

## 4. Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

# Optimization problem in standard form

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- $x \in \mathbf{R}^n$  is the optimization variable
- $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$  is the objective or cost function
- $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$ ,  $i = 1, \dots, m$ , are the inequality constraint functions
- $h_i : \mathbf{R}^n \rightarrow \mathbf{R}$  are the equality constraint functions

**optimal value:**

$$p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, \ i = 1, \dots, m, \ h_i(x) = 0, \ i = 1, \dots, p\}$$

- $p^* = \infty$  if problem is infeasible (no  $x$  satisfies the constraints)
- $p^* = -\infty$  if problem is unbounded below

# Optimal and locally optimal points

$x$  is **feasible** if  $x \in \text{dom } f_0$  and it satisfies the constraints

a feasible  $x$  is **optimal** if  $f_0(x) = p^*$ ;  $X_{\text{opt}}$  is the set of optimal points

$x$  is **locally optimal** if there is an  $R > 0$  such that  $x$  is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{array}$$

**examples** (with  $n = 1$ ,  $m = p = 0$ )

- $f_0(x) = 1/x$ ,  $\text{dom } f_0 = \mathbf{R}_{++}$ :  $p^* = 0$ , no optimal point
- $f_0(x) = -\log x$ ,  $\text{dom } f_0 = \mathbf{R}_{++}$ :  $p^* = -\infty$
- $f_0(x) = x \log x$ ,  $\text{dom } f_0 = \mathbf{R}_{++}$ :  $p^* = -1/e$ ,  $x = 1/e$  is optimal
- $f_0(x) = x^3 - 3x$ ,  $p^* = -\infty$ , local optimum at  $x = 1$

# Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i,$$

- we call  $\mathcal{D}$  the **domain** of the problem
- the constraints  $f_i(x) \leq 0$ ,  $h_i(x) = 0$  are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints ( $m = p = 0$ )

**example:**

$$\text{minimize } f_0(x) = -\sum_{i=1}^k \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints  $a_i^T x < b_i$

# Feasibility problem

$$\begin{array}{ll}\text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

can be considered a special case of the general problem with  $f_0(x) = 0$ :

$$\begin{array}{ll}\text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- $p^* = 0$  if constraints are feasible; any feasible  $x$  is optimal
- $p^* = \infty$  if constraints are infeasible

# Convex optimization problem

## standard form convex optimization problem

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p\end{array}$$

- $f_0, f_1, \dots, f_m$  are convex; equality constraints are affine
- problem is *quasiconvex* if  $f_0$  is quasiconvex (and  $f_1, \dots, f_m$  convex)

often written as

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

important property: feasible set of a convex optimization problem is convex

## example

$$\begin{array}{ll}\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0\end{array}$$

- $f_0$  is convex; feasible set  $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$  is convex
- not a convex problem (according to our definition):  $f_1$  is not convex,  $h_1$  is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0\end{array}$$

## Local and global optima

any locally optimal point of a convex problem is (globally) optimal

**proof:** suppose  $x$  is locally optimal and  $y$  is optimal with  $f_0(y) < f_0(x)$

$x$  locally optimal means there is an  $R > 0$  such that

$$z \text{ feasible, } \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)$$

consider  $z = \theta y + (1 - \theta)x$  with  $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$ , so  $0 < \theta < 1/2$
- $z$  is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$  and

$$f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$$

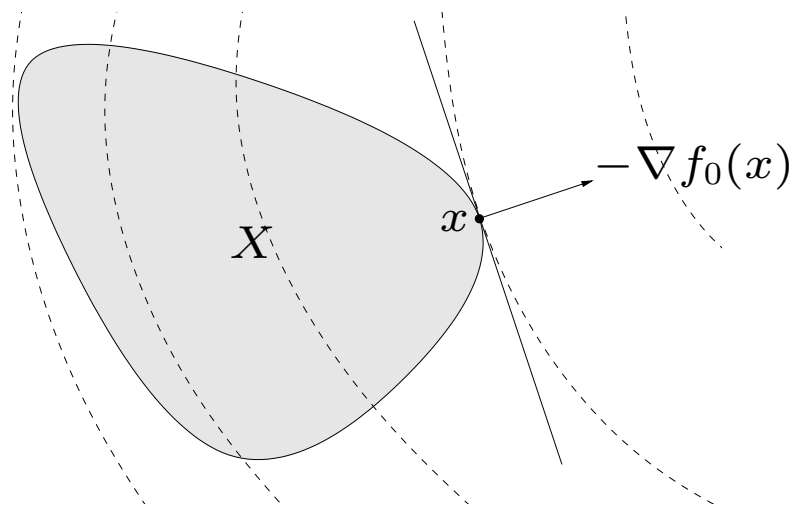
which contradicts our assumption that  $x$  is locally optimal



