ is its solution satisfying $x(t_0-t_0)=x_0$.

This property is called shift-invariant. As a result, we can always assume the initial time to be zero.

稳定性定义

For non-autonomous systems $\dot{x} = f(x,t), \qquad t \geq t_0$

Let $x(t)$ be its solution satisfying $x(t_0)=x_0$ and let $y(t)=x(t-t_0)$. Then

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t-t_0)}{\mathrm{d}t} = f(x(t-t_0),\ \ t-t_0) = f(y(t), t-t_0) \neq f(y(t),t)$$

That is, $x(t-t_0)$ may not be its solution.

➡ The initial time matters, and the shift-invariant property does not hold.

That's why we need to include t_0 in the definitions of stability analysis for non-autonomous systems and introduce the concept of uniformity.

稳定性定义

The equilibrium point $x=0$ of (2.1) is :

(1) Stable: if for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, t_0) > 0$, such that:

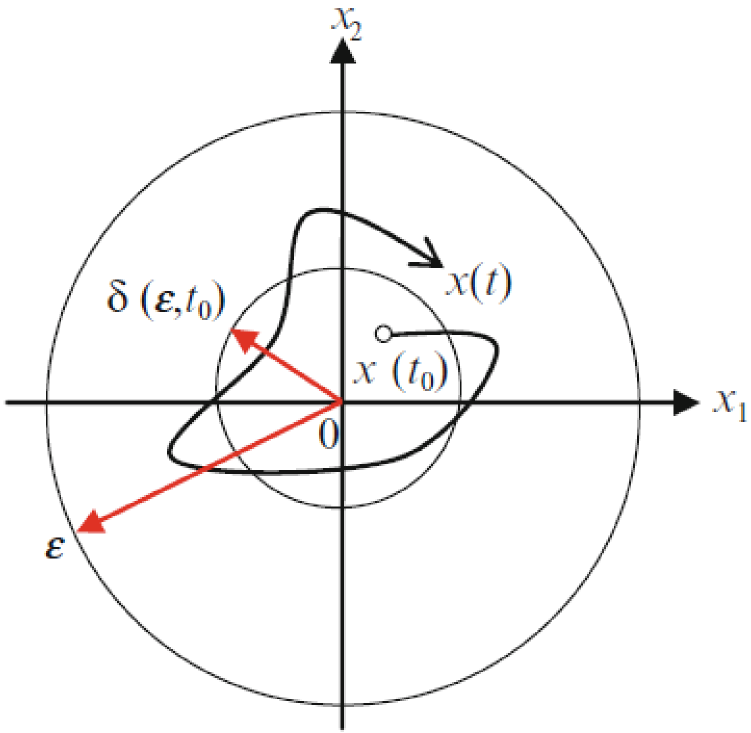
$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0$$

(2) Uniformly stable: if $\delta = \delta(\varepsilon) > 0$ is independent of t_0

(3) Unstable: if it is not stable

- 对于任意的 $\varepsilon>0$ 和任意初始时刻 t_0,
- 都对应存在一个实数 $\delta(\varepsilon,t_0)>0$,
- 使得对于任意位于平衡点 $x=0$ 的球域 $S(\mathbf{0},\delta)$的初始状态 $\mathbf{x}_0$,
- 当从此初始状态$\mathbf{x}_0$出发的状态方程的解$\mathbf{x}$都位于球域$S(\mathbf{0},\varepsilon)$内，则称系统的平衡点 $x=0$ 是李亚普诺夫意义下稳定的。
- 若实数$\delta(\varepsilon,t_0)$与初始时刻t_0无关,则称稳定的平衡态$\mathbf{x}_e$是李亚普诺夫意义下一致稳定的.

稳定性定义



Geometric interpretation of Lyapunov stability for two-dimensional dynamics

稳定性定义

- 对于自治系统来说,上述定义中的实数$\delta(\varepsilon,t_0)$与初始时刻t_0必定无关，故其稳定性与一致稳定性两者等价.
  - 但对于非自治系统来说，则这两者的意义很可能不同.

稳定性定义

Example:

$$\dot{x}(t) = 2t\,(3\,\sin(t) - 1)\,x(t)$$

System solution:

$$x(t) = x(t_0)\ \exp\left[\int_{t_0}^{t} 2\,\tau\,(3\,\sin(\tau) - 1)\,d\tau\right]$$

$$= x(t_0)\ \exp\left(6\ \sin t - 6\,t\ \cos t - t^2 - 6\ \sin t_0 + 6\,t_0\ \cos t_0 + t_0^2\right)$$

Note that: $6\sin t - 6t\cos t - t^2 \le 6 + 6|t| - t^2 = -(|t| - 3)^2 + 15 \le 15$

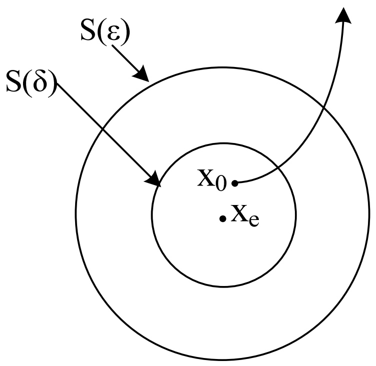
We have: $|x(t)| \le |x(t_0)|\,c(t_0)$ $\Rightarrow$ stable

let $t_0 = 2\,k\,\pi$, $\quad x((2k+1)\,\pi) = x(2\,k\,\pi)\ \exp((4\,k+1)\,(6-\pi)\,\pi)$

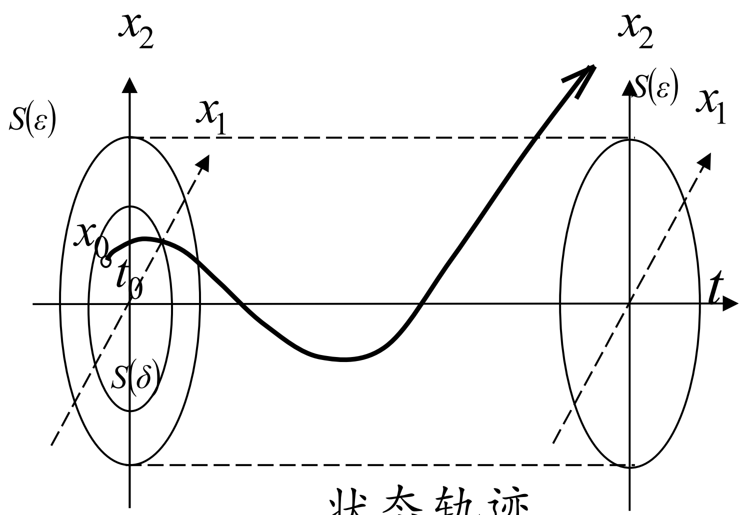
$$\frac{x((2k+1)\,\pi)}{x(2\,k\,\pi)} = \exp((4\,k+1)\,(6-\pi)\,\pi) \to \infty, \quad \text{as} \quad k \to \infty.$$

Unstability(不稳定性)

如果对于某个实数$\varepsilon>0$和任一个实数$\delta>0$，不管这两个实数多么小，在$S(\delta)$内总存在一个状态x_0，使得始于这一状态的轨迹最终会脱离开$S(\varepsilon)$，那么平衡状态$x_e=0$称为不稳定的。



(c)
平衡状态



状态轨迹

不稳定性的几何表示

Asymptotical Stability(渐近稳定性)

- 上述稳定性定义只强调了系统在稳定平衡点附近的解总是在该平衡点附近的某个有限的球域内，相当于状态响应有界，并未强调系统的最终状态稳定于何处.

  - 下面我们给出强调系统最终状态稳定性的李亚普诺夫意义下的一致渐近稳定性定义.

Asymptotical Stability(渐近稳定性)

The equilibrium point $x = 0$ of (2.1) is :

(1) Stable: if for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, t_0) > 0$, such that:

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0$$

(2) Uniformly stable: if $\delta = \delta(\varepsilon) > 0$ is independent of t_0

(3) Unstable: if it is not stable

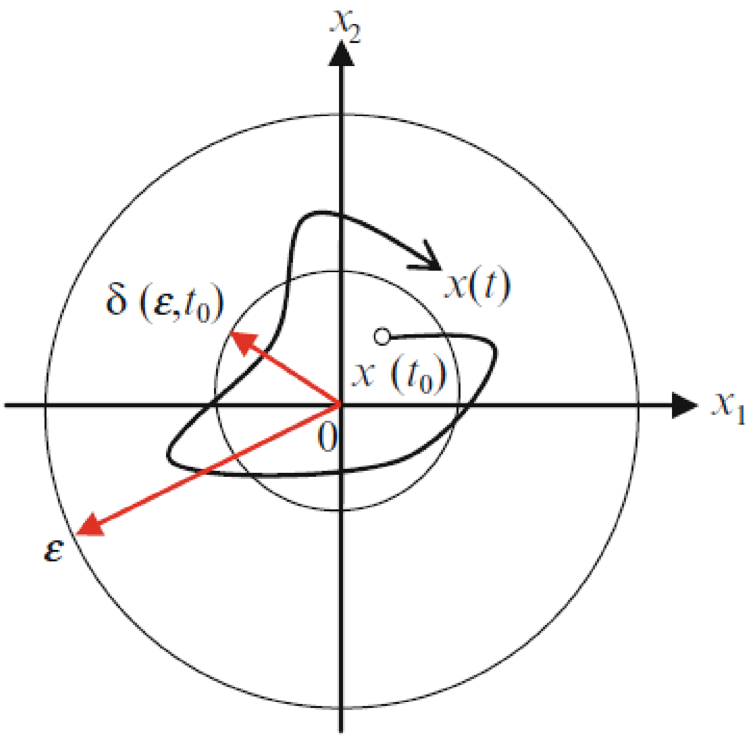
(4) Asymptotically stable: if it is stable and there exists $c = c(t_0) > 0$, such that:

$$\lim_{t \to \infty} x(t) = 0$$

for all $\|x(t_0)\| < c$

(5) Uniformly asymptotically stable: if c is independent of t_0

Asymptotical Stability(渐近稳定性)



