Chapter 5 Linear Programming

- Linear Programming, a. k. a. Linear Optimization, is referred to as the case when both the objective function and constraints are linear functions of the variables.
- As shown in Chapter 1, the feasible region of the linear programming problem is an enclosed convex polygon or polytope. The optimal solution is often located at one or more vertex points.
- The mathematical model of the linear programming problem is also comprised by the objective function, variables, and the constraints. The constraints usually consist of a series of equalities or inequalities, as well as a series of non-negative constraints for all the variables.

5.1 Mathematical Formulation of Linear Programming

- **General Form:**

$$\min\ (\max)\ f(X) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \Sigma c_jx_j$$

$$(j = 1, 2, \cdots, n)$$

$$s.t. \quad \Sigma a_{ij}x_j = b_i \qquad (i = 1, 2, \cdots, m)$$

$$x_j \geq 0 \qquad (j = 1, 2, \cdots, n)$$

- **Vectorial Form:**

$$\min\ (\max)\ f(X) = C^T X$$

$$s.t. \quad AX = b$$

$$x_i \geq 0 \qquad (i = 1, 2, \cdots, n)$$

5.1 Mathematical Formulation of Linear Programming (II)

where: $X = [x_1, x_2, \cdots, x_n]^{\mathrm{T}}$ $\quad C = [c_1, c_2, \cdots, c_n]^{\mathrm{T}}$

$b = [b_1, b_2, \cdots, b_m]^{\mathrm{T}}$

$$A = A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A is the coefficient matrix; $AX = b$ are the constraint equations; $x_i \geq 0$ are the nonnegativity constraints.

5.1 Mathematical Formulation of Linear Programming (III)

- **In general, it should have $m < n$ since the constraint equations can have multiple solutions only when $m < n$ sustains. The goal of the linear optimization is to find the optimal one from these multiple solutions to make the objective function achieve the optimal value.**
- **In the standard math form of the linear programming problem, all the constraints ought to be equalities except the non-negative constraints for the variables.**
- **If in the practical problems, there exists other inequalities, then for each of them, a non-negative variable should be employed into these inequalities to make them equalities. This non-negative variables are called the slack variables.**

5.1 Mathematical Formulation of Linear Programming (IV)

e.g. (exempli gratia):

$$\max\ f(X) = 60x_1 + 120x_2$$

$$s.t.\quad 9x_1 + 4x_2 \le 360 \qquad 3x_1 + 10x_2 \le 300$$

$$4x_1 + 5x_2 \le 200 \qquad x_1, x_2 \ge 0$$

Introducing 3 new slack variables x_3, x_4, x_5, then the problem becomes:

$$\max\ f(X) = 60x_1 + 120x_2$$

$$s.t.\quad 9x_1 + 4x_2 + x_3 = 360 \qquad 3x_1 + 10x_2 + x_4 = 300$$

$$4x_1 + 5x_2 + x_5 = 200 \qquad x_1, x_2, x_3, x_4, x_5 \ge 0$$

5.2 Solution for the L. P. Problem

- The solutions for the linear programming problems can be classified as three categories:

1. **Basic solution**: only satisfies the constraint equations.
2. **Basic feasible solution**: satisfies both the constraint equations and the non-negative constraints for variables.
3. **Optimal solution**: the basic feasible solution which makes the objective function achieve the optimal value.

