Optimal Design for Synchronization of Cooperative Systems: State Feedback, Observer and Output Feedback

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***Abstract*—This technical note studies synchronization of identical general linear systems on a digraph containing a spanning tree. A leader node or command generator is considered, which generates the desired tracking trajectory. A framework for cooperative tracking control is proposed, including full state feedback control, observer design and dynamic output feedback control. The classical system theory notion of duality is extended to networked systems. It is shown that unbounded synchronization regions that achieve synchronization on arbitrary digraphs containing a spanning tree can be guaranteed by using linear quadratic regulator based optimal control and observer design methods at each node.**

***Index Terms*—Cooperative control, duality, linear systems, observer, output feedback, synchronization.**

I. Introduction

Research on behaviors of networked cooperative systems (or multi-agent systems) has received extensive attention in the past two decades, mainly due to its widespread applications in spacecraft, unmanned air vehicles (UAVs), mobile robots, sensor networks, networked autonomous team, etc. (see [1], [2] for surveys). Early work was done in the control systems community by [3]–[7], which by now are well known.

Consensus (or leaderless consensus) problem [8] requires all nodes converge to a common value, which is not prescribed. This restricts the dynamical response of the closed-loop systems. On the other hand, the problem of consensus with a leader allows all nodes to track desired leader node trajectories. This is also known as cooperative tracking control [9], synchronization to a leader, pinning control [10], [11], or model reference consensus [12].

In literature, much attention has been paid to consensus of first order or second order integrator dynamics [2]. However, networked systems with general linear systems [3], [13]–[16] are more interesting, and include the integrator dynamics (of any order) as a special case. Most work on cooperative control has focused on static consensus protocols that rely on the receipt of full state information from one's neighbors. In many applications, full state information is not always available for controller design, thus output feedback design is required. Output feedback using a compensator was discussed in [3]. Consensus using output feedback and a dynamic compensator was also given in [13], and consensus region was analyzed. A low gain approach to dynamic output feedback compensator design for consensus was given in [16]. In [17] distributed observers were designed at each node to estimate the state of the leader node.

Of particular relevance to this note are the works [13] and [18]. Reference [13] proposed an observer-type consensus protocol based on measured output information. They designed the "observer" gain using Finsler's lemma and linear matrix inequality (LMI) technique, and showed the consensus region is unbounded. In [18], linear quadratic regulator (LQR) based optimal control approach was used for leaderless consensus and only full state feedback was considered.

Motivated by the above facts, in this note we study cooperative tracking or synchronization problem of general linear systems using measured output information. An active leader node or control node generates the desired tracking trajectories. An optimal design framework for both cooperative state variable feedback (SVFB) control and output feedback (OPFB) control is proposed. Specifically, three types of output feedback synchronization algorithms are proposed. We show that unbounded synchronization regions can be guaranteed by using LQR based optimal control and observer design methods at each node. This guarantees synchronization on arbitrary digraphs containing a spanning tree. It is seen that LQR based optimal design provides a straightforward way to construct controllers and observers that guarantee synchronization. It is also worth mentioning that by using the optimal design approach, the control gain and observer gain design are decoupled from the communication graph structure. We also show the duality of cooperative controller design and cooperative observer design, and this extends the classical system theory notion of duality to systems on graphs.

This note is organized as follows. Some preliminaries are presented in Section II. LQR based optimal cooperative SVFB design is proposed in Section III and synchronization region is analyzed. Section IV shows that the cooperative observer design is dual to the cooperative SVFB control design. Then we propose three types of OPFB tracking protocols in Section V. A conclusion is drawn in Section VI.

II. Preliminaries

Notations: $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. $\mathbf{1}_n \in \mathbb{R}^n$ is the vector with all entries be one. Kronecker product is denoted by $\otimes$. $A = [a_{ij}]$ is a matrix with a_{ij} be the entry of ith row and jth column. $A = diag\{a_i\}$ is a diagonal matrix with diagonal entries a_i. Matrix $P > 0 (P < 0)$ means P is positive (negative) definite. For $\lambda \in \mathbb{C}$, $Re(\lambda)$ is the real part of λ. The conjugate transpose of matrix A is denoted as A^*.