Geometric interpretation of Lyapunov stability for two-dimensional dynamics

Asymptotical Stability(渐近稳定性)

- 对于李亚普诺夫渐近稳定性，还有如下说明：
  - 稳定和渐近稳定，两者有很大的不同。
    - ✓ 对于稳定而言，只要求状态轨迹永远**不会跑出球域** $S(\mathbf{x}_e,\varepsilon)$，至于在球域内如何变化不作任何规定。
    - ✓ 而对渐近稳定，不仅要求状态的运动轨迹不能跑出球域，而且还要求最终收敛或无限趋近平衡状态$\mathbf{x}_e$。
  - 从工程意义来说，**渐近稳定性比经典控制理论中的稳定性更为重要**.
    - ✓ 由于渐近稳定性是个平衡态附近的**局部性概念**,只确定平衡态渐近稳定性,并不意味着整个系统能稳定地运行.

Global Asymptotical Stability(全局渐近稳定性)

The equilibrium point $x = 0$ of (2.1) is :

(1) Stable: if for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, t_0) > 0$, such that:

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0$$

(2) Uniformly stable: if $\delta = \delta(\varepsilon) > 0$ is independent of t_0

(3) Unstable: if it is not stable

(4) Asymptotically stable: if it is stable and there exists $c = c(t_0) > 0$, such that:

$$\lim_{t \to \infty} x(t) = 0$$

for all $\|x(t_0)\| < c$

(5) Uniformly asymptotically stable: if c is independent of t_0

(6) Globally uniformly asymptotically stable: if it is uniformly asymptotically stable and $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$

Global Asymptotical Stability(全局渐近稳定性)

- 对于n维状态空间中的所有状态,如果由这些状态出发的状态轨线都具有渐近稳定性,那么平衡点$\mathbf{x}_e$称为李雅普诺夫意义下全局渐近稳定的.
  - 换句话说,若状态方程在任意初始状态下的解,当t无限增长时都趋于平衡点,则该平衡点为全局渐近稳定的.
  - 显然,全局渐近稳定性的必要条件是系统在整个状态空间中只有一个平衡点.
    - ✓对于线性定常系统, 如果其平衡点是渐近稳定的,则一定是全局渐近稳定的.
    - ✓但对于非线性系统则不然, 渐近稳定性是一个局部性的概念,而非全局性的概念.

Lyapunov Stability Theorems(稳定性定理)

❑李亚普诺夫第一法（又称为间接法）

- ❑ 李亚普诺夫第一法是研究动态系统的一次近似数学模型(线性化模型)稳定性的方法.它的基本思路是:
  - ➢ 首先, 对于非线性系统, 可先将非线性状态方程在平衡态附近进行线性化,
    - ✓ 即在平衡态求其一次Taylor展开式,
    - ✓ 然后利用这一次展开式表示的线性化方程去分析系统稳定性.
  - ➢ 其次,解出线性化状态方程组或线性状态方程组的特征值,然后根据全部特征值在复平面上的分布情况来判定系统在零输入情况下的稳定性.

Lyapunov Stability Theorems(稳定性定理)

- 下面将讨论李亚普诺夫第一法的结论以及在判定系统的状态稳定性中的应用.

- 设所讨论的非线性动态系统的状态方程为自治系统

$$\dot{x} = f(x)$$

其中 $f(x)$ 为与状态向量x同维的关于x的非线性向量函数,其各元素对x有连续的偏导数.

Lyapunov Stability Theorems(稳定性定理)

- 欲讨论系统在平衡态x^*的稳定性,先必须将非线性向量函数$f(x)$在平衡态附近展开成Taylor级数,即有

$$
\begin{aligned}
\dot{x} &= f(x^*) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*}(x-x^*) + R(x-x^*) \\
&= A(x-x^*) + R(x-x^*)
\end{aligned}
$$

其中A为n×n维的向量函数$f(x)$与x间的雅可比矩阵; $R(x-x^*)$为Taylor展开式中包含$x-x^*$的二次及二次以上的余项. 雅可比矩阵A定义为

$$
A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*} = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}_{x=x^*}
$$

Lyapunov Stability Theorems(稳定性定理)

- 上述线性化方程的右边第一项 $A(x-x^*)$ 代表原非线性状态方程的一次近似式, 如果用该一次近似式来表达原非线性方程的近似动态方程,即可得如下线性化的状态方程:

$$\dot{x}=A(x-x^*)$$

  - 由于对如上式所示的状态方程总可以通过n维状态空间中的坐标平移,将平衡态x^*移到原点.
  - 因此,上式又可转换成如下原点平衡态的线性状态方程:

$$\dot{x}=Ax$$

- 判别非线性系统平衡态 x^*稳定性的李亚普诺夫第一法的思想即为:
  - 通过线性化,将讨论非线性系统平衡态稳定性问题转换到讨论线性系统 $\dot{x}=Ax$ 的稳定性问题.

Lyapunov Stability Theorems(稳定性定理)

- 李亚普诺夫第一法的基本结论是:

1. 若线性化系统的状态方程的系统矩阵A 的所有特征值都具有负实部,则原非线性系统的平衡态 x^* 渐近稳定,而且系统的稳定性与高阶项$R(\mathbf{x})$ 无关.
2. 若线性化系统的系统矩阵A 的特征值中至少有一个具有正实部,则原非线性系统的平衡态 x^* 不稳定,而且该平衡态的稳定性与高阶项$R(\mathbf{x})$ 无关.
3. 若线性化系统的系统矩阵A 除有实部为零的特征值外,其余特征值都具有负实部,则原非线性系统的平衡态 x^* 的稳定性由高阶项$R(\mathbf{x})$ 决定.

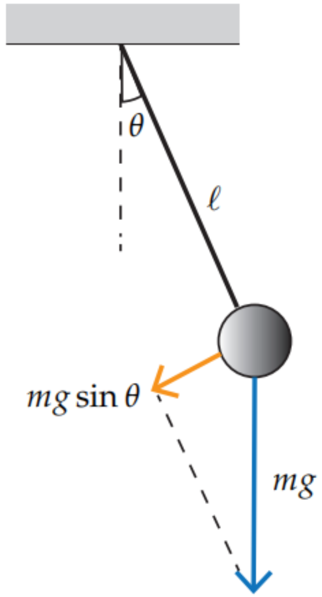
Lyapunov Stability Theorems(稳定性定理)

- 由上述李亚普诺夫第一法的结论可知，该方法与经典控制理论中稳定性判据的思路一致，需求解线性化状态方程或线性状态方程的特征值，根据特征值在复平面的分布来分析稳定性.
  - 值得指出的区别是：
    - ✓ 经典控制理论讨论的是输出稳定性问题，而李亚普诺夫方法讨论状态稳定性问题.
  - 由于李亚普诺夫第一法需要求解线性化后系统的特征值，因此该方法也仅能适用于非线性定常系统或线性定常系统，而不能推广至时变系统.

Example：Pendulum

$$\ell m\ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1$$

Equilibria: $(0,0)$ and $(\pi,0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{\ell} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases}$$

Lyapunov Stability Theorems(稳定性定理)

❑ 李亚普诺夫第二法（又称为直接法）

➢ 它是在用能量观点分析稳定性的基础上建立起来的.

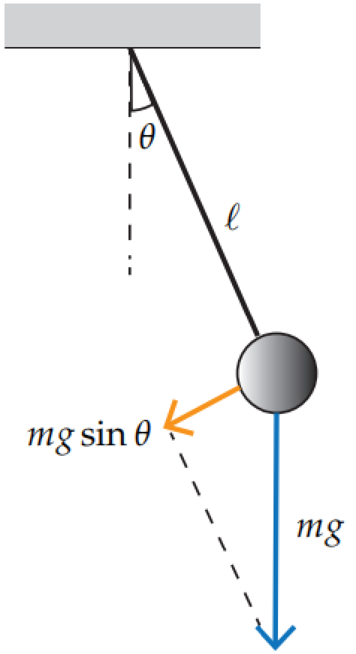
✓ 若系统平衡态渐近稳定,则系统经激励后,其储存的能量将随着时间推移而衰减.当趋于平衡点时,其能量达到最小值.

✓ 反之,若平衡点不稳定,则系统将不断地从外界吸收能量,其储存的能量将越来越大.

➢ 基于这样的观点,只要能找出一个能合理描述动态系统的n维状态的某种形式的能量正性函数,通过考察该函数随时间推移是否衰减,就可判断系统平衡态的稳定性.