5.2.1 Basic Solution of L. P. Problems

- **The constraint of the linear programming problems actually is a linear equation set consisting of n variables and m equations ($n>m$). Since the number of variables is more than the number of equations, there are multiple solutions for this equation set.**
- **If setting n-m variables to zeros, then the constraint becomes an equation set with m variables and m equations. This equation set can be solved to get the values of these m variables. These m values and the n-m ZERO values comprise a solution of this equation set, which is referred to as the Basic Solution of the linear programming problem.**
- **In this solution, the n-m variables which have ZERO values are called Non-basic Variables, and the other m variables are called Basic Variables. Therefore, a basic solution is composed by n-m non-basic variables and m basic variables.**

5.2.1 Basic Solution (II)

- **The *n-m* non-basic variables can be chosen randomly and set as zero, then solve the other *m* basic variables to form a basic solution. Thus the number of basic solutions can be determined as:**

$$C_n^m = \frac{n!}{m!(n-m)!}$$

- **The augmented matrix can be obtained by combining the coefficient matrix A and the constant vector *b*:**

$$[A \mid b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

5.2.1 Basic Solution: Conversion

Make transformations to the augmented matrix such that it becomes:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & a_{1,m+1} & \cdots & a_{1n} & b_1^{'} \\ 0 & 1 & \cdots & 0 & a_{2,m+1} & \cdots & a_{2n} & b_2^{'} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{m,m+1} & \cdots & a_{mn} & b_m^{'} \end{bmatrix} \quad \textbf{(5-5)}$$

Let the n-m variables (from x_{m+1} to x_n) be non-basic variables, i.e., set their values to be zero, then the equation turns to be:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & a_{1,m+1} & \cdots & a_{1n} \\ 0 & 1 & \cdots & 0 & a_{2,m+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{'} \\ b_2^{'} \\ \vdots \\ b_m^{'} \end{bmatrix} \quad \textbf{(5-6)} \Rightarrow \textbf{Basic Solution } X_0 = \begin{bmatrix} b_1^{'} \\ b_2^{'} \\ \vdots \\ b_m^{'} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

5.2.1 Conversion of BS (II)

- **These transformations of the augmented matrix can be made via the Gaussian Elimination, of which the basic procedures are described as follows:**

1. Choose the principal element a_{lk} which needs to be changed to be "1", where subscripts *l* and *k* denoting the row and the column on which the principal element is, respectively;

2. Divide all the elements on the principal row by a_{lk} such that the principal element a_{lk}=1;

3. Make all the elements on the principal column zero except the principal element.

5.2.1 Conversion of BS (III)

- **The formula for Gaussian Elimination:**

$$
\begin{cases}
a_{l,j}' = \dfrac{a_{l,j}}{a_{l,k}} & (i = l) \\[2ex]
a_{l,j}' = a_{i,j} - a_{i,k}\dfrac{a_{l,j}}{a_{l,k}} & (i \neq l) \\[2ex]
b_l' = \dfrac{b_l}{a_{l,k}} & (i = l) \\[2ex]
b_i' = b_i - a_{i,k}\dfrac{b_l}{a_{l,k}} & (i \neq l)
\end{cases}
\qquad
\begin{matrix}
(i = 1, 2, \cdots, m) \\
(j = 1, 2, \cdots, n)
\end{matrix}
$$

- **Hence, a basic solution can be obtained by making *m* Gaussian eliminations to the augmented matrix. The principal element can be chosen from any column.**

5.2.1 Conversion of BS (IV)

- **Once a basic solution is acquired, another basic solution can be obtained via transformations on the augmented matrix. For example, in Eq.(5-5), choose a_{2m+1} as the new principal element and make some transformations, we can get:**

$$\begin{bmatrix} 1 & a_{12}'' & 0 & \cdots & 0 & 0 & a_{1,m+2}'' & \cdots & a_{1n}'' & b_1'' \\ 0 & a_{22}'' & 0 & \cdots & 0 & (1) & a_{2,m+2}'' & \cdots & a_{2n}'' & b_2'' \\ 0 & a_{32}'' & 1 & \cdots & 0 & 0 & a_{3,m+2}'' & \cdots & a_{3n}'' & b_3'' \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{m2}'' & 0 & \cdots & 1 & 0 & a_{m,m+2}'' & \cdots & a_{mn}'' & b_m'' \end{bmatrix} \Rightarrow \textbf{Basic Solution} \quad X_1 = \begin{bmatrix} b_1'' \\ 0 \\ b_3'' \\ \vdots \\ b_m'' \\ b_2'' \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- **The difference between X_1 and X_0 is that, x_2 has changed from a basic variable to a non-basic variable, while the non-basic variable x_{m+1} becomes a basic variable.**