## Optimality criterion for differentiable $f_0$

$x$  is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$



if nonzero,  $\nabla f_0(x)$  defines a supporting hyperplane to feasible set  $X$  at  $x$

- **unconstrained problem:**  $x$  is optimal if and only if

$$x \in \mathbf{dom} f_0, \quad \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimize } f_0(x) \quad \text{subject to } Ax = b$$

$x$  is optimal if and only if there exists a  $\nu$  such that

$$x \in \mathbf{dom} f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**

$$\text{minimize } f_0(x) \quad \text{subject to } x \succeq 0$$

$x$  is optimal if and only if

$$x \in \mathbf{dom} f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

## Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m\end{array}$$

where  $F$  and  $x_0$  are such that

$$Ax = b \iff x = Fz + x_0 \text{ for some } z$$

- **introducing equality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \dots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x, y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \dots, m\end{array}$$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x, s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \dots, m \\ & s_i \geq 0, \quad i = 1, \dots, m\end{array}$$

- **epigraph form:** standard form convex problem is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, t) & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- **minimizing over some variables**

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

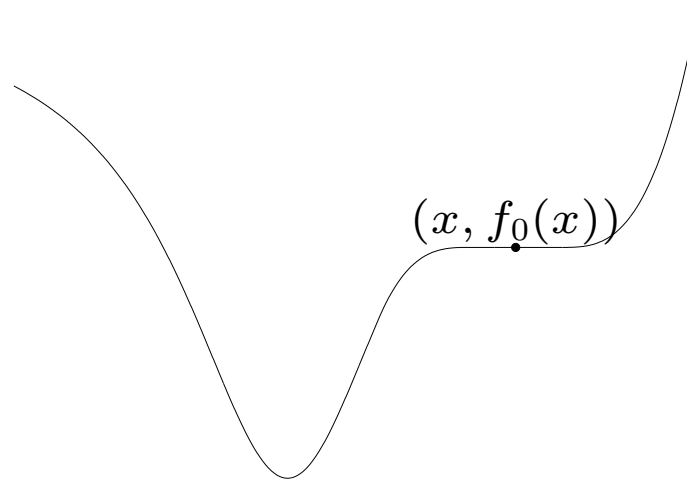
where  $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

# Quasiconvex optimization

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

with  $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$  quasiconvex,  $f_1, \dots, f_m$  convex

can have locally optimal points that are not (globally) optimal



## convex representation of sublevel sets of $f_0$

if  $f_0$  is quasiconvex, there exists a family of functions  $\phi_t$  such that:

- $\phi_t(x)$  is convex in  $x$  for fixed  $t$
- $t$ -sublevel set of  $f_0$  is 0-sublevel set of  $\phi_t$ , i.e.,

$$f_0(x) \leq t \iff \phi_t(x) \leq 0$$

### example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with  $p$  convex,  $q$  concave, and  $p(x) \geq 0$ ,  $q(x) > 0$  on  $\text{dom } f_0$

can take  $\phi_t(x) = p(x) - tq(x)$ :

- for  $t \geq 0$ ,  $\phi_t$  convex in  $x$
- $p(x)/q(x) \leq t$  if and only if  $\phi_t(x) \leq 0$

## quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \leq 0, \quad f_i(x) \leq 0, \quad i = 1, \dots, m, \quad Ax = b \quad (1)$$

- for fixed  $t$ , a convex feasibility problem in  $x$
- if feasible, we can conclude that  $t \geq p^*$ ; if infeasible,  $t \leq p^*$

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*Bisection method for quasiconvex optimization*

**given**  $l \leq p^*$ ,  $u \geq p^*$ , tolerance  $\epsilon > 0$ .

**repeat**

1.  $t := (l + u)/2$ .

2. Solve the convex feasibility problem (1).

3. **if** (1) is feasible,  $u := t$ ; **else**  $l := t$ .