Lyapunov Stability Theorems(稳定性定理)

- 考察如右图所示单摆系统平衡状态(0,0)的稳定性.
  - Define the following energy function (kinetic and potential energy):



$$V = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m\,(l\dot{\theta})^2 + mgl\,(1-\cos\theta\,)$$

$$= \frac{1}{2}m\,(lx_2)^2 + mgl\,(1-\cos x_1)$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1$$

The derivative of V with respect to t:

$$\dot{V} = ml^2 x_2 \dot{x}_2 + mgl \sin x_1 \dot{x}_1$$
$$= -kl^2 x_2^2 \le 0$$

- 从直观物理意义的角度,也非常易于理解: 由于物体运动所受到的摩擦力作负功,由能量守恒定律可知,物体的能量将随物体运动减少, 即其导数(变化趋势)为负.

Lyapunov Stability Theorems(稳定性定理)

李亚普诺夫稳定性定理的直观意义

- 从平衡状态的定义可知,平衡状态是使得系统静止不动(导数为零,即运动变化的趋势为零)的状态.
  - 从能量的观点来说, 静止不动即不存在运动变化所需要的能量, 即变化所需的能量为零.
  - 通过分析状态变化所反映的能量变化关系可以分析出状态的变迁或演变,可以分析出平衡态是否稳定或不稳定.

Positive Definite(正定性)

A function $V : D \to \mathbb{R}$ is said to be

- positive definite if $V(0) = 0$ and $V(x) > 0, \ \forall x \neq 0$
- positive semidefinite if $V(0) = 0$ and $V(x) \geq 0, \ \forall x \neq 0$
- negative definite (resp. negative semi definite) if $-V(x)$ is definite positive (resp. definite semi positive).

$$x_1^2 + 2x_2^2 \qquad (x_1 - 2x_2)^2 + x_2^2$$

$$-x_1^2 - 2x_2^2 \qquad -(x_1 + 2x_2)^2 - 5x_1^2$$

$$2x_2^2 \qquad (x_1 - 2x_2)^2$$

$$-3x_1^2 \qquad -(x_1 + 2x_2)^2$$

$$x_1^2 + x_2^2 + 1 \qquad x^2(4 - x^2)$$

Quadratic Form(二次型)

二次型函数的一般概念

定义：代数式中一种多项式函数，每一项的次数都是二次，则称该函数为二次型函数（标量函数）。

二次型函数的表示形式

以三阶系统为例：

代数式： $$v(x) = dx_1x_2 + ax_1^2 + ex_1x_3 + bx_2^2 + cx_3^2 + fx_2x_3$$

矩阵式：

$$v(x) = x^T P x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0.5d & 0.5e \\ 0.5d & b & 0.5f \\ 0.5e & 0.5f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Quadratic Form(二次型)

对于线性系统，通常可用二次型函数 $x^T Px$ 作为李雅普诺夫函数。

二次型的矩阵表示（通式）：

$$V(x)=x^T Px=\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

通常*P*为对称正定矩阵。

Quadratic Form(二次型)

二次型函数的符号性质 $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$.

- 正定：当 $\begin{cases} V(x) > 0, & x \neq 0 \\ V(x) = 0, & x = 0 \end{cases}$ (P正定)时，则函数 $V(x)$ 正定。

- 半正定：当 $\begin{cases} V(x) \geq 0, & x \neq 0 \\ V(x) = 0, & x = 0 \end{cases}$ (P半正定)时，则函数 $V(x)$ 半正定。

- 负定：当 $\begin{cases} V(x) < 0, & x \neq 0 \\ V(x) = 0, & x = 0 \end{cases}$ (P负定)时，则函数 $V(x)$ 负定。

- 半负定：当 $\begin{cases} V(x) \leq 0, & x \neq 0 \\ V(x) = 0, & x = 0 \end{cases}$ (P半负定)时，则函数 $V(x)$ 半负定。

Lyapunov Stability Theorems(稳定性定理)

Autonomous Systems: $\dot{x} = f(x)$

Theorem 2.1 - Lyapunov's stability theorem

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0,\ V(x) > 0, \quad \forall x \in D \setminus \{0\}$
- $\dot{V}(x) \leq 0, \quad \forall x \in D$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0, \forall x \in D \setminus \{0\}$$

then $x = 0$ is asymptotically stable.

Examples

Consider the Pendulum example without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin(x_1)$$

Assume the following energy function

$$V(x) = a(1-\cos(x_1)) + \frac{1}{2}x_2^2.$$

Clearly, $V(0) = 0$ and $V(x) > 0$, $-2\pi < x_1 < 2\pi$, $x_1 \neq 0$.

$$\begin{aligned}\dot{V}(x) &= a\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= a\sin(x_1)x_2 - x_2 a\sin(x_1) = 0\end{aligned}$$

Thus the origin is stable. Since $\dot{V}(x) = 0$, we can also conclude that the origin is not asymptotically stable.

Lyapunov Stability Theorems(稳定性定理)

Pendulum equation, but this time with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2\end{aligned}$$

Consider

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Then,

$$\dot{V}(x) = -bx_2^2 \leq 0$$

which is negative semidefinite.

why? because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1.

We can only conclude that the origin is stable!

Lyapunov Stability Theorems(稳定性定理)

However, we know that is asymptotically stable. Let us try

$$V(x) = \frac{1}{2}x^T P x + a(1 - \cos(x_1))$$

where P given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

is positive definite if $P_{11} > 0$ and $P_{11}P_{22} - P_{12}^2 > 0$.

Computing $\dot{V}$ and taking $P_{22} = 1$, $P_{11} = bP_{12}$, $P_{12} = b/2$, yields

$$\dot{V} = -\frac{1}{2}a\,b\,x_1 \sin(x_1) - \frac{1}{2}b\,x_2^2$$

The term $x_1 \sin(x_1) > 0$ for all $0 < |x_1| < \pi$.
Taking $D = \{x \in \mathbb{R}^2 : |x_1| < \pi\}$ we conclude that V is a Lyapunov function and the origin is asymptotically stable.

This example emphasizes an important feature:

The Lyapunov theorem's conditions are only sufficient!

Lyapunov Stability Theorems(稳定性定理)

Region of attraction (吸引域):

Let $\phi(t,x)$ be the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$. Then, the $\boxed{\underline{\text{region of attraction}}}$ is defined as the set of all points x such that $\phi(t,x)$ is defined for all $t \geq 0$ and

$$\lim_{t \to \infty} \phi(t,x) = 0.$$

If the Lyapunov function satisfies the conditions of asymptotic stability over a domain D, then the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\} \subset D$$

is an estimate of the region of attraction.

When the region of attraction is $\mathbb{R}^n$?
That is, when $x = 0$ is globally asymptotically stabe (GAS)?
Clearly $D = \mathbb{R}^n$, but is this enough?

Lyapunov Stability Theorems(稳定性定理)

To ensure that Ω_c is bounded *for all values* of $c > 0$ we need the radially unbounded condition (径向无界):

$$V(x) \to \infty \ as \ \|x\| \to \infty$$

Theorem 2 . 2 - GAS

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$ and $V(x) > 0, \quad \forall x \neq 0$
- $\|x\| \to \infty \Rightarrow V(x) \to \infty$
- $\dot{V}(x) < 0, \quad \forall x \neq 0.$

Then $x = 0$ is globally asymptotically stable (GAS).

Lyapunov Stability Theorems(稳定性定理)

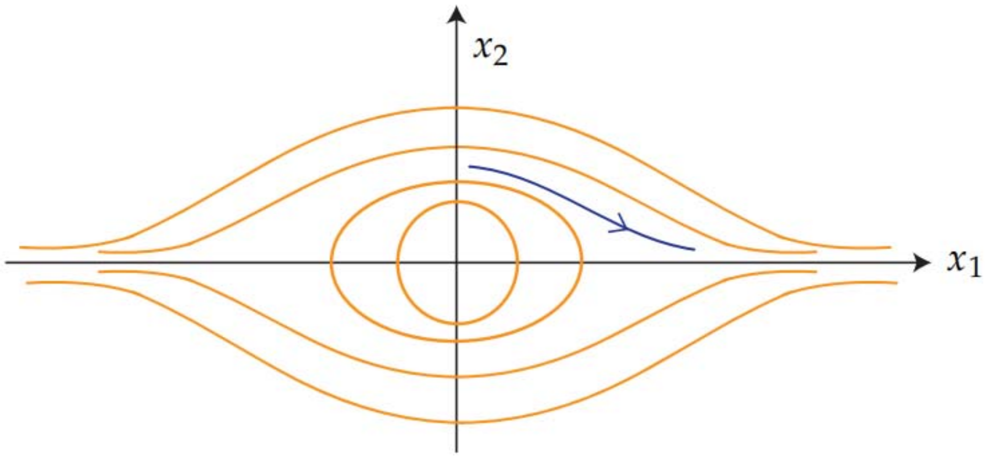
Global asymptotic stability:

Why do we need radial unboundedness?

Example:

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

Set $x_2 = 0$, let $x_1 \to \infty$: $V(x) \to 1$ (not radially unbounded).



Therefore, $x_1(t)$ may grow unbounded while $V(x(t))$ is decreasing.

Lyapunov Stability Theorems(稳定性定理)

Pendulum equation, but this time with friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2 \end{aligned}$$

Consider

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Then,

$$\dot{V}(x) = -bx_2^2 \leq 0$$

which is negative semidefinite.

why? because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1.