5.2.1 Example

Ex: Solve the L.P. problem $\min f(X) = x_1 + x_2$

$$s.t.\quad 5x_1 + 4x_2 + 13x_3 - 2x_4 + x_5 = 30$$

$$x_1 + x_2 + 5x_3 - x_4 + x_5 = 8 \quad x_1, x_2, \cdots, x_5 \geq 0$$

Solution: Construct the augmented matrix

$$\begin{bmatrix} 5 & 4 & 13 & -2 & 1 & 30 \\ 1 & 1 & 5 & -1 & 1 & 8 \end{bmatrix}$$

Choose a_{11} and a_{22} as principal elements

$$\Rightarrow \begin{bmatrix} 1 & 0 & -7 & 2 & -3 & -2 \\ 0 & 1 & 12 & -3 & 4 & 10 \end{bmatrix}$$

The basic solution: $X_0 = [-2, 10, 0, 0, 0, 0]^T$, $f(X_0) = 8$

Basic variables: $x_1 = -2$, $x_2 = 10$;

Non-basic variables: $x_3 = x_4 = x_5 = 0$.

5.2.1 Example (Cont'd)

In the new augmented matrix, choose $a_{25} = 4$ as the principal element, make transformations to get:

$$\begin{bmatrix} 1 & 0.75 & 2 & -0.25 & 0 & 5.5 \\ 0 & 0.25 & 3 & -0.75 & (1) & 2.5 \end{bmatrix}$$

Therefore another basic solution can be acquired:

$$X_1 = \begin{bmatrix} 5.5 & 0 & 0 & 0 & 2.5 \end{bmatrix}^T, \qquad f(X_1) = 5.5$$

- **It can be seen that X_0 is a basic solution but NOT a basic feasible solution; X_1 is not only a basic solution but a basic feasible solution. Continue transforming can get more basic feasible solutions, including the optimal solution.**

5.2.2 Basic Feasible Solution

- **The basic feasible solution is a solution satisfying both constraint equations and non-negative constraints. The optimal solution lies in the basic feasible solutions.**
- **It can be proven that there are two cases for the acquirement of an initial basic feasible solution:**

1. If all the constraint equations are the inequalities with "$\leq$" except the non-negative constraints, and the components in the constant vector are also positive numbers, then a basic feasible solution can be acquired by introducing the slack variables and using them as the basic variables.

5.2.2 Basic Feasible Solution (II)

e.g., for the following constraints:

$$\begin{aligned} x_1 - x_2 + x_3 &\le 4 \\ x_1 + 2x_2 - x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned} \quad \Rightarrow$$

Introducing slack variables x_4 and x_5:

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 4 \\ x_1 + 2x_2 - x_3 + x_5 &= 8 \\ x_1, x_2, \ldots, x_5 &\ge 0 \end{aligned}$$

The corresponding augmented matrix is:

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 4 \\ 0 & 2 & -1 & 0 & 1 & 8 \end{bmatrix}$$

The solution is $X_0 = [0, 0, 0, 4, 8]^T$. Obviously it is a basic feasible solution where the basic variables are the slack variables are x_4 and x_5.

5.2.2 Basic Feasible Solution (III)

2. If there exists equalities in the constraints, a variable similar to the slack variable can be introduced into each inequality constraint, called the artificial variable. Then an assistant LP program can be established and a basic feasible solution can be obtained by solving this problem.

- The objective function of the assistant LP program is the sum of all the artificial variables. The constraints include the equality constraints with the artificial variables and the non-negative constraints for all variables (including the artificial variables).
- During the solving process of the assistant LP problem, when the value of the objective function is zero, the solution is found, where the basic feasible solution is the part without the artificial variables.

