We then propose the following control algorithm

$$u = \hat{k}_1(t)x + \hat{k}_2(t)u_c(t) - \hat{\theta}_1(t)\phi_1(x) + \hat{\theta}_2(t)\phi_2(x), \tag{2}$$

where $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are estimates of θ_1 and $\frac{\theta_2}{b}$, respectively.

By assuming that $\hat{k}_1$ and $\hat{k}_2$ are constants and by ignoring the terms associated with θ_1 and θ_2, we can get the following matching condition

$$k_1^* = \frac{a_{ref} + a}{b}, \qquad k_2^* = \frac{b_{ref}}{b}. \tag{3}$$

Using (2), (1) can be written as

$$\begin{aligned} \dot{x} &= a_{ref}x + b_{ref}u_c - a_{ref}x - b_{ref}u_c - ax + b\hat{k}_1x + b\hat{k}_2u_c \\ &\quad - b\bar{\theta}_1\phi_1(x) + b\bar{\theta}_2\phi_2(x) \\ &= a_{ref}x + b_{ref}u_c + b\bar{k}_1x + b\bar{k}_2u_c - b\bar{\theta}_1\phi_1(x) + b\bar{\theta}_2\phi_2(x), \end{aligned} \tag{4}$$

where $\bar{\theta}_1 \triangleq \hat{\theta}_1 - \theta_1$, $\bar{\theta}_2 \triangleq \hat{\theta}_2 - \theta_2/b$, $\bar{k}_1 = \hat{k}_1 - k_1$, and $\bar{k}_2 = \hat{k}_2 - k_2$.

Define the tracking error

$$e = x - x_{ref}. \tag{5}$$

Then the error dynamics is given by

$$\dot{e} = a_{ref}e + b\bar{k}_1x + b\bar{k}_2u_c - b\bar{\theta}_1\phi_1(x) + b\bar{\theta}_2\phi_2(x). \tag{6}$$

Note that $b > 0$. Consider the following Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{b}{2\gamma_1}\bar{k}_1^2 + \frac{b}{2\gamma_2}\bar{k}_2^2 + \frac{b}{2\gamma_3}\bar{\theta}_1^2 + \frac{b}{2\gamma_4}\bar{\theta}_2^2. \tag{7}$$

where γ_i, $i = 1, \ldots, 4$, are positive constants. Its time derivative along (6) is given by

$$\begin{aligned} \dot{V} &= e\dot{e} + \frac{b}{\gamma_1}\bar{k}_1\dot{\bar{k}}_1 + \frac{b}{\gamma_2}\bar{k}_2\dot{\bar{k}}_2 + \frac{b}{\gamma_3}\bar{\theta}_1\dot{\bar{\theta}}_1 + \frac{b}{\gamma_4}\bar{\theta}_2\dot{\bar{\theta}}_2 \\ &= a_{ref}e^2 + b\bar{k}_1xe + b\bar{k}_2u_ce - b\bar{\theta}_1\phi_1(x)e + b\bar{\theta}_2\phi_2(x)e \\ &\quad + \frac{b}{\gamma_1}\bar{k}_1\dot{\bar{k}}_1 + \frac{b}{\gamma_2}\bar{k}_2\dot{\bar{k}}_2 + \frac{b}{\gamma_3}\bar{\theta}_1\dot{\bar{\theta}}_1 + \frac{b}{\gamma_4}\bar{\theta}_2\dot{\bar{\theta}}_2 \\ &= a_{ref}e^2 + \frac{b}{\gamma_1}\bar{k}_1(\dot{\bar{k}}_1 + \gamma_1xe) + \frac{b}{\gamma_2}\bar{k}_2(\dot{\bar{k}}_2 + \gamma_2u_ce) \\ &\quad + \frac{b}{\gamma_3}\bar{\theta}_1(\dot{\bar{\theta}}_1 - \gamma_3\phi_1(x)e) + \frac{b}{\gamma_4}\bar{\theta}_2(\dot{\bar{\theta}}_2 + \gamma_4\phi_2(x)e). \end{aligned} \tag{8}$$