We can only conclude that the origin is stable!

The Invariant Principle (不变集原理)

Definitions

- P is said to be a $\boxed{\textit{positive limit point}}$ of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $x \to \infty$ such that $x(t_n) \to p$ as $n \to \infty$.
- The set of all positive limit points is called $\boxed{\textit{positive limit set.}}$
- A set M is said to be a $\boxed{\textit{positively invariant set}}$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

- $x(t)$ approaches a set M as $t \to \infty$, if

$$\forall \epsilon > 0\, \exists T > 0 : \text{dist}(x(t), M) < \epsilon, \quad \forall t > T$$

where $\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$.

Examples:

Equilibrium points

The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The Invariant Principle (不变集原理)

Theorem 2 .3 - La Salle's Theorem

Let

- $\Omega \subset D$ be a compact positively invariant set.
- $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω.
- $E = \{x \in \Omega : \dot{V}(x) = 0\}$.
- M be the largest invariant set in E.

Then every solution starting in Ω approaches M as $t \to \infty$.

The Invariant Principle (不变集原理)

When E is the origin?

Corollary 2.1

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1 positive definite function containing the origin $x = 0$ such that $\dot{V} \leq 0$ in D. Let $S = \{x \in D : \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then, the origin is asymptotically stable.

Corollary 2.2

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1, radially unbounded, positive definite function such that $\dot{V} \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n : \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then $x = 0$ is GAS.

The Invariant Principle (不变集原理)

Examples:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2)\end{aligned}$$

where $h_i(0) = 0$ and $y\,h_i(y) > 0, \forall y \neq 0,\ i = 1, 2$. Also assume that $\int_0^y h_1(z)dz \to \infty$ as $\|y\| \to \infty$.

Consider

$$V(x) = \int_0^{x_1} h_1(y)dy + \frac{1}{2}x_2^2$$

Clearly $V(0) = 0,\ V(x) > 0, \forall x \neq 0$ and $V(x) \to \infty$ as $|x| \to \infty$.

$$\begin{aligned}\dot{V}(x) &= h_1(x_1)\dot{x}_1 + x_2\dot{x}_2\\ &= h_1(x_1)x_2 - x_2h_1(x_1) - x_2h_2(x_2)\\ &= -x_2h_2(x_2) \leq 0\end{aligned}$$

i.e, negative semidefinite.

Is it GAS?

The Invariant Principle (不变集原理)

Examples:

Pendulum equation, but this time with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2\end{aligned}$$

Consider

$$V(x) = a(1-\cos(x_1)) + \frac{1}{2}x_2^2$$

Then,

$$\dot{V}(x) = -bx_2^2 \leq 0$$

which is negative semidefinite.

The Invariant Principle (不变集原理)

Examples:

$$S = \{x \in \mathbb{R}^2 : \dot{V}(x) = 0\} = \{x \in \mathbb{R}^2 : x_2 = 0\}$$

Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1 = 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) = 0$. Thus, $x = 0$ is GAS.

The Invariant Principle (不变集原理)

Pendulum equation, but this time with friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2 \end{aligned}$$

Consider

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Then,

$$\dot{V}(x) = -bx_2^2 \leq 0$$

which is negative semidefinite.

Lyapunov Stability Theorems(稳定性定理)

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

Theorem 2.4

A matrix A is Hurwitz if and only if for every $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Lyapunov Stability Theorems(稳定性定理)

Nonautonomous Systems: $\dot{x} = f(t, x), \quad f(t, 0) = 0$

Definition $V(t,x)$ is said to be:

(1) **Positve semi-definte**: if $V(t,x) \geq 0, V(t,0) = 0$.

(2) **Positve definte:** if $V(t,x) \geq \omega_1(x)$ for some PD function $\omega_1(x)$.

(3) **Radially unbounded:** if $\omega_1(x)$ is so, and decrescent if $V(t,x) \leq \omega_2(x)$ for some PD function $\omega_2(x)$.

Lyapunov Stability Theorems(稳定性定理)

Theorem (Lyapunov Theorem)

- ***Stability:*** *if in a ball B_R around the equilibrium point 0, there exists a scalar function $V(x,t)$ with continuous partial derivatives such that*

  1. *V is positive definite*
  2. *$\dot{V}$ is negative semi-definite*

  then the equilibrium point is ***stable****.*

- ***Uniform stability and uniform asymptotic stability:*** *If furthermore,*

  3. *V is decrescent*

  then the origin is ***uniformly stable****. If condition 2 is strengthened by requiring that $\dot{V}$ be negative definite, then the equilibrium is* ***uniformly asymptotically stable****.*

- ***Global uniform asymptotic stability:*** *If B_R is replaced by the whole state space, and condition 1, the strengthened condition 2, condition 3 and the condition*

  4. *$V(x,t)$ is radially unbounded*

  are all satisfied, then the equilibrium point at 0 is ***globally uniformly asymptotically stable****.*

Lyapunov Stability Theorems(稳定性定理)

let V satisfy:

$$w_1(x) \leq V(t,x) \leq w_2(x)$$

$w_1(x)$ and $w_2(x)$ are positive definite functions.

$$\dot{V}(t,x) \leq -w_3(x)$$

(1) If $w_3(x)$ is PSD, $x=0$ is uniformly stable

(2) If $w_3(x)$ is PD, $x=0$ is uniformly asymptotically stable

Finally, if $D \subset \mathbb{R}^n$ and $w_1(x)$ is radically unbounded $x=0$ is globally uniformly stable.

Lyapunov Stability Theorems(稳定性定理)

- 定理：若 $V(t,x)$ 满足：

$$w_1(x) \le V(t,x) \le w_2(x)$$

其中 $w_1(x)$ 和 $w_2(x)$ 都是正定的，且

$$\dot{V}(t,x) \le -w_3(x)$$

那么

(1) 如果 $w_3(x)$ 是半正定的，那么系统在原点平衡状态处是稳定的。

(2) 如果 $w_3(x)$ 是正定的，那么系统在原点平衡状态处是渐近稳定的。

(3) 如果 $w_1(x)$ 是径向无界的，那么稳定性是全局的。

Lyapunov Stability Theorems(稳定性定理)

Example: Consider the scalar system:

$$\dot{x} = -[1 + g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$.

Consider the following Lyapunov function candidate:

$$V(x) = \frac{1}{2}x^2$$

Blackboard

$$\dot{V} = x\dot{x} = -[1 + g(t)]x^4 \leq -x^4$$

take $w_1(x) = w_2(x) = \frac{1}{2}x^2$, $w_3(x) = -x^4$

Lyapunov Stability Theorems(稳定性定理)

Example: Consider the system:

$$\dot{x}_1 = -x_1 - g(t)x_2$$
$$\dot{x}_2 = x_1 - x_2$$

where $g(t)$ is continuous differentiable and satisfies $0 \le g(t) \le k$ and $\dot{g}(t) \le g(t)$, $\forall t \ge 0$.

Consider the following Lyapunov function candidate:

$$V(t, x) = x_1^2 + [1 + g(t)]x_2^2$$

Blackboard

take $w_1(x) = x_1^2 + x_2^2$, $w_2(x) = x_1^2 + (1+k)x_2^2$

$$\begin{aligned}
\dot{V}(t, x) &= 2x_1\dot{x}_1 + 2[1 + g(t)]x_2\dot{x}_2 + \dot{g}(t)x_2^2 \\
&= 2x_1[-x_1 - g(t)x_2] + 2[1 + g(t)]x_2(x_1 - x_2) + \dot{g}(t)x_2^2 \\
&= -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \\
&\le -2x_1^2 + 2x_1x_2 - [2 + g(t)]x_2^2 \\
&\le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x_1^2 - x_2^2 - (x_1 - x_2)^2
\end{aligned}$$

Barbalat's Lemma

Barbalat's Lemma talks about asymptotical convergence of functions and their derivatives

$$\dot{f} \to 0$$

- $\dot{f} \to 0 \;\not\Rightarrow\; f$ convergence

$$f(t) = \ln t \qquad \dot{f}(t) = \frac{1}{t}$$

- f convergence $\;\not\Rightarrow\; \dot{f} \to 0$

$$f(t) = e^{-t}\sin(e^{2t}) \qquad \dot{f}(t) = -e^{-t}\sin(e^{2t}) + 2e^{t}\cos(e^{2t})$$

Barbalat's Lemma

Barbalat's Lemma

If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

Definition Uniformly Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be uniformly continuous if:

$$\begin{aligned} &\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall |t_2 - t_1| \le \delta \\ &\Rightarrow |f(t_2) - f(t_1)| \le \varepsilon \end{aligned}$$

A very simple sufficient condition for a differentiable function to be uniformly continuous is that its derivative be bounded.

Lyapunov-Like Lemma

Lyapunov-Like Lemma

If a scalar function $V(x,t)$ satisfies the following conditions

- $V(x,t)$ is lower bounded
- $\dot{V}(x,t)$ is negative semi-definite
- $\dot{V}(x,t)$ is uniformly continuous in time

then $\dot{V}(x,t) \to 0$ as $t \to \infty$.

