

1. Working With Data

1.1 Introduction

- Programming language - a way of communicating a set of instructions to a computer.
- program - the *text of the instructions* (programming source code/ code) we wish to tell the computer to execute.
- language
 - syntax (format) - the rules governing what constitutes a valid program in the language
 - semantics (meaning) - specifies what the computer should do for each instruction
- Python interpreter
 - takes programs written in the Python programming language and executes the instructions
 - a mediator between the programmer and the computer hardware
- writing code
 - write full Python programs and save them
 - run the Python interpreter in an *interactive mode* (**Python console/ Python shell**)
 - benefits - get quick feedback
 - drawbacks - interactions are lost every time

1.2 Using the Python Console

- console prompt >>>
- expression
 - a piece of Python code that produces a value
 - **evaluating** - calculating the value of an expression
- literal
 - the simplest kind of Python expression

- a piece of code that represents the exact value as written
- **operator**
 - a symbol that represents a specific computation to perform
 - **binary** operators - used to combine two expressions together
- the expression `4 * 5`
 - two smaller expressions, the literals 4 and 5
 - operator *
- error message
 - something went wrong when the interpreter tried to execute the code
 - syntax error `SyntaxError`

1.3 Representing Data I: Numbers

- **data type**
 - allowed **values** for a piece of data of that type
 - allowed **operations** we perform on a piece of data of that type
- **abstract data types** - independent of programming language
- numeric data
 - natural number - $\mathbb{N} = \{0, 1, 2, \dots\}$ → **0 is a natural number**
 - integer - $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
 - real number - \mathbb{R}
 - rational number - $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$
 - irrational number - $\overline{\mathbb{Q}} = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$
- data type
 - `int` - represent natural numbers and integers
 - `float` - represent arbitrary real numbers
 - cannot always represent exactly
 - approximates the value of the real number
- arithmetic operations

- **modulo operation** % - produces the remainder
- power **
- **floored/ integer** division //

 - **rounded down** the result to the nearest integer (quotient of dividing x by y)
 - **always** get an **integer**

- **exact** division / - get a **float** number (**always**)
- comparison operations
 - equality ==
 - inequality !=
 - greater or equal >=
 - less or equal <=
- int and float
 - for ints, all operations **except** / always produce a int
 - for floats, all operations **including** // always produce a float
 - for int and float, all operations always produce a float
- different data type but equal values

```
>>> 3.0 == 3
True
```

1.4 Representing Data II: Booleans and Strings

- **boolean** - a value from the set {True, False}
 - bool - only two literal values: True, False
 - operations
 - **not** - reverses the value of a boolean (\neg)
 - **and** - produces **True** when **both** of the values are **True**, and False otherwise (\wedge)
 - **or** - produces **True** when **at least one** of the values is **True**, and False otherwise (\vee) \leftarrow inclusive or

- **inclusive or** - produces **True** when **both** of the given boolean values are **True**
- exclusive or - produces **False** when **both** of the given boolean values are **True**
- **string** - a sequence of characters
 - str - a sequence of characters surrounded by single-quotes ''
 - operations
 - **string equality** - use ==
 - uppercase and lowercase letters are not equal
 - exact match of all characters in both strings
 - **substring search** - in
 - every string is a substring of itself
 - the **empty string** is a substring of every string

```

>>> 'm' in 'computer' # single character search
True
>>> 'x' in 'computer'
False
>>> 'put' in 'computer' # multiple character search
True
>>> 'pur' in 'computer' # False, because characters must appear
consecutively
False
>>> 'computer' in 'computer' # every string is a substring of
itself
True
>>> '' in 'computer' # the empty string is a substring of every
string
True

```

- **string concatenation** - use +

```
>>> 'David' + 'Mario'  
'DavidMario'  
>>> 'David ' + 'Mario' # Extra space after David  
'David Mario'  
>>> 'Mario' + 'David ' # Order matters!  
'MarioDavid '
```

- **string indexing** $s[i]$ - produces the $(i + 1)$ -th character of a string $s \leftarrow$ index starts at 0

1.5 Representing Data III: Collections

- **set** - a collection of zero or more distinct values, where order does not matter
 - set - **not duplicated** elements, **no order**
 - operations
 - **set equality** - use ==, contain the exact same elements
 - **subset** - {subset} \leq {set}
 - **element checking** - in
- **list** - a sequence of zero or more values that may **contain duplicates**
 - list - may **contain duplicated** elements, a **specified order**
 - operations
 - list equality - use ==
 - element checking - in
 - list concatenation - use +
 - list indexing - list[i] (i starts at 0)
- mapping - a collection of association pairs, where each pair consists of a **key** and **associated value** for that key
 - dict - **key** must be **unique**, a key cannot exist in the mapping without a corresponding value
 - operations
 - mapping equality - use ==

- **key checking** - in (WARNING: cannot check value).
- key lookup - dict[key]
- **homogeneous** collection - every element has the same type
heterogeneous collection - there are elements of different types
- empty collection
 - list - []
 - dict - {}
 - set - set()

Abstract data type	set	list	mapping
Python data type	set	list	dict
Description	collection of elements	sequence of elements	collection of association(key-value) pairs
Example	{1,2,3}	[1,2,3]	{1: 'one', 2:'two'}
Empty	set()	[]	{}
Order matters?	no	yes	no
May contain duplicates?	no	yes	no duplicate keys, but possibly duplicate values
Definition of homogeneous	all elements have same type	all elements have same type	all keys have same type and all values have same type
Equality checking	==	==	==
Element checking	in	in	in (for keys only)

Abstract data type	set	list	mapping
Other operations	subset checking <=	concatenation +, indexing...[...]	key lookup ...[...]

1.6 Variable

- **variable** - a piece of code consisting of a name that refers to a value
- **statement** - a piece of code representing an instruction to the computer
 - **expression** - every expression is a statement
 - **assignment statement**

```
<variable> = <expression>
```

- evaluated the expression on the right-hand side
- the value is assigned to the variable on the left-hand side
- **memory model** - a structured representation of variables and data in a program
- **value-based** memory model - represent the association between each variable and their associated value

Variable	Value
distance	2.34345
total_distance	10

1.7 Comprehension

- **set comprehension**

```
{<expression> for <variable> in <collection>
    {x ** 2 for x in numbers}
== {1 ** 2, 2 ** 2, 3 ** 2, 4 ** 2, 5 ** 2} # Replacing x with 1, 2, 3,
4, and 5.
```

- **list comprehension**

```
[ <expression> for <variable> in <collection> ]
>>> [x + 4 for x in {10, 20, 30}]
[14, 24, 34]
>>> [x * 3 for x in [100, 200, 300]]
[300, 600, 900]
```

- when a list comprehension generates elements from a set, the results can be unexpected

```
>>> [x for x in {20, 10, 30}]
[10, 20, 30]
```

- **dictionary comprehension**

```
{ <key_expr>: <value_expr> for <variable> in <collection> }
>>> {x : x ** 2 + 1 for x in {1, 2, 3, 4, 5}}
{1: 2, 2: 5, 3: 10, 4: 17, 5: 26}
```

- `range(start, end)` - produce a collection of the numbers from `start` to `end-1` ← include start number but exclude the end number
- **Cartesian product** - $A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$
 - if $A = \{1, 2, 3\}$ and $B = \{10, 20, 30\}$, then
 $A \times B = \{(1, 10), (1, 20), (1, 30), (2, 10), (2, 20), (2, 30), (3, 10), (3, 20), (3, 30)\}$

```
>>> nums1 = [1, 2, 3]
>>> nums2 = [10, 20, 30]
>>> [(x, y) for x in nums1 for y in nums2]
[(1, 10), (1, 20), (1, 30), (2, 10), (2, 20), (2, 30), (3, 10), (3,
20), (3, 30)]
```

2. Function

2.1 Built-In Functions

- **function** - a mapping from elements in the **domain** to elements in the **codomain**
- function call
 - **argument** - input expressions
 - arguments are passed to the function when evaluating a function call
 - a function call returns the value when producing its output value

```
<function>(<argument>, <argument>, ...)
```

- built-in functions
 - `abs(x)` - returns the absolute value
 - `round(x)` - returns the `int` that equals `x` rounded to the nearest integer
 - `round(x,d)` - returns the `float` value of `x` rounded to `d` decimal places
 - `len(x)` - returns the size of its input
 - `sum(x)` - takes a collection of numbers and returns the sum of the numbers
 - `sorted(x)` - takes a collection and returns a sorted `list` (default ascending order)
 - `type` - returns its data type
 - `help` - displays help documentation for the argument
 - `max` (similar `min`)
 - two or more inputs - inputs must be numeric, returns the largest one

- a single argument - a non-empty collection of numbers, returns the largest number in the collection

```
>>> max(2, 3)
3
>>> max({2, 3})
3
```

2.2 Define Function

- function **header**
 - function's **name**
 - the number and type of **parameters**
 - function's **return type**
 - **type contract** - restriction of types

```
def <function_name>(<parameter_name>: <parameter_type>) ->
<return_type>:

    #multiple parameter
    def <function_name>(<para1_name>: <para1_type>, <para2_name>:
        <para2_type>...) -> <return_type>:
```

- function **docstring** - communicate what the function does
 - start and end with """
 - function description
 - doctest examples
- function body - the code that is executed
 - can call another function inside the function body
 - return statement

```
return <expression>
```

- function call
 - evaluate the argument, assign its value to the function parameter
 - evaluate the function body, stop and return

```
def my_function (para_1: type_1, para_2: type_2) -> return_type
    # docstring begin
    """
    This function does ... # function description
    >>> my_function(x, y)    # doctest example
    ...
    """
    # doctrine end

    <statement> # function body
    return ...    #return statement
```

2.3 Local Variables & Function Scope

- **local variable** - limited to the **function body** (scope: function body)
 - can **only** be accessed in the function body it is **defined**
- scope - the places where that variable can be accessed
- allow using the **same** variable name in two different scopes (**different functions**)

2.4 Methods

- function
 - **method** - a function that is defined as part of a data type
 - top-level functions - functions are not methods
- string method
 - `str.lower()` - returning a new string uppercase turned into lowercase

- `str.split()` - splits a string into words contained in a list
- set method
 - `set.union()` - return the new set $C = A \cup B$
 - `set.intersection()` - return a new set $C = A \cap B$
- list method
 - `list.count()` - returns the number of times that value appears in the list

2.5 Modules

- modules - not automatically loaded Python code files
 - use **import statement** to load

```
import <module_name>
```

- use `dir` to see a list of functions and variables defined in the module
- `datetime` module
 - `datetime.date` - new data types for representing time-based data
 - `datetime.date()` - create a new date
 - `datetime.date.weekday()` - return the day of the week of the date (0 = Monday)
 - `datetime.timedelta` - data type
 - instance `days` represent an interval of time
 - use `==, <, ...` to compare date

```
>>> import datetime
>>> canada_day = datetime.date(1867, 7, 1)
>>> david_fake_birthday = datetime.date(2000, 3, 22)
>>> david_fake_birthday - canada_day
datetime.timedelta(days=48469)
```

2.6 Type Conversion Function

- int(), float(), bool(), list(), set(), dict(), str()

```
>>> int('10')
10
>>> float('10')
10.0
>>> bool(1000)
True
>>> bool(0)
False
>>> list({1, 2, 3})
[1, 2, 3]
>>> set([1, 2, 3])
{1, 2, 3}
>>> set() # Giving set no arguments results in the empty set
set()
>>> dict([('a', 1), ('b', 2), ('c', 3)])
{'a': 1, 'b': 2, 'c': 3}
```

- conversion errors
 - ValueError - invalid literal
 - TypeError - invalid type
- range
 - a type conversion function
 - a data type that is distinct from lists
 - can be converted into a list

```
>>> five_to_nine = range(5, 10)
>>> type(five_to_nine)
<class 'range'>
>>> five_to_nine == [5, 6, 7, 8, 9]
False
>>> list(five_to_nine)
[5, 6, 7, 8, 9]
```

2.7 The Function Design Recipe

1. Write example uses
 - what the function does
 - expected returned values
2. Write the function header - meaningful name, type contract
3. Write the function description
4. Implement the function body
5. Test the function
 - function header and docstring serve as documentation for the function

2.8 Testing Functions I: `doctest` and `pytest`

- manual testing - requires human interaction to complete
- `doctest`

```
if __name__ == '__main__':
    import doctest # import the doctest library
    doctest.testmod(verbose=True) # run the tests and display all results
    (pass or fail)
```

- exception
 - must start with a trackback header → containing the exception type & details

- pytest - write tests in a separate file

```
# At the bottom of test_trues.py

if __name__ == '__main__':
    import pytest
    pytest.main(['test_trues.py'])
```

- **test suite** - a collection of tests that check the behaviour of a function or set of functions
- **unit test** - a block of code
 - check for the correct behaviour of a function for one specific input
 - name starts with the prefix `test_`
 - return type is `none`
 - contains an `assert` statement instead of a `return` statement

`assert <expression>`

- evaluates `<expression>`
- True - continue onto next statement
- False - raise `AssertionError` → test **fails**

3. Formal Logic in Computer Science

3.1 Propositional Logic

- propositions - either True or False
- propositional variables - represent propositions
- propositional operator - whose arguments must all be either True or False
- propositional formula - expression that is built up from propositional variables in combination with propositional operators

- basic operator
 - **NOT** - negates the value of a boolean (\neg)
 - **AND** - produces **True** when **both** of the values are **True**, and False otherwise (\wedge)
 - **OR** - produces **True** when **at least one** of the values is **True**, and False otherwise (\vee)
 - ← inclusive or

p	q	$\neg p$	$p \wedge q$	$p \vee q$
False	False	True	False	False
False	True	True	False	True
True	False	False	False	True
True	True	False	True	True

6. Modifying Values and Variables

6.1 Variable Reassignment, Revisited

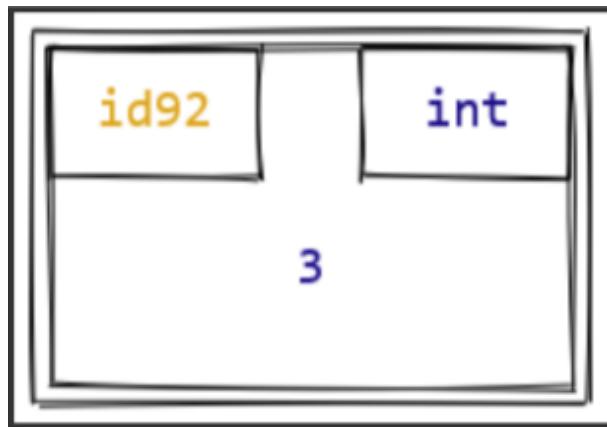
- **variable reassignment** - assigns a value to a variable when that variable already refers to a value
 - only changes the immediate variable being reassigned
 - does not change any other variables or values, even ones that were defined using the variable being reassigned
 - changes what value a variable refers to
- **augmented assignment statement**

```
<variable> += <expression> # augmented assignment
<variable> = <variable> + <expression> #reassignment
```

1. evaluates `<expression>`
2. Adds the value of `<expression>` to the current value of `<variable>` and reassigns `<variable>` to the result (**except** when `<variable>` refers to a list)

6.2 Objects & Object Mutation

- **object** - where every piece of data is stored
 - variables refer to objects
 - three fundamental components
 - **id** - unique int representation of the memory address of the object (**unchangeable**)
 - data type (**unchangeable**)
 - value - may change (depending on the data type)



- **object mutation** - an operation that changes the value of an existing object

	object mutation	variable reassignment
result	same	same
object	refer to the same object	create a new object
value	mutate the value of the object	assign the new object to the variable

- universal statements make code easier to reason about and variable reassignment and object mutation weaken this property

6.3 Mutable Data Types

- **mutable** - supports at least one mutation operation
 - list, set, dict, data classes