(a) Note that the pair $(A, B\Lambda)$ is controllable, there exists a matrix K^* such that $A + B\Lambda K^*$ is Hurwitz. Since the control objective is to regulate x towards zero, the reference model can be designed as

$$\dot{x}_{ref} = A_{ref}x_{ref}, \tag{14}$$

where $A_{ref} \triangleq A + B\Lambda K^*$ is Hurwitz.
We propose the following control algorithm

$$u = \widehat{K}(t)x - \widehat{\Theta}^T\Phi(x), \tag{15}$$

where $\widehat{\Theta}$ is the estimate of Θ.
Using (15), we obtain

$$\begin{aligned}\dot{x} &= A_{ref}x - A_{ref}x + Ax + B\Lambda\widehat{K}(t)x - B\Lambda\widetilde{\Theta}^T\Phi(x)\\ &= A_{ref}x + B\Lambda\widetilde{K}(t)x - B\Lambda\widetilde{\Theta}^T\Phi(x),\end{aligned} \tag{16}$$

where $\widetilde{\Theta} \triangleq \widehat{\Theta} - \Theta$ and $\widetilde{K} \triangleq \widehat{K} - K^*$.
Since A_{ref} is Hurwitz, there exists a $P > 0$ such that

$$PA_{ref} + A_{ref}^T P = -Q < 0.$$

We then consider the following Lyapunov function candidate

$$V = x^T Px + tr(\widetilde{K}^T|\Lambda|\widetilde{K}\Gamma_1^{-1}) + tr(\widetilde{\Theta}|\Lambda|\widetilde{\Theta}^T\Gamma_2^{-1}) \tag{17}$$

where Γ_1 and Γ_2 are positive definite constant matrixes, and

$$|\Lambda| \triangleq \Lambda sgn(\Lambda) = \Lambda \begin{pmatrix} sgn(\lambda_1) & & \\ & \ddots & \\ & & sgn(\lambda_m) \end{pmatrix}.$$

Its time derivative along (16) is given by

$$\begin{aligned}\dot{V} =& -x^T Qx + 2tr(\widetilde{K}^T\Lambda B^T Pxx^T) - 2tr(\widetilde{\Theta}\Lambda B^T Px\Phi^T)\\ &+ 2tr(\widetilde{K}^T|\Lambda|\dot{\widehat{K}}\Gamma_1^{-1}) + 2tr(\widetilde{\Theta}|\Lambda|\dot{\widehat{\Theta}}^T\Gamma_2^{-1})\end{aligned} \tag{18}$$

We then choose the adaptive updating laws as

$$\dot{\widehat{K}} = -sgn(\Lambda)B^T Pxx^T\Gamma_1, \tag{19}$$

$$\dot{\widehat{\Theta}} = \Gamma_2\Phi(x)x^T PBsgn(\Lambda). \tag{20}$$

As a result, we have

$$\dot{V} = -x^T Qx \le 0, \tag{21}$$

Solutions:

(a) *Barbalat's Lemma: If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\lim_{t\to\infty} \dot{f}(t) = 0$.*
Proof of Corollary 0.1:
Define

$$f(t) = \int_0^t \|x(\tau)\|^2 d\tau$$

It follows from the fact $x \in \mathbb{L}_2$ that $f(t)$ has a finite limit and from the fact $x, \dot{x} \in \mathbb{L}_\infty$ that $\dot{f}$ is uniformly continuous. We can then get from Barbalat's Lemma that $\lim_{t\to\infty} \dot{f}(t) = 0$.