A. Graph Theory

Consider a weighted digraph (directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a nonempty finite set of N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, a set of edges or arcs $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and the associated adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. In this note, the digraph is assumed to be time-invariant, i.e., $\mathcal{A}$ is constant. An edge rooted at node j and ended at node i is denoted by (v_j, v_i), which means information can flows from node j to node i. a_{ij} is the weight of edge (v_j, v_i) and $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. We assume there are no repeated edges and no self loops, i.e., $a_{ii} = 0, \forall\, i \in \mathcal{N}$ with $\mathcal{N} = \{1, 2, \ldots, N\}$. Node j is called a neighbor of node i if $(v_j, v_i) \in \mathcal{E}$. The set of neighbors of node i is denoted as $N_i = \{j | (v_j, v_i) \in \mathcal{E}\}$. Define the in-degree matrix as $D = diag\{d_i\} \in \mathbb{R}^{N\times N}$ with $d_i = \sum_{j\in N_i} a_{ij}$ and the Laplacian matrix as $L = D - \mathcal{A}$. It is obvious that $L\mathbf{1}_N = 0$. A sequence of successive edges in the form $\{(v_i, v_k), (v_k, v_l), \ldots, (v_m, v_j)\}$ is a direct path from node i to node j. A digraph is said to have a spanning tree, if there is a node i_r (called the root), such that there is a directed path from the root to every other nodes in the graph.

Definition 1: [19], [20] The *reverse*, *transpose*, or *converse* of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is another digraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}', \mathcal{A}')$ with the same set of vertices $\mathcal{V}$ and all edges reversed. That is $(v_i, v_j) \in \mathcal{E} \Longleftrightarrow (v_j, v_i) \in \mathcal{E}'$ and $\mathcal{A}' = \mathcal{A}^T$. ■

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B. Problem Formulation

Consider a group of N nodes, distributed on a communication graph $\mathcal{G}$, with identical dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad \forall\, i \in \mathcal{N} \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the input, $y_i \in \mathbb{R}^p$ is the measured output. (A, B, C) is assumed to be stabilizable and detectable throughout this note. It is worth mentioning that system (1) is a general linear system, which includes the first order, second order and high-order integrators (order ≥ 2) as special cases, which are mainly concerned in the literature of cooperative control [2].

The dynamics of the leader or control node, labeled 0, is given by

$$\dot{x}_0 = Ax_0, \quad y_0 = Cx_0$$

where $x_0 \in \mathbb{R}^n$ is the state, $y_0 \in \mathbb{R}^p$ is the measured output. It can be considered as an exosystem or a command generator, which generates the desired target trajectory. The leader node can be observed from a small subset of nodes in graph $\mathcal{G}$. If node i observes the leader, an edge (v_0, v_i) is said to exist with weighting gain $g_i > 0$. We refer to node i with $g_i > 0$ as a pinned or controlled node. Denote the pinning matrix as $G = diag\{g_i\} \in \mathbb{R}^{N\times N}$.

Remark 1: In literature of leaderless consensus, e.g., $g_i = 0, \forall\, i \in \mathcal{N}$, it is often required that all eigenvalues of A sit in the closed left-half complex plane [15], [16]. In this note, however, A can be either stable, marginally stable, or even unstable, as long as (A, B) is stabilizable. ■

Definition 2: The *cooperative tracking problem* is to design local control protocols u_i for all the nodes $i(\forall\; i \in \mathcal{N})$ in $\mathcal{G}$, such that all nodes synchronize to the state trajectory of the leader node, i.e., $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, \forall\, i \in \mathcal{N}$. ■

Denote the global state as $x = \left[x_1^T, x_2^T, \ldots, x_N^T\right]^T \in \mathbb{R}^{nN}$, $\underline{x}_0 = \mathbf{1}_N \otimes x_0 \in \mathbb{R}^{nN}$. Define the *global synchronization error* (cf. the disagreement vector in [5]) as

$$\delta = x - \underline{x}_0 \in \mathbb{R}^{nN}$$

then the cooperative tracking problem is solved if $\lim_{t\to\infty} \delta(t) = 0$.

The following assumption of the graph topology holds throughout this note.

Assumption 1: The digraph $\mathcal{G}$ contains a spanning tree and the root node i_r can observe information from the leader node, i.e., $g_{i_r} > 0$. ■

III. State Variable Feedback Control: An Optimal Design Approach

In this section, we assume the full state information is available, and design the cooperative sate variable feedback (SVFB) control using the linear quadratic regulator (LQR) optimal design approach. In Section IV, we consider estimating the states from measured output information using an observer.