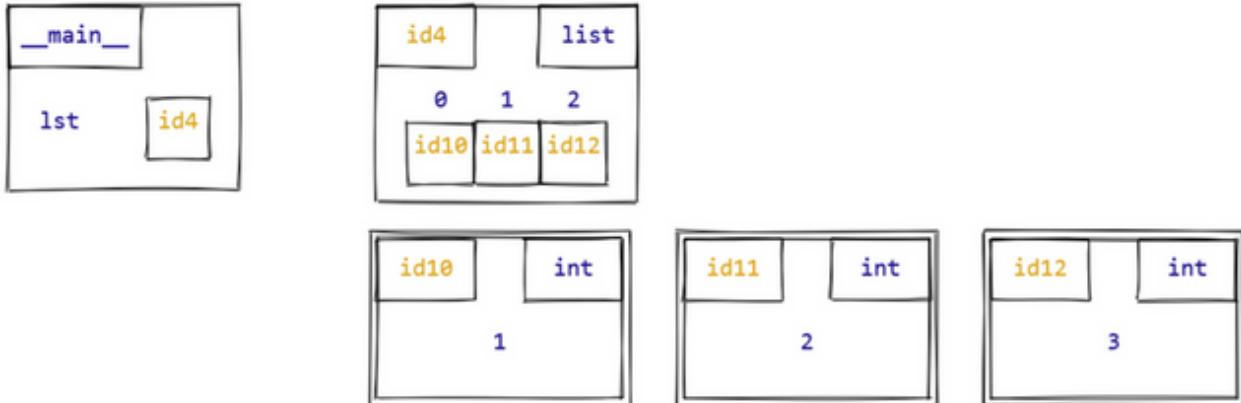
immutable - int, float, bool, str, tuple

- **mutating function** - a function that mutates one of its arguments
 - **NOTE:** the return type is None - **do no return anything except** list.pop and dict.pop)
- mutating lists
 - list.append, list.insert, list.extend, list.sort, list.pop
 - **list indexing** - list[] = <expression>, list[] += <expression>
- mutating sets
 - set.add, set.remove
- mutating dicts
 - dict.pop
 - dictionary key lookup - dict[] = <expression>, dict[] += <expression>
- mutating data classes
 - use dot notation
- **tips**
 - **do** use reassignment/ mutation in loops
 - **do** use **mutation** for collecting accumulators (mutation is faster than reassignment)
 - **do** use mutation when changing part of a larger object
 - **avoid** unnecessary reassignments/mutation
 - **avoid** changing a variable's value in many different places
 - **avoid** mutating function arguments unless described in the function docstring

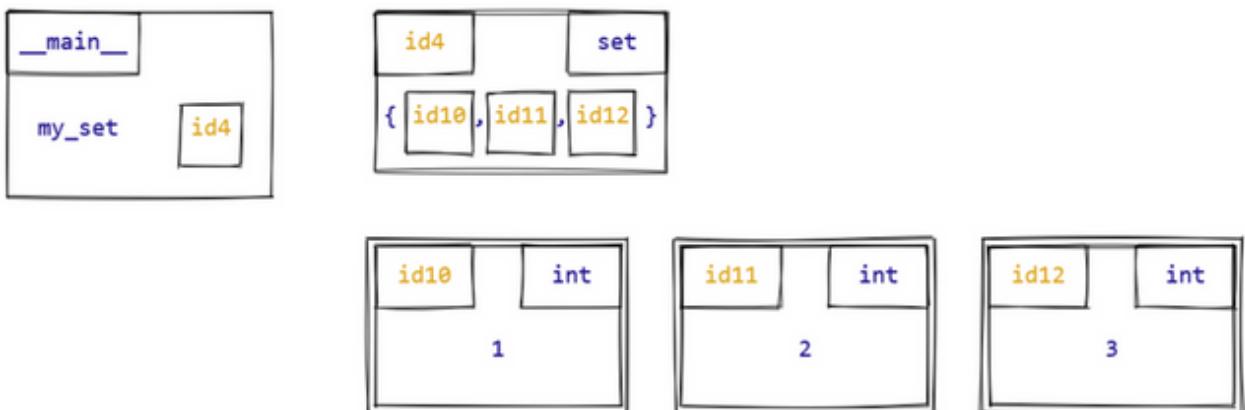
6.4 Python Memory Model: Introduction

- The variables don't store values, they store the **id** of an object
- expression evaluation
 - **literal** - creates an object of the appropriate type to hold the value
 - binary operation - create a new object to hold the resulting value
 - **variable**
 - exists - **produce** the 'id' stored in the variable
 - not exists - NameError is raised

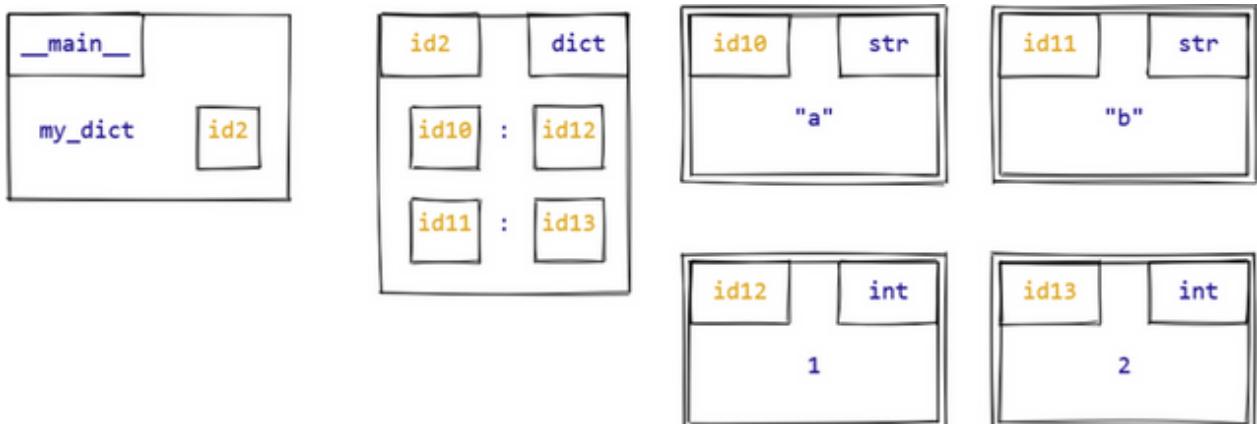
- assignment statement
 - evaluating the expression on the RHS
 - if the variable on the LHS doesn't exist, create it
 - store the id from the expression in the variable
- compound data type - list, set, dict, data classes
 - an instance of a compound data type stores the **ids** of other objects
 - list - `lst = [1, 2, 3]`



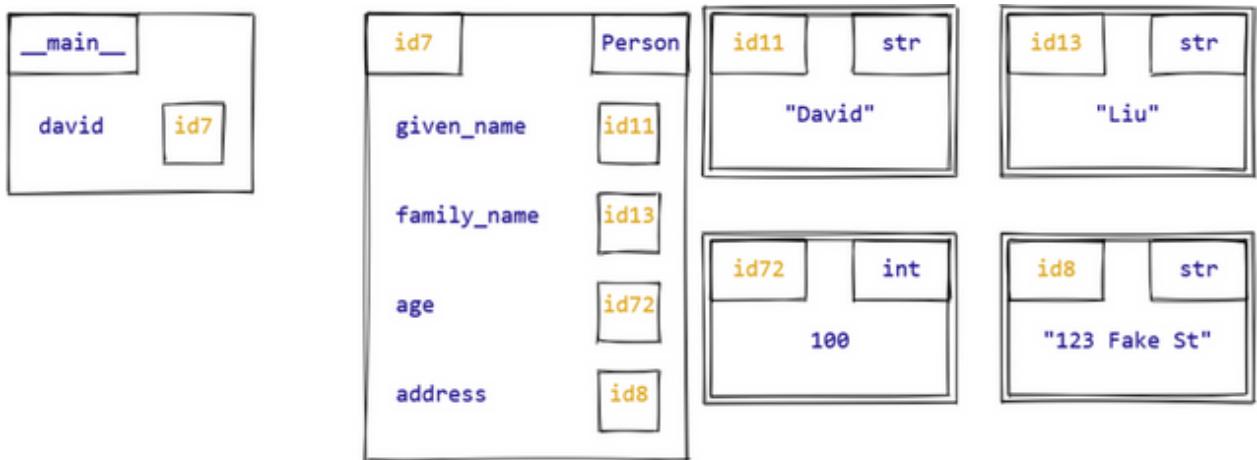
- set - `my_set = {1, 2, 3}`



- dict - `my_dict = {'a' : 1, 'b' : 2 }`



- data classes

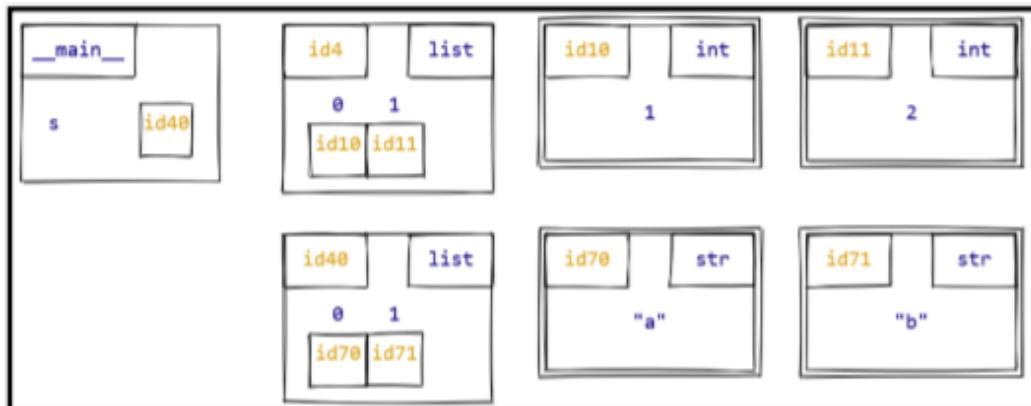


- variable reassignment - refers to a **new** object with a new value

- before reassignment - `s = [1, 2]`



- after reassignment - `s = ['a', 'b']`

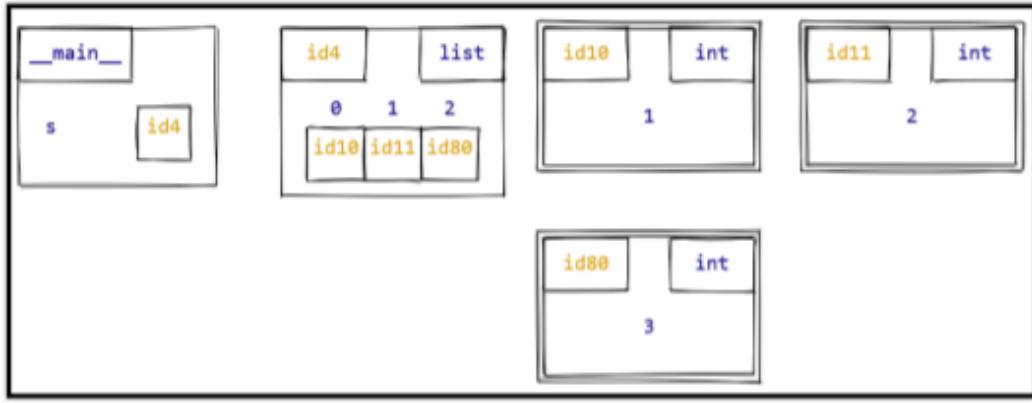


- object mutation - refers to the **same** object, but with a new value

- before mutation - `s = [1, 2]`



- after mutation - `list.append(s, 3)`



- `list[] = <expression>` is a form of reassignment - reassign an **id** instead of a variable

6.5 Aliasing

- alias - more than one variables refer to the **same** object
 - performing an object mutation operation on one **affects the other**

```
>>> x = [1, 2, 3]
>>> z = x
>>> z[0] = -999
>>> x
[-999, 2, 3]
```

- reassign one will **break the aliasing** and have no effect on others

```
>>> x = [1, 2, 3]
>>> z = x
>>> z = [1, 2, 3, 4]
>>> x
[1, 2, 3]
```

- loop variables
 - **element-based** - reassign the loop variable, **no effect** on the initial object

```
>>> numbers = [5, 6, 7]
>>> for number in numbers:
...     number = number + 1
...
>>> numbers
[5, 6, 7]
```

- **index-based** - mutate the initial object and **change** its value

```
>>> numbers
[5, 6, 7]
>>> for i in range(0, len(numbers)):
...     numbers[i] = numbers[i] + 1
...
>>> numbers
[6, 7, 8]
```

- **equality**

- **value equality** == - check whether two objects have the same **value**

```
>>> x = [1, 2, 3]
>>> y = [1, 2, 3]
>>> z = x
>>> x == y
True
>>> x == z
True
```

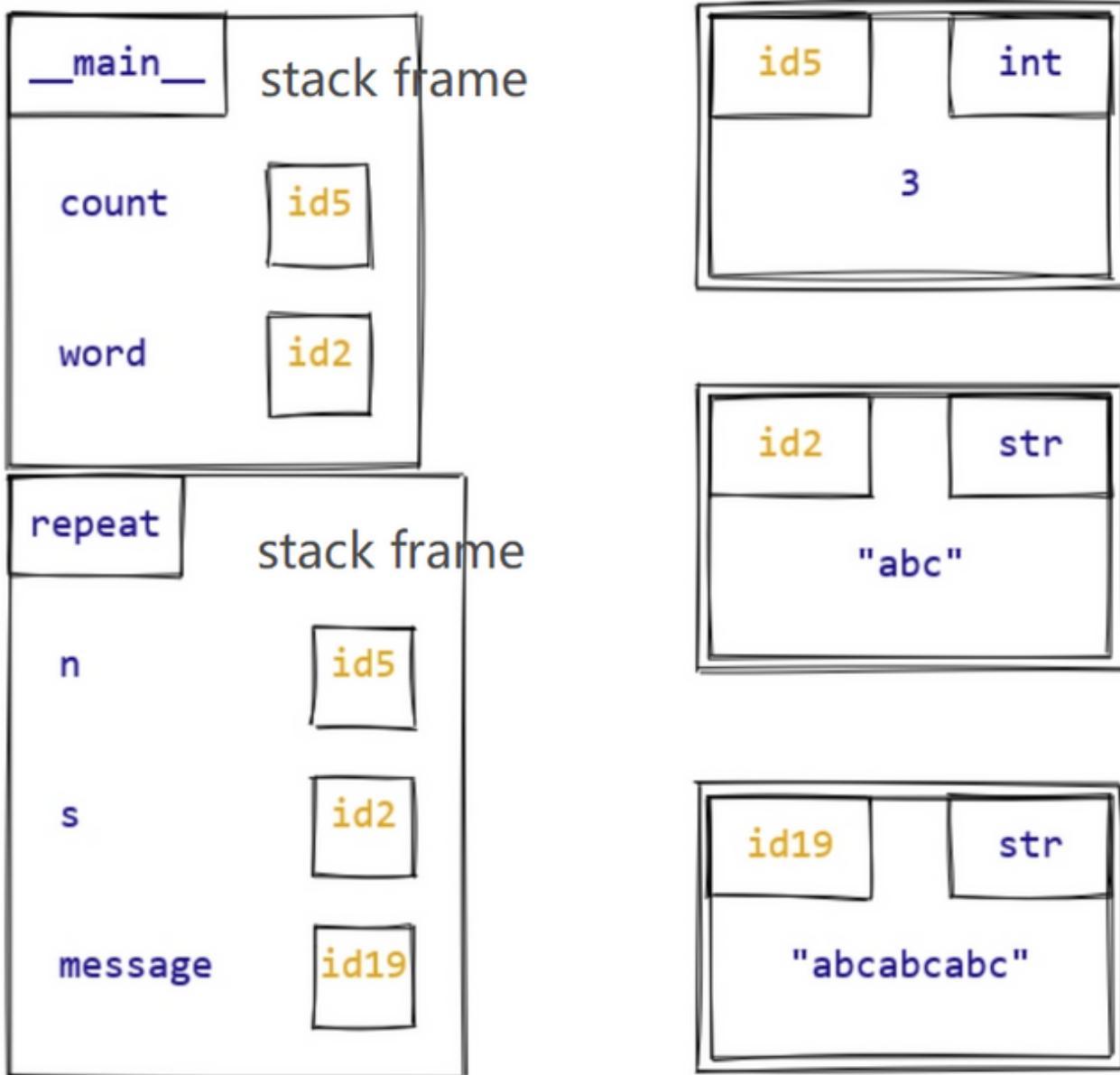
- **identity equality** is - check whether two objects have the same **id**

```
>>> x = [1, 2, 3]
>>> y = [1, 2, 3]
>>> z = x
>>> x is y
False
>>> x is z
True
```

- `<expr1> is <expr2>` implies `<expr1> == <expr2>` (**converse is not true**)
- if two objects have the same value, they need not have the same id
- automatic aliasing
 - if two variables have the **same immutable value**, the program's behaviour **does not** depend on whether the two variables are aliases or not
 - automatically creates aliases for "small" immutable values
- rules
 - boolean - use `is` (`<expr> is True` equals to `<expr>`, `<expr> is False` equals to `not <expr>`)
 - non-boolean immutable - use `==`
 - mutable - use `==` to compare value equality, use `is` to check for identity equality

6.6 Python Memory Model: Function Calls

- **stack frame** - a piece of data used to keep track of function calls and their local variables
- **function call stack** - collection of all stack frames



- call a function
 1. create a new stack frame and add it to the call stack
 2. evaluate the arguments in the function, yielding the ids of objects
 3. execute the body of the function
 4. return statement executed, the id of the return object is saved and the stack frame for the function call is **removed** from the call stack
- argument passing and aliasing
 - all function parameters are **aliases** for the function's arguments when it is called

```

def emphasize(words: list[str]) -> None:
    """Add emphasis to the end of a list of words."""
    new_words = ['believe', 'me!']
    list.extend(words, new_words)

# In the Python console
>>> sentence = ['winter', 'is', 'coming']
>>> emphasize(sentence)
>>> sentence
['winter', 'is', 'coming', 'believe', 'me!']

