Chapter 6
Constrained Optimization Methods

- Constrained optimization methods are for nonlinear constrained optimization problems, which have the following generalized form:

$$\min f(\mathrm{X})$$

$$\begin{aligned} s.t. \quad & g_u(\mathrm{X}) \leq 0 \qquad && (u = 1, 2, \cdots, p) \\ & h_v(\mathrm{X}) = 0 \qquad && (v = 1, 2, \cdots, m) \end{aligned}$$

- The optimal solution of the constrained problem is not only related with the objective function, but restricted with constraint conditions. Hence the solving method is much more complicated than that of unconstrained optimization problems.

Classification of Constrained Optimization Methods

- The key issue of solving nonlinear constrained problems is to deal with the constraints. The solving methods can be classified into direct methods and indirect methods.
- In direct methods, the iteration points are always confined in the feasible region, after considering all the constraints. e.g., Feasible Direction Method.
- In indirect methods, constraints are put into the objective function such that constrained problems can be converted to unconstrained problem, or nonlinear problems are converted to relatively simple quadratic programming problems. e. g. Penalty Function Method, SQP (Sequential Quadratic Programming) Method.

Effective Constraints

- The constraint conditions can be divided as **effective constraints** and **ineffective constraints**, depending on whether the investigated point is on the constraint boundary or not.
- The number of effective constraints of the investigated point X_k and the corresponding sequence number can be presented in terms of set and expressed as:

$$I_k = \{u \mid g_u(X_k) = 0 \quad (u = 1, 2, \cdots, p)\}$$

where, I_k represents the subscript set of the effective constraints of point X_k (as shown below).

Extremum Conditions (ECs) for Constrained Problems

- There are various cases for the extremum values of constrained problems.

1. When the minimum point of the objective function is located inside the feasible region, it is the minimum point of the constrained problem;
2. When the minimum point is outside the feasible region, the minimum point of the constrained problem is located on the constraint boundary - it is the tangential point of the boundary and one of the contours of the objective function.

ECs for Constrained Problems (II)

1) ECs for equality-constrained problems:

For optimization problems with equality constraints:

$$\min f(\mathrm{X})$$

$$s.t. \quad h_v(\mathrm{X}) = 0 \qquad (v = 1, 2, \cdots, m)$$

The Lagrange function can be established as follows:

$$L(\mathrm{X}, \lambda) = f(\mathrm{X}) + \sum \lambda_v h_v(\mathrm{X})$$

where, $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_m]^{\mathrm{T}}$ is the Lagrangian multiplier vector.

ECs for Constrained Problems (III)

Set $\nabla L(\mathrm{X}, \lambda) = 0$, then there has:

$\nabla f(\mathrm{X}) + \sum \lambda_v \nabla h_v(\mathrm{X}) = 0$ λ_v are not all zeros (1)

- This is the necessary condition for the equality-constrained problem to achieve the extremum value at point X.
- It presents that at the extremum point, the negative gradient of the objective function equals the non-zero linear combination of the gradients of all constraints.

ECs for Constrained Problems (IV)

2) ECs for inequality-constrained problems:

$$\min f(X)$$

$$s.t. \quad g_u(X) \le 0 \qquad (u = 1, 2, \cdots, p)$$

By introducing p slack variables $x_{n+u} \ge 0$ ($u = 1, 2, \cdots, p$), the following inequality-constrained problem becomes a inequality-constrained problem:

$$\min f(X)$$

$$s.t. \quad g_u(X) + x_{n+u}^2 = 0 \qquad (u = 1, 2, \cdots, p)$$

Establish the Lagrange Function for the new problem:

$$L(X, \lambda, \mathsf{X}) = f(X) + \sum \lambda_u [g_u(X) + x_{n+u}^2]$$

where, $\mathsf{X} = [x_{n+1}, x_{n+2}, \cdots, x_{n+p}]^T$ is a vector comprised by the slack variables.