A. LQR-Based Cooperative SVFB Control

Definition 3: [21] For cooperative tracking problem, *neighborhood synchronization error* of node i is defined as

$$\varepsilon_i = \sum_{j\in N_i} a_{ij}(x_j - x_i) + g_i(x_0 - x_i).$$

■

We consider a SVFB control protocol for each node $i(i \in \mathcal{N})$ to be

$$u_i = cK\varepsilon_i \tag{2}$$

with scalar coupling gain $c > 0$ and feedback control gain matrix $K \in \mathbb{R}^{m\times n}$. These controllers are thoroughly distributed in the sense that they are implemented at each node using only the neighborhood synchronization error information ε_i.

Then the overall closed-loop system dynamics is

$$\dot{x} = A_c x + B_c \underline{x}_0,$$

where

$$A_c = I_N \otimes A - c(L+G)\otimes BK,$$
$$B_c = c(L+G)\otimes BK.$$

The global synchronization error dynamics is given as

$$\dot{\delta} = \dot{x} - \dot{\underline{x}}_0 = A_c\delta. \tag{3}$$

A_c reflects the local control matrix $A - BK$ as modified on the graph structure $L + G$.

The next result (cf. [3, Th. 3]) provides a necessary and sufficient condition for asymptotic stability of the closed-loop dynamics using SVFB control law (2).

Lemma 1: Let $\lambda_i (i \in \mathcal{N})$ be the eigenvalues of $(L + G)$. Then the synchronization error dynamics (3) is asymptotically stable if and only if all the matrices

$$A - c\lambda_i BK, \quad \forall\, i \in \mathcal{N}$$

are Hurwitz, i.e., asymptotically stable. ■

Proof: Under Assumption 1, all the eigenvalues of matrix $L+G$ have positive real parts [13], [14]. Then the conclusion follows from similar development as in [3]. □

Lemma 1 implies that an arbitrary SVFB control gain K that stabilizes $A - BK$ may fail to achieve synchronization for a prescribed graph structure. The next result shows how to select SVFB control gain K to guarantee stability on arbitrary digraphs containing a spanning tree by using LQR based optimal design [22] and proper choice of the coupling gain c.

Theorem 1: Let design matrices $Q = Q^T \in \mathbb{R}^{n\times n}$ and $R = R^T \in \mathbb{R}^{m\times m}$ be positive definite. Design the SVFB control gain K as

$$K = R^{-1}B^TP \tag{4}$$

where P is the unique positive definite solution of the control algebraic Riccati equation (ARE)

$$0 = A^TP + PA + Q - PBR^{-1}B^TP. \tag{5}$$

Then under Assumption 1, the synchronization error dynamics (3) is asymptotically stable if the coupling gain

$$c \geq \frac{1}{2\underline{\lambda}_R} \tag{6}$$

with $\underline{\lambda}_R = min_{i\in\mathcal{N}} Re(\lambda_i)$. ■

Proof: Let the eigenvalues of $L+G$ be $\lambda_i = \alpha_i + j\beta_i$, where α_i, $\beta_i \in \mathbb{R}$ and j is the imaginary unit with the property $j^2 = -1$. Then under Assumption 1, $\alpha_i > 0, \forall\, i \in \mathcal{N}$ [13, Lemma 5]. Considering (4) and (5), straightforward computation gives the Lyapunov equation

$$(A - c\lambda_i BK)^*P + P(A - c\lambda_i BK) = -Q - (2c\alpha_i - 1)K^TRK.$$

Since $P > 0$ and $Q > 0$, by Lyapunov theory [23], $A - c\lambda_i BK$ is asymptotically stable if $c \geq 1/2\alpha_i, \forall\, i \in \mathcal{N}$. Then Lemma 1 completes the proof. □

The importance of this result is that it decouples the SVFB gain design from the details of the communication graph structure. The SVFB gain is designed using the control ARE (5), and then the graph topology comes into the choice of the coupling gains c through condition (6).