```

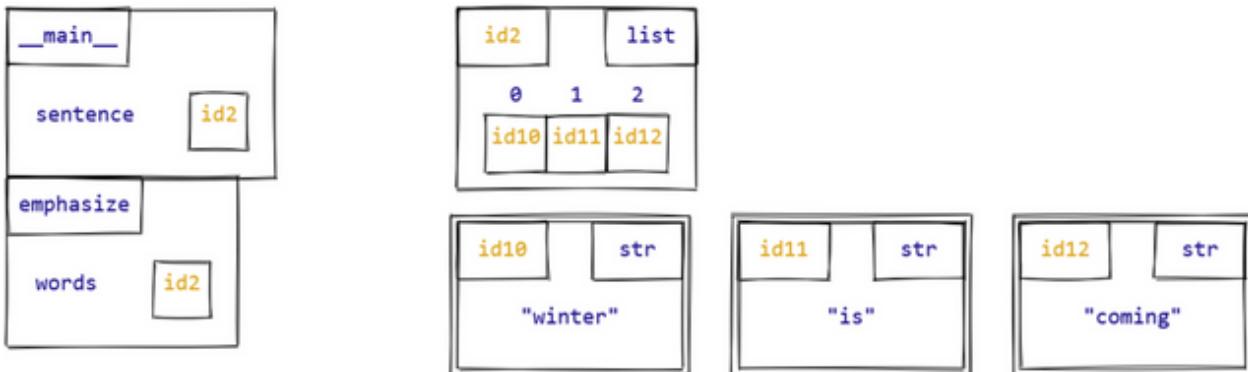
```

def emphasize_v2(words: list[str]) -> None:
    """Add emphasis to the end of a list of words."""
    new_words = ['believe', 'me!']
    words = words + new_words

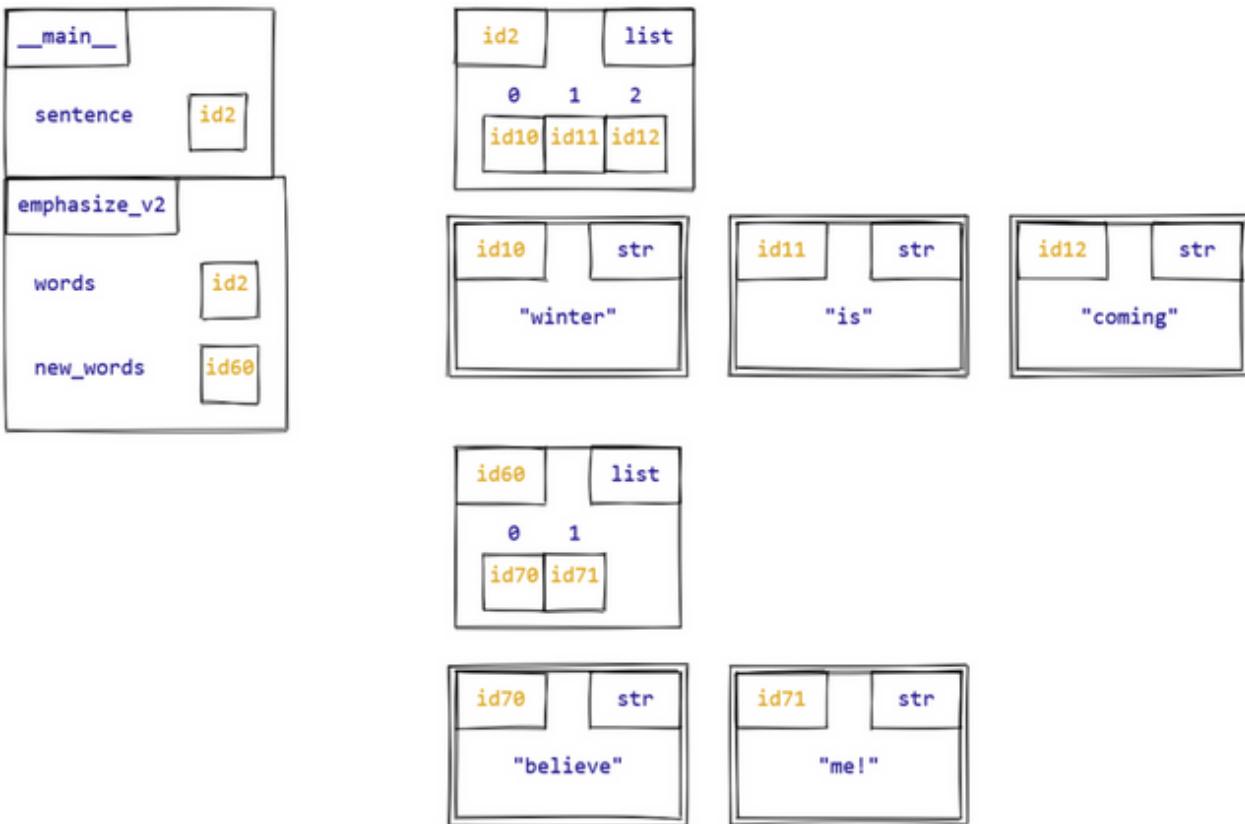
# In the Python console
>>> sentence = ['winter', 'is', 'coming']
>>> emphasize_v2(sentence)
>>> sentence
['winter', 'is', 'coming']

```

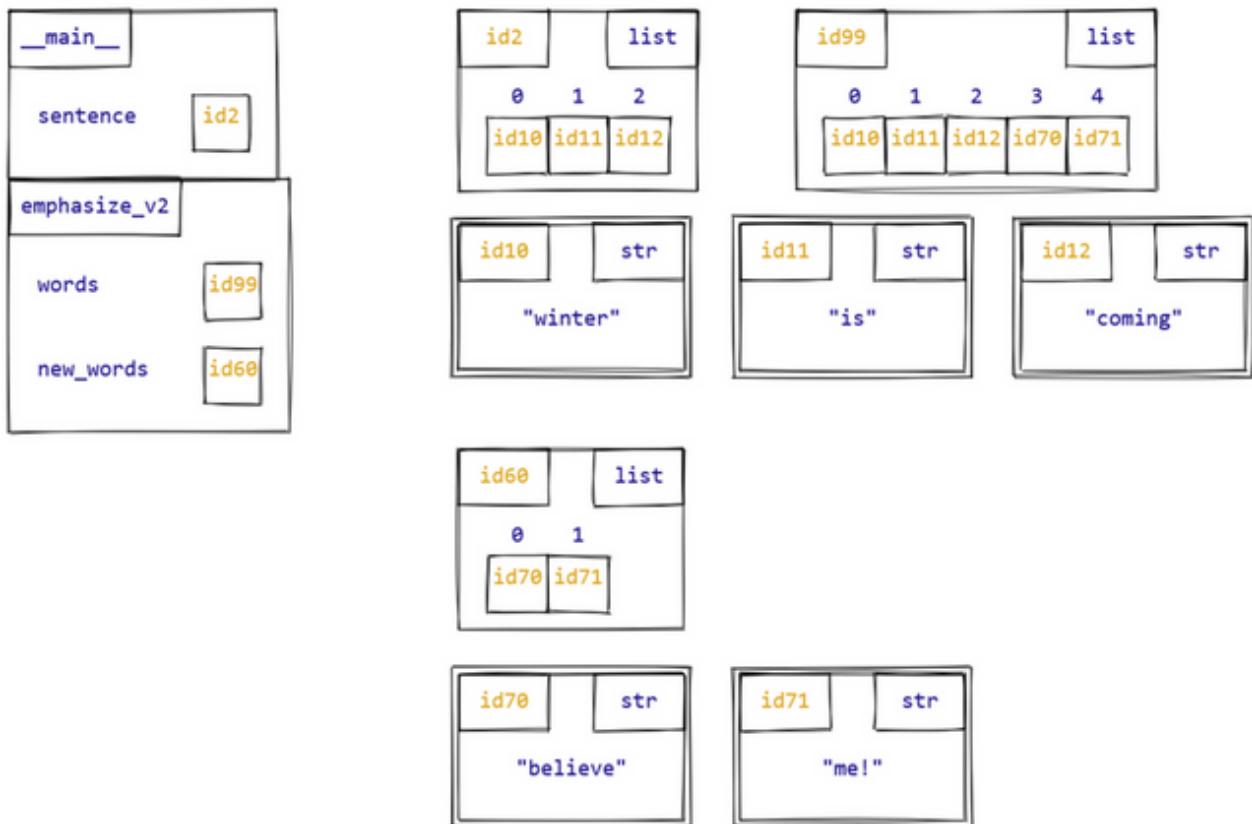
- before function call



- after setting new_words



- before returning - words and sentence are no longer aliases



6.7 Testing Function III: Testing Mutation

- function description
 - without mutation - return ...
 - does mutation - modify ...
- testing for **no** mutation
 1. create an object and copy it
 2. **compare** them after calling the function (no change)

```
def squares(numbers: list[int]) -> list[int]:  
    """Return a list of the squares of the given numbers."""  
    squares_so_far = []  
  
    for number in numbers:  
        list.append(squares_so_far, number * number)  
  
    return squares_so_far  
  
def test_squares_no_mutation_1() -> None:  
    """Test that squares does not mutate the list it is given.  
    """  
    lst = [1, 2, 3]  
    squares(lst)  
  
    assert lst == [1, 2, 3]  
  
def test_squares_no_mutation_2() -> None:  
    """Test that squares does not mutate the list it is given.  
    """  
    lst = [1, 2, 3]  
    lst_copy = list.copy(lst) # Create a copy of lst (not an alias!)  
    squares(lst)  
  
    assert lst == lst_copy  
  
from hypothesis import given  
from hypothesis.strategies import lists, integers
```

```

@given(lst=lists(integers()))
def test_squares_no_mutation_general(lst: list[int]) -> None:
    """Test that squares does not mutate the list it is given.
    """
    lst_copy = list.copy(lst) # Create a copy of lst (not an alias!)
    squares(lst)

    assert lst == lst_copy

```

- testing for **mutation**

- mutation function return None → **cannot** compare the result and the original object
 1. create an object and copy
 2. verify the relationship between corresponding elements

```

def square_all(nums: list[int]) -> None:
    """Modify nums by squaring each of its elements."""
    for i in range(0, len(nums)):
        nums[i] = nums[i] * nums[i]

def test_square_all_1() -> None:
    """Test that square_all mutates the list it is given correctly.
    """
    lst = [1, 2, 3]
    result = square_all(lst)

    assert result == [1, 4, 9]

def test_square_all_mutation_2() -> None:
    """Test that square_all mutates the list it is given correctly.
    """
    lst = [1, 2, 3]
    square_all(lst)

    assert lst == [1, 4, 9]

@given(lst=lists(integers()))

```

```

def test_square_all_mutation_general(lst: list[int]) -> None:
    """Test that square_all mutates the list it is given correctly.
    """
    lst_copy = list.copy(lst)
    square_all(lst)

    assert all({lst[i] == lst_copy[i] ** 2 for i in range(0, len(lst))})

```

7. Number Theory

7.1 Introduction

- d divides n ($d|n$)

$$\exists k \in \mathbb{Z}, n = kd \text{ (where } d, n \in \mathbb{Z})$$

```

def divides(d: int, n: int) -> bool:
    """Return whether d divides n."""
    if d == 0:
        return n == 0
    else:
        return n % d == 0

def divides(d: int, n: int) -> bool:
    """Return whether d divides n."""
    possible_divisors = range(-abs(n), abs(n) + 1)
    return any({n == k * d for k in possible_divisors})

```

- synonymous
 - n is divisible by d
 - d is a factor of n
 - n is a multiple of d
- **prime** - an integer p is prime when it is greater than 1 and the only natural numbers that divide it are 1 and itself

$$IsPrime(P) : p > 1 \wedge (\forall d \in \mathbb{N}, d|p \implies d = 1 \vee d = p), \quad \text{where } p \in \mathbb{Z}$$

```
from math import floor, sqrt

def is_prime_v2(p: int) -> bool:
    """Return whether p is prime."""
    possible_divisors = range(2, floor(sqrt(p)) + 1)
    return (
        p > 1 and
        all({not divides(d, p) for d in possible_divisors})
    )
```

- **greatest common divisor** ($gcd(m, n)$) - The largest integer that divides both m and n (where $m, n \in \mathbb{Z}$).

$$\forall m, n, d \in \mathbb{Z}, (m = 0 \wedge n = 0 \implies d = 0) \wedge \\ (m \neq 0 \vee n \neq 0 \implies d|m \wedge d|n \wedge (\forall e \in \mathbb{N}, e|m \wedge e|n \implies e \leq d))$$

- $gcd(0, 0) = 0, gcd(n, 0) = n$
- 1 divides **every** integer
- **coprime** - m and n are coprime when $gcd(m, n) = 1$ (where $m, n \in \mathbb{Z}$)
- **Quotient-Remainder Theorem**

$$\forall n \in \mathbb{Z}, d \in \mathbb{Z}^+, \exists q \in \mathbb{Z}, r \in \mathbb{N}, s.t., (n = qd + r) \wedge (0 \leq r < |d|)$$

```
q, r = divmod(n, d)
```

- q and r are unique for a given n and d
 - q is the **quotient** and r is the **remainder** when n is divided by d
 - a is **equivalent** to b **modulo** n ($a \equiv b \pmod{n}$)
- $n|a - b$, or equivalently, $n|b - a$ where $a, b, n \in \mathbb{Z}$ and $n \neq 0$
- **theorem** - if and only if a and b have the **same remainder** when divided by n

$$\forall a, b, n \in \mathbb{Z}, n \neq 0 \implies (a \equiv b \pmod{n}) \Leftrightarrow (a \% n = b \% n)$$

- **theorem** - Let $a, b, c, n \in \mathbb{Z}$ with $n \neq 0$.

$$a \equiv a \pmod{n}$$

$$a \equiv b \pmod{n} \implies b \equiv a \pmod{n}$$

$$(a \equiv b \pmod{n}) \wedge (b \equiv c \pmod{n}) \implies a \equiv c \pmod{n}$$

- **theorem** - Let $a, b, c, d, n \in \mathbb{Z}$ with $n \neq 0$. If $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$

$$a + b \equiv c + d \pmod{n}$$

$$a - b \equiv c - d \pmod{n}$$

$$a \times b \equiv c \times d \pmod{n}$$

7.2 Greatest Common Divisor

- **theorem** - if p and q are distinct primes, then p and q are coprime, meaning $\gcd(p, q) = 1$

$$\forall p, q \in \mathbb{Z}, (\text{IsPrime}(p) \wedge \text{IsPrime}(q) \wedge p \neq q) \implies \gcd(p, q) = 1$$

Let $p, q \in \mathbb{Z}$.