ECs for Constrained Problems (V)

Set $L(\mathrm{X}, \lambda, \mathcal{X}) = 0$，then there is:

$$\cdots\ \cdots\ \cdots$$

$$\nabla f(\mathrm{X}) + \sum \lambda_i \nabla g_i(\mathrm{X}) = 0$$

$$\lambda_i \geq 0 \qquad (\, i \in I_k \,) \qquad\qquad (2)$$

It is the necessary condition for the inequality-constrained problem to achieve the extremum value at point X. It shows that at the extremum point of the inequality problem, the negative gradient of the objective function is equal to the non-negative linear combination of the gradients of effective constraints.

(1) And (2) are called Kuhn-Tucker Condition, or K-T Condition. They are the necessary conditions for a constrained problem to achieve an extremum value.

Example 1

- **Solve the optimization problem with equality constraints:**

$$\min f(X) = x_1^2 - 2x_2^2$$
$$\text{s.t. } x_1 + 2x_2 + 1 = 0$$

- **Solution:**

$$\nabla f(X) = \begin{bmatrix} 2x_1 \\ -4x_2 \end{bmatrix}, \quad \nabla h(X) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{K}-\text{T condition} \Rightarrow \begin{bmatrix} 2x_1 \\ -4x_2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -\lambda / 2 \\ x_2 = \lambda / 2 \end{cases}$$

$$x_1 + 2x_2 + 1 = 0 \Rightarrow x_1 = 1, \; x_2 = -1, \; \lambda = -2$$

$$\text{Since } \lambda \neq 0 \quad \Rightarrow \text{X}^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is the optimal solution of } f(\text{X}).$$

Example 2

- **Identify whether $X_k = [2, 0]^T$ is the minimum point of the optimization problem or not:**

$$\begin{aligned} \min\ & f(X) = (x_1 - 3)^2 + x_2^2 \\ \text{s.t.}\ & x_1^2 + x_2 - 4 \le 0 \\ & -x_2 \le 0 \\ & -x_1 \le 0 \end{aligned}$$

Solution:

$g_1(X_k) = 0 \quad g_2(X_k) = 0 \quad g_3(X_k) = -2 \neq 0$

Thus the effective constraints are $g_1(X)$ and $g_2(X)$.

Example 2 (Cont'd)

$$\nabla f(X_k) = \begin{bmatrix} 2(x_1 - 3) \\ 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\nabla g_1(X_k) = \begin{bmatrix} 2x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\nabla g_2(X_k) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\text{K} - \text{T condition} \quad \Rightarrow \lambda_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} -2 \\ 0 \end{bmatrix} \quad \Rightarrow \lambda_1 = \lambda_2 = 0.5$$

$$\text{Since } \lambda_1 = \lambda_2 > 0 \quad \Rightarrow \text{X}^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ is the optimal solution of } f(\text{X}).$$

6.1 Feasible Direction Method

- **Feasible Direction Method is a direct method for solving inequality-constrained problems, which are generally formulated as:**

$$\min f(X)$$
$$s.t.\quad g_u(X) \le 0 \qquad (u = 1, 2, \cdots, p)$$

- **The basic idea of this method can be described as: Starting from a feasible point inside the feasible region, choose a suitable search direction S_k and step length α_k such that the next iteration point:**

$$X_{k+1} = X_k + \alpha_k S_k$$

is still inside the feasible region while make the objective function value decrease mostly.

6.1 Feasible Direction Method (II)

- **Thus, the new iteration point satisfies:**

$$g_u(X) \le 0 \qquad (u = 1, 2, \cdots, p) \qquad (1)$$

and $$f(X_{k+1}) - f(X_k) \le 0 \qquad (2)$$

The direction satisfying (1) is called Feasible Direction, while the one satisfying (2) is Descent Direction. If the direction satisfies both (1) and (2), then it is called Descent Feasible Direction.