Lyapunov Stability Theorems(稳定性定理)

Example In adaptive control we often encounter the following non-autonomous system:

$$\begin{aligned}\dot{e} &= -e + \tilde{\theta}\varphi(t) \\ \dot{\tilde{\theta}} &= -e\varphi(t)\end{aligned} \tag{2.21}$$

Consider the quadratic scalar function:

$$V = \frac{1}{2}(e^2 + \tilde{\theta}^2) \tag{2.22}$$

Blackboard

Example

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$$\dot{e} = -e + \tilde{\theta}\varphi(t)$$
$$\dot{\tilde{\theta}} = -e\varphi(t)$$

Consider the quadratic scalar function:

$$V = \frac{1}{2}(e^2 + \tilde{\theta}^2)$$

clearly, V is lower bounded:

$$\begin{aligned}\dot{V}(t) =& e\dot{e} + \tilde{\theta}\dot{\tilde{\theta}} \\ =& -e^2 + e\tilde{\theta}\varphi(t) - \tilde{\theta}e\varphi(t) \\ =& -e^2 \leq 0\end{aligned}$$

then we have:

$$\begin{aligned}\ddot{V} &= -2e\dot{e} = -2e(-e + \tilde{\theta}\varphi(t)) \\ &= 2e^2 - 2e\tilde{\theta}\varphi(t)\end{aligned}$$

Let $\varphi(t)$ is bounded, then $\ddot{V}$ is bounded, so we can reach the conclusion that:

$$\dot{V} \to 0, e \to 0$$

Lyapunov Stability Theorems(稳定性定理)

Uniform Ultimate Boundedness(一致最终有界)

考虑以下用常微分方程表示的系统:

$$\dot{x} = f(t,x) + \xi(t), x(t_0) = x_0$$

其中 $t \in \mathbb{R}^+$ 表示时间, $x \in \mathbb{R}^n$ 表示状态, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$,

$f(t,0) = 0, \forall t \geq 0.$ $\xi(t) \in \mathbb{R}^n$ $\|\xi(t)\| \leq \xi_{max}.$

Uniform Ultimate Boundedness(一致最终有界)

例： 系统状态方程

$$\dot{x} = -x + \delta \sin t, x(t_0) = a, a > \delta > 0$$

系统的解： $$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau d\tau$$

$$\begin{aligned} |x(t)| &\le e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\ &\le e^{-(t-t_0)}a + \delta(1 - e^{-(t-t_0)}) \\ &\le \delta + (a-\delta)e^{-(t-t_0)} \\ &\le \delta + a - \delta = a \end{aligned}$$

$\delta < b < a,$ $\Longrightarrow$ $|x(t)| \le b, \forall t \ge t_0 + \ln(\dfrac{a-\delta}{b-\delta})$

Lyapunov Stability Theorems(稳定性定理)

Uniform Ultimate Boundedness(一致最终有界)

一致有界：如果存在独立于初始时刻的正常数 c ，对任意 $a \in (0, c)$，存在 $\beta = \beta(a) > 0$ ，使得

$$\|x(t_0)\| \le a \Rightarrow \|x(t)\| \le \beta$$

一致最终有界：如果存在独立于初始时刻的正常数 b 和 c ，对任意 $a \in (0, c)$ ，存在 $T = T(a, b)$，使得

$$\|x(t_0)\| \le a \Rightarrow \|x(t)\| \le b, \forall t \ge T + t_0$$

Robot

The term *robot* is nowadays used to denote animated *autonomous* machines.

These machines may be roughly classified as follows:

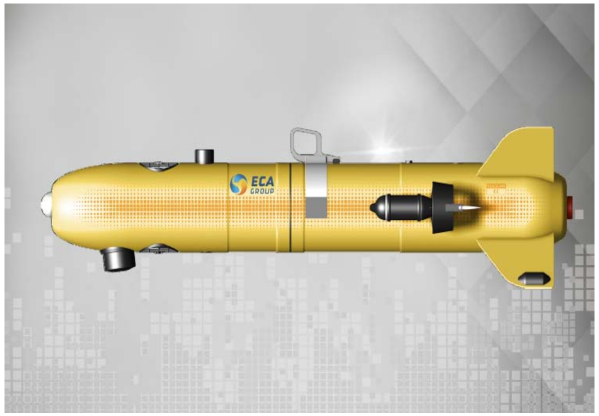
- Robot manipulators
- Mobile robots
  - Ground robots
    - Wheeled robots
    - Legged robots
  - Submarine robots
  - Aerial robots

Mobile Robots

AmigoBot

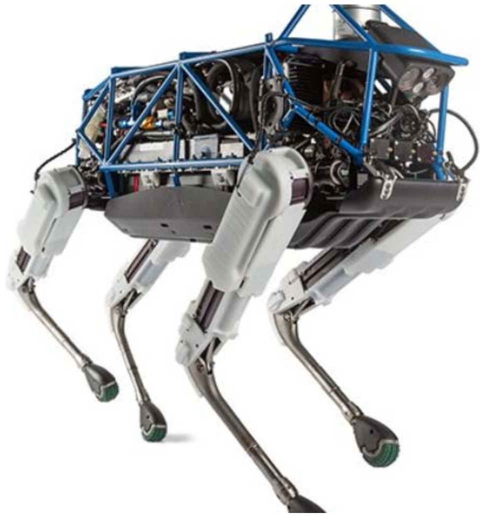
P3-AT

Xiaomi



SeaScan MK2

M100



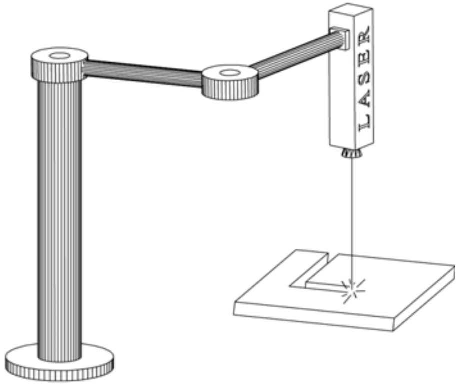
Spot

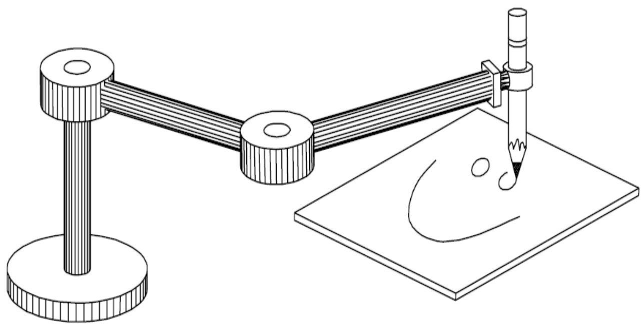
Robot Manipulator

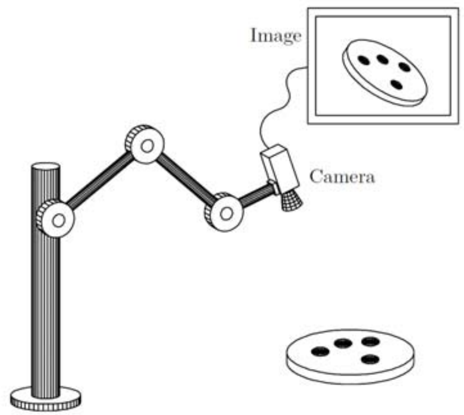
UR

KUKA

Kawasaki







Euler-Lagrange Equation

A dynamical system with p degrees of freedom can be described by the EL equations as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau, \tag{1}$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates, $M(q) \in \mathbb{R}^{p\times p}$ is the symmetric positive definite inertia matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal forces, $g(q)$ is the vector of gravitational force, and $\tau \in \mathbb{R}^p$ is the vector of control force.

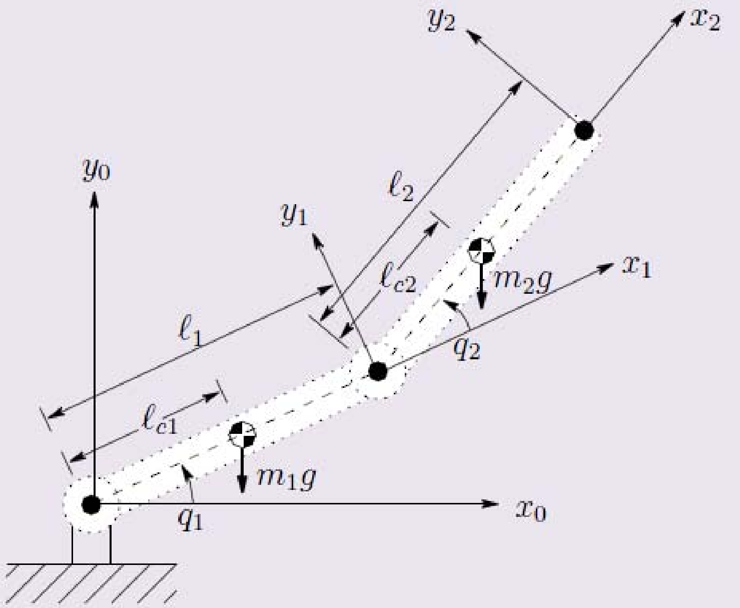
Properties:

1) $M(q)$ is positive definite and $k_{\underline{m}} x^T x \leq x^T M x \leq k_{\overline{m}} x^T x$; $\|C(x,y)z\| \leq k_C \|y\| \|z\|$.

2) $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric.

