Homework 2

December 9, 2023

1. Consider the system defined by the following equations:

$$
\begin{aligned}
\dot{x}_1 &= \frac{2}{3}x_2 \\
\dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)
\end{aligned}
$$

   (a) Show that the points defined by (i) $x = (0,0)$ and (ii) $1-(3x_1^2+2x_2^2) = 0$ are invariant sets.

   (b) Study the stability of the origin and the invariant set $1-(3x_1^2+2x_2^2) = 0$, respectively, using LaSalle's Invariant Theorem.

2. It is known that a given dynamical system with the state $x = (x_1, x_2)$ has an equilibrium point at the origin. For this system, a function $V(\cdot)$ have been proposed, and its derivative $\dot{V}(\cdot)$ has been computed. Assuming that $V(\cdot)$ and $\dot{V}(\cdot)$ are given below you are asked to classify the origin, in each case, as (a) stable, (b) locally uniformly asymptotically stable, and/or (c) globally uniformly asymptotically stable. Explain you answer in each case.

   (i) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -x_1^2$.

   (ii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.

   (iii) $V(x,t) = x_1^2 + x_2^2$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{t}$.

   (iv) $V(x,t) = (x_1^2 + x_2^2)e^{t}$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + \sin^2 t)$.

   (v) $V(x,t) = (x_1^2 + x_2^2)e^{-t}$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)$.

   (vi) $V(x,t) = (x_1^2 + x_2^2)(1 + e^{-t})$, $\dot{V}(x,t) = -x_1^2 e^{-t}$.

   (vii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)e^{-t}$.

   (viii) $V(x,t) = (x_1^2 + x_2^2)(1 + \cos^2 t)$, $\dot{V}(x,t) = -(x_1^2 + x_2^2)(1 + e^{-t})$.

3. A pendulum with time-varying friction is represented by

$$
\dot{x}_1 = x_2, \tag{1}
$$

$$
\dot{x}_2 = -\sin x_1 - g(t)x_2. \tag{2}
$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \text{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1) \tag{3}$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that

$$\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1 - \cos x_1) + O(\|x\|^3), \tag{4}$$

where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

4. We denote by $|x|$ the absolute value of x if x is scalar and the euclidean norm of x is x is a vector. For functions of time, the L_2 norm is given by

$$\|x\|_p = \left(\int_0^\infty |x(\tau)|^p \mathrm{d}\tau \right)^{\frac{1}{p}}, \tag{5}$$

for $p \in [1, \infty]$, while

$$\|x\|_\infty = \sup_{t \geq 0} |x(t)|. \tag{6}$$

We say that $x \in \mathbb{L}_p$ when $\|x\|_p < \infty$.

- Write down the Barbalat's lemma, the Lyapunov-like lemma, and the Lashalle-Yoshizawa theorem.
- Use Barbalat's lemma to prove the Lyapunov-like lemma, Lashalle-Yoshizawa theorem, and the following corollary.

Corollary 0.1 *If* $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ *and* $\dot{x} \in \mathbb{L}_\infty$, *then* $\lim_{t \to \infty} x(t) = 0$.

5. Consider the following multi-dimensional system

$$\dot{x} = Ax + B(u + \Theta^T \Phi(x))$$

where $x \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$ are known matrices, $u \in \mathbb{R}^m$ is the control input, $\Phi(x) \in \mathbb{R}^k$ is a bounded function, and $\Theta \in \mathbb{R}^{k \times m}$ is an unknown constant matrix. Assume that (A, B) is contrllable.

- Design an adaptive control law to stabilize the system.

- Design an adaptive control law with adaptive $\sigma-$ modification to stabilize the system.