5.2.2 Basic Feasible Solution (IV)

e.g., for linear programming problem:

$$\min f(X) = x_1 + x_2 + x_3$$
$$\begin{aligned} s.t. \quad & x_1 - x_2 + 2x_3 = 4 \\ & x_1 + 2x_2 - x_3 = 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

After introducing the artificial variables, the assistant LP problem is set as:

$$\min \phi(X) = x_4 + x_5$$
$$\begin{aligned} s.t. \quad & x_1 - x_2 + 2x_3 + x_4 = 4 \\ & x_1 + 2x_2 - x_3 + x_5 = 8 \\ & x_1, x_2, \ldots, x_5 \geq 0 \end{aligned}$$

5.2.3 Conversion of BFS

- **Since basic feasible solutions are a part of basic solutions, the conversion of BFS can still be done using the same strategy, i. e., transformations via Gaussian Elimination.**
- **In order to obtain the optimal solution by the conversion, three problems need to be solved:**

(1) Optimal Condition: after conversion, the objective function value should not only decrease, but have the biggest reduction.

(2) Nonnegativity Condition: the solution after the conversion is still a basic feasible solution, i.e., the values of the constants are greater or equal to zero.

(3) Judgement of the optimal solution: need to judge if the solution is indeed an optimal solution.

5.2.3 Conversion of BFS (II)

1. Optimal Condition

The objective function of the LP problem can be written as:

$$f(X)=\sum_{i=1}^{m}c_i x_i+\sum_{j=m+1}^{n}c_j x_j$$

where, x_i represents the basic variable and x_j denotes the non-basic variable.

According to Eq.(5-6),each basic variable x_i can be presented by non-basic variables x_j as:

$$x_i=b_i-\sum_{j=m+1}^{n}a_{ij}x_j \qquad (i=1,\ 2,\cdots,m)$$

5.2.3 Conversion of BFS (III)

Thus, the objective function can be expressed as:

$$
\begin{aligned}
f(X) &= \sum_{i=1}^{m} c_i (b_i - \sum_{j=m+1}^{n} a_{ij} x_j) + \sum_{j=m+1}^{n} c_j x_j \\
&= \sum_{i=1}^{m} c_i b_i + \sum_{j=m+1}^{n} (c_j - \sum_{i=1}^{m} c_i a_{ij}) x_j \\
&= \sum_{i=1}^{m} c_i b_i + \sum_{j=m+1}^{n} \sigma_j x_j
\end{aligned}
$$

where, $\Sigma c_i b_i$ represents the objective function value before the conversion, and $\sigma_j = c_j - \Sigma c_i a_{ij}$ represents the Judgement Number of the column j.

5.2.3 Conversion of BFS (IV)

- **During the conversion of the basic solution, the essence of the transformations via Gaussian Elimination is the switch between a basic variable and a non-basic variable.**
- **During the conversion of the basic feasible solution, the difference is that the non- basic variable (with the value of zero) is required to become a non-negative basic variable.**
- **Among the n-m non-basic variables x_j, there is only one needs to become a non-negative basic variable, the others remains to be 0. In order to make the objective function value decrease after the conversion, the judgment number corresponding to this variable should be negative, i.e., $\sigma_j < 0$.**

5.2.3 Conversion of BFS (V)

- **Judgment Number Criterion: convert the non-basic variable with the Smallest judgment number into a basic variable. That is, choose the column with the Smallest judgment number to be the principal column for next transformation. The smaller the judgment number is, the more the function value decreases.**

$$\sigma_k = \min \{\sigma_j \mid \sigma_j < 0, j = 1, 2, \ldots, n\}$$

If all the judgment numbers are non-negative, i.e.:

$$\sigma_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

Then the function value can't decrease any more, which means the current solution is the optimal solution. Thus, the judgment number can be used to determine whether the current basic feasible solution is the optimal solution or not. If not, it can also be used to identify the principal column for next transformation.