3) $M(q)x + C(q,\dot{q})y + g(q) = Y(q,\dot{q},y,x)\Theta$, where $Y(q,\dot{q},y,x)$ is the regressor and Θ is an unknown but constant vector.

One example: Two-link Manipulator



$$M(q) = \begin{bmatrix} \Theta_1 + 2\Theta_2 \cos(q_2) & \Theta_3 + \Theta_2 \cos(q_2) \\ \Theta_3 + \Theta_2 \cos(q_2) & \Theta_3 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} -\Theta_2 \sin(q_2)\dot{q}_2 & -\Theta_2 \sin(q_2)(\dot{q}_2 + \dot{q}_1) \\ \Theta_2 \sin(q_2)\dot{q}_1 & 0 \end{bmatrix},$$

$$g(q) = \begin{bmatrix} \Theta_4 g \cos(q_1) + \Theta_5 g \cos(q_1 + q_2) \\ \Theta_5 g \cos(q_1 + q_2) \end{bmatrix},$$

$$\Theta = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5] = [m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + J_1 + J_2, m_2 l_1 l_{c2}, m_2 l_{c2}^2 + J_2, m_1 l_{c1} + m_2 l_1, m_2 l_{c2}],$$

$$Y = \begin{bmatrix} x_1 & \cos(q_2)(2x_1 + x_2) - \sin(q_2)[y_1\dot{q}_2 + y_2(\dot{q}_1 + \dot{q}_2)] & x_2 & g\cos(q_1) & g\cos(q_1 + q_2) \\ 0 & \cos(q_2)x_1 + \sin(q_2)y_1\dot{q}_1 & x_1 + x_2 & 0 & g\cos(q_1 + q_2) \end{bmatrix}.$$

Consider the following system for a robotic manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u$$

1. Position Control.

Control objective: $q \to q_d$, $\dot{q}_d = 0$

Define the position error: $\tilde{q} = q - q_d$, $\dot{\tilde{q}} = \dot{q} - \dot{q}_d = \dot{q}$

The error dynamics is:

$$M(q)\cdot\ddot{\tilde{q}} + C(q,\dot{q})\,\dot{\tilde{q}} = u$$

$$\Rightarrow \quad M(\tilde{q}+q_d)\,\ddot{\tilde{q}} + C(\tilde{q}+q_d,\dot{\tilde{q}})\,\dot{\tilde{q}} = u$$

$$\begin{pmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{pmatrix} \to 0.$$

Define $x_1 = \tilde{q}$, $x_2 = \dot{\tilde{q}}$

$$\dot{x} = f(x) \Leftarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = M^{-1}(x_1 + q_d)\left[u - C(x_1 + q_d, x_2) \cdot x_2\right] \end{cases}$$

$\rightarrow$ autonomous.

Design the following control input

$u = -k_1 x_1 - k_2 x_2$

$$u = -K_p \tilde{q} - K_d \dot{\tilde{q}} = -K_p(q - q_d) - K_d \dot{q} \qquad (7.3)$$

where K_p and K_d are positive definite matrices. Then the closed-loop system is

$$M(\tilde{q} + q_d)\ddot{\tilde{q}} + C(\tilde{q} + q_d, \dot{\tilde{q}})\dot{\tilde{q}} = -K_p \tilde{q} - K_d \dot{\tilde{q}} \qquad (7.4)$$

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2} x_2^T M x_2 = \frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} \dot{\tilde{q}}^T M \dot{\tilde{q}}$$

Its derivative is

$$\dot{V}_1 = \dot{\tilde{q}}^T M \ddot{\tilde{q}} + \frac{1}{2} \dot{q}^T \dot{M} \dot{q}$$

$$= \dot{\tilde{q}}^T \left(-C \dot{\tilde{q}} - K_p \tilde{q} - K_d \dot{\tilde{q}}\right) + \frac{1}{2} \dot{q}^T \dot{M} \dot{q}$$

$$= \frac{1}{2} \dot{\tilde{q}}^T \underbrace{(\dot{M} - 2C)}_{=0} \dot{\tilde{q}} - \dot{\tilde{q}}^T \underbrace{K_p}_{?} \tilde{q} - \dot{\tilde{q}}^T \underbrace{K_d}_{\checkmark} \dot{\tilde{q}}$$

Nonlinear Control of Robotic Manipulators

Note that V_1 is only positive semidefinite. We consider

$$V = V_1 + \frac{1}{2}\tilde{q}^T K_p \tilde{q} = \frac{1}{2}\dot{\tilde{q}}^T M \dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_p \tilde{q}$$

Then we have

$$\dot{V} = \dot{V}_1 + \dot{\tilde{q}}^T K_p \tilde{q}$$

$$= -\dot{\tilde{q}}^T K_d \dot{\tilde{q}} \leq 0 \quad \text{NSD.} \quad \rightarrow \text{stable.}$$

Note that the closed-loop system is autonomous. We have

$$E \triangleq \{(x_1, x_2) \mid \dot{V} = 0\} \Rightarrow \{(x_1, x_2) \mid x_2 \equiv 0\}.$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0.$$

Therefore, the only solution that can stay identically in E is the origin. Thus, from LaSalle's Theorem, the origin is asymptotically stable, i.e., $\lim_{t\to\infty} q(t) = q_d$, $\lim_{t\to\infty} \dot{q}(t) = 0$.

2. Tracking Control.

objective: $q(t) \to q_d(t)$, $\dot{q}(t) \to \dot{q}_d(t)$. $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ are bounded.

Define the tracking error: $\tilde{q}(t) = q(t) - q_d(t)$, $\dot{\tilde{q}}(t) = \dot{q}(t) - \dot{q}_d(t)$.

The error dynamics: $M(q)\left(\ddot{\tilde{q}}(t) + \ddot{q}_d(t)\right) + C(q,\dot{q})\left(\dot{\tilde{q}}(t) + \dot{q}_d(t)\right) = u$

$\Rightarrow M\left(\tilde{q}(t) + q_d(t)\right)\ddot{\tilde{q}}(t) + C\left(\tilde{q}(t) + q_d(t), \dot{\tilde{q}}(t) + \dot{q}_d(t)\right)\dot{\tilde{q}}(t) = u - M\ddot{q}_d(t) - C\dot{q}_d(t)$

Nonlinear Control of Robotic Manipulators

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Design the following controller

$$u = M\ddot{q}_d(t) + C\dot{q}_d(t) - K_p\tilde{q} - K_d\dot{\tilde{q}} \qquad (7.5)$$

Then the closed-loop system is

$$M(q)\ddot{\tilde{q}}(t) + C(q,\dot{q})\dot{\tilde{q}}(t) = -K_p\tilde{q} - K_d\dot{\tilde{q}} \qquad (7.6)$$

$\tilde{q}(t) + q_d(t)$ $\qquad$ $\dot{\tilde{q}}(t) + \dot{q}_d(t)$

Actually, (7.6) is nonautonomous due to the presence of $q_d(t)$ and $\dot{q}_d(t)$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\dot{\tilde{q}}^T M \dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_p \tilde{q} \qquad (7.7)$$

Its derivative

$$\dot{V} = \dot{\tilde{q}}^T M \ddot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T \dot{M} \dot{\tilde{q}} + \dot{\tilde{q}}^T K_p \tilde{q}.$$

$$= -\dot{\tilde{q}}^T K_d \dot{\tilde{q}} \leq 0 \quad \text{NSD}. \qquad (7.8)$$

Since $\dot{V} \le 0$, $V(t) \le V(0)$. From (7.7), $\dot{\tilde{q}}, \tilde{q} \in L_\infty$. Note that

$$\ddot{V} = -2\dot{\tilde{q}}^T K_d \ddot{\tilde{q}} \in L_\infty .$$

Then from Barbalat's Lemma, $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} \dot{\tilde{q}}(t) = 0$.

Unfortunately, from the study sketched above, it is not possible to derive any immediate conclusion about the asymptotic behavior of the position error $\tilde{q}$.