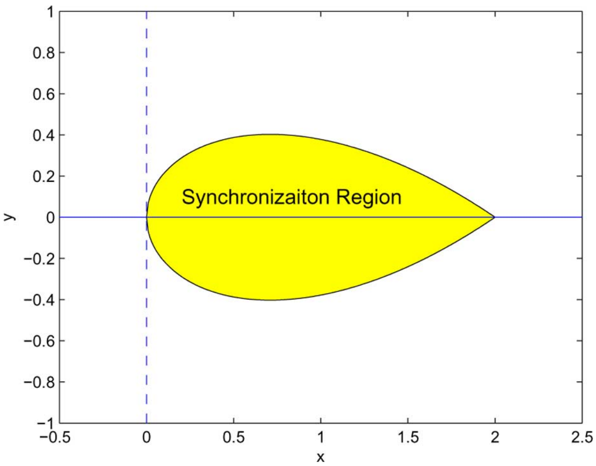

Fig. 1. Bounded synchronization region for arbitrary stabilizing SVFB gain.

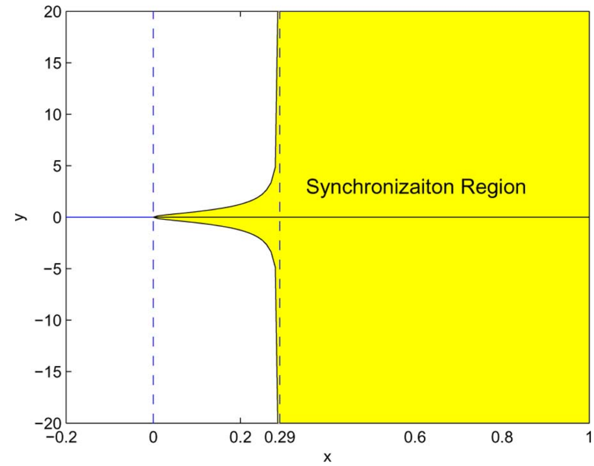

Fig. 2. Unbounded synchronization region for LQR based SVFB gain.

This theorem highlights the importance of optimal SVFB control design in guaranteeing synchronization on arbitrary graphs containing a spanning tree.

B. Region of Synchronization

One way to evaluate the performance of synchronization control protocols is to show how synchronizability depends on structural parameters of the communication graph by using the concept of "region of synchronization or consensus" [13], [24].

Definition 4: [13] Consider the SVFB control protocol (2), the synchronization region is a complex region defined as $\mathcal{S} \triangleq \{s \in \mathbb{C} | A - sBK \text{ is Hurwitz}\}$. ■

By Lemma 1 and Definition 4, the synchronization is achieved if $c\lambda_i \in \mathcal{S}$ for all $i \in \mathcal{N}$. An unbounded consensus region is a desired property of a consensus protocol [13], since by Lemma 1, it implies synchronization on arbitrary graphs containing a spanning tree.

Corollary 1: For protocol (2) with the LQR based control gain (4), the synchronization region is unbounded. More specifically, a conservative synchronization region is $\mathcal{S} = \{\alpha + j\beta | \alpha \in [1/2, \infty), \beta \in (-\infty, \infty)\}$. ■

Proof: Let $s = \alpha + j\beta$. Using the same development as in Theorem 1, we have

$$(A - sBK)^* P + P(A - sBK) = -Q - (2\alpha - 1)K^T RK.$$

Since $P > 0$, $Q > 0$ and $R > 0$, $A - sBK$ is Hurwitz if and only if $-Q - (2\alpha - 1)K^T RK < 0$. A sufficient condition is $\alpha \geq 1/2$. This completes the proof. □

Note that the synchronization region $\mathcal{S}$ in Corollary 1 is larger than the one given in [13, Proposition 2].

The following example shows that a random stabilizing control gain K may yield a bounded synchronization region, while the LQR based control gain (4) renders an unbounded synchronization region.

Example 1: Consider the following system [13]:

$$A = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For an arbitrary stabilizing feedback gain, for example, $K = [0.5, 0.5]$, the synchronization region is $\mathcal{S}_1 = \{x + jy | x < 2; x/2(1-(x/2))^2 - (1-(x/8))y^2 > 0\}$, which is shadowed in Fig. 1. Now consider the optimal feedback gain $K = [1.544, 1.8901]$ provided by Theorem 1 with $Q = I_2$ and $R = 1$. The synchronization region is $\mathcal{S}_2 = \{x + jy | 1 + 1.544x > 0; (5.331x - 1.5473)y^2 + 2.2362(1 + 1.544x)^2 x > 0\}$, which is unbounded as shadowed in Fig. 2. Lemma 4 in [13] is used in computing the synchronization region. Note that the region indicated in Fig. 2 is the actual synchronization region determined from solving the inequality given. The region given by Corollary 1 is the region $\{x + jy | x \in [1/2, \infty), y \in (-\infty, \infty)\}$, which is a conservative estimate of $\mathcal{S}_2$. ■

IV. Cooperative Observer Design: A Dual Problem

As is well known in classical control theory, controller design and observer design are dual problems [23]. In this section, we extend this important result to networked cooperative linear systems. We show the cooperative SVFB controller design and the cooperative observer design are also dual problems.