Assume p and q are both prime and that $p \neq q$.

By the definition of prime,

we know that $p \neq 1$,

and the only positive divisors of q are 1 and q itself.

Then since $p \neq q$ and $p \neq 1$,

we know that $p \nmid q$.

Since 1 divides every number, so 1 is the only positive common divisor of p and q

.

We have proved $\gcd(p, q) = 1$ as desired.

- linear combination of m and n - exist $p, q \in \mathbb{Z}$ such that $a = pm + qn$
- **Divisibility of Linear Combinations**

$$\forall d, m, n, p, q \in \mathbb{Z}, (d|m \wedge d|n) \implies d|pm + qn$$

- **GCD Characterization** - $\gcd(m, n)$ is the smallest positive integer that is a linear combination of m and n
- **theorem** - if d divides both m and n then d also divides $\gcd(m, n)$

$$\forall m, n, d \in \mathbb{Z}, d|m \wedge d|n \implies d|\gcd(m, n)$$

Let $m, n, d \in \mathbb{Z}$.

Assume $d|m$ and $d|n$.

Case 1: assume $m = 0$ and $n = 0$

By the definition of gcd, we know $\gcd(m, n) = 0$

So $d | \gcd(m, n)$ since d divides m and n , which are 0

Case 2: assume $m \neq 0$ or $n \neq 0$

By the GCD Characterization Theorem,

there exist integers $p, q \in \mathbb{Z}$ such that $\gcd(m, n) = pm + qn$

By the Divisibility of Linear Combinations Theorem,

since $d|m$ and $d|n$, we know that $d | pm + qn$.

Therefore $d | \gcd(m, n)$

- **GCD and Remainders Theorem**

$$\forall m, n \in \mathbb{Z}, m \neq 0 \implies \gcd(n, m) = \gcd(m, n \% m)$$

- key idea - even if n is very large, $n \% m < |m|$.

7.3 Proof and Algorithms III: Computing the Greatest Common Divisor

- naive algorithm

```

def naive_gcd(m: int, n: int) -> int:
    """Return the gcd of m and n."""
    if m == 0:
        return abs(n)
    elif n == 0:
        return abs(m)
    else:
        possible_divisors = range(1, min(abs(m), abs(n)) + 1)
        return max({d for d in possible_divisors if divides(d, m) and
                    divides(d, n)})

```

- **GCD and Remainders Theorem**

$$\forall m, n \in \mathbb{Z}, m \neq 0 \implies \gcd(n, m) = \gcd(m, n \% m)$$

- key idea - even if n is very large, $n \% m < |m|$.

- **Euclidean Algorithm**

- Given: non-negative integers a and b
- Returns: $\gcd(a, b)$

1. Initialize two variables x, y to the given numbers a and b
2. Let r be the remainder when x is divided by y
3. Reassign x and y to y and r , respectively
4. Repeat steps 2 and 3 until y is 0
5. At this point, x refers to the $\gcd(a, b)$

- **while loop** - repeats its body as long as its <condition> is **True**

```

while <condition>:
    <statement>

```

- **parallel assignment statement**

- Evaluates **every** expression on the right-hand side
- Then, each object is assigned to the corresponding variable on the left-hand side
 - order **doesn't** matter

```
x, y = y, r

# or, same
y, x = r, y
```

- loop invariant** - a property about loop variables that must be true at the start and end of each loop iteration

```
def euclidean_gcd(a: int, b: int) -> int:
    """Return the gcd of a and b.

    Preconditions:
    - a >= 0
    - b >= 0
    """
    # Step 1: initialize x and y
    x, y = a, b

    while y != 0: # Step 4: repeat Steps 2 and 3 until y is 0
        assert math.gcd(x, y) == math.gcd(a, b) # (NEW) Loop invariant

        # Step 2: calculate the remainder of x divided by y
        r = x % y

        # Step 3: reassign x and y
        x, y = y, r

    # Step 5: x now refers to the gcd of a and b
    return x
```

- The extended Euclidean Algorithm

- **key** - x and y are **always** linear combinations of a and b at every loop iteration
- Given: non-negative integers a and b
- Returns: $\text{gcd}(a, b)$, p, q such that $\text{gcd}(a, b) == p * a + q * b$

```

def extended_euclidean_gcd(a: int, b: int) -> tuple[int, int, int]:
    """Return the gcd of a and b, and integers p and q such that

    gcd(a, b) == p * a + b * q.

    Preconditions:
    - a >= 0
    - b >= 0

    >>> extended_euclidean_gcd(13, 10)
    (1, 7, -9)
    """
    x, y = a, b

    # NEW: more loop variables
    px, qx = 1, 0 # Since x == a == 1 * a + 0 * b
    py, qy = 0, 1 # Since y == b == 0 * a + 1 * b

    while y != 0:
        assert math.gcd(x, y) == math.gcd(a, b) # Loop invariant 1
        assert x == px * a + qx * b           # Loop invariant 2
        assert y == py * a + qy * b           # Loop invariant 3

        q, r = divmod(x, y) # quotient and remainder when a is divided
                            # by b

        x, y = y, r

        # NEW: update the new loop variables
        px, qx, py, qy = py, qy, px - q * py, qx - q * qy

    return x, px, qx

```

- example - `extended_euclidean_gcd(100, 13)`

Iteration	x	px	qx	y	py	qy
0	100	1	0	13	0	1
1	13	0	1	9	1	-7
2	9	1	-7	4	-1	8
3	4	-1	8	1	3	-23
4	1	3	-23	0	-13	100

7.4 Modular Arithmetic

- **modular inverse** - if $\gcd(a, n) = 1$ then there exists $ap \equiv 1 \pmod{n} \rightarrow p$ is a **modular inverse** of a modulo n

$$\forall n \in \mathbb{Z}^+, \forall a \in \mathbb{Z}, \gcd(a, n) = 1 \implies (\exists p \in \mathbb{Z}, ap \equiv 1 \pmod{n})$$

Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$. Assume $\gcd(a, n) = 1$

Since $\gcd(a, n) = 1$

By the GCD Characterization Theorem, we know that there exist integers p and q such that $pa + qn = \gcd(a, n) = 1$

We can get $pa - 1 = qn$ and so $n \mid pa - 1 \implies pa \equiv 1 \pmod{n}$

- **modular division** - if $\gcd(a, n) = 1$ then for all integer b there exists an integer k such that $ak \equiv b \pmod{n}$

$$\forall n \in \mathbb{Z}^+, \forall a \in \mathbb{Z}, \gcd(a, n) = 1 \implies (\forall b \in \mathbb{Z}, \exists k \in \mathbb{Z}, ak \equiv b \pmod{n})$$

Let $n \in \mathbb{Z}^+$ and $a, b \in \mathbb{Z}$. Assume $\gcd(a, n) = 1$.

Using the modular inverses theorem, since $\gcd(a, n) = 1$,

we know that there exists $p \in \mathbb{Z}$ such that $ap \equiv 1 \pmod{n}$.

And we know that $apb \equiv b \pmod{n}$.

We let $k = pb$, and we have that $ak \equiv b \pmod{n}$

7.5 Modular Exponentiation and Order

- the **order** of a modulo n - the **smallest positive integer** k such that $a^k \equiv 1 \pmod{n}$
 - $a \equiv b \pmod{n} \rightarrow a^k \equiv b^k \pmod{n}$
- Fermat's Little Theorem** - p is prime and that $p \nmid a$ then $a^{p-1} \equiv 1 \pmod{p}$

$$\forall a, p \in \mathbb{Z}, \text{IsPrime}(p) \wedge (\forall k \in \mathbb{Z}, a \neq kp) \implies a^{p-1} \equiv 1 \pmod{p}$$

- Euler totient function**

$$\varphi : \mathbb{Z}^+ \rightarrow \mathbb{N} \quad \varphi(n) = |\{a \mid a \in \{1, \dots, n-1\} \text{ and } \gcd(a, n) = 1\}|$$

- theorem** - for all prime numbers $p, q \in \mathbb{Z}^+$, $\varphi(pq) = (p-1)(q-1)$
- Euler's Theorem**

$$\forall a \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+, \gcd(a, n) = 1 \implies a^{\varphi(n)} \equiv 1 \pmod{n}$$

8. Cryptography

8.1 Introduction

- symmetric-key cryptosystem**
 - plaintext** - set \mathcal{P} of possible original messages
 - ciphertext** - set \mathcal{C} of possible encrypted messages
 - shared secret keys** - set \mathcal{K} known by both Alice and Bob but no one else
 - encrypt**: $\mathcal{K} \times \mathcal{P} \rightarrow \mathcal{C}$ - Alice encrypts a plaintext message $m \in \mathcal{P}$ using k to obtain a ciphertext $c \in \mathcal{C}$ and sends to Bob
 - decrypt**: $\mathcal{K} \times \mathcal{C} \rightarrow \mathcal{P}$ - Bob decrypts the ciphertext c using k to obtain the original plaintext message m
 - properties**
 - correctness** - for all $k \in \mathcal{K}$ and $m \in \mathcal{P}$, $\text{Decrypted}(k, \text{Encrpted}(k, m)) = m$
 - security** - for all $k \in \mathcal{K}$ and $m \in \mathcal{P}$, if an eavesdropper only knows the value of $c = \text{Encrypt}(k, m)$ but does not know k , it is computationally infeasible to find m
- Caesar Cipher**

- plaintext and ciphertext messages are strings of ASCII characters
- secret key k is a numeric shift of each letter
- $c[i] = (m[i] + k) \% 128$
- security - not secure
 - Any cryptosystem based on character substitution reveals information about the structure of the original message

```
def encrypt_ascii(k: int, plaintext: str) -> str:
    """Return the encrypted message using the Caesar cipher with key k.

    Preconditions:
        - all({ord(c) < 128 for c in plaintext})
        - 1 <= k <= 127

    >>> encrypt_ascii(4, 'Good morning!')
    'Kssh$qsvr$mrk%'

    ciphertext = ''

    for letter in plaintext:
        ciphertext = ciphertext + chr((ord(letter) + k) % 128)

    return ciphertext


def decrypt_ascii(k: int, ciphertext: str) -> str:
    """Return the decrypted message using the Caesar cipher with key k.

    Preconditions:
        - all({ord(c) < 128 for c in ciphertext})
        - 1 <= k <= 127

    >>> decrypt_ascii(4, 'Kssh$qsvr$mrk%')
    'Good morning!'

    plaintext = ''

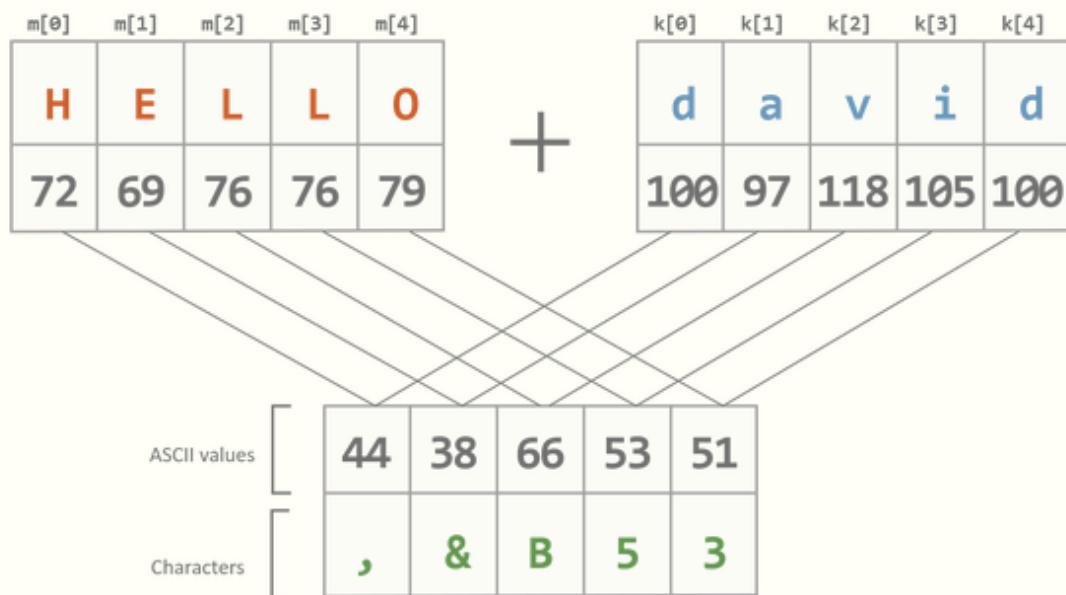
    for letter in ciphertext:
```

```
    plaintext += chr((ord(letter) - k) % 128)
```

```
return plaintext
```

8.2 The One-Time Pad and Perfect Secrecy

- one-time pad
 - using a **string** of length greater than or equal to the length of the plaintext message
 - $c[i] = (m[i] + k[i]) \% 128$
 - perfect secrecy - the ciphertext yields **no** information about the plaintext
 - limitation
 - the **length** of the secret key must be \geq the length of the plaintext
 - **cannot reuse** the secret key (reused \rightarrow no perfect secrecy)

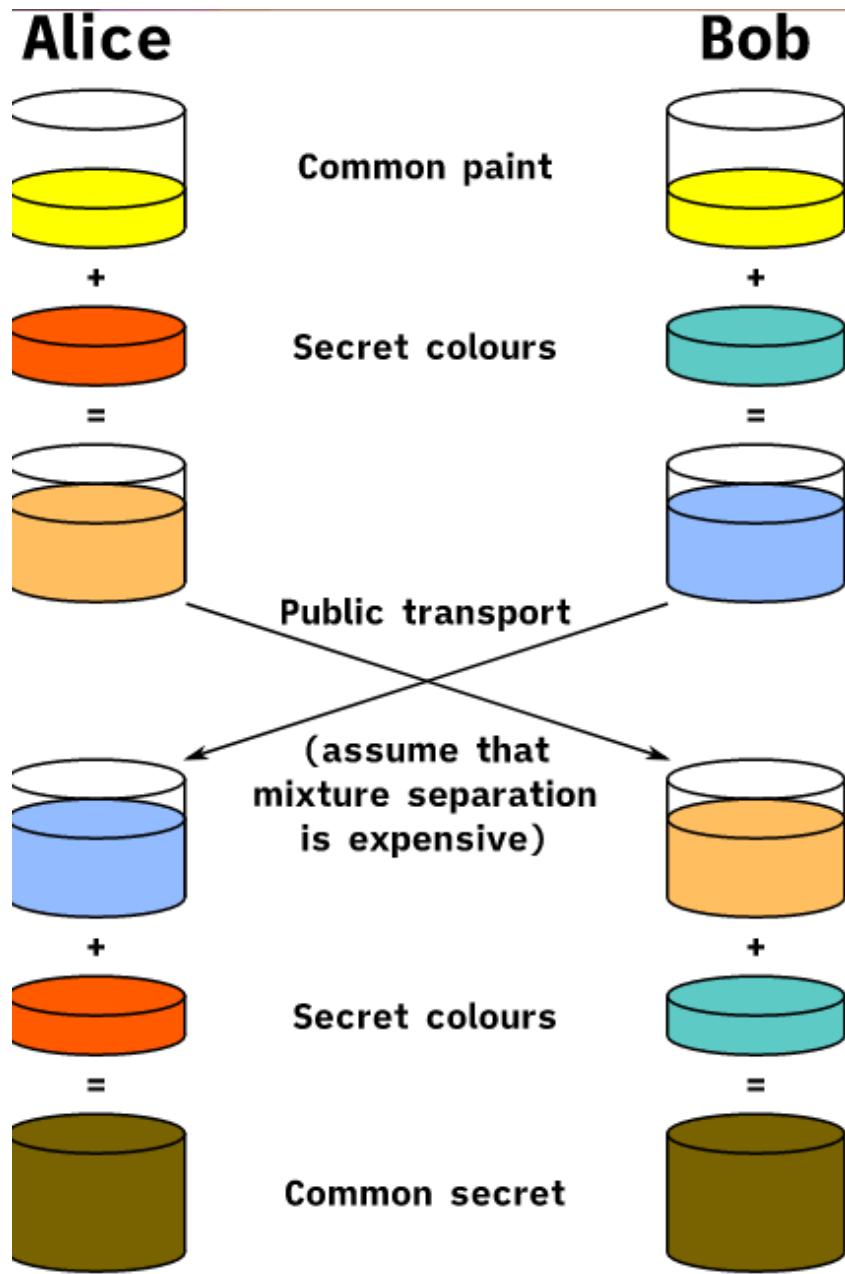


- **stream ciphers** - based on the one-time pad, but use a small shared key
 - both parties use an algorithm to generate an arbitrary number of new random characters