Quadrotor Control

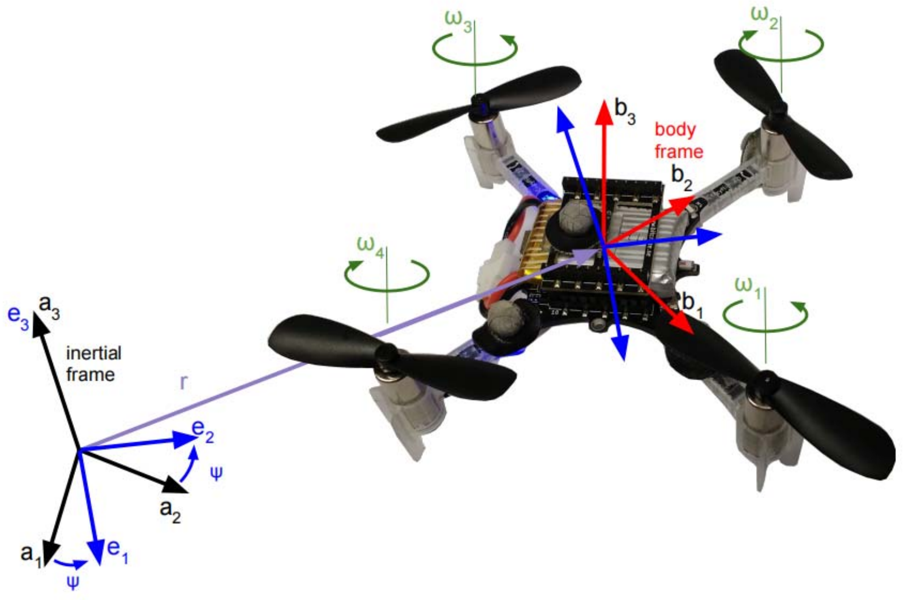
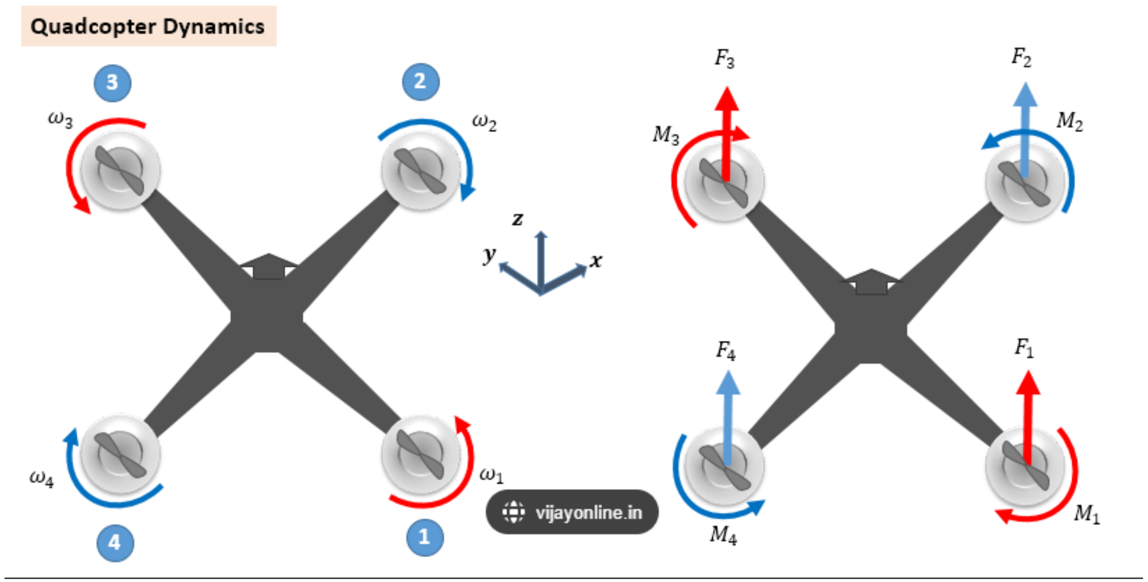


Figure 1: The CrazyFlie 2.0 robot. Note the reflective motion capture markers attached. A pair of motors spins counter clockwise while the other pair spins clockwise, such that when all propellers spin at the same speed, the net torque in the yaw direction is zero. The pitches on the corresponding propellers are reversed so that the thrust is always pointing in the $\mathbf{b_3}$ direction for all propellers. Shown also is the transformation from the inertial frame to the body-fixed frame. First a rotation by ψ around the $\mathbf{a_3}$ axis (leading to coordinate system $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$) is performed, followed by a translation $\mathbf{r}$ to the center of mass C of the robot. Subsequent rotations by ϕ and θ generate the final body-fixed coordinate system $\mathcal{B}$, where the axes $\mathbf{b_1}$ and $\mathbf{b_2}$ are aligned with the arms, and $\mathbf{b_3}$ is perpendicular to them.

Quadrotor Control



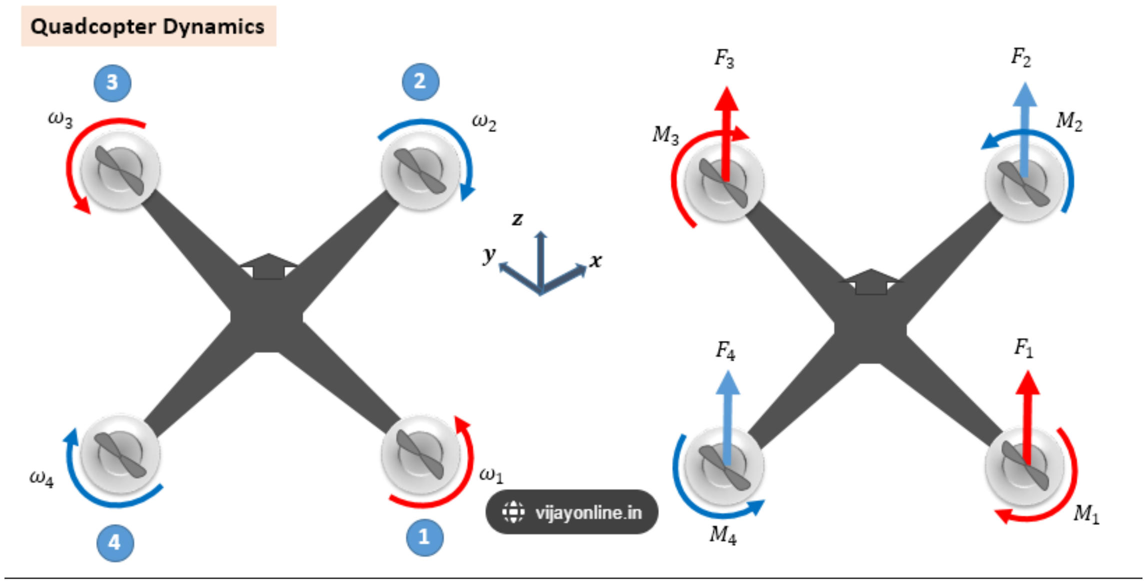
(a) mass: $m = 0.030\,\text{kg}$;

(b) the distance from the center of mass to the axis of a motor: $L = 0.046\,\text{m}$; and

(c) the components of the inertia dyadic using $\mathbf{b}_i$ as the SRT:

$$[I_C]^{\mathbf{b}_i} = \begin{bmatrix} 1.43 \times 10^{-5} & 0 & 0 \\ 0 & 1.43 \times 10^{-5} & 0 \\ 0 & 0 & 2.89 \times 10^{-5} \end{bmatrix}.$$

Quadrotor Control



Each rotor has an angular speed ω_i and produces a vertical force F_i according to

$$F_i = k_F \omega_i^2. \tag{1}$$

Experimentation with a fixed rotor at steady-state shows that $k_F \approx 6.11 \times 10^{-8}\, \mathrm{N/rpm^2}$. The rotors also produce a moment according to

$$M_i = k_M \omega_i^2. \tag{2}$$

The constant, k_M, is determined to be about $1.5 \times 10^{-9}\, \mathrm{Nm/rpm^2}$ by matching the performance of the simulation to the real system.

Quadrotor Control

$$R(\psi)=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix} R(\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix} R(\phi)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ \sin\phi & 0 & \cos\phi\end{bmatrix}$$

$$\begin{aligned} R_b^e &= R(\psi)R(\theta)R(\phi)\\ &= \begin{bmatrix}\cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi-\sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi+\sin\psi\sin\phi\\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi-\cos\psi\sin\phi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}.\end{aligned}$$

- 姿态运动学

用 w_b 代表无人机在机体坐标系下的速度，用 Θ_e 代表三个欧拉角 ϕ ， θ ， ψ 。 w_b 和 Θ_e 之间存在如下关系 $\dot{\Theta}_e = W\cdot w_b$

$$\Theta=[\phi \quad \theta \quad \psi]^T, W=\begin{bmatrix}1 & \tan\theta\sin\phi & \tan\theta\cos\phi\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta\end{bmatrix}, w_b=[w_{bx}, w_{by}, w_{by}]$$

$$\begin{bmatrix}\omega_{xb}\\ \omega_{yb}\\ \omega_{zb}\end{bmatrix}=R^{-1}(\phi)R^{-1}(\theta)\begin{bmatrix}0\\0\\ \dot{\psi}\end{bmatrix}+R^{-1}(\phi)\begin{bmatrix}0\\ \dot{\theta}\\ 0\end{bmatrix}+I_3\begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix},$$

Quadrotor Control

- 位置动力学

  对于无人机受到的力，我们肯定会首先想到重力和螺旋桨的升力，同时也会受到一些阻力的作用。简单起见，这里只考虑重力和螺旋桨升力的作用。根据 $f = ma$ 得

$$m\dot{V}_e = mg - f_e$$

因为空间坐标系的z轴是向下的，所以 f_e 前面是负号。注意，这里的 g 是矢量，只在 z_e 有分量； f_e 也是矢量，但无法直接得到，需要用机体坐标系下的 f_b 表示 f_e ，它们之间有如下关系式。

$$f_e = R_b^e f_b$$

将其带入，则有

$$m\dot{V}_e = mg - R_b^e f_b$$

$$\dot{V}_e = g - R_b^e \frac{f_b}{m}$$

- 姿态动力学

  无人机螺旋桨产生的拉力会对机体坐标系的三个轴产生力矩，同时对于旋转的东西还会存在一个陀螺力矩 $\boldsymbol{G}_a$ 。因此姿态动力学的方程如下

$$\boldsymbol{G}_a + \boldsymbol{\tau} = \boldsymbol{J}\dot{\boldsymbol{\omega}}^b + \boldsymbol{\omega}^b \times \boldsymbol{J}\boldsymbol{\omega}^b$$