Consider system (1). Denote $\hat{x}_i \in \mathbb{R}^n$ as the estimate of the state x_i, $\hat{y}_i = C\hat{x}_i$ as the consequent estimate of the output y_i, and $\tilde{y}_i = y_i - \hat{y}_i$ as the output estimation error for node i.

Definition 5: In the design of cooperative observer, the *neighborhood state estimation error* is defined as

$$\epsilon_i = \sum_{j \in N_i} a_{ij}(\hat{x}_j - \hat{x}_i) + g_i(\hat{x}_0 - \hat{x}_i).$$

The *neighborhood output estimation error* is defined as

$$\zeta_i = \sum_{j \in N_i} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i).$$

■

The cooperative observer for each node $i (i \in \mathcal{N})$ is designed in the following form

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\zeta_i$$

where $c > 0$ is the coupling gain, $F \in \mathbb{R}^{n \times p}$ is the observer gain. These observers are thoroughly distributed in the sense that each observer only requires its own output estimation error information and that of its neighbors. Note that this observer is similar to that in [13]. However, the "observer" [13, eq. (31)] is not an actual observer, in a sense that the "state estimates" v_i do not converge to the real states x_i. v_i only act as intermediate variables for the controller design.

Since the leader node 0 acts as a command generator, it is reasonable to assume that the leader node knows its states, i.e., $\hat{x}_0 = x_0$ and $\tilde{y}_0 = 0$. Then the global observer dynamics is

$$\dot{\hat{x}} = A_o\hat{x} + c[(L + G) \otimes F]y + (I_N \otimes B)u$$

where $\hat{x} = [\hat{x}_1^T, \hat{x}_2^T, \ldots, \hat{x}_N^T]^T$, $y = [y_1^T, y_2^T, \ldots, y_N^T]^T$, $u = [u_1^T, u_2^T, \ldots, u_N^T]^T$ and

$$A_o = I_N \otimes A - c(L+G) \otimes FC.$$

A_o reflects the local observer matrix $A - FC$ as modified on the graph structure $L+G$. Let the global state estimation error be $\tilde{x} = x - \hat{x}$, then straightforward computation gives the dynamics of the state estimation error

$$\dot{\tilde{x}} = A_o \tilde{x}. \tag{7}$$

The next result is dual to Lemma 1 and provides a necessary and sufficient condition for asymptotic stability of the observer dynamics.

Lemma 2: Let $\lambda_i (i \in \mathcal{N})$ be the eigenvalues of $(L+G)$. Then the observer error dynamics (7) is asymptotically stable if and only if all the matrices

$$A - c\lambda_i FC, \quad \forall\, i \in \mathcal{N}$$

are Hurwitz, i.e., asymptotically stable. ■

Dual to Theorem 1, LQR based observer gain F and the coupling gain c are designed in the following theorem.

Theorem 2: Let design matrices $Q = Q^T \in \mathbb{R}^{n\times n}$ and $R = R^T \in \mathbb{R}^{p\times p}$ be positive definite. Design the observer gain F as

$$F = PC^T R^{-1}$$

where P is the unique positive definite solution of the observer ARE

$$0 = AP + PA^T + Q - PC^T R^{-1} CP.$$

Then the estimation error dynamics (7) are asymptotically stable if the coupling gain satisfies

$$c \geq \frac{1}{2\underline{\lambda}_R} \tag{8}$$

where $\underline{\lambda}_R = min_{i\in\mathcal{N}} Re(\lambda_i)$. ■

It can be seen from (6) and (8) that the coupling gain c in observer design can take the same value as that in the controller design.