- **It can be inferred that start from any point inside the feasible region, the optimal point of the constrained problem can be approached via linear search along the descent feasible direction while the constraints are considered.**

6.1.1 Descent Feasible Direction

- **Descent Direction**:

As we know, the direction having an obtuse angel with the gradient make the function value decrease. That is, the descent direction S at point X_k satisfies:

$$[\nabla f(X_k)]^{\mathrm{T}} S < 0$$

- **Feasible Direction**: **the direction pointing inward to the feasible region.**

(1) When X_k is inside the feasible region, any direction starting from X_k is a feasible direction (Fig. 6-1a).

6.1.1 Descent Feasible Direction

(2) When X_k is on the boundary of one effective constraint $g_i(X) = 0$, direction S starting from X_k has to satisfy:

$$[\nabla g_i(X_k)]^T S < 0 \qquad (i \in I_k)$$

(as shown in Fig. 6-1b)

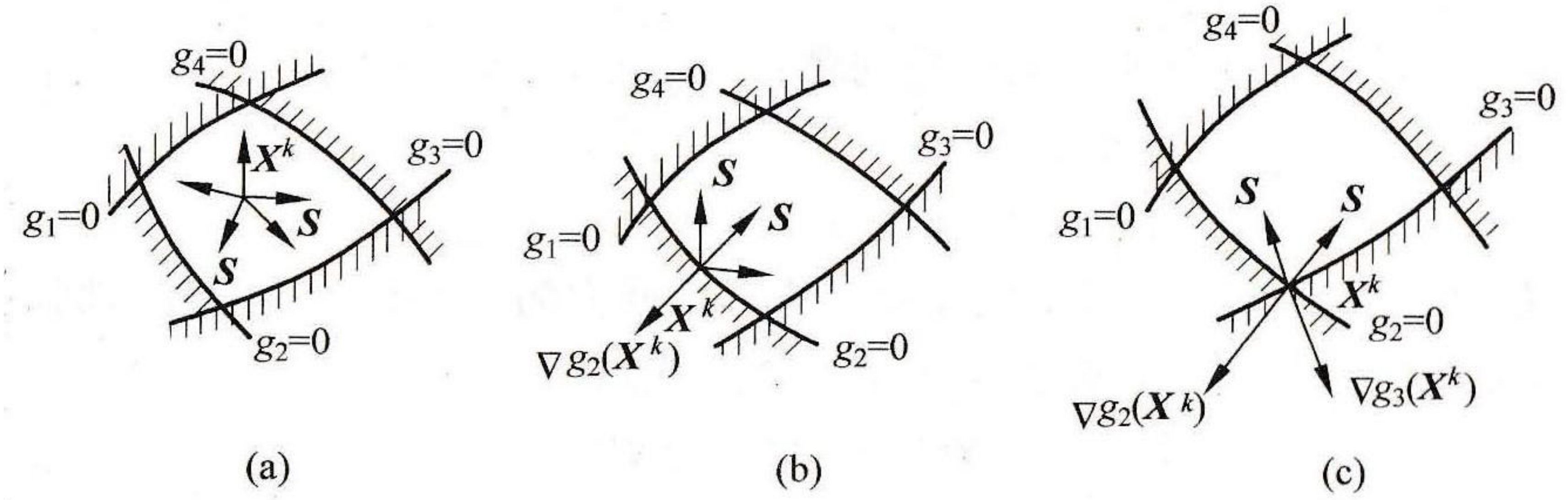


Fig. 6-1

6.1.1 Descent Feasible Direction (II)

(3) When X_k is located on the intersection of several boundaries of effective constraints $g_i(X) = 0$ ($i \in I_k$), the feasible direction S must have obtuse angels with gradients of each effective constraint (Fig. 6-1c).

- From above discussion, a descent feasible direction S, which not only decreases the function value but points inward the feasible region, must satisfy:

$$[\nabla f(X_k)]^T S < 0 \qquad (6\text{-}3a)$$

$$[\nabla g_i(X_k)]^T S < 0 \quad (i \in I_k) \qquad (6\text{-}3b)$$

where I_k is the subscript set of the effective constraints.