8.3 Computing Shared Secret Keys

- Diffie-Hellman key exchange algorithm - establish a shared secret key while **only communicating publicly**
 - computationally secure
 - for large enough primes, there is no computationally efficient way of determining the secret key from just the public communication
1. Alice chooses prime number p greater than two and $g \in \{2, 3, \dots, p-1\}$. Then send both numbers (p, g) to Bob
 2. Alice chooses a secret number $a \in \{1, 2, \dots, p-1\}$ and sends Bob $A = g^a \% p$
 3. Bob chooses a secret number $b \in \{1, 2, \dots, p-1\}$ and sends Alice $B = g^b \% p$
 4. Alice computes $k_A = B^a \% p$ and Bob computes $k_b = A^b \% p$
 5. Then $k_A = k_B$ and this is the secret key

1. Choose $p = 23, g = 2 \rightarrow$ send $(23, 2)$
2. Choose $a = 5 \rightarrow$ send $A = 2^{5\%} 23 = 9$
3. Choose $b = 14 \rightarrow$ send $B = 2^{14\%} 23 = 8$
4. $k_A = 8^{5\%} 23 = 16$ and $k_B = 9^{14\%} 23 = 16$



- Correctness of Diffie-Hellman key exchange

$$\forall p, g, a, b \in \mathbb{Z}^+, (g^b \% p)^a \% p = (g^a \% p)^b \% p$$

Let $p, g, a, b \in \mathbb{Z}^+$.

Let $A = g^a \% p$ and $B = g^b \% p$

We know that $g^a \equiv A \pmod{p}$ and $g^b \equiv B \pmod{p}$

So $A^b \equiv (g^a)^b \equiv g^{ab} \pmod{p}$ and $B^a \equiv (g^b)^a \equiv g^{ba} \pmod{p}$

Since $g^{ab} = g^{ba}$. We can conclude that $A^b \equiv B^a \pmod{p}$

So $A^b \% p = B^a \% p$

- **discrete logarithm problem** - given $p, g, A \in \mathbb{Z}^+$, find $a \in \mathbb{Z}^+$ such that $g^a \equiv A \pmod{p}$, if such an a exists
 - there is **no known efficient algorithm** for solving the discrete logarithm problem

8.4 The RSA Cryptosystem

- **Public-key cryptosystem**
 - plaintext - set \mathcal{P} of possible original messages
 - ciphertext - set \mathcal{C} of possible encrypted messages
 - **public key** - set \mathcal{K}_1 of possible public keys
 - **private key** - set \mathcal{K}_2 of possible private keys
 - **encrypt**: $\mathcal{K}_1 \times \mathcal{P} \rightarrow \mathcal{C}$
 - **decrypt**: $\mathcal{K}_2 \times \mathcal{C} \rightarrow \mathcal{P}$
 - properties
 - **correctness** - for all $(k_1, k_2) \in \mathcal{K}$ and $m \in \mathcal{P}$,
 $Decrypted(k_2, Encrpted(k_1, m)) = m$
 - **security** - for all $(k_1, k_2) \in \mathcal{K}$ and $m \in \mathcal{P}$, if an eavesdropper only knows the value of the public key k_1 and the ciphertext $c = Encrypt(k_1, m)$ but does not know k_2 , it is computationally infeasible to find m
- **Rivest-Shamir-Adleman (RSA) cryptosystem**
 - in the (basic) RSA cryptosystem, plaintexts and ciphertexts are positive integers
 - properties
 - **correctness** - for all $(p, q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ be private key and $(n, e) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ its corresponding public key. Let $m, c, m' \in \{1, \dots, n-1\}$ be the original plaintext message, ciphertext and decrypted message, respectively. Then $m' = m$

(prove in the **special case** when $gcd(m, n) = 1$)

From the encryption step, we know

$$c \equiv m^e \pmod{n}$$

From the decryption step, we know

$$m' \equiv c^d \equiv (m^e)^d \equiv m^{ed} \pmod{n}$$

We know that $de \equiv 1 \pmod{\varphi(n)}$, i.e

$\exists k \in \mathbb{Z}$ such that $ed - 1 = k \times \varphi(n)$

Thus

$$\begin{aligned} m &\equiv m^{ed} \pmod{n} \\ &\equiv m^{k \times \varphi(n) + 1} \pmod{n} \\ &\equiv (m^{\varphi(n)})^k \times m \pmod{n} \\ &\equiv 1^k \times m \pmod{n} \quad (\text{by Euler's Theorem}) \\ &\equiv m \pmod{n} \end{aligned}$$

Since $1 \leq m, m' \leq n - 1$, we can conclude $m' = m$

- **security** - eavesdropper knows: public key (n, e) , ciphertext $c = m^e$
 - can't efficiently compute m directly since we don't have an efficient way of computing $e - th$ roots in modular arithmetic
 - can't efficiently compute d directly since we know n not $\varphi(n)$
 - can't efficiently compute $\varphi(n)$ for large n
 - can't find p, q since there is no known efficient algorithm for factoring large numbers - **integer factorization problem**

- **Phase 1 - key generation**

- pick two distinct primes p and q
- compute $n = pq$
- choose integer $e \in \{2, 3, \dots, \varphi(n)\}$ such that $\gcd(e, \varphi(n)) = 1$ (NOTE: $\varphi(n) = (p-1)(q-1)$)
- compute $d \in \{2, 3, \dots, \varphi(n)\}$ such that $ed \equiv 1 \pmod{\varphi(n)}$ (NOTE: d is an inverse of e modulo $\varphi(n)$)
- **public key** - (n, e) **private key** - (p, q, d)

1. $p = 23, q = 31$

2. $n = pq = 23 \times 31 = 713$

3. $\varphi(713) = (23-1)(31-1) = 660, e = 547$

4. $d \cdot 547 \equiv 1 \pmod{660} \implies d = 403$

5. public key $(713, 547)$, private key $(23, 31, 403)$

- **Phase 2 - message encryption**

- compute $c = m^e \% n$

want to sent 42

$$c = 42^{547} \% 713 = 106$$

- **Phase 3 - message decryption**

- compute $m' = c^d \% n$

$$m = 106^{403} \% 713 = 42$$

8.5 Implementing RSA in Python

```
def rsa_generate_key(p: int, q: int) -> \
    tuple[tuple[int, int, int], tuple[int, int]]:
    """Return an RSA key pair generated using primes p and q.
```

The return value is a tuple containing two tuples:

1. The first tuple is the private key, containing (p, q, d).
2. The second tuple is the public key, containing (n, e).

Preconditions:

- p and q are prime
- p != q

"""

Compute the product of p and q

n = p * q

Choose e such that gcd(e, phi_n) == 1.

phi_n = (p - 1) * (q - 1)

Since e is chosen randomly, we repeat the random choice

until e is coprime to phi_n.

e = random.randint(2, phi_n - 1)

while math.gcd(e, phi_n) != 1:

e = random.randint(2, phi_n - 1)

```

# Choose d such that e * d % phi_n = 1.
# Notice that we're using our modular_inverse from our work in the last
chapter!
d = modular_inverse(e, phi_n)

return ((p, q, d), (n, e))

def rsa_encrypt(public_key: tuple[int, int], plaintext: int) -> int:
    """Encrypt the given plaintext using the recipient's public key.

    Preconditions:
        - public_key is a valid RSA public key (n, e)
        - 0 < plaintext < public_key[0]
    """
    n, e = public_key[0], public_key[1]

    encrypted = (plaintext ** e) % n

    return encrypted

def rsa_decrypt(private_key: tuple[int, int, int]  ciphertext: int) -> int:
    """Decrypt the given ciphertext using the recipient's private key.

    Preconditions:
        - private_key is a valid RSA private key (p, q, d)
        - 0 < ciphertext < private_key[0] * private_key[1]
    """
    p, q, d = private_key[0], private_key[1], private_key[2]
    n = p * q

    decrypted = (ciphertext ** d) % n

    return decrypted

def rsa_encrypt_text(public_key: tuple[int, int], plaintext: str) -> str:
    """Encrypt the given plaintext using the recipient's public key.

    Preconditions:
        - public_key is a valid RSA public key (n, e)
        - all({0 < ord(c) < public_key[0] for c in plaintext})

```

```

"""
n, e = public_key

encrypted = ''
for letter in plaintext:
    # Note: we could have also used our rsa_encrypt function here instead
    encrypted = encrypted + chr((ord(letter) ** e) % n)

return encrypted


def rsa_decrypt_text(private_key: tuple[int, int, int], ciphertext: str) ->
str:
    """Decrypt the given ciphertext using the recipient's private key.

    Preconditions:
        - private_key is a valid RSA private key (p, q, d)
        - all({0 < ord(c) < private_key[0] * private_key[1] for c in
ciphertext})
    """

    p, q, d = private_key
    n = p * q

    decrypted = ''
    for letter in ciphertext:
        # Note: we could have also used our rsa_decrypt function here instead
        decrypted = decrypted + chr((ord(letter) ** d) % n)

    return decrypted

```

8.6 Application: Securing Online Communications

- HTTPS is a **communication protocol**
 - **HTTP** (Hypertext Transfer Protocol) - how data is formatted
 - **TLS** (Transfer Layer Security) - how formatted data is encrypted
- **digital signatures**
 - for **encryption/decryption** - encrypt with public key, decrypts with private key

- for **signing/verification** - signs with private key, verify with public key

10. Abstraction, Classes and Software Design

10.1 Introduction

- **interface**
 - for entity - how to use, public side
 - for creator and user - boundary
 - **creator** - responsible for the design of the interface
 - **user** - responsible for **learning** the interface
- we are users of programming languages, built-in function
we are creators of new functions, data types and modules
we are both users and creators of mathematical statement
- specific interface
 - function - header and docstring
 - data class - everything we write to define
 - module - the collection of interfaces of the functions and data
- every interface is a contract between creator and user
 - creator - responsible to make the interface easy and intuitive for users
 - hard to change once released

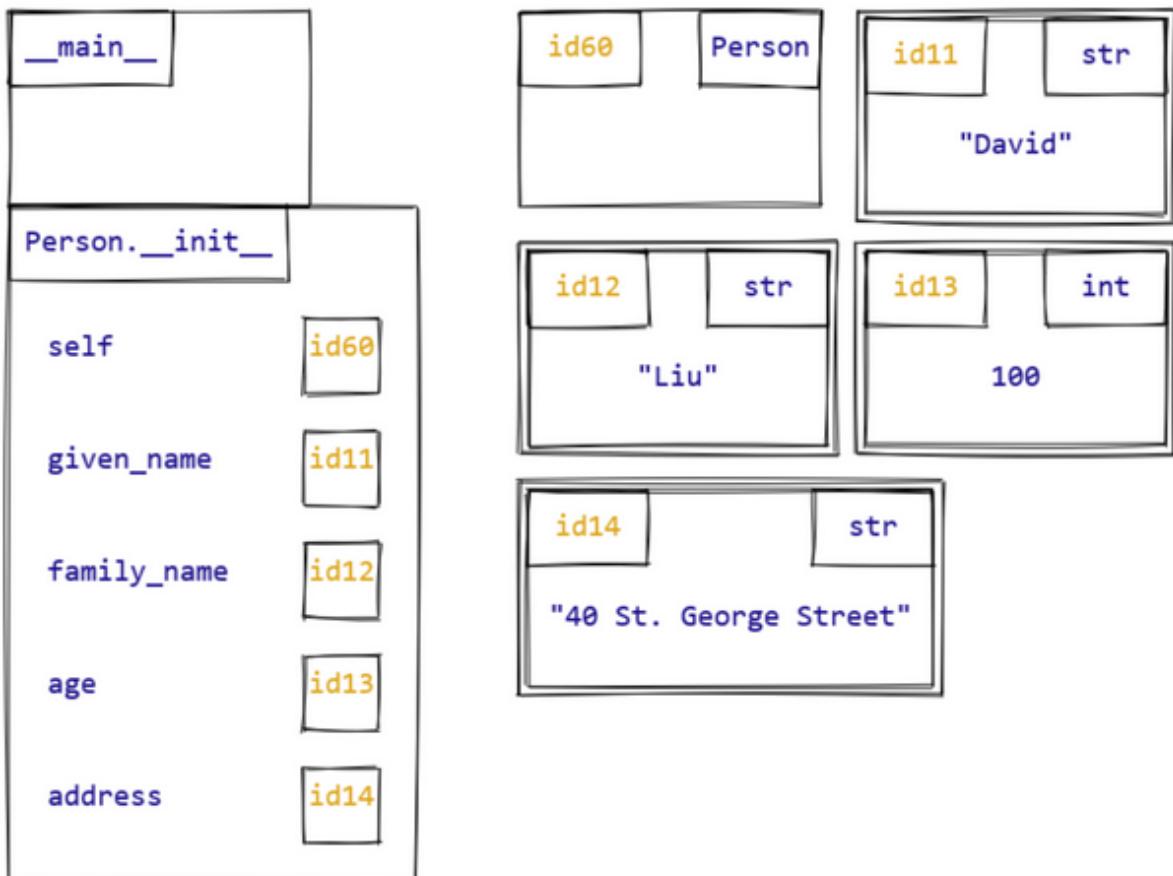
10.2 Defining Our Own Data Types III

- without `@dataclass`
 - `TypeError: <data class> takes no arguments` raised
 - `AttributeError: <dataclass> object has no attribute <instance>` raise - haven't specified values for any of the instance attributes
- **initializer _init_ - initialize all** of the instance attributes for the new object
 - this method definition is intended so that it is inside the body of the class definition