Theorem 3: (Duality of cooperative systems on graphs)

Given a networked system of N identical linear dynamics (A, B, C) on a communication graph $\mathcal{G}$. Suppose the SVFB gain K stabilizes the synchronization error dynamics (3). Then the observer gain K^T stabilizes the state estimation error dynamics (7) for a networked dual system of N identical linear dynamics (A^T, C^T, B^T) on a reverse communication graph $\mathcal{G}'$. ■

Proof: Consider the networked system with (A, B, C) and graph $\mathcal{G}$, under SVFB gain K, the synchronization error dynamics is

$$\dot{\delta} = [I_N \otimes A - c(L+G) \otimes BK]\delta = A_c \delta. \tag{9}$$

For the networked dual system with (A^T, C^T, B^T) and reverse graph $\mathcal{G}'$, under the observer gain K^T, observer estimation error dynamics is

$$\dot{\tilde{x}} = [(I_N \otimes A^T) - c(L^T + G) \otimes (K^T B^T)]\tilde{x} = A_o^{\mathcal{G}'} \tilde{x}.$$

Since $G = diag\{g_i\}$, it is obvious that $A_o^{\mathcal{G}'} = (A_c)^T$. Thus proves the duality. □

Duality of SVFB control and OPFB control is shown in [18] for leaderless consensus on the same graph. This is different from the duality presented in Theorem 3, i.e., duality of SVFB control design and observer design for cooperative tracking problems on reverse graphs. Theorem 3 extends the classical concept of duality in control theory [23] to networked systems on graphs.

V. Cooperative Output Feedback Tracking Protocols

In many practical applications, full state information is not always available for controller design. In this section, we propose three cooperative output feedback tracking protocols. The LQR based control gain (4) or LQR based observer gain (9) or both are used in the output feedback tracking protocols, which yield unbounded synchronization regions. This justifies the importance of optimal design approach in cooperative control of networked systems.

A. Neighborhood Controller and Neighborhood Observer

In this protocol, both controllers and observers are designed using the neighborhood information, i.e.

$$u_i = cK\epsilon_i, \tag{10}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\zeta_i. \tag{11}$$

Assume $\hat{x}_0 = x_0$. Considering the identity $(L \otimes BK)\underline{x}_0 = 0$ which roots in the fact $L\mathbf{1}_N = 0$, the complete global closed-loop dynamics can be written as

$$\begin{aligned}\dot{x} &= (I_N \otimes A)x - c[(L+G) \otimes BK](\hat{x} - \underline{x}_0),\\ \dot{\hat{x}} &= [I_N \otimes A - c(L+G) \otimes FC]\hat{x} + c[(L+G) \otimes F]y\\ &\quad - c[(L+G) \otimes BK](\hat{x} - \underline{x}_0).\end{aligned} \tag{12}$$

Theorem 4: Select the SVFB gain K and coupling gain c according to Theorem 1 and the observer gain F according to Theorem 2. Then the cooperative dynamic OPFB tracking protocol given by (10) and (11) synchronizes all nodes $i (i \in \mathcal{N})$ to the leader node 0. ■

Proof: Equation (12) can be written as

$$\dot{x} = A_c x + B_c(\tilde{x} - \underline{x}_0).$$

Further computation gives

$$\dot{\delta} = A_c \delta + B_c \tilde{x}.$$

The global observer error dynamics is

$$\dot{\tilde{x}} = [I_N \otimes A - c(L+G) \otimes FC]\tilde{x} = A_o \tilde{x}.$$

Then one has

$$\begin{bmatrix}\dot{\delta}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix} A_c & B_c\\ 0 & A_o\end{bmatrix}\begin{bmatrix}\delta\\ \tilde{x}\end{bmatrix}.$$

Stability of A_c and A_o follow Theorem 1 and Theorem 2 respectively. This completes the proof. □

Under this type of tracking control, synchronization region can also be shown unbounded, using the same development as in Section III.

B. Neighborhood Controller and Local Observer

This type of tracking protocol uses the neighborhood state estimation information ϵ_i for the controller design and the local output estimation error information $\tilde{y}_i$ for the observer design. For node i, the controller and the observer are designed as

$$\begin{aligned} u_i &= cK\epsilon_i = cK\left(\sum_{j\in N_i} a_{ij}(\hat{x}_j - \hat{x}_i) + g_i(\hat{x}_0 - \hat{x}_i)\right),\\ \dot{\hat{x}}_i &= A\hat{x}_i + Bu_i - cF\tilde{y}_i.\end{aligned}$$

It can be shown that

$$\begin{bmatrix}\dot{\delta}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix} A_c & B_c\\ 0 & I_N \otimes (A + cFC)\end{bmatrix}\begin{bmatrix}\delta\\ \tilde{x}\end{bmatrix}. \tag{13}$$

It is clear from dynamics (13), that synchronization to the leader node can be achieved if both A_c and $I_N \otimes (A + cFC)$ are Hurwitz. The former is guaranteed by picking K and c as in Theorem 1, and the latter can also be guaranteed by picking an appropriate F, since (A, C) is detectable. Unbounded synchronization region can also be shown.