6.1.2 Optimal Desc. Feas. Direction

- **Among all the descent feasible directions of X_k, the one makes the function value decrease most is called Optimal Descent Feasible Direction (ODFD).**
- **When X_k is in the feasible region, the negative gradient at this point is the ODFD; when X_k is on the crossing line of several boundaries of effective constraints, (6-3) provides the existence range of all the descent feasible directions.**

6.1.2 ODFD (II)

- **ODFD can be obtained via minimizing the directional derivative of the objective function, on the premise of satisfying feasible conditions.**
- **Thus, the problem of seeking the optimal descent feasible direction S can be described as follows:**

$$\min\ [\nabla f(X_k)]^{\mathrm{T}} S$$

$$\text{s.t.}\quad [\nabla g_i(X_k)]^{\mathrm{T}} S < 0 \qquad (i \in I_k)$$

in which $S = [s_1, s_2, \cdots, s_n]^{\mathrm{T}}$

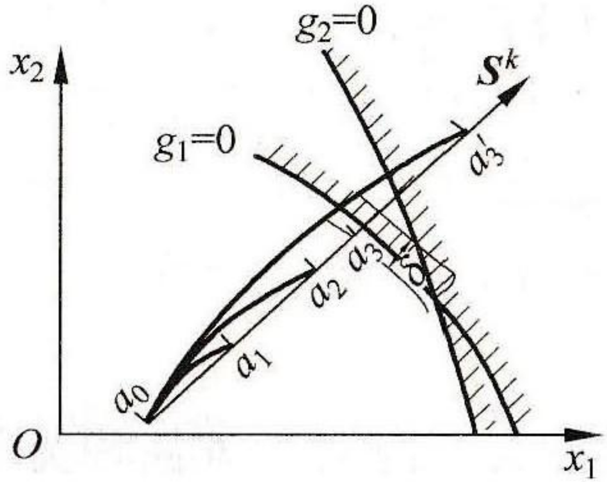
$$-1 \le s_j \le 1 \qquad (j = 1, 2, \cdots, n)$$

6.1.2 ODFD (III)

- **Since $\nabla f(X_k)$ and $\nabla g_i(X_k)$ ($i \in I_k$) are known vectors, both the objective function and the constraint functions are linear functions of variables $s_1, s_2, \cdots, s_n$. Thus this problem turns out to be a typical linear optimization problem. Hence the optimal descent direction S^* can be obtained by using the Simplex Method.**
- **Let S^* be the search direction, i.e., $S_k = S^*$, then a 1-D search can be conducted along this search direction to acquire the optimal solution of this nonlinear optimization problem, considering the constraints.**
- **This method is called "Feasible Direction Method" and This 1-D search is called "Constrained 1-D Search".**

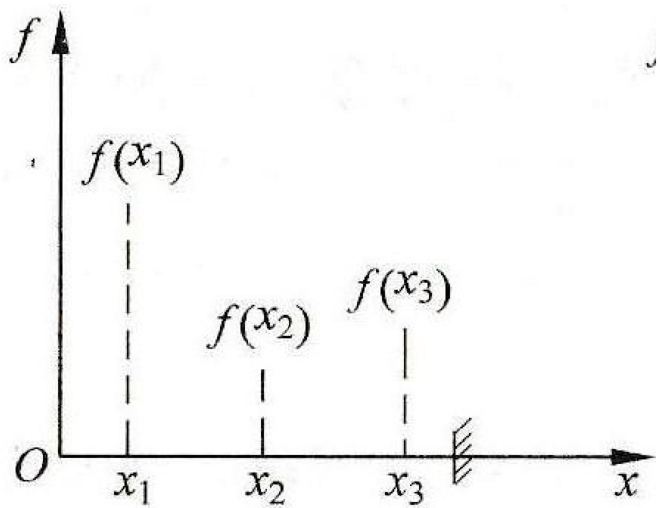
6.1.3 Constrained 1-D Search

- **Different from unconstrained 1-D search, the feasibility of every iteration point generated by the constrained 1-D search should be checked. If one or some constraint conditions are violated, then the step length ought to be reduced to make sure the new search point locate on the closest constraint boundary or within an allowed range of the constraint boundary. (as illustrated)**