(b) *Define $d(t) = \max_{t\geq 0}\{d_1(t), d_2(t), d_3(t)\}$ and $\phi(x) = 1 + x + x^2$. We have*

$$\dot{x} = u + d(t)\phi(x).$$

Clearly, $\|d(t)\| \leq d_{\max}$. We propose the following control algorithm

$$\begin{aligned} u =& - kx - \hat{d}(t)sgn(x)\|\phi(x)\|, \\ \dot{\hat{d}}(t) =& \gamma\|x\|\|\phi(x)\|, \end{aligned}$$

where γ is a positive constant.
Consider the following Lyapunov function candidate

$$V = \frac{1}{2}x^2 + \frac{1}{2\gamma}(\hat{d}(t) - d_{max})^2. \tag{29}$$

Its derivative can be written as

$$\begin{aligned} \dot{V} =& x\dot{x} + \frac{1}{\gamma}(\hat{d}(t) - d_{max})\dot{\hat{d}}(t) \\ =& - kx^2 - \hat{d}(t)\|x\|\|\phi(x)\| + d(t)x\phi(x) + (\hat{d}(t) - d_{\max})\|x\|\|\phi(x)\| \\ \leq& - kx^2 \leq 0. \end{aligned}$$

Therefore, we can get that $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ and $\dot{x} \in \mathbb{L}_\infty$, then $\lim_{t\to\infty} x(t) =$ 0.

Final Examination-Standard Solutions

May 11, 2016

1. (20 points) Answer the following questions:

   (a) What are the definitions of indirect and direct adaptive control?

   (b) What are the four methods for robust adaptive control mentioned in our class?

   Solutions:

   (a) In indirect adaptive control, the plant parameters are estimated online and are used to calculate the controller parameters. In direct adaptive control, the plant model is parameterized in terms of the desired controller parameters, which are then estimated directly without intermediate calculations involving plant parameter estimates.

   (b) The four methods for robust adaptive control include dead-zone modification, σ-modification, e-modification, and projection-based design.

2. (20 points) Consider the first-order plant

$$\dot{x} = -ax + b[u + \theta_1\phi_1(x)] - \theta_2\phi_2(x),$$

where a, b, θ_1 and θ_2 are unknown constants with $b > 0$, while $\phi_1(x)$ and $\phi_2(x)$ are Lipschitz-continuous in x.

Design u, such that all signals in the closed-loop system are bounded and x tracks the state x_{ref} of the following reference model given by

$$\dot{x}_{ref} = a_{ref}x_{ref} + b_{ref}u_c(t),$$

where $a_{ref} < 0$ and b_{ref} are known, $u_c(t)$ is the input command which is bounded and piecewise continuous.

Solutions:

The plant can be written as

$$\dot{x} = -ax + b[u + \theta_1\phi_1(x) - \frac{\theta_2}{b}\phi_2(x)]. \tag{1}$$

We propose the following adaptive updating laws

$$\dot{\tilde{k}}_1 = -\gamma_1 x e, \tag{9}$$
$$\dot{\tilde{k}}_2 = -\gamma_2 u_c e, \tag{10}$$
$$\dot{\tilde{\theta}}_1 = \gamma_3 \phi_1(x) e, \tag{11}$$
$$\dot{\tilde{\theta}}_2 = -\gamma_4 \phi_2(x) e. \tag{12}$$

We then get from (8) that

$$\dot{V} = a_{ref} e^2 \leq 0, \tag{13}$$

which implies that $V(t) \leq V(0), \forall t \geq 0$, and consequently, $e, \tilde{k}_1, \tilde{k}_2, \tilde{\theta}_1, \tilde{\theta}_2 \in \mathbb{L}_\infty$. Since $a_{ref} < 0$ and $u_c \in \mathbb{L}_\infty$, we have $x_{ref} \in \mathbb{L}_\infty$. Thus $x \in \mathbb{L}_\infty$ and thus $\phi_1(x)$ and $\phi_2(x)$ are bounded since they are Lipschitz continuous in x. We can then get from (2) that $u \in \mathbb{L}_\infty$ and get from (6) that $\dot{e} \in \mathbb{L}_\infty$. Overall, all signals in the system are bounded.