5.2.3 Conversion of BFS (VI)

2. Non-negativity Condition

After converting one basic feasible solution to another one, all the components in the constant vector b', after the conversion, must be non-negative, i.e.:

$$b' = \frac{b_l}{a_{lk}} \geq 0 \quad \text{and} \quad b_i' = b_i - a_{ik}\frac{b_l}{a_{lk}} \geq 0 \quad (i = 1,\ 2, \cdots,\ m;\ \ i \neq l)$$

$$\Rightarrow \quad a_{lk} > 0 \qquad \frac{b_i}{a_{ik}} \geq \frac{b_l}{a_{lk}}$$

Hence, after the principal column k has been identified by the judgment number criterion, one can choose the principal row l according to the following criterion, to ensure the solution after conversion is still a basic feasible solution.

$$\frac{b_l}{a_{lk}} = \min\{\frac{b_i}{a_{ik}} \mid a_{ik} > 0 \quad (i = 1,2,\cdots,m)\}$$

5.3 The Simplex Method

- **The Simplex Method** is a LP algorithm based on the conversion of the basic feasible solution. In general the transformations are developed in a table called the simplex table.
- The simplex table is a transformation table based on the augmented matrix of the constraints. It contains all the information needed for solving the LP problem.

SIMPLEX TABLE		x_1	x_2	…	x_n	x_{n+1}	x_{n+2}	…	x_{n+m}	b_i
Basic Variable	Coefficient	c_1	c_2	…	c_n	0	0	…	0	c_0
x_{n+1}	0	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	b_{n+1}
x_{n+2}	0	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	b_{n+2}
:	:	:	:		:	0	0		0	:
x_{n+m}	0	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_{n+m}
Judgment number σ_j		σ_1	σ_2	…	σ_n	0	0	…	0	$f(X)$

5.3 The Simplex Method (II)

- **About the transformation in the simplex table:**

(1) Each simplex table corresponds to a basic feasible solution, which is comprised by *m* non-negative values of the basic variables and *n-m* zeros of the non-basic variables.

(2) When the judgment numbers in the bottom line of the simplex table are all non-negative, the current solution is the optimal solution of the LP problem. When there exists negative numbers, further transformation is needed.

5.3 The Simplex Method (III)

(3) The principal column k for next transformation is the column with the smallest judgment number; the principal row l is the row having the smallest quotient of b_i / a_{ik}, where a_{ik} is the positive coefficient in the principal column and b_i is the corresponding term in the constant vector.

(4) The judgment number for the basic variable is always zero; The judgment number for the non-basic variable equals $\sigma_j = c_j - \Sigma c_i a_{ij}$, where c_j is the coefficient on the top of the column and c_i is the coefficient on the left end. The number at the bottom right corner represents the function value f(X).

5.3 The Simplex Method (IV)

(5) When there are equalities in the constraints, an artificial variable can be introduced into each equality, and an assistant LP problem can be constructed with the sum of the artificial variables as the objective function.

(6) The assistant LP problem can be solved via the simplex table. When the number on the right bottom corner in the simplex table, i.e., the function value *f*(X), is equal to zero, the assistant problem is solved and a basic feasible solution of the original problem is found. Then the simplex table of the original problem can be established, and the optimal solution can be found by continuing the transformations on the simplex table.