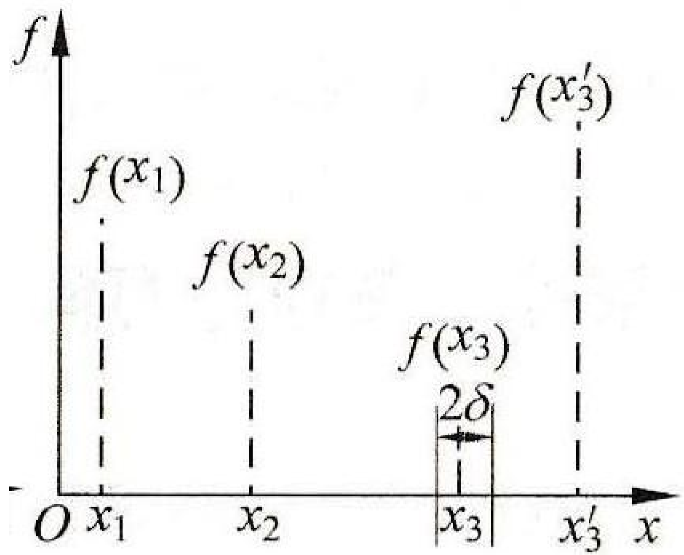
6.1.3 Constrained 1-D Search

- **If three adjacent search points are all feasible pints and a convex shape can be formed by these three points, the initial search interval can be determined. This initial search interval is then gradually shortened to acquire the minimum point of this constrained optimization problem. (as shown)**



6.1.3 Constrained 1-D Search

- **If the obtained search point is located within the allowance (±δ) of the constraint boundary, and the function value of this search point is smaller than that of the previous search point (as shown below), then this obtained point is the minimum point of this constrained optimization problem.**



6.1.4 Algorithm of FDM

1) The initial inner point X_0, convergence accuracy ε and the constraint allowance δ are provided. Set $k = 0$.

2) Identify the effective constraints of point X_k :

$$I_k(X_k, \delta) = \{u \mid -\delta \le g_u(X_k) \le \delta \ (u = 1, 2, \cdots, p)\}$$

3) Convergence Judgement:

When I_k is a null set and X_k is within the feasible region, if $\|\nabla f(X_k)\| \le \varepsilon$, then the algorithm can be terminated and $X^* = X_k$, $f^* = f(X_k)$; Otherwise, let $S_k = -\nabla f(X_k)$, go to Step 6.

When I_k is a non-null set, go to Step 4.

6.1.4 Algorithm of FDM

4) Convergence Judgement:

If X_k satisfies the K-T condition:

$$\nabla f(X_k) + \sum \lambda_u \nabla g_u(X_k) = 0 \qquad \lambda_u \geq 0$$

Then the algorithm can be terminated and there is $X^* = X_k$, $f^* = f(X_k)$; Otherwise, go to Step 5.

5) Solve the linear optimization problem:

$$\min\ [\nabla f(X_k)]^T S$$
$$\text{s.t.}\quad [\nabla g_u(X_k)] S \leq 0 \quad (u \in I_k)$$
$$-1 \leq s_j \leq 1 \quad (j = 1, 2, \cdots, n)$$

Then let $S_k = S^*$ = the optimal search direction.

6) Along the search direction S_k, conduct the 1-D constrained search to acquire X_{k+1}, let $k=k+1$, go to Step 2.

Example 6-1

Solve the following nonlinear constrained problem using the FDM:

$$\min f(X) = x_1^2 + x_2^2 - x_1x_2 - 2x_1 - 3x_2$$

$$\text{s.t.} \quad g_1(X) = x_1 + x_2 - 2 \le 0$$

$$g_2(X) = x_1 + 5x_2 - 5 \le 0$$

$$g_3(X) = -x_1 \le 0$$

$$g_4(X) = -x_2 \le 0$$

Solution to Example 6-1

(1) Choose $X_0 = [0, 0]^T$, then:

$f(X_0) = 0$, $\nabla f(X_0) = [-2, -3]^T$, $I_k = \{3, 4\}$

$\nabla g_3(X) = [-1, 0]^T$, $\nabla g_4(X) = [0, -1]^T$

Convergence judgement: at $X_0 = [0, 0]^T$

$\lambda_1 = -2$ and $\lambda_2 = -3$, K-T NOT satisfied, continue.