**until**  $u - l \leq \epsilon$ .

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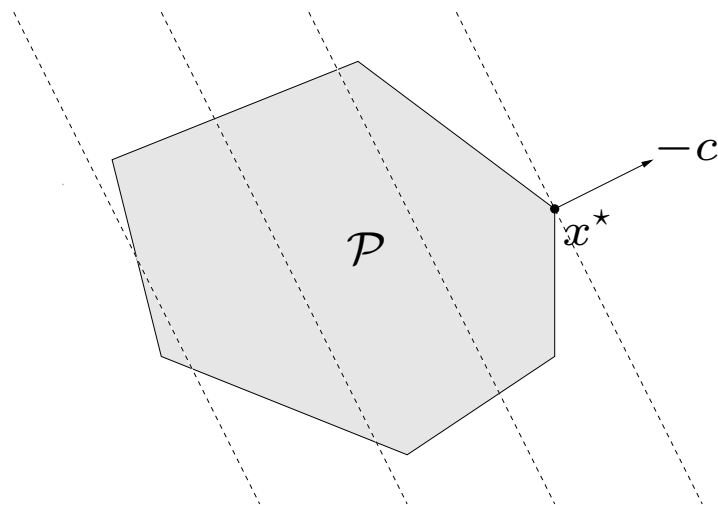
requires exactly  $\lceil \log_2((u - l)/\epsilon) \rceil$  iterations (where  $u, l$  are initial values)



# Linear program (LP)

$$\begin{array}{ll}\text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



## Examples

**diet problem:** choose quantities  $x_1, \dots, x_n$  of  $n$  foods

- one unit of food  $j$  costs  $c_j$ , contains amount  $a_{ij}$  of nutrient  $i$
- healthy diet requires nutrient  $i$  in quantity at least  $b_i$

to find cheapest healthy diet,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0\end{array}$$

### piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\dots,m} (a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \dots, m\end{array}$$

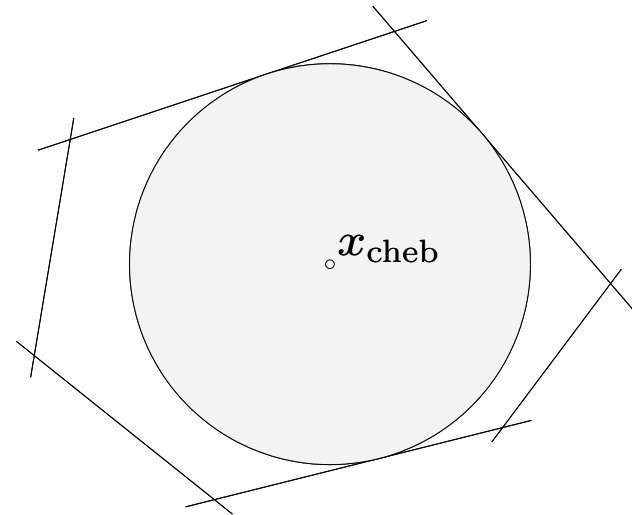
## Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i, \ i = 1, \dots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$



- $a_i^T x \leq b_i$  for all  $x \in \mathcal{B}$  if and only if

$$\sup\{a_i^T (x_c + u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

- hence,  $x_c, r$  can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \dots, m \end{array}$$

## (Generalized) linear-fractional program

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

### linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \quad \text{dom } f_0(x) = \{x \mid e^T x + f > 0\}$$

- a quasiconvex optimization problem; can be solved by bisection
- also equivalent to the LP (variables  $y, z$ )

$$\begin{array}{ll}\text{minimize} & c^T y + dz \\ \text{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0\end{array}$$

## generalized linear-fractional program

$$f_0(x) = \max_{i=1,\dots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \quad \text{dom } f_0(x) = \{x \mid e_i^T x + f_i > 0, i = 1, \dots, r\}$$

a quasiconvex optimization problem; can be solved by bisection

**example:** Von Neumann model of a growing economy

$$\begin{array}{ll} \text{maximize (over } x, x^+) & \min_{i=1,\dots,n} x_i^+ / x_i \\ \text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax \end{array}$$

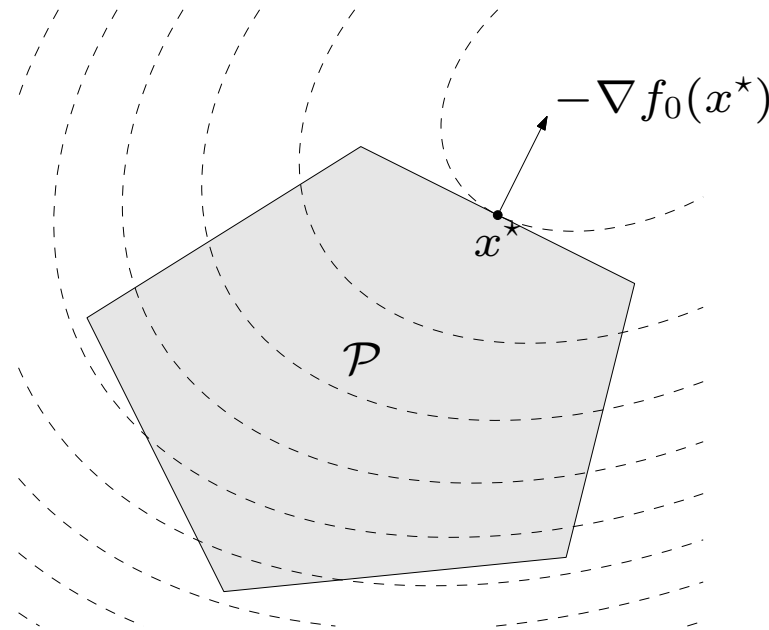
- $x, x^+ \in \mathbf{R}^n$ : activity levels of  $n$  sectors, in current and next period
- $(Ax)_i, (Bx^+)_i$ : produced, resp. consumed, amounts of good  $i$
- $x_i^+ / x_i$ : growth rate of sector  $i$