C. Local Controller and Neighborhood Observer

This type of tracking protocol (cf. [13, eq. (31)]) uses the local "state estimation" for controller design and the neighborhood output information for the "observer" design. For node i, the tracking protocol is designed as

$$u_i = K(\hat{x}_i - \hat{x}_{0i}), \tag{14}$$

$$\begin{cases} \dot{\hat{x}}_i = (A + BK)\hat{x}_i \\ \qquad - cF\left(\sum_{j\in N_i} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(y_0 - C\hat{x}_{0i} - \tilde{y}_i)\right), \\ \dot{\hat{x}}_{0i} = (A + BK)\hat{x}_{0i}. \end{cases} \tag{15}$$

Define $\mu = \left[\mu_1^T, \mu_2^T, \ldots, \mu_N^T\right]^T$ with $\mu_i = x_i - x_0$, $\eta = \left[\eta_1^T, \eta_2^T, \ldots, \eta_N^T\right]^T$ with $\eta_i = \hat{x}_i - \hat{x}_{0i}$, and $\theta = [\mu^T, \eta^T]^T$. Then asymptotic stability of θ implies that $lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$ and $lim_{t\to\infty}(\hat{x}_i(t) - \hat{x}_{0i}(t)) = 0, \forall\, i \in \mathcal{N}$. Let $\hat{x}_{0i}(t_0) = \hat{x}_{0j}(t_0)$, then $\hat{x}_{0i}(t) = \hat{x}_{0j}(t), \forall\, i, j \in \mathcal{N}$ and $\forall\, t \geq t_0$. Straightforward computation yields the dynamics of θ as

$$\dot{\theta} = \begin{bmatrix} I_N \otimes A & I_N \otimes BK \\ c(L+G) \otimes FC & I_N \otimes BK + A_o \end{bmatrix} \theta = A_\theta \theta. \tag{16}$$

Matrix A_θ is similar to the matrix

$$\begin{bmatrix} I_N \otimes (A + BK) & I_N \otimes BK \\ 0 & A_o \end{bmatrix}.$$

Therefore, θ is asymptotically stable if and only if both $I_N \otimes (A + BK)$ and A_o are Hurwitz, which are equivalent to the conditions that $A + BK$ and $A - c\lambda_i FC (\forall\, i \in \mathcal{N})$ are Hurwitz (see Lemma 2). The design of the "observer" gain F and the coupling gain c are provided by Theorem 2. Synchronization region is also unbounded. Note that the derivation of (16) requires the conditions $\hat{x}_{0i}(t_0) = \hat{x}_{0j}(t_0), \forall\, i$, $j \in \mathcal{N}$. A special case is when $\hat{x}_{0i}(t_0) = 0, \forall\, i \in \mathcal{N}$. Then $\hat{x}_{0i}(t) = 0$, $\forall\, i \in \mathcal{N}, t \geq t_0$ and the protocol (14) and (15) are simplified as

$$\begin{aligned} u_i &= K\hat{x}_i, \\ \dot{\hat{x}}_i &= (A + BK)\hat{x}_i \\ &\quad - cF\left(\sum_{j\in N_i} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(y_0 - \tilde{y}_i)\right). \end{aligned}$$

Remark 2: The "observer" (15) was proposed in [13] and it is not a real observer, in the sense that $\hat{x}_i \nrightarrow x_i$ and $\hat{x}_{0i} \nrightarrow x_0$. $\hat{x}_i$ and $\hat{x}_{0i}$ only take roles of intermediate variables in the controller design. We have presented the local protocol (14) to obtain the convergence result. ∎

VI. Conclusion

This note solved a cooperative tracking problem of networked systems with identical general linear systems, which include integrator dynamics of any order as a special case. We showed a framework of designing distributed synchronization protocols first by using full state information and then using measured output information. LQR based optimal design approach was applied in the design of control gain and observer gain. This yields an unbounded synchronization region, which implies that the same optimal controller can also synchronize a networked system on arbitrary graphs containing a spanning tree, by only adjusting the coupling gain c. Duality of the cooperative controller design and cooperative observer design on reverse graphs was presented.

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