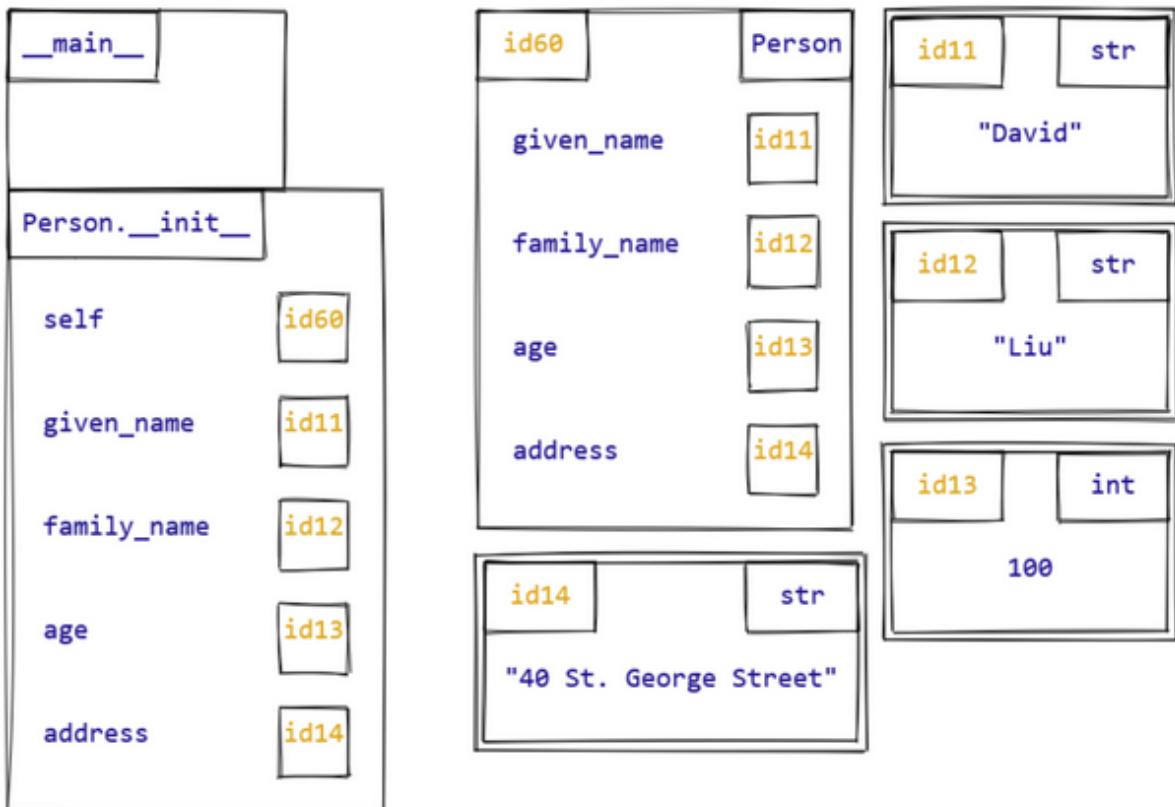
- use `self` refers to the instance that has just been created and is to be initialized
- <parameter> - parameter of the initialize
`self.<parameter>` - instance attribute

```
class Person:  
    """A custom data type that represents data for a person."""  
    given_name: str  
    family_name: str  
    age: int  
    address: str  
  
    def __init__(self, given_name: str, family_name: str, age: int,  
address: str) -> None:  
        """Initialize a new Person object."""  
        self.given_name = given_name  
        self.family_name = family_name  
        self.age = age  
        self.address = address
```

- `_init_` first called - create and initialize the instance attributes



- immediately before the initializer returns



- after called



- actually happen when calling <class>
 - calling <class> **doesn't just cause** `_init_` to be called
 1. create a new <class> object behind the scenes
 2. call <class>`._init_` with the new <class> object passed as the parameter `self`, along with the other arguments (WARNING: **object is not directly returned from the call in this step**)
 3. return the new object
 - `_init_` is a **helper function** whose task is only to initialize attributes for an object.
 - creating the object beforehand and returning the new object **after** `_init_` has been called
- every data type has an initializer (type conversion)

10.3 Defining Our Own Methods

- any function that operates on an instance of a class can be converted into a method
 - indent the function → underneath the instance attributes
 - ensure that the **first parameter** is `self` (NOTE: **without type notation**)

```

class <ClassName>:
    """
    ...
    <instance attributes/types omitted>

    def <method_name>(self, <param>: <type>, ...) -> <return type>:
        """Method docstring"""
        <statement>
        ...

```

- method calls

```

type(obj).method(obj, arg1, arg2, ..., argn)

obj.method(arg1, arg2, ..., argn) # shortcut syntax

```

- reason
 - matches other languages with an object-oriented style of programming
 - only the "object dot notation" style of method call support **inheritance**

10.4 Data Types, Abstract and Concrete

- **concrete data types** (class) - have a concrete implementation in code (int, str,...)
 - every data class is a **class** but **not** every class is a data class
- **abstract data type (ADT)** - an abstract(no code) definition of a data type
 - what data is **stored**
what **operations** can be performed on the data
 - independent of programming language
 - pure interface - only concern with the **what** and **not the how**
- **Set**
 - Data - a collection of unique elements (order doesn't matter)

- Operations - get size, insert a value (without introducing duplicates), remove a specified value, check membership in the set

- **List**

- Data - a sequence of elements (which may or may not be unique)
- Operations - get size, access element by index, insert a value at a given index, remove a value at a given index

- **Mapping**

- Data - a collection of key-value pairs, where each key is unique and associated with a single value
- Operations - get size, lookup a value for a given key, insert a new key-value pair, remove a key-value pair, update the value associated with a given key

- **Iterable**

- Data - a collection of values(may or may not be unique)
- Operations - iterate through the elements of the collection one at a time
- WARNING - there is **not** a one-to-one correspondence between abstract and concrete data type
 - list can be used to implement the Set ADT
 - dict can be used to implement the List ADT
 - every concrete data type can be used to implement multiple ADTs

10.5 Stacks

- **Stack**

- Data - a collection of items
- Operations
 - determine whether the stack is empty
 - add an item (push)
 - remove the **most recently-added** item (pop)
- **Last-In-First-Out (LIFO)** - the first item added is the last removed
- application - **keeping track** of the history of actions so that can undo

```
class Stack1:
    """A last-in-first-out (LIFO) stack of items.

    Stores data in first-in, last-out order. When removing an item from
    the stack, the most recently-added item is the one that is removed.

    >>> s = Stack1()
    >>> s.is_empty()
    True
    >>> s.push('hello')
    >>> s.is_empty()
    False
    >>> s.push('goodbye')
    >>> s.pop()
    'goodbye'
    """
    # Private Instance Attributes:
    #   - _items: The items stored in the stack. The END of the list
    #             represents the top of the stack.
    _items: list

    def __init__(self) -> None:
        """Initialize a new empty stack.
        """
        self._items = [] # Theta(1)

    def is_empty(self) -> bool:
        """Return whether this stack contains no items.
        """
        return self._items == [] # Theta(1)

    def push(self, item: Any) -> None:
        """Add a new element to the top of this stack.
        """
        self._items.append(item) # Theta(1)

    def pop(self) -> Any:
        """Remove and return the element at the top of this stack.
        """
```

```

    Preconditions:
        - not self.is_empty()
    """
    return self._items.pop() # Theta(1)

```

```

class Stack2:
    """A last-in-first-out (LIFO) stack of items.
    Stores data in a last-in, first-out order. When removing an item
    from the
        stack, the most recently-added item is the one that is removed.

    >>> s = Stack()
    >>> s.is_empty()
    True
    >>> s.push('hello')
    >>> s.is_empty()
    False
    >>> s.push('goodbye')
    >>> s.pop()
    'goodbye'
    """

    # Private Instance Attributes:
    #     - items: a list containing the items in the stack. The FRONT of
    #         the list represents the top of the stack.

    _items: list

    def __init__(self) -> None:
        """Initialize a new empty stack."""
        self._items = []

    def is_empty(self) -> bool:
        """Return whether this stack contains no items.
        """
        return self._items == []

    def push(self, item: Any) -> None:
        """Add a new element to the top of this stack."""
        self._items.insert(0, item)
        # Theta(n), where n is the size of the stack

```

```

def pop(self) -> Any:
    """Remove and return the element at the top of this stack.

    Preconditions:
        - not self.is_empty()
    """
    return self._items.pop(0)
# Theta(n), where n is the size of the stack

```

```

# Exercise 1

def size_v1(s: Stack) -> int:
    """Return the number of items in s.
    """
    count = 0
    for _ in s:      # PROBLEM: stacks don't support "for loop iteration"
        count = count + 1
    return count

def size_v2(s: Stack) -> int:
    """Return the number of items in s.
    """
    count = 0
    while not s.is_empty(): # or, while not Stack.is_empty(s)
        s.pop()              # PROBLEM: mutates s!
        count = count + 1
    return count

def size_v3(s: Stack) -> int:
    """Return the number of items in s.
    """
    return len(s._items)

# PROBLEM: tries to access instance attribute _items, which isn't part
# of a Stack!

def size_v4(s: Stack) -> int:
    """Return the number of items in s.
    """
    s_copy = s      # PROBLEM: this creates an alias of s, not a copy!
    count = 0
    while not s_copy.is_empty():

```

```

        s_copy.pop()    # PROBLEM: so mutating s_copy also will mutate s
    (they refer to the same object)
        count += 1
    return count

def size(s: Stack) -> int:
    """Return the number of items in s.
    >>> s = Stack()
    >>> size(s)
    0
    >>> s.push('hi')
    >>> s.push('more')
    >>> s.push('stuff')
    >>> size(s)
    3
    """
    temp_stack = Stack()
    # Count the items in s by popping them off, but store them in
    temp_stack
    count = 0
    while not s.is_empty():
        item = s.pop()
        temp_stack.push(item)
        temp_stack.push(s.pop())
        count = count + 1
    # Restore the items in s by popping them off of temp_stack
    while not temp_stack.is_empty():
        item = temp_stack.pop()
        s.push(item)
    # Return the count
    return count

```

- **public instance attribute** - part of their public interface → users can access this attribute
- **private instance attribute** - not a part of a class's interface → accessed by only the classes' methods
 - move the description from the docstring to the comments in class body
 - name with leading underscore _ (such as _item)

- in Python, PIA can still be accessed by external code
- reduces the cognitive load on the user of the class ("one less thing to worry about") gives freedom to implement for the class to modify or remove this attribute at a later point

10.6 Exceptions as a Part of the Public Interface

- custom exception - descriptive, does not reveal any implementation details
- exception message

```

class <ExceptionClass>(Exception)
    """Exception raised when ..."""

    def __str__(self) -> str
        """Return a string representation of this error."""
        return '<message>'

class EmptyStackError(Exception):
    """Exception raised when calling pop on an empty stack."""

    def __str__(self) -> str:
        """Return a string representation of this error."""
        return 'pop may not be called on an empty stack'

>>> s = Stack()
>>> s.pop()
Traceback (most recent call last):
  File "<input>", line 1, in <module>
    File "...", line 60, in pop
      raise EmptyStackError
EmptyStackError: pop may not be called on an empty stack

```

- testing exception

```
def test_<name>() -> None:
```

```

"""Test that ..."""
<statement>

with pytest.raises(<ExceptionClass>):
    (call function)
# test past when that exception is raised

import pytest

def test_empty_stack_error() -> None:
    """Test that popping from an empty stack raises an exception."""
    s = Stack()

    with pytest.raises(EmptyStackError):
        s.pop()

```

- handling exception
 1. the block of code indented with the try is executed
 2. if no exception occurs when executing this block, the except part is skipped
 3. if an exception occurs when executing this block
 - if the exception has type <ExceptionClass>, the block under the except is executed, and then continues
 - if the exception is a different type, this **does stop** the normal program execution

```

try:
    <statement>
    ...
except <ExceptionClass>:
    <statement>
    ...

def second_from_top(s: Stack) -> Optional[str]:
    """Return the item that is second from the top of s.

    If there is no such item in the Stack, returns None.
    """

```

```

try:
    hold1 = s.pop()
except EmptyStackError:
    # In this case, s is empty. We can return None.
    return None

try:
    hold2 = s.pop()
except EmptyStackError:
    # In this case, s had only one element.
    # We restore s to its original state and return None.
    s.push(hold1)
    return None

    # If we reach this point, both of the previous s.pop() calls
succeeded.

    # In this case, we restore s to its original state and return the
second item.

    s.push(hold2)
    s.push(hold1)

return hold2

```

10.7 Queues

- Queue
 - Data - a collection of items
 - Operations
 - determine whether the queue is empty
 - add an item (enqueue)
 - remove the **least recently-added** item (dequeue)
 - **First-In-First-On (FIFO)** - items come out in the order in which they enter

```

class Queue:
    """A first-in-first-out (FIFO) queue of items.

```

Stores data in a first-in, first-out order. When removing an item from the queue, the most recently-added item is the one that is removed.

```
>>> q = Queue()
>>> q.is_empty()
True
>>> q.enqueue('hello')
>>> q.is_empty()
False
>>> q.enqueue('goodbye')
>>> q.dequeue()
'hello'
>>> q.dequeue()
'goodbye'
>>> q.is_empty()
True
"""

# Private Instance Attributes:
#   - _items: The items stored in this queue. The front of the list
#             represents the front of the queue.
_items: list

def __init__(self) -> None:
    """Initialize a new empty queue."""
    self._items = [] # Theta(1)

def is_empty(self) -> bool:
    """Return whether this queue contains no items.
    """
    return self._items == [] # Theta(1)

def enqueue(self, item: Any) -> None:
    """Add <item> to the back of this queue.
    """
    self._items.append(item)
    # or: list.append(self._items, item)

    # (Analysis) The list.append method takes constant time (Theta(1)).

    # VERSION 2 (insert at front)
    # self._items.insert(0, item)
```

```

# (Analysis) In general, list.insert takes Theta(n - i) time.
# Here, i = 0, so this takes Theta(n) time.

def dequeue(self) -> Any:
    """Remove and return the item at the front of this queue.

    Raise an EmptyQueueError if this queue is empty.
    """

    if self.is_empty():
        raise EmptyQueueError
    else:
        return self._items.pop(len(self._items) // 2)
        # or: list.pop(self._items, 0)

    # (Analysis)
    # In general, list.pop takes Theta(n - i) time.
    # Here, i = 0, so this takes Theta(n) time.

    # VERSION 2 (remove from end)
    # return self._items.pop()
    # Theta(1)

```

Queue Operation	"Front of list" runtime	"Back of list" runtime
enqueue	$\Theta(1)$	$\Theta(n)$
dequeue	$\Theta(n)$	$\Theta(1)$

10.8 Priority Queues

- Priority Queues
 - Data - a collection of times and their priorities
 - Operations
 - determine whether the priority queue is empty
 - add an item with a given priority (enqueue)
 - remove the item with **highest priority** (dequeue)

```

class PriorityQueueSorted:
    """A queue of items that can be dequeued in priority order.

    When removing an item from the queue, the highest-priority item is the
    one that is removed.

    """

    # Private Instance Attributes:
    #   - _items: a list of the items in this priority queue
    _items: list[tuple[int, Any]] = []

    def __init__(self) -> None:
        """Initialize a new and empty priority queue."""
        self._items = []

    def is_empty(self) -> bool:
        """Return whether this priority queue contains no items.

        """
        return self._items == []

    def enqueue(self, priority: int, item: Any) -> None:
        """Add the given item with the given priority to this priority queue.

        """
        i = 0
        while i < len(self._items) and self._items[i][0] < priority:
            # at most n iterations
            # exactly i iterations
            # Loop invariant: all items in self._items[0:i]
            # have a lower priority than <priority>.
            i = i + 1  # constant time

        self._items.insert(i, (priority, item))
        # take at most n steps
        # take exactly n-i steps
        # total 1+n+n = 2n+1 -> Big-O(n)
        # total 1+i+(n-i)=n+1 -> Theta(n)

        # version 2 (this version works if there are no ties in priorities)
        # self._items.append((priority, item))
        # self._item.sort()

    def dequeue(self) -> Any:

```

```

    """Remove and return the item with the highest priority.

    Raise an EmptyPriorityQueueError when the priority queue is empty.
    """

    if self.is_empty():
        raise EmptyPriorityQueueError
    else:
        _priority, item = self._items.pop() # Theta(1)
        return item

class EmptyPriorityQueueError(Exception):
    """Exception raised when calling pop on an empty priority queue."""

    def __str__(self) -> str:
        """Return a string representation of this error."""
        return 'You called dequeue on an empty priority queue.'