allocate activity to maximize growth rate of slowest growing sector

# Quadratic program (QP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- $P \in \mathbf{S}_{+}^n$ , so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



# Examples

## least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution  $x^* = A^\dagger b$  ( $A^\dagger$  is pseudo-inverse)
- can add linear constraints, *e.g.*,  $l \preceq x \preceq u$

## linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- $c$  is random vector with mean  $\bar{c}$  and covariance  $\Sigma$
- hence,  $c^T x$  is random variable with mean  $\bar{c}^T x$  and variance  $x^T \Sigma x$
- $\gamma > 0$  is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

# Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $P_i \in \mathbf{S}_+^n$ ; objective and constraints are convex quadratic
- if  $P_1, \dots, P_m \in \mathbf{S}_{++}^n$ , feasible region is intersection of  $m$  ellipsoids and an affine set



# Second-order cone programming

$$\begin{array}{ll}\text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \dots, m \\ & Fx = g\end{array}$$

$$(A_i \in \mathbf{R}^{n_i \times n}, F \in \mathbf{R}^{p \times n})$$

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i+1}$$

- for  $n_i = 0$ , reduces to an LP; if  $c_i = 0$ , reduces to a QCQP
- more general than QCQP and LP

# Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m,\end{array}$$

there can be uncertainty in  $c$ ,  $a_i$ ,  $b_i$

two common approaches to handling uncertainty (in  $a_i$ , for simplicity)

- deterministic model: constraints must hold for all  $a_i \in \mathcal{E}_i$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \dots, m,\end{array}$$

- stochastic model:  $a_i$  is random variable; constraints must hold with probability  $\eta$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \dots, m\end{array}$$

## deterministic approach via SOCP

- choose an ellipsoid as  $\mathcal{E}_i$ :

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\} \quad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n \times n})$$

center is  $\bar{a}_i$ , semi-axes determined by singular values/vectors of  $P_i$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \dots, m \end{array}$$

is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \dots, m \end{array}$$

(follows from  $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$ )

## stochastic approach via SOCP

- assume  $a_i$  is Gaussian with mean  $\bar{a}_i$ , covariance  $\Sigma_i$  ( $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$ )
- $a_i^T x$  is Gaussian r.v. with mean  $\bar{a}_i^T x$ , variance  $x^T \Sigma_i x$ ; hence

$$\mathbf{prob}(a_i^T x \leq b_i) = \Phi \left( \frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2} \right)$$

where  $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x e^{-t^2/2} dt$  is CDF of  $\mathcal{N}(0, 1)$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \dots, m, \end{array}$$

with  $\eta \geq 1/2$ , is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \dots, m \end{array}$$

# Geometric programming

## monomial function

$$f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}, \quad \text{dom } f = \mathbf{R}_{++}^n$$

with  $c > 0$ ; exponent  $\alpha_i$  can be any real number

**posynomial function:** sum of monomials

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom } f = \mathbf{R}_{++}^n$$

## geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \dots, m \\ & h_i(x) = 1, \quad i = 1, \dots, p \end{array}$$

with  $f_i$  posynomial,  $h_i$  monomial

## Geometric program in convex form

change variables to  $y_i = \log x_i$ , and take logarithm of cost, constraints

- monomial  $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$  transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = a^T y + b \quad (b = \log c)$$