On the other hand, note that $\ddot{V} = 2a_{ref} e\dot{e} \in \mathbb{L}_\infty$. We can get from Barbalat's Lemma that $\lim_{t\to\infty} \dot{V}(t) = 0$, i.e., $\lim_{t\to\infty} e(t) = 0$.

3. (30 points) Consider a linear system with nonlinear matched uncertainties in the form

$$\dot{x} = Ax + B\Lambda[u + \Theta^T \Phi(x)] + \varepsilon(t),$$

where $x \in \mathbb{R}^{n\times n}$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\Lambda^{m\times m}$, $\Theta \in \mathbb{R}^{N\times m}$ are constant matrices, and $\varepsilon(t) \in \mathbb{R}^n$ is the disturbance. Assume that the pair $(A, B\Lambda)$ is controllable. $\Phi(x) = (\phi_1(x), \dots \phi_n(x))^T \in \mathbb{R}^N$ denotes the known regressor vector, whose components $\phi_i(x)$ are assumed to be Lipschitz-continuous in x.

(a) Assume that $\varepsilon(t) = 0$, B is known, while A and Λ are unknown. In addition, it is assumed that Λ is diagonal with m nonzero diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_m$, and the signs of all λ_i are known. Design and analyze a directed MRAC scheme that can stabilize the system and regulate x towards zero.

(b) Assume that $\|\varepsilon(t)\| \leq \varepsilon_f$, $\forall t > 0$, where $\varepsilon_f > 0$, and A, B, Λ are all known, while Λ is a positive diagonal matrix. Design a σ-modification robust control algorithm that can stabilize the system and regulate x towards the neighborhood of zero.

Solutions:

which implies that $V(t) \le V(0)$, $\forall t \ge 0$, *and consequently*, $x, \widetilde{K}, \widetilde{\Theta} \in \mathbb{L}_\infty$. *We can get that* $\Phi(x) \in \mathbb{L}_\infty$ *since it is Lipschitz continuous in* x. *We can then get from* (15) *that* $u \in \mathbb{L}_\infty$ *and get from* (16) *that* $\dot{x} \in \mathbb{L}_\infty$. *Overall, all signals in the system are bounded.*

On the other hand, note that $\ddot{V} = -2x^T Q\dot{x} \in \mathbb{L}_\infty$. *We can get from Barbalat's Lemma that* $\lim_{t\to\infty} \dot{V}(t) = 0$, *i.e.*, $\lim_{t\to\infty} x(t) = 0$.

(b) *Since* A, B, Γ *are all known, we propose the following control algorithm*

$$u = K^* x - \widehat{\Theta}^T \Phi(x), \tag{22}$$

where $\widehat{\Theta}$ *is the estimate of* Θ.

Using (22), *we then obtain*

$$\begin{aligned} \dot{x} &= A_{ref}x - A_{ref}x + Ax + B\Lambda K^* x - B\Lambda\widetilde{\Theta}^T\Phi(x) + \varepsilon(t) \\ &= A_{ref}x - B\Lambda\widetilde{\Theta}^T\Phi(x) + \varepsilon(t). \end{aligned} \tag{23}$$

Motivated by the results in (a), we proposing the following the adaptive updating laws with σ*-modification*

$$\dot{\widehat{\Theta}} = [\Phi(x)x^T PB\Lambda - \sigma\widehat{\Theta}]\Gamma, \tag{24}$$

where σ *is a positive constant.*

We then consider the following Lyapunov function candidate

$$V = x^T Px + tr(\widetilde{\Theta}\Gamma^{-1}\widetilde{\Theta}^T) \tag{25}$$

where Γ *is positive definite constant matrix.*

Its time derivative along (23) *is given by*

$$\begin{aligned} \dot{V} =& -x^T Qx - 2tr(\Phi x^T PB\Lambda\widetilde{\Theta}^T) + 2x^T P\varepsilon(t) \\ &+ 2tr(\dot{\widehat{\Theta}}\Gamma^{-1}\widetilde{\Theta}^T) \\ =& -x^T Qx - 2\sigma tr(\widehat{\Theta}\widetilde{\Theta}^T) + 2x^T P\varepsilon(t). \end{aligned} \tag{26}$$