```

- `self._items[i][0] < priority` and `self._items[i][0] <= priority`

```

>>> pqueue = PriorityQueue()
>>> pqueue.enqueue(1, 'a')
>>> pqueue.enqueue(5, 'b')
>>> pqueue.enqueue(5, 'c')
>>> pqueue.enqueue(5, 'hi')

self._items[i][0] < priority

pqueue = [(1, 'a'), (5, 'hi'), (5, 'c'), (5, 'b')]

self._items[i][0] <= priority

pqueue = [(1, 'a'), (5, 'b'), (5, 'c'), (5, 'hi')]

```

- second version

```

class PriorityQueueUnsorted:

    ...

def enqueue(self, priority: int, item: Any) -> None:
    """Add the given item with the given priority to this priority
queue.

    """
    self._items.append((priority, item))
    # Note the two pairs of parentheses!
    # Theta(1)

def dequeue(self) -> Any:
    """Remove and return the element of this priority queue with the
highest priority.

    Preconditions:
        - not self.is_empty()

    """
    # 1. Find the maximum priority element of the list
    max_index_so_far = 0 # index of the maximum priority pair so far

    for i in range(1, len(self._items)):
        # If the priority at index i is > the max priority so far,
        # update max_index_so_far

        # current_tuple = self._items[i]
        # max_tuple_so_far = self._items[max_index_so_far]

        if self._items[i][0] > self._items[max_index_so_far][0]:
            max_index_so_far = i

    # 2. Get the pair and remove it from the list.
    max_pair = self._items.pop(max_index_so_far)

    # 3. Return the item
    return max_pair[1]

# (ROUGH analysis, skipping lines 167/183)
# The for loop takes n - 1 steps

```

```

# The list.pop takes (n - max_index_so_far) steps

# So in total, # steps is
# (n - 1) + (n - max_index_so_far)
# = 2n - 1 - max_index_so_far
#
# Since max_index_so_far is a valid index into self._items,
# 0 <= max_index_so_far < n
# So then 2n - 1 - max_index_so_far is Theta(n)

```

Operation	runtime	runtime
enqueue	$\Theta(1)$	$\Theta(n)$
dequeue	$\Theta(n)$	$\Theta(1)$

10.9 Inheritance

- **abstract method** - not implemented
 - raised `NotImplementedError`
 - if implemented → **concrete**
- **abstract class** - a class with **at least** one abstract method (opposite: **concrete class**)
- **inheritance** - subclass and superclass
 - a subclass **must** implement **all abstract** method from the superclass
 - can call the superclass methods on instances of the subclass (without changing public interface)

```
from typing import Any
```

```
class Stack:
    """A last-in-first-out (LIFO) stack of items.
```

This is an abstract class. Only subclasses should be instantiated.
 """

```
def is_empty(self) -> bool:
    """Return whether this stack contains no items.
    """
    raise NotImplementedError

def push(self, item: Any) -> None:
    """Add a new element to the top of this stack.
    """
    raise NotImplementedError

def pop(self) -> Any:
    """Remove and return the element at the top of this stack.

    Raise an EmptyStackError if this stack is empty.
    """
    raise NotImplementedError

class EmptyStackError(Exception):
    """Exception raised when calling pop on an empty stack."""

class Stack1(Stack):
    def __init__(self) -> None:
        """Initialize a new empty stack.
        """
        self._items = []

    def is_empty(self) -> bool:
        """...
        """
        return self._items == []

    def push(self, item: Any) -> None:
        """...
        """
        self._items.append(item)

    def pop(self) -> Any:
        """...
        """
        return self._items.pop()

class Stack2(Stack):
```

```

def __init__(self) -> None:
    """Initialize a new empty stack.
    """
    self._items = []

def is_empty(self) -> bool:
    """...
    return self._items == []

def push(self, item: Any) -> None:
    """...
    self._items.insert(0, item)

def pop(self) -> Any:
    """...
    return self._items.pop(0)

```

- Stack is the **superclass** or **parent class**
- Stack1 and Stack2 are the **subclass** or **child class**
- Stack1 and Stack2 **inherit** from Stack
- Stack defines a **shared public interface** that is **implemented** by Stack1 and Stack2
- **polymorphic** - it can take as input values of different concrete data types

```

def push_and_pop(stack: Stack, item: Any) -> None:
    """Push and pop the given item onto the stack stack."""
    stack.push(item)
    stack.pop()

>>> stack1 = Stack1()
>>> push_and_pop(stack1) # call function in Stack1
>>> stack2 = Stack2()
>>> push_and_pop(stack2) # call function in Stack2

```

- use **specific** class name would make function **not polymorphic**

```
def push_and_pop_alt1(stack: Stack, item: Any) -> None:
    """Push and pop the given item onto the stack stack."""
    Stack.push(stack, item) # NotImplemented error raised
    Stack.pop(stack)

def push_and_pop_alt2(stack: Stack, item: Any) -> None:
    """Push and pop the given item onto the stack stack."""
    Stack1.push(stack, item) # guaranteed to work on Stack1 instances
    Stack1.pop(stack)       # not any other Stack subclass
```

- `type(x) is t` returns whether `x` is an **object** of type `t`
- `isinstance(x, t)` returns whether `x` is an object of type `t` or **any subclass of t**

```
>>> my_stack = Stack1()

>>> type(my_stack) is Stack1
True
>>> isinstance(my_stack, Stack1)
True

>>> type(my_stack) is Stack
False
>>> isinstance(my_stack, Stack)
True
```

10.10 The `object` Superclass

- `object` - the **default superclass** for all classes
 - `object` class is concrete
 - `_init_, _str_ and _eq_` acts as a **default** for all classes
- method inheritance** - `B` is a subclass of `A`

- if A has a method `m` and B **does not** implement the same method, then B inherits the method `m` from A
- All instances of B can call `A.m`
- **method overriding** - B is a subclass of A
 - if A has a method `m` and B **implements** the same method, then B **overrides** the method `m` from A
 - All instances of B can call `B.m`
- **Application Programming Interface** - a public interface that an application provides to allow other programs to interact with it

11. Case Study: Building a Simulation

entities.py

```
# entities.py
from dataclasses import dataclass
from typing import Optional # Needed for the type annotation
import datetime # Needed for the start and end times of the order

@dataclass
class Vendor:
    """A vendor that sells groceries or meals.

    This could be a grocery store or restaurant.

    Instance Attributes:
        - name: the name of the vendor
        - address: the address of the vendor
        - menu: the menu of the vendor with the name of the food item mapping
        to its price
        - location: the location of the vendor as (latitude, longitude)

    Representation Invariants:
        - self.name != ''
        - self.address != ''
        - all(self.menu[item] >= 0 for item in self.menu)
```

```

    - -90.0 <= self.location[0] <= 90.0
    - -180.0 <= self.location[1] <= 180.0
"""

name: str
address: str
menu: dict[str, float]
location: tuple[float, float]

@dataclass
class Customer:
    """A person who orders food.

Instance Attributes:
    - name: the name of the customer
    - location: the location of the customer as (latitude, longitude)
"""

name: str
location: tuple[float, float]

@dataclass
class Courier:
    """A person who delivers food orders from vendors to customers.

Instance Attributes:
    - name: the name of the courier
    - location: the location of the courier as (latitude, longitude)
    - current_order: the order that the courier is currently assigned to,
                      or None if the courier is not assigned to any order

Representation Invariants:
    - (in English)
        IF self.current_order is not None,
        THEN the order's courier is equal to self
        (p => q) <=> ((not p) or q)
    - (in Python)
        (self.current_order is None) or
        (self.current_order.courier == self)

        (self.current_order is None) or
        (self.current_order.courier is self)  <-
            This one is better, because it's checking whether the objects are
            the same (in memory)

```

```

"""
name: str
location: tuple[float, float]
current_order: Optional[Order] = None

@dataclass
class Order:
    """A food order from a customer.

Instance Attributes:
    - customer: the customer who placed this order
    - vendor: the vendor that the order is placed for
    - food_items: a mapping from names of food to the quantity being
ordered
    - start_time: the time the order was placed
    - courier: the courier assigned to this order (initially None)
    - end_time: the time the order was completed by the courier (initially
None)

Representation Invariants:
    - self.food_items != {}
    - all(self.food_items[item] >= 0 for item in self.food_items)
"""

customer: Customer
vendor: Vendor
food_items: dict[str, int]
start_time: datetime.datetime
courier: Optional[Courier] = None    # of type Courier or None
end_time: Optional[datetime.datetime] = None

```

food_delivery_system.py

```

# food_delivery_system.py
import datetime
from typing import Optional

from entities import Courier, Customer, Order, Vendor

```

```

class FoodDeliverySystem:
    """A system that maintains all entities (vendors, customers, couriers,
and orders).

    Representation Invariants:
        - all(vendor == self._vendors[vendor].name for vendor in
self._vendors)
        - all(customer == self._customers[customer].name for customer in
self._customers)
        - all(courier == self._couriers[courier].name for courier in
self._couriers)
    """

    # Private Instance Attributes:
    #     - _vendors: a mapping from vendor name to Vendor object.
    #         This represents all the vendors in the system.
    #     - _customers: a mapping from customer name to Customer object.
    #         This represents all the customers in the system.
    #     - _couriers: a mapping from courier name to Courier object.
    #         This represents all the couriers in the system.
    #     - _orders: a list of all orders (both open and completed orders).
    _vendors: dict[str, Vendor]
    _customers: dict[str, Customer]
    _couriers: dict[str, Courier]
    _orders: list[Order]

    def __init__(self) -> None:
        """Initialize a new food delivery system.

        The system starts with no entities.
        """
        self._vendors = {}
        self._customers = {}
        self._couriers = {}
        self._orders = []

    def add_vendor(self, vendor: Vendor) -> bool:
        """Add the given vendor to this system.

        Do NOT add the vendor if one with the same name already exists.

        Return whether the vendor was successfully added to this system.
        """

```

```
        if vendor.name in self._vendors:
            # Vendor with the same name already exists
            return False
        else:    # Vendor doesn't already exist
            self._vendors[vendor.name] = vendor
            return True
```

```
def add_customer(self, customer: Customer) -> bool:
    """Add the given customer to this system.
```

Do NOT add the customer if one with the same name already exists.

Return whether the customer was successfully added to this system.

```
"""
```

```
if customer.name in self._customers:
    return False
else:
    self._customers[customer.name] = customer
    return True
```

```
def add_courier(self, courier: Courier) -> bool:
    """Add the given courier to this system.
```

Do NOT add the courier if one with the same name already exists.

Return whether the courier was successfully added to this system.

```
"""
```

```
if courier.name in self._couriers:
    return False
else:
    self._couriers[courier.name] = courier
    return True
```

```
def place_order(self, order: Order) -> bool:
    """Add an order to this system.
```

Do NOT add the order if no couriers are available (i.e., are already assigned orders).

- If a courier is available, add the order and assign it a courier, and return True.

```

    - Otherwise, do not add the order, and return False.

"""

# Find an available courier and assign them to the order.
# If there are no couriers available, None is returned.
courier = self._assign_courier(order)

if courier is None:
    return False
else:
    # Only add the order if it was assigned a courier.
    self._orders.append(order)
return True

def _assign_courier(self, order: Order) -> Optional[Courier]:
    """Find an available courier and assign the order to them.

    Return the courier assigned to the order, or None if no courier was
    available.

    Note that this is a private method: it is only meant to be used as a
    helper method for place_order.

    """

    for name in self._couriers:
        courier = self._couriers[name]
        if courier.current_order is None:
            order.courier = courier
            courier.current_order = order

    return courier

    return None

# NOTE: this method does not use "self", so technically could be a top-
level function.

def complete_order(self, order: Order, timestamp: datetime.datetime) ->
None:
    """Record that the given order has been delivered successfully at the
    given timestamp.

    Make the courier who was assigned this order available to take a new
    order.

```

```

Preconditions:
    - order in self._orders
    - order.end_time is None
    - order.start_time < timestamp
"""

# Set the courier's current order back to None
order.courier.current_order = None

# Alternate version:
# assigned_courier = order.courier
# assigned_courier.current_order = None

# Set the order's end time.
order.end_time = timestamp

#####
# Additional methods used for events
#####

def get_vendors(self) -> list[Vendor]:
    """Return a list of all vendors registered with this system."""
    return list(self._vendors.values())

def get_customers(self) -> list[Customer]:
    """Return a list of all customers registered with this system."""
    return list(self._customers.values())

#####
# Additional methods used for example runners
#####

def get_customer(self, name: str) -> Customer:
    """Return the customer with the given name.

    Preconditions:
        - a customer with the given name exists
    """

    return self._customers[name]

def get_vendor(self, name: str) -> Vendor:
    """Return the food vendor with the given name.

```

```

    Preconditions:
    - a vendor with the given name exists
    """
    return self._vendors[name]

def get_courier(self, name: str) -> Courier:
    """Return the courier with the given name.

    Preconditions:
    - a courier with the given name exists
    """
    return self._couriers[name]

```

event.py

```

# event.py
from __future__ import annotations

import datetime
import random

from entities import Order
from food_delivery_system import FoodDeliverySystem

class Event:
    """An abstract class representing an event in a food delivery simulation.

    Instance Attributes:
    - timestamp: the start time of the event
    """
    timestamp: datetime.datetime

    def __init__(self, timestamp: datetime.datetime) -> None:
        """Initialize this event with the given timestamp."""
        self.timestamp = timestamp

    def handle_event(self, system: FoodDeliverySystem) -> list[Event]:
        """Mutate the given food delivery system to process this event.

```

```
(NEW) Return a new list of new events created by processing
this event.

"""

raise NotImplementedError

class NewOrderEvent(Event):
    """An event representing when a customer places an order at a vendor."""

    # Private Instance Attributes:
    #   _order: the new order to be added to the FoodDeliverySystem
    _order: Order

    def __init__(self, order: Order) -> None:
        """Initialize a NewOrderEvent for the given order."""
        self._order = order  # This initializes self._order
        Event.__init__(self, order.start_time)
        # This initializes self.timestamp

        # This works, but is not the best practice
        # self.timestamp = order.start_time

    def handle_event(self, system: FoodDeliverySystem) -> list[Event]:
        """Mutate system by placing an order.

        (NEW) Return a new list of new events created by processing
        this event.

        """

        # NOTE: we should modify this code in case system.place_order returns
        False!

        # What might we do in that case?
        system.place_order(self._order)
        future_complete_time = self.timestamp + datetime.timedelta(minutes=1)
        complete_event = CompleteOrderEvent(future_complete_time, \
                                             self._order)
        return [complete_event]

    def __str__(self) -> str:
        """Return a string representation for this event.

        Useful if we want to call print on the event.

        """


```

```

        return f'{self.timestamp}: New order from {self._order.customer.name}\nfor {self._order.vendor.name}'

class CompleteOrderEvent(Event):
    """An event representing when an order is delivered to a customer by a
    courier."""
    # Private Instance Attributes:
    #   _order: the order to be completed by this event
    _order: Order

    def __init__(self, timestamp: datetime.datetime, order: Order) -> None:
        Event.__init__(self, timestamp)
        self._order = order

    def handle_event(self, system: FoodDeliverySystem) -> list[Event]:
        """Mutate the system by recording that the order has been delivered
        to the customer.

        (NEW) Return a new list of new events created by processing
        this event.
        """
        system.complete_order(self._order, self.timestamp)
        return [] # Trigger no new events

        # Other possibilities:
        # Place an order at the same restaurant in the future
        # Place 5 orders at a competing restaurant

    def __str__(self) -> str:
        """Return a string representation for this event.
        Useful if we want to call print on the event.
        """
        return f'{self.timestamp}: Order from {self._order.customer.name}' \
            f' was completed by {self._order.courier.name}'

class GenerateOrdersEvent(Event):
    """An event that causes a random generation of new orders.

    Private Representation Invariants:
        - self._duration > 0
    """
    # Private Instance Attributes:

```

```

# - _duration: the number of hours to generate orders for
_duration: int

def __init__(self, timestamp: datetime.datetime, duration: int) -> None:
    """Initialize this event with timestamp and the duration in hours.

    Preconditions:
        - duration > 0
    """
    Event.__init__(self, timestamp)
    self._duration = duration

def handle_event(self, system: FoodDeliverySystem) -> list[Event]:
    """Generate new orders for this event's timestamp and duration."""
    # Conceptual idea (simple)
    # new_events = []
    #
    # for i in range(0, self._duration):
    #     new_event = NewOrderEvent(...)
    #     new_events.append(new_event)
    #
    # return new_events

    # Actual implementation
    customers = system.get_customers()
    vendors = system.get_vendors()

    events = [] # Event accumulator
    current_time = self.timestamp
    end_time = self.timestamp + datetime.timedelta(hours=self._duration)

    while current_time < end_time:
        # Create a randomly-generated Order called new_order.
        # Note the use of random.choice, which returns a random element
        # from its argument list
        customer = random.choice(customers)
        vendor = random.choice(vendors)
        food_items = {} # This is a simple version
        new_order = Order(customer=customer, vendor=vendor,
                          food_items=food_items, start_time=current_time)
        new_order_event = NewOrderEvent(new_order)
        events.append(new_order_event)