(2) Solve for the optimal direction at X_0:

$$\min \nabla f(X_0)^T S = -2s_1 - 3s_2$$

$$s.t. \quad -s_1 \le 0 , \quad -s_2 \le 0$$

$$-1 \le s_j \le 1 \qquad (j = 1, 2)$$

to have $S^* = [1, 1]^T$. Then, $S_0 = [1, 1]^T$.

Solution to Ex. 6-1 (Cont'd)

(3) Conduct a 1-D constrained search along the direction S_0 :

$$X_1 = X_0 + \alpha S_0 = [\alpha, \alpha]^T, \quad f(X_1) = \alpha_0^2 - 5\alpha_0$$

$$df(X_1)/d\alpha_0 = 0 \;\Rightarrow\; \alpha_0 = 5/2 \;\Rightarrow\; X_1 = [5/2, 5/2]^T .$$

Note that X_1 is within boundaries of constraints of $g_3(X)$ and $g_4(X)$, but out of those of $g_1(X)$ and $g_2(X)$, so it is NOT the constrained minimum point along S_0 .

Hence $X_1 = [\alpha, \alpha]^T$ is put into $g_1(X) = 0$ and $g_2(X) = 0$ to solve out $\alpha_1 = 1$ and $\alpha_2 = 5/6$. Thus, $X^* = [5/6, 5/6]^T$ is the constrained minimum point along the direction S_0.

Solution to Ex. 6-1 (Cont'd)

(4) Convergence judgement:

at $X_1 = [5/6, 5/6]^T$:

$$\nabla f(X_1) = \begin{bmatrix} -7/6 \\ -13/6 \end{bmatrix}, \qquad \nabla g_2(X_1) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\text{K}-\text{T condition} \;\Rightarrow\; \begin{bmatrix} -7/6 \\ -13/6 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Rightarrow\; \text{No Solution!}$$

$$\Rightarrow \; \text{X}_1 = \begin{bmatrix} 5/6 \\ 5/6 \end{bmatrix} \text{ is NOT the optimal solution of } f\text{ (X).}$$

Solution to Ex. 6-1 (Cont'd)

(5) Solve for the optimal direction at X_1 :

$$\min \nabla f(X_1)^T S = -(7/6)s_1 - (13/6)s_2$$

$$s.t. \quad s_1 + 5s_2 \leq 0$$

$$-1 \leq s_j \leq 1 \quad (j = 1, 2)$$

to obtain $S^* = [1, -0.2]^T$. Thus, $S_1 = [1, -0.2]^T$.

(6) Conduct a 1-D search along direction S_1 to get:

$$X_2 = [5/6 + \alpha_1,\ 5/6 - \alpha_1/5\,]^T, \quad df(X_2)/d\alpha_1 = 0 \quad \Rightarrow$$

$$\alpha_1 = 55/186,\ \ X_2 = [35/31, 24/31]^T,\ \ f(X_2) = -111/31$$

which satisfies all the constraints.

Solution to Ex. 6-1 (Cont'd)

(7) Convergence judgement:

at $X_2 = [35/31, 24/31]^T$:

$$\nabla f(X_2) = \begin{bmatrix} -16/31 \\ -80/31 \end{bmatrix}, \qquad \nabla g_2(X_2) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\text{K} - \text{T condition} \Rightarrow \begin{bmatrix} -16/31 \\ -80/31 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda = \frac{16}{31} > 0$$

$$\Rightarrow \text{X}_2 = \begin{bmatrix} 35/31 \\ 24/31 \end{bmatrix} \text{ is the optimal solution of } f(\text{X}).$$

$$\Rightarrow f_{\min} = f(\text{X}_2) = -\frac{111}{31}$$