Quadrotor Control

综上所述，多旋翼的刚体模型包含了下述四个公式

$$
\begin{aligned}
\dot{P}_e &= V_e \\
\dot{V}_e &= g - R_b^e \frac{f_b}{m} \\
\dot{\Theta}_e &= W \cdot w_b \\
\boldsymbol{\tau} &= \boldsymbol{J} \dot{\boldsymbol{\omega}}^b
\end{aligned}
$$

根据转速得出机体系升力 f_e 和作用在机体上的力矩 τ 。升力 f_e 和力矩 τ 与转速 ϖ 有如下关系。**输入的转速单位是弧度每秒**。

$$
\begin{bmatrix} f_b \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} c_{\mathrm{T}} & c_{\mathrm{T}} & c_{\mathrm{T}} & c_{\mathrm{T}} \\ -\frac{\sqrt{2}}{2} d c_{\mathrm{T}} & \frac{\sqrt{2}}{2} d c_{\mathrm{T}} & \frac{\sqrt{2}}{2} d c_{\mathrm{T}} & -\frac{\sqrt{2}}{2} d c_{\mathrm{T}} \\ \frac{\sqrt{2}}{2} d c_{\mathrm{T}} & -\frac{\sqrt{2}}{2} d c_{\mathrm{T}} & \frac{\sqrt{2}}{2} d c_{\mathrm{T}} & -\frac{\sqrt{2}}{2} d c_{\mathrm{T}} \\ c_{\mathrm{M}} & c_{\mathrm{M}} & -c_{\mathrm{M}} & -c_{\mathrm{M}} \end{bmatrix} \begin{bmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \end{bmatrix}
$$

电机的模型可以表示为一个一阶系统。关系式如下

$$
\begin{aligned}
\varpi_{ss} &= C_R \sigma + \varpi_b \\
\varpi &= \frac{1}{T_m s + 1} \varpi_{ss} \\
&= \frac{1}{T_m s + 1} (C_R \sigma + \varpi_b)
\end{aligned}
$$

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Attitude Stabilization of a VTOL Quadrotor Aircraft

Abdelhamid Tayebi and Stephen McGilvray

The dynamical model described in [4], ignoring aerodynamic effects,[1] with a slight modification of the gyroscopic torques expression due to the fact that the pair of rotors 1–3 rotate in opposite direction of the pair 2–4, is given as follows:

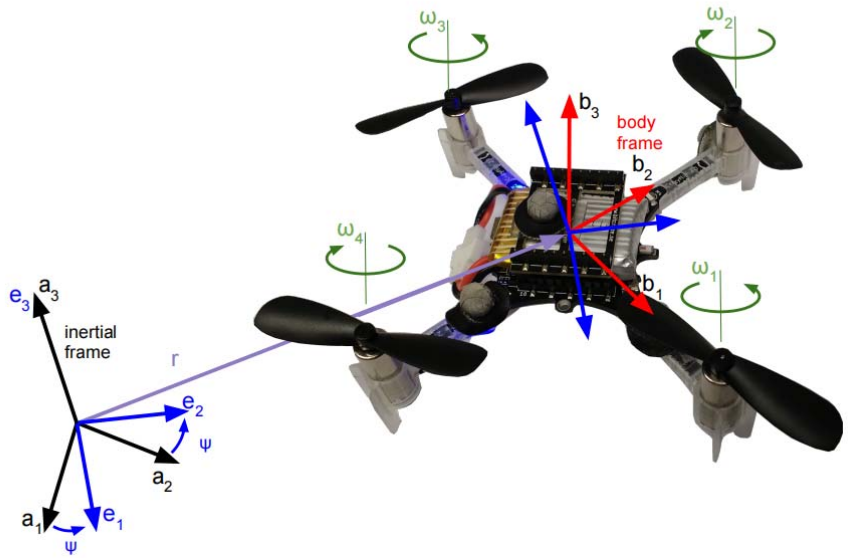
$$\dot{\xi} = v \tag{1}$$

$$\dot{v} = g e_z - \frac{1}{m} T R e_z \tag{2}$$

$$\dot{R} = R S(\Omega) \tag{3}$$

$$I_f \dot{\Omega} = -\Omega \times I_f \Omega - G_a + \tau_a \tag{4}$$

$$I_r \dot{\omega}_i = \tau_i - Q_i, \quad i \in \{1, 2, 3, 4\} \tag{5}$$



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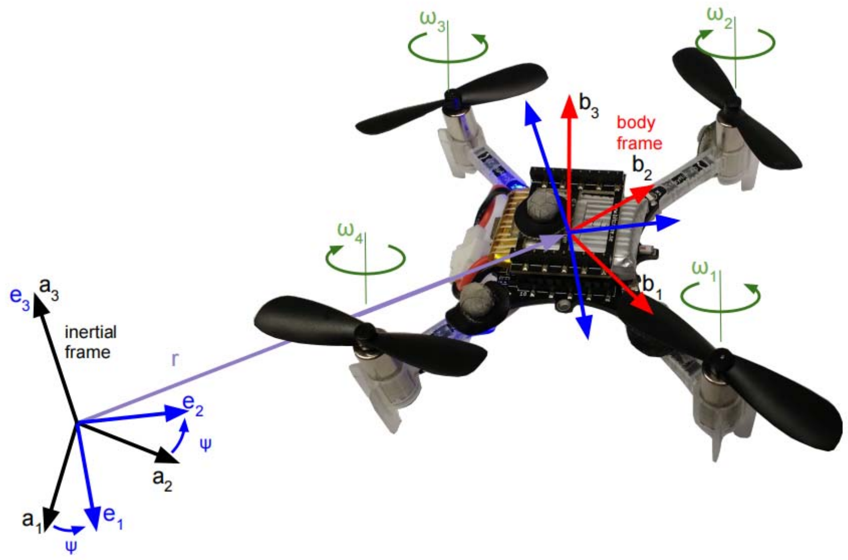
Quadrotor Control



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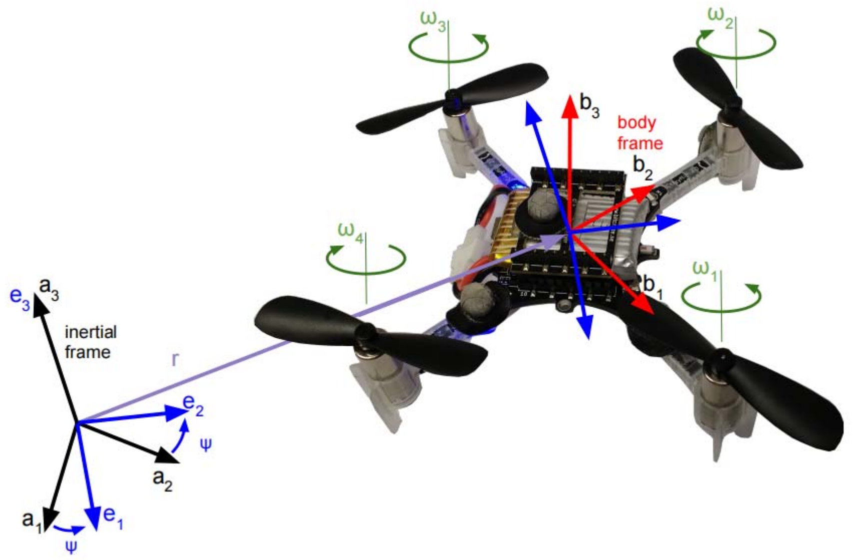

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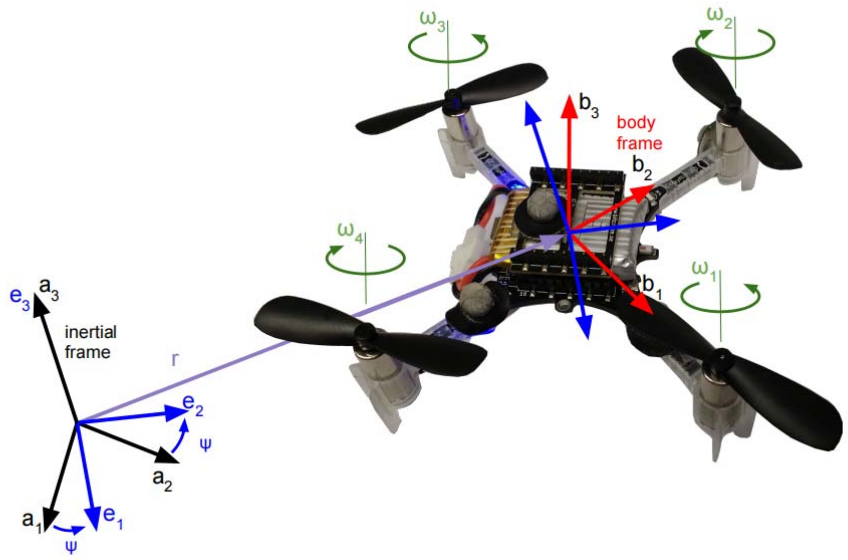

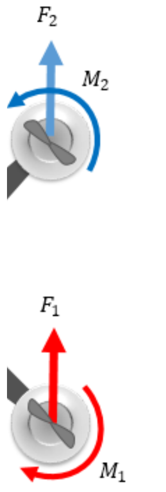

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Thank You !