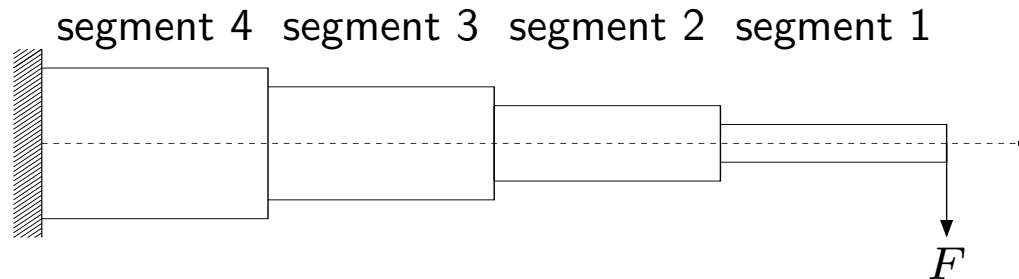
- posynomial  $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$  transforms to

$$\log f(e^{y_1}, \dots, e^{y_n}) = \log \left( \sum_{k=1}^K e^{a_k^T y + b_k} \right) \quad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$\begin{aligned} &\text{minimize} && \log \left( \sum_{k=1}^K \exp(a_{0k}^T y + b_{0k}) \right) \\ &\text{subject to} && \log \left( \sum_{k=1}^K \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \dots, m \\ &&& Gy + d = 0 \end{aligned}$$

# Design of cantilever beam



- $N$  segments with unit lengths, rectangular cross-sections of size  $w_i \times h_i$
- given vertical force  $F$  applied at the right end

## design problem

minimize    total weight  
subject to    upper & lower bounds on  $w_i, h_i$   
                  upper bound & lower bounds on aspect ratios  $h_i/w_i$   
                  upper bound on stress in each segment  
                  upper bound on vertical deflection at the end of the beam

variables:  $w_i, h_i$  for  $i = 1, \dots, N$

## objective and constraint functions

- total weight  $w_1 h_1 + \cdots + w_N h_N$  is posynomial
- aspect ratio  $h_i/w_i$  and inverse aspect ratio  $w_i/h_i$  are monomials
- maximum stress in segment  $i$  is given by  $6iF/(w_i h_i^2)$ , a monomial
- the vertical deflection  $y_i$  and slope  $v_i$  of central axis at the right end of segment  $i$  are defined recursively as

$$v_i = 12(i - 1/2) \frac{F}{E w_i h_i^3} + v_{i+1}$$
$$y_i = 6(i - 1/3) \frac{F}{E w_i h_i^3} + v_{i+1} + y_{i+1}$$

for  $i = N, N - 1, \dots, 1$ , with  $v_{N+1} = y_{N+1} = 0$  ( $E$  is Young's modulus)

$v_i$  and  $y_i$  are posynomial functions of  $w, h$



## formulation as a GP

$$\begin{aligned} \text{minimize} \quad & w_1 h_1 + \cdots + w_N h_N \\ \text{subject to} \quad & w_{\max}^{-1} w_i \leq 1, \quad w_{\min} w_i^{-1} \leq 1, \quad i = 1, \dots, N \\ & h_{\max}^{-1} h_i \leq 1, \quad h_{\min} h_i^{-1} \leq 1, \quad i = 1, \dots, N \\ & S_{\max}^{-1} w_i^{-1} h_i \leq 1, \quad S_{\min} w_i h_i^{-1} \leq 1, \quad i = 1, \dots, N \\ & 6iF\sigma_{\max}^{-1} w_i^{-1} h_i^{-2} \leq 1, \quad i = 1, \dots, N \\ & y_{\max}^{-1} y_1 \leq 1 \end{aligned}$$

note

- we write  $w_{\min} \leq w_i \leq w_{\max}$  and  $h_{\min} \leq h_i \leq h_{\max}$

$$w_{\min}/w_i \leq 1, \quad w_i/w_{\max} \leq 1, \quad h_{\min}/h_i \leq 1, \quad h_i/h_{\max} \leq 1$$

- we write  $S_{\min} \leq h_i/w_i \leq S_{\max}$  as

$$S_{\min} w_i / h_i \leq 1, \quad h_i / (w_i S_{\max}) \leq 1$$

# Minimizing spectral radius of nonnegative matrix

## Perron-Frobenius eigenvalue $\lambda_{\text{pf}}(A)$

- exists for (elementwise) positive  $A \in \mathbf{R}^{n \times n}$
- a real, positive eigenvalue of  $A$ , equal to spectral radius  $\max_i |\lambda_i(A)|$
- determines asymptotic growth (decay) rate of  $A^k$ :  $A^k \sim \lambda_{\text{pf}}^k$  as  $k \rightarrow \infty$
- alternative characterization:  $\lambda_{\text{pf}}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0\}$

## minimizing spectral radius of matrix of posynomials

- minimize  $\lambda_{\text{pf}}(A(x))$ , where the elements  $A(x)_{ij}$  are posynomials of  $x$
- equivalent geometric program:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \sum_{j=1}^n A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \dots, n \end{array}$$

variables  $\lambda, v, x$

# Generalized inequality constraints

**convex problem with generalized inequality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$  convex;  $f_i : \mathbf{R}^n \rightarrow \mathbf{R}^{k_i}$   $K_i$ -convex w.r.t. proper cone  $K_i$
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