5.3 Examples for Simplex Method

Ex 5-1: Solving the following linear programming problem using the simplex method.

$$\min f(X) = -60x_1 - 120x_2$$

$$\text{s.t.} \quad 9x_1 + 4x_2 \leq 360$$

$$3x_1 + 10x_2 \leq 300$$

$$4x_1 + 5x_2 \leq 200$$

$$x_1, x_2 \geq 0$$

5.3 Solution for Example 5-1 (I)

Solution:

1. Introduce the slack variables x_3, x_4, x_5, and the problem becomes:

$$\min f(X) = -60x_1 - 120x_2$$

$$\text{s.t.} \quad 9x_1 + 4x_2 + x_3 = 360$$

$$3x_1 + 10x_2 + x_4 = 300$$

$$4x_1 + 5x_2 + x_5 = 200$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

5.3 Solution for Example 5-1 (II)

2. Establish the initial simplex table:

Basic Variables	Coefficients	x_1	x_2	x_3	x_4	x_5	b_i
		-60	-120	0	0	0	0
x_3	0	9	4	1	0	0	360
x_4	0	3	(10)	0	1	0	300
x_5	0	4	5	0	0	1	200
σ_j		-60	-120	0	0	0	0

5.3 Solution for Example 5-1 (III)

3. The initial basic feasible solution can be obtained:

$$X_0 = [0,\ 0,\ 360,\ 300,\ 200]^T,\quad f(X_0) = 0$$

Since the judgments σ_1 and σ_2 are less than zero, X_0 is not the optimal solution.

$$\min\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\} = \sigma_2 = -120$$

$$\min\{b_i/a_{i,2} \mid a_{i,2} > 0\ \ (i=1, 2, 3)\} = b_2/a_{2,2} = 300/10 = 30$$

$$\Rightarrow\quad k = 2,\quad l = 2$$

Thus choose $a_{2,2}$ as the principal element for the next transformation.

5.3 Solution for Example 5-1 (IV)

4. Choose $a_{2,2}$ and the new simplex table becomes:

Basic Variables	Coeffic-ients	x_1	x_2	x_3	x_4	x_5	b_i
		-60	-120	0	0	0	0
x_3	0	7.8	0	1	-0.4	0	240
x_2	-120	0.3	1	0	0.1	0	30
x_5	0	(2.5)	0	0	-0.5	1	50
σ_j		-24	0	0	12	0	-3600

The corresponding basic feasible solution is:

$$X_1 = [0,\ 30,\ 240,\ 0,\ 50]^T, \qquad f(X_1) = -3600$$

Since $\sigma_1 = -24 < 0$, X_1 is not the optimal solution.

5.3 Solution for Example 5-1 (V)

5. min $\{b_i/a_{i,1} \mid a_{i,1} > 0 \ (i=1, 2, 3)\} = b_3/a_{3,1} = 50/2.5 = 20$

$\Rightarrow$ choose $a_{3,1} = 2.5$ as the principal element for next transformation and the new simplex table becomes:

Basic Variables	Coeffic-ients	x_1	x_2	x_3	x_4	x_5	b_i
		-60	-120	0	0	0	0
x_3	0	0	0	(1)	1.16	-3.12	84
x_2	-120	0	1	0	0.16	-0.12	24
x_1	-60	1	0	0	-0.2	0.4	20
σ_j		0	0	0	7.2	9.6	-4080

5.3 Solution for Example 5-1 (VI)

6. The corresponding basic feasible solution is:

$$X_2 = [20,\ 24,\ 84,\ 0,\ 0]^T, \qquad f(X_2) = -4080$$

Since all the judgment numbers are all non-negative, X_2 is the optimal solution.

After removing the slack variables, the optimal solution for the original problem is:

$$X^* = [20, 24]^T, \qquad f(X^*) = -4080$$

5.3 Another Example for Simplex

Ex 5-2: Solving the problem using the simplex method.

$$\min f(X) = x_1 + x_2$$

$$\text{s.t.}\quad 2x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 3x_2 + x_3 = 3$$

$$x_1, x_2, x_3 \geq 0$$

5.3 Solution for Example 5-2 (I)

Solution:

1. Introduce the artificial variables x_4 and x_5, establish the assistant problem as:

$$\min \phi(\underline{X}) = x_4 + x_5$$

$$\text{s.t.} \quad 2x_1 + x_2 + 2x_3 + x_4 = 4$$

$$3x_1 + 3x_2 + x_3 + x_5 = 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

5.3 Solution for Example 5-2 (II)

2. Establish the simplex table for the assistant problem:

Basic Variables	Coefficients	x_1	x_2	x_3	x_4	x_5	b_i
		0	0	0	1	1	0
x_4	1	2	1	2	1	0	4
x_5	1	(3)	3	1	0	1	3
σ_j		-5	-4	-3	0	0	7

The corresponding basic feasible solution is:

$$\underline{X}_0 = [0,\ 0,\ 0,\ 4,\ 3]^T$$

$\underline{X}_0$ is not the optimal solution. Choose $a_{2,1}$=3 as the principal element for the next transformation.

5.3 Solution for Example 5-2 (III)

3. Choose $a_{2,1}$ as the principal element for the next transformation, the new simplex table becomes:

Basic Variables	Coeffic-ients	x_1	x_2	x_3	x_4	x_5	b_i
		0	0	0	1	1	0
x_4	1	0	-1	4/3	1	-2/3	2
x_1	0	(1)	1	1/3	0	1/3	1
σ_j		0	1	-4/3	0	5/3	2

The corresponding basic feasible solution is:

$$\underline{X}_1 = [1,\ 0,\ 0,\ 2,\ 0]^T$$

$\underline{X}_1$ is not the optimal solution since $\sigma_3 = -4/3 < 0$.

5.3 Solution for Example 5-2 (IV)

4. $\min\{\sigma_k \mid \sigma_k < 0\} = \sigma_3 = -4/3$

   $\min\{b_i/a_{i,1} \mid a_{i,1} > 0 \ \ (i=1, 2)\} = b_1/a_{1,3} = 1.5$

$\Rightarrow$ choose $a_{1,3} = 4/3$ as the principal element for next transformation and the new simplex table becomes:

Basic Variables	Coeffic-ients	x_1	x_2	x_3	x_4	x_5	b_i
		0	0	0	1	1	0
x_3	0	0	-3/4	1	3/4	-1/2	3/2
x_1	0	1	5/4	0	-3/4	1/2	1/2
σ_j		0	0	0	1	1	0

5.3 Solution for Example 5-2 (V)

5. The corresponding basic feasible solution is:

$$\underline{X}_2 = [1/2,\ 0,\ 3/2,\ 0,\ 0]^T, \qquad \phi(\underline{X}_2) = x_4 + x_5 = 0$$

Now all the artificial variables and the objective function are all zeros, so $\underline{X}_2$ with the artificial variables removed is an initial basic feasible solution for the original LP problem.

That is, the initial basic feasible solution for the original problem is:

$$X_0 = [1/2,\ 0,\ 3/2]^T$$

5.3 Solution for Example 5-2 (VI)

6. Now solve the original LP problem:

The simplex table corresponding to X_0 is:

Basic Variables	Coefficients	x_1	x_2	x_3	b_i
		1	1	0	0
x_3	0	0	-3/4	1	3/2
x_1	1	1	5/4	0	1/2
σ_j		0	-1/4	0	1/2

Since σ_2 = -1/4 < 0, X_0 is not the optimal solution.

It is easy to choose $a_{2,2}$ as the element for next transf.

5.3 Solution for Example 5-2 (VII)

3. The new simplex table is:

Basic Variables	Coefficients	x_1	x_2	x_3	b_i
		1	1	0	0
x_3	0	3/5	0	1	9/5
x_2	1	4/5	1	0	2/5
σ_j		1/5	0	0	2/5

The corresponding solution is:

$$X_1 = [0,\ 0.4,\ 1.8]^T \qquad f(X_1) = 0.4$$

$X^* = X_1$ is the optimal solution for the original problem since all the judgment numbers are non-negative now.