Note that

$$\begin{aligned} -2tr(\widehat{\Theta}\widetilde{\Theta}^T) &= -2tr((\Theta + \widetilde{\Theta})\widetilde{\Theta}^T) \\ &= -2tr(\Theta\widetilde{\Theta}^T) - 2tr(\widetilde{\Theta}\widetilde{\Theta}^T) \\ &\le tr(\Theta\Theta^T) + tr(\widetilde{\Theta}\widetilde{\Theta}^T) - 2tr(\widetilde{\Theta}\widetilde{\Theta}^T) \\ &= -tr(\widetilde{\Theta}\widetilde{\Theta}^T) + tr(\Theta\Theta^T), \end{aligned}$$

$\lambda_{min}(Q)\|x\|^2 \le x^T Q x \le \lambda_{max}(Q)\|x\|^2$

and

$$\begin{aligned} 2x^T P\varepsilon(t) &\le 2\lambda_{\max}(P)\|x\|\|\varepsilon(t)\| \\ &\le \frac{\lambda_{\min}(Q)}{2}x^T x + \frac{2\lambda^2_{\max}(P)\varepsilon_f^2}{\lambda_{\min}(Q)} \end{aligned}$$

$\le \frac{\lambda_{min}(Q)\|x\|^2}{2} + \frac{2\lambda^2_{max}(P)\varepsilon_f^2}{\lambda_{min}}$

Therefore, we have

$\lambda_{\min}(P)$

$$\begin{aligned}\dot V \le& -\frac{\lambda_{\min}(Q)}{2}x^T x - \sigma tr(\tilde\Theta\tilde\Theta^T) + \sigma tr(\Theta\Theta^T) + \frac{2\lambda_{\max}^2(P)\varepsilon_f^2}{\lambda_{\min}(Q)} \\ \le& -\beta V + C,\end{aligned} \tag{27}$$

where

$$\beta \triangleq \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\min}(P)}, \sigma\lambda_{\min}(\Gamma)\right\},$$

$$C \triangleq \sigma tr(\Theta\Theta^T) + \frac{2\lambda_{\max}^2(P)\varepsilon_f^2}{\lambda_{\min}(Q)}.$$

We then have

$$V(t) \le e^{-\beta t}V(0) + C(1 - e^{-\beta t}). \tag{28}$$

Clearly, V(t) is bounded and thus $x, \tilde\Theta \in \mathbb{L}_\infty$.

4. (30 points) We denote by $\|x\|$ the absolute value of x if x is a scalar and the Euclidean norm of x if x is a vector. For functions of time, the L_p norm is given by

$$\|x\|_p = \left(\int_0^\infty \|x(\tau)\|^p d\tau\right)^{\frac{1}{p}},$$

for $p \in [1, \infty)$, while

$$\|x\|_\infty = \sup_{t\ge 0}\|x(t)\|.$$

We say that $x \in \mathbb{L}_p$ when $\|x\|_p < \infty$.

(a) Write down Barbalat's lemma and use it to prove the following corollary.

Corollary 0.1 *If $x \in \mathbb{L}_2 \bigcap \mathbb{L}_\infty$ and $\dot x \in \mathbb{L}_\infty$, then* $\lim_{t\to\infty} x(t) = 0$.

(b) Consider the following first-order system

$$\dot x = u + d_1(t) + d_2(t)x + d_3(t)x^2,$$

where $d_1(t), d_2(t)$, and $d_3(t)$ are time-varying continuous functions satisfying

$$\max_{t\ge 0}\{\|d_1(t)\|, \|d_2(t)\|, \|d_3(t)\|\} \le d_{\max},$$

for some unknown positive constant $d_{\max}$. Design a control algorithm combing the adaptive and sliding control to stabilize the system and regulate x towards to zero. (Use the corollary in (a) to prove the result.)