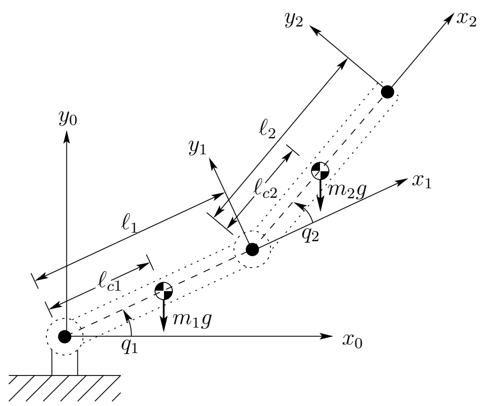
6. The dynamic equations of a robot manipulator in closed form are always written in the form of Euler-Lagrange equation. A dynamical system with p degrees of freedom can be described by the EL equations as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau, \tag{7}$$

where $q \in \mathbb{R}^p$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the symmetric positive definite inertia matrix, $C(q, \dot{q})\dot{q} \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal forces, $g(q)$ is the vector of gravitational force, and $\tau_i \in \mathbb{R}^p$ is the vector of control force. And it has the following properties:

Properties:
1) M_q is positive definite and $k_{\underline{m}} x^T x \leq x^T M x \leq k_{\overline{m}} x^T x$; $\|C(x, y)z\| \leq k_C\|y\|\|z\|$.
2) $\dot{M}(q) - 2C(q, \dot{q})$ is skew symmetric.
3) $M(q)x + C(q, \dot{q})y + g(q) = Y(q, \dot{q}, y, x)\Theta$, where $Y(q, \dot{q}, y, x)$ is the regressor and Θ is an unknown but constant vector.

The following shows a two-link robotic manipulator and its corresponding dynamics.



$$M(q) = \begin{bmatrix} \Theta_1 + 2\Theta_2\cos(q_2) & \Theta_3 + \Theta_2\cos(q_2) \\ \Theta_3 + \Theta_2\cos(q_2) & \Theta_3 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} -\Theta_2\sin(q_2)\dot{q}_2 & -\Theta_2\sin(q_2)(\dot{q}_2 + \dot{q}_1) \\ \Theta_2\sin(q_2)\dot{q}_1 & 0 \end{bmatrix},$$

$$g(q) = \begin{bmatrix} \Theta_4 g\cos(q_1) + \Theta_5 g\cos(q_1 + q_2) \\ \Theta_5 g\cos(q_1 + q_2) \end{bmatrix},$$

$$\Theta = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5]$$
$$= [m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + J_1 + J_2, m_2 l_1 l_{c2}, m_2 l_{c2}^2 + J_2, m_1 l_{c1} + m_2 l_1, m_2 l_{c2}],$$

$$Y = \begin{bmatrix} x_1 & \cos(q_2)(2x_1+x_2) - \sin(q_2)[y_1\dot{q}_2 + y_2(\dot{q}_1+\dot{q}_2)] & x_2 & g\cos(q_1) & g\cos(q_1+q_2) \\ 0 & \cos(q_2)x_1 + \sin(q_2)y_1\dot{q}_1 & x_1+x_2 & 0 & g\cos(q_1+q_2) \end{bmatrix}.$$

The masses of links 1 and 2 of the revolute joint arm are, respectively, $m_1 = 1$ kg and $m_2 = 1.5$ kg, the lengths of links 1 and 2 are, respectively, $l_1 = 0.2$ m and $l_2 = 0.3$ m, the distances from the previous joint to the center of mass of links 1 and 2 are, respectively, $l_{c1} = 0.1$ m and $l_{c2} = 0.15$ m. The moments of inertia of links 1 and 2 are, respectively, $J_1 = 0.013$kg m^2 and $J_2 = 0.045$ kg m^2. Use the following control law

$$\tau = -K_1(q - q_d) - K_2(\dot{q} - \dot{q}_d) + M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + g(q) \qquad (8)$$

where K_1 and K_2 are positive definite matrices, q_d is the desired tracking trajectory. Perform simulation in the following cases and draw the errors $q - q_d$ and $\dot{q} - \dot{q}_d$.

(a) $q_d = [0.3, 0.1], \dot{q}_d = [0, 0], q(0) = \dot{q}(0) = [0, 0]$;

(b) $q_d = [0.3+0.02\sin t, 0.1+0.01\cos t], \dot{q}_d = [0.02\cos t, -0.01\sin t], q(0) = \dot{q}(0) = [0, 0]$;