**conic form problem:** special case with affine objective and constraints

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b\end{array}$$

extends linear programming ( $K = \mathbf{R}_+^m$ ) to nonpolyhedral cones

# Semidefinite program (SDP)

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b\end{array}$$

with  $F_i, G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

# LP and SOCP as SDP

## LP and equivalent SDP

$$\begin{array}{ll} \text{LP:} & \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array} \end{array} \qquad \begin{array}{ll} \text{SDP:} & \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Ax - b) \preceq 0 \end{array} \end{array}$$

(note different interpretation of generalized inequality  $\preceq$ )

## SOCP and equivalent SDP

$$\begin{array}{ll} \text{SOCP:} & \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \dots, m \end{array} \end{array}$$

$$\begin{array}{ll} \text{SDP:} & \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \dots, m \end{array} \end{array}$$

# Eigenvalue minimization

$$\text{minimize } \lambda_{\max}(A(x))$$

where  $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$  (with given  $A_i \in \mathbf{S}^k$ )

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables  $x \in \mathbf{R}^n$ ,  $t \in \mathbf{R}$
- follows from

$$\lambda_{\max}(A) \leq t \iff A \preceq tI$$

## Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left( \lambda_{\max}(A(x)^T A(x)) \right)^{1/2}$$

where  $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$  (with given  $A_i \in \mathbf{S}^{p \times q}$ )

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

- variables  $x \in \mathbf{R}^n$ ,  $t \in \mathbf{R}$
- constraint follows from

$$\begin{aligned} \|A\|_2 \leq t & \iff A^T A \preceq t^2 I, \quad t \geq 0 \\ & \iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

# Vector optimization

## general vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) \leq 0, \quad i = 1, \dots, p \end{array}$$

vector objective  $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}^q$ , minimized w.r.t. proper cone  $K \in \mathbf{R}^q$

## convex vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

with  $f_0$   $K$ -convex,  $f_1, \dots, f_m$  convex

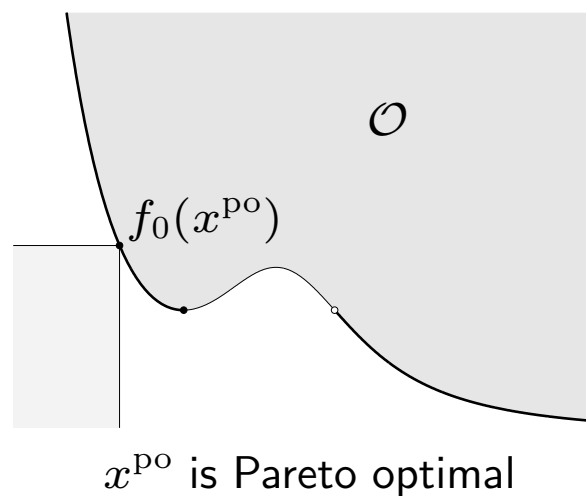
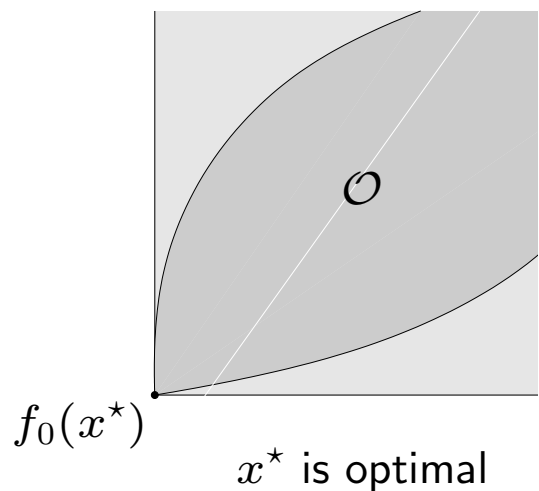


# Optimal and Pareto optimal points

set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible  $x$  is **optimal** if  $f_0(x)$  is a minimum value of  $\mathcal{O}$
- feasible  $x$  is **Pareto optimal** if  $f_0(x)$  is a minimal value of  $\mathcal{O}$



# Multicriterion optimization

vector optimization problem with  $K = \mathbf{R}_+^q$

$$f_0(x) = (F_1(x), \dots, F_q(x))$$

- $q$  different objectives  $F_i$ ; roughly speaking we want all  $F_i$ 's to be small
- feasible  $x^*$  is optimal if

$$y \text{ feasible} \implies f_0(x^*) \preceq f_0(y)$$

if there exists an optimal point, the objectives are noncompeting

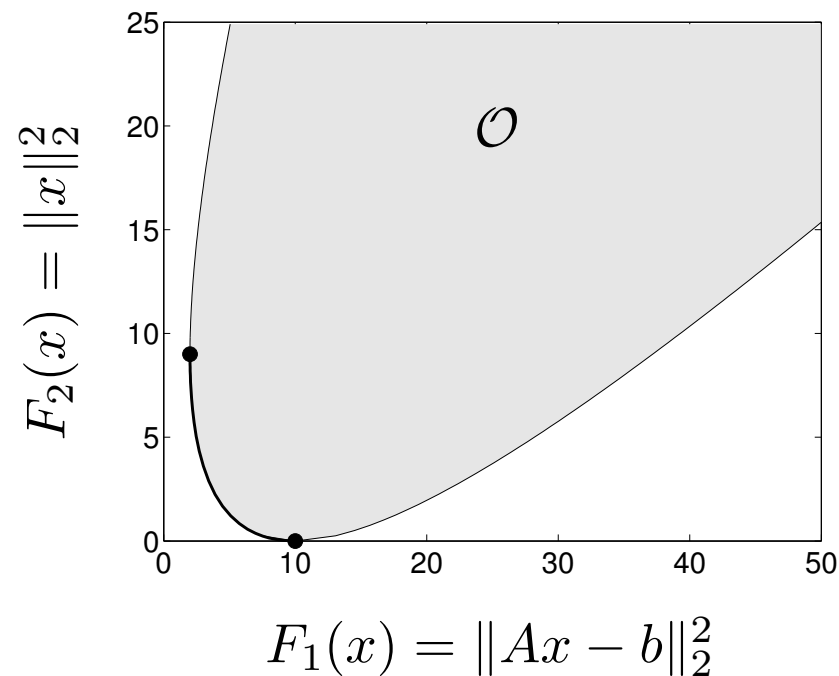
- feasible  $x^{\text{po}}$  is Pareto optimal if

$$y \text{ feasible, } f_0(y) \preceq f_0(x^{\text{po}}) \implies f_0(x^{\text{po}}) = f_0(y)$$

if there are multiple Pareto optimal values, there is a trade-off between the objectives

# Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$



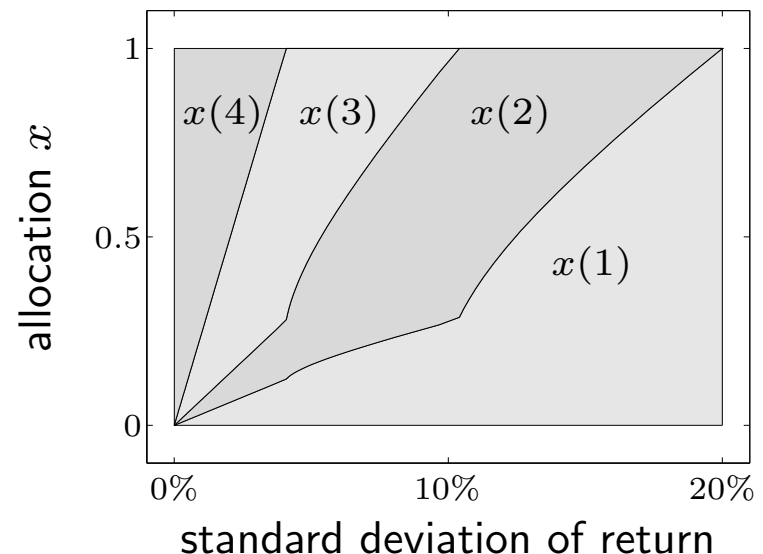
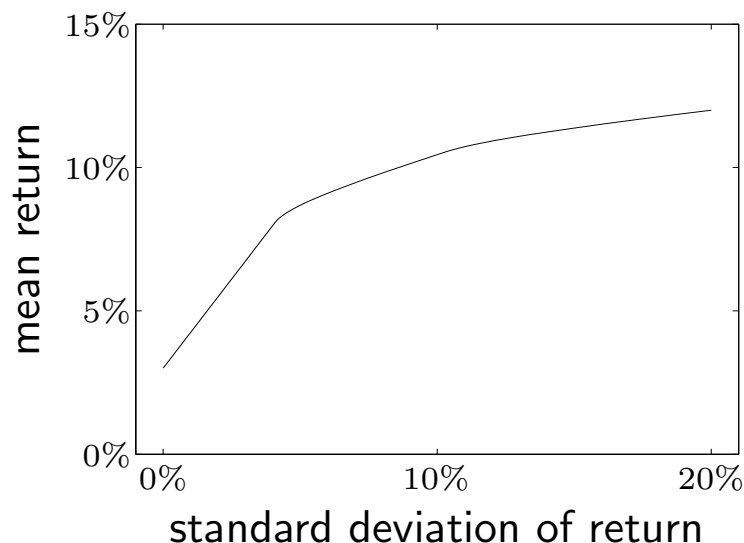
example for  $A \in \mathbf{R}^{100 \times 10}$ ; heavy line is formed by Pareto optimal points

# Risk return trade-off in portfolio optimization

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{R}_+^2) & (-\bar{p}^T x, x^T \Sigma x) \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

- $x \in \mathbf{R}^n$  is investment portfolio;  $x_i$  is fraction invested in asset  $i$
- $p \in \mathbf{R}^n$  is vector of relative asset price changes; modeled as a random variable with mean  $\bar{p}$ , covariance  $\Sigma$
- $\bar{p}^T x = \mathbf{E} r$  is expected return;  $x^T \Sigma x = \mathbf{var} r$  is return variance

## example

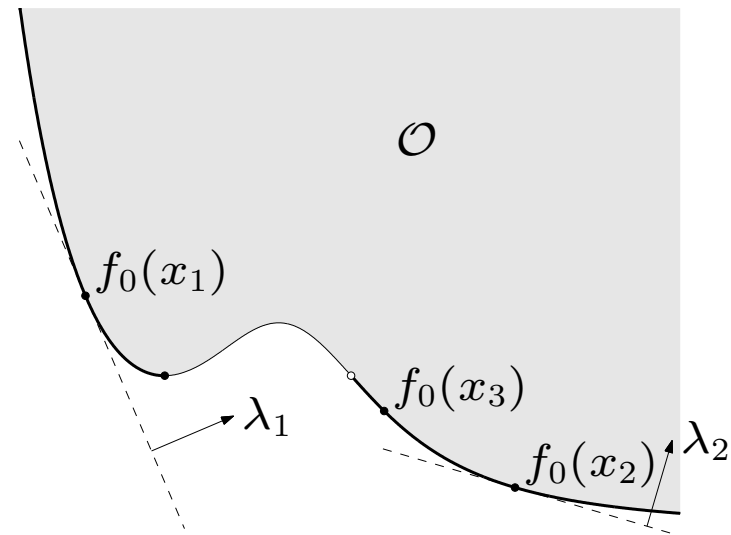


# Scalarization

to find Pareto optimal points: choose  $\lambda \succ_{K^*} 0$  and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

if  $x$  is optimal for scalar problem,  
then it is Pareto-optimal for vector  
optimization problem



for convex vector optimization problems, can find (almost) all Pareto  
optimal points by varying  $\lambda \succ_{K^*} 0$

# Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

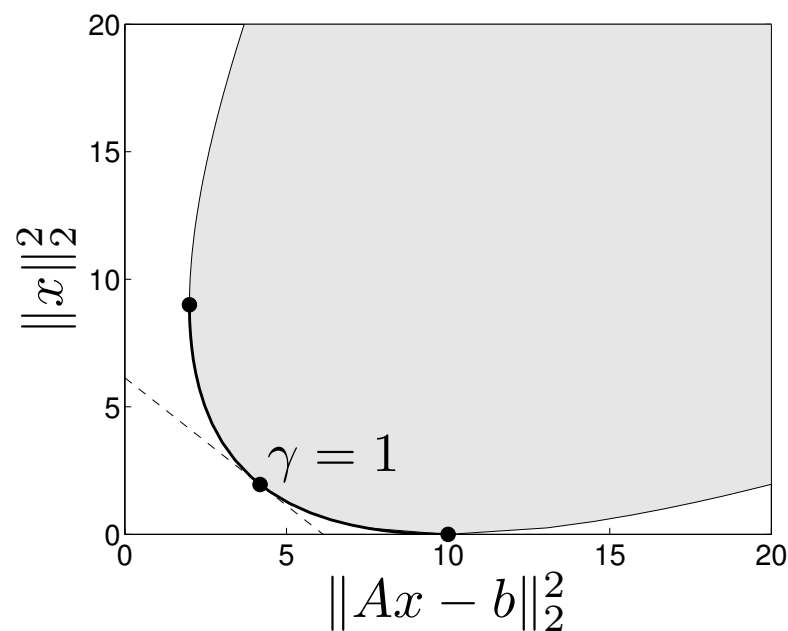
## examples

- regularized least-squares problem of page 4–43

take  $\lambda = (1, \gamma)$  with  $\gamma > 0$

minimize  $\|Ax - b\|_2^2 + \gamma \|x\|_2^2$

for fixed  $\gamma$ , a LS problem



- risk-return trade-off of page 4–44

$$\begin{array}{ll}\text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0\end{array}$$

for fixed  $\gamma > 0$ , a quadratic program