```

```

        # Update current_time
        current_time = current_time + \
datetime.timedelta(minutes=random.randint(1, 60))

    return events

def __str__(self) -> str:
    """Return a string representation for this event.
    Useful if we want to call print on the event.
    """
    return f'{self.timestamp}: Generating new orders (up to'\
f' {self._duration} hours)'

```

event_queue.py

```

import datetime

from events import Event

class EventQueue:
    """A priority queue of events.

    Events are dequeued in timestamp order (earlier timestamp = higher
    priority).
    """

    def is_empty(self) -> bool:
        """Return whether this event queue contains no items."""
        raise NotImplementedError

    def enqueue(self, event: Event) -> None:
        """Add event to this event queue, sorted by its timestamp."""
        raise NotImplementedError

    def dequeue(self) -> Event:
        """Remove and return the earliest event in this event queue.

        Preconditions:
        - not self.is_empty()
        """

```

```
raise NotImplementedError

class EventQueueList(EventQueue):
    """A queue of events that can be dequeued in timestamp order.
    Note: this is related to the "PriorityQueueSorted" class
    we discussed last week.
    """
    # Private Instance Attributes:
    #     _events: a list of the events in this queue

    _events: list[Event]

    def __init__(self) -> None:
        """Initialize a new and empty event queue."""
        self._events = []

    def is_empty(self) -> bool:
        """Return whether this event queue contains no items."""
        return self._events == []

    def enqueue(self, event: Event) -> None:
        """Add event to this event queue."""
        index = 0
        while index < len(self._events) and \
                self._events[index].timestamp > event.timestamp:
            index = index + 1
        self._events.insert(index, event)

    def dequeue(self) -> Event:
        """Remove and return the earliest event in this event queue.
        Preconditions:
        - not self.is_empty()
        """
        return self._events.pop()
```

example_runner.py

```
import datetime
import random

from entities import Courier, Customer, Order, Vendor
from food_delivery_system import FoodDeliverySystem

def run_example() -> FoodDeliverySystem:
    """This is an example for creating objects in our system."""
    # Create the system
    system = FoodDeliverySystem()

    # Add customers, vendors, and couriers
    for i in range(0, 100):
        customer = Customer(
            name = f'Customer {i}',
            location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0))
        )
        system.add_customer(customer)

        vendor = Vendor(
            name = f'Vendor {i}',
            address = f'{random.randint(1, 1000)} College St.',
            location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0)),
            menu = {'Chocolate': random.uniform(1.0, 100.0)}
        )
        system.add_vendor(vendor)

        courier = Courier(
            name = f'Courier {i}',
            location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0))
        )
        system.add_courier(courier)

    # Place some orders
    for _ in range(0, 10):
```

```

# Pick a random customer and vendor
customer = system.get_customer(f'Customer {random.randint(0, 99)}')
vendor = system.get_vendor(f'Vendor {random.randint(0, 99)}')

# Equivalent to:
# customer = system._customers[f'Customer {random.randint(0, 99)}']
# vendor = system._vendors[f'Vendor {random.randint(0, 99)}']

# Create the order
new_order = Order(
    customer = customer,
    vendor = vendor,
    food_items = {'Chocolate': random.randint(0, 10)},
    start_time = datetime.datetime(2022, 12, 1)
)
system.place_order(new_order)

return system

#####
# Second example, with an event queue
#####
from events import Event, GenerateOrdersEvent
from event_queue import EventQueueList

def run_simulation(initial_events: list[Event],
                   system: FoodDeliverySystem) -> None:
    """Main function for running a simulation."""
    events = EventQueueList()
    for event in initial_events:
        events.enqueue(event)

    while not events.is_empty():
        event = events.dequeue()
        print(event) # This is just for showing each event as it's processed

        new_events = event.handle_event(system)
        for new_event in new_events:
            events.enqueue(new_event)

def run_example_2() -> FoodDeliverySystem:

```

```
"""This is an example for running a discrete event simulation
"""

# Create the system (this part is the same as before)
system = FoodDeliverySystem()

# Add customers, vendors, and couriers
for i in range(0, 100):
    customer = Customer(
        name = f'Customer {i}',
        location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0))
    )
    system.add_customer(customer)

    vendor = Vendor(
        name = f'Vendor {i}',
        address = f'{random.randint(1, 1000)} College St.',
        location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0)),
        menu = {'Chocolate': random.uniform(1.0, 100.0)}
    )
    system.add_vendor(vendor)

    courier = Courier(
        name = f'Courier {i}',
        location = (random.uniform(42.0, 44.0), random.uniform(78.0, \
80.0))
    )
    system.add_courier(courier)

# NEW: create an initial event
initial_event = GenerateOrdersEvent(datetime.datetime(2022, 12, 1), 100)
run_simulation([initial_event], system)

return system
```

Note

- **class composition** - indicates a "has-a" relationship
- When B is a subclass of A and A defines its own `_init_` method:
 - B.`_init_` must call A.`_init_` to initialize all common attributes
 - B.`_init_` is responsible for initializing **any additional** attributes that are **specific** to B

12. Interlude: Nifty Python Features

12.1 Sequences Revisited: Ranges, Indexing, and Slicing

- `range`
 - step `range(start, stop, step)`
 - optional start - the start argument is optional and default `0`
 - `range(i)` is equivalent to `range(0, i)`

```
>>> [x for x in range(0, 10, 2)]
[0, 2, 4, 6, 8]
```

```
>>> [x for x in range(0,5)]
[0, 1, 2, 3, 4]
```

```
>>> [x for x in range(5)]
[0, 1, 2, 3, 4]
```

- negative indexing - access elements offset from the end of the sequence
 - `seq[-i]` is equivalent to `seq[len(seq)-1]`

```
>>> seq = [10, 20, 30, 40]
>>> seq[-1]
```

40

- sequence slicing `seq[i:j]`
 - has the same data type as `seq`
 - equivalent [`seq[x] for x in range(i, j)`]
 - `seq[start:stop:step]`
 - `seq[::-1]` for reversing

```
>>> [10, 20, 30, 40][1:3]
[20, 30]

>>> 'Hello'[:3]
'Hel'

>>> 'David is cool'[10:0:-1]
'oc si diva'
```

12.2 String Interpolation with f-strings

- in an **f-string**, any text surrounded by `{...}` is interpreted as a Python [expression](#)

```
>>> family_name = 'Liu'
>>> given_name = 'David'
>>> student_number = 123456789

>>> f'{family_name}, {given_name} ({student_number})'
'Liu, David (123456789)'

>>> f'{family_name.upper()}'
```

'LIU'

12.3 Function with Optional Parameters

```
# parameter definition with default value syntax
def ...(<parameter_name>: <parameter_type> = <default_value>, ...) -> ...:
```

```
def increment(n: int, step: int = 1) -> int:
    """Return n incremented by step.

    If the step argument is omitted, increment by 1 instead.
    """
    return n + step

>>> increment(10, 2)  # n = 10, step = 2
12
>>> increment(10)      # n = 10
11
```

- in the function header, optional parameters must be written **after mandatory parameters**
- **WARNING** - do not use mutable objects as default values
- every default value is an object that is created when the function is **defined**, not when the function is **called**
- can use None

```
# wrong
def add_num(num: int, numbers: list[int] = []) -> list[int]:
```



```
# right
def add_num(num: int, numbers: Optional[list[int]] = None) -> list[int]:
```

13. Linked List

13.1 Introduction

- use `_<class_name>` to indicate that this entire class is private

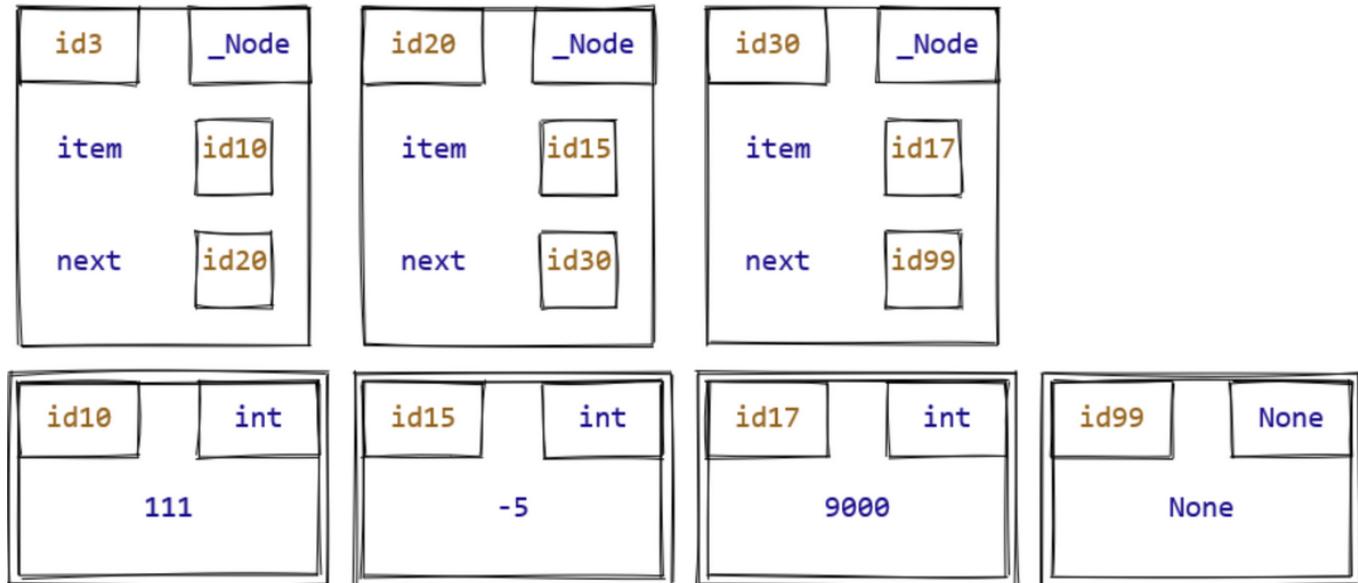
```
from dataclasses import dataclass
from typing import Optional

@dataclass
class _Node:
    """A node in a linked list.

    Instance Attributes:
        - item: The data stored in this node.
        - next: The next node in the list, if any.
    """

    item: Any
    next: Optional[_Node] = None
    # By default, this node does not link to any other node
```

- `_Node` represents a single element of a list



```
class LinkedList:
    """A linked list implementation of the List ADT.
    """

    # Private Instance Attributes:
    # - _first: The first node in this linked list, or None if this list is
    #   empty.
    _first: Optional[_Node]

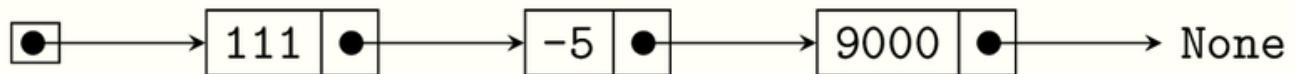
    def __init__(self) -> None:
        """Initialize an empty linked list.
        """

        self._first = None

    >>> linky = LinkedList() # linky is an empty linked list
    >>> linky._first is None
    True
    >>> node1 = _Node(111) # New node with item 111
    >>> node2 = _Node(-5) # New node with item -5
    >>> node3 = _Node(9000) # New node with item 900
    >>> node1.item
    111
    >>> node1.next is None # By default, new nodes do not link to another node
    True
    >>> node1.next = node2 # Let's set some links
    >>> node2.next = node3
    >>> node1.next is node2 # Now node1 links to node2!
    True
    >>> node1.next.item
    -5
    >>> node1.next.next is node3
    True
    >>> node1.next.next.item
    9000
    >>> linky._first = node1 # Finally, set linky's first node to node1
    >>> linky._first.item # linky now represents the list [111, -5, 9000]
    111
    >>> linky._first.next.item
    -5
    >>> linky._first.next.next.item
    9000
```

- node1 is an object `_Node`, `node.item` is the value stored in node

`_first`



13.2 Traversing Linked List

```

# 1. Initialize curr to the start of the list.
curr = my_linked_list._first
# 2. curr is None if we've reached the end of the list.
while curr is not None:
    # 3. Do something with the current *element*, curr.item.
    ... curr.item ...
    # 4. "Increment" curr, assigning it to the next node.
    curr = curr.next

class LinkedList:
    def print_items(self) -> None:
        """Print out each item in this linked list."""
        curr = self._first
        while curr is not None:
            print(curr.item)
            curr = curr.next

    def to_list(self) -> list:
        """Return a built-in Python list containing the items of this linked
list.
    """

```

The items in this linked list appear in the same order in the returned list.

```

"""
items_so_far = []

curr = self._first
while curr is not None:
    items_so_far.append(curr.item)
    curr = curr.next

```

```
    return items_so_far
```

```
from __future__ import annotations
from dataclasses import dataclass
import math
from typing import Any, Optional

@dataclass
class _Node:
    """A node in a linked list.

    Note that this is considered a "private class", one which is only meant
    to be used in this module by the LinkedList class, but not by client
    code.

    Instance Attributes:
        - item: The data stored in this node.
        - next: The next node in the list, if any.
    """
    item: Any
    next: Optional[_Node] = None
    # By default, this node does not link to any other node

class LinkedList:
    """A linked list implementation of the List ADT.

    """
    # Private Instance Attributes:
    #     - _first: The first node in this linked list, or None if this list is
    #       empty.
    _first: Optional[_Node]
    def __init__(self) -> None:
        """Initialize an empty linked list.
        """
        self._first = None

    def sum_items(self) -> int:
        """Return the sum of the items in this linked list.

        Preconditions:
            - all items in this linked list are ints
        """
        return items_so_far
```

```

    sum_so_far = 0
    curr = self._first
    while curr is not None: # or, while not (curr is None):
        sum_so_far = sum_so_far + curr.item
        curr = curr.next
    # Contrast with:
    # i = 0
    # while i < len(self):
    #     sum_so_far = sum_so_far + self[i]
    #     i = i + 1
    return sum_so_far

#####
# Exercise 1: Linked List Traversal
#####

def maximum(self) -> float:
    """Return the maximum element in this linked list.

    Preconditions:
        - every element in this linked list is a float
        - this linked list is not empty
    """
    >>> linky = LinkedList()
    >>> node3 = _Node(30.0)
    >>> node2 = _Node(-20.5, node3)
    >>> node1 = _Node(10.1, node2)
    >>> linky._first = node1
    >>> linky.maximum()
    30.0
    """
    # Implementation note: as usual for compute maximums,
    # import the math module and initialize your accumulator
    # to -math.inf (negative infinity).
    max_so_far = -math.inf
    # Comment: could also initialize to self._first.item
    curr = self._first

    while curr is not None: # or, while not (curr is None):
        if curr.item > max_so_far:
            max_so_far = curr.item
        # Or,
        # max_so_far = max(max_so_far, curr.item)
        curr = curr.next

```

```
    return max_so_far

def __contains__(self, item: Any) -> bool:
    """Return whether item is in this list.

    >>> linky = LinkedList()
    >>> linky.__contains__(10)
    False
    >>> node2 = _Node(20)
    >>> node1 = _Node(10, node2)
    >>> linky._first = node1
    >>> linky.__contains__(20)
    True
    """
    curr = self._first
    while curr is not None:
        # We should be comparing the node's item with item,
        # not the node itself.
        # As written, this comparison will always be False
        # (assuming item isn't a _Node).
        # if curr == item:
        if curr.item == item:
            # We've found the item and can return early.
            return True
        curr = curr.next
    # If we reach the end of the loop without finding the item,
    # it's not in the linked list.
    return False

def __getitem__(self, i: int) -> Any:
    """Return the item stored at index i in this linked list.

    Raise an IndexError if index i is out of bounds.

    Preconditions:
        - i >= 0
    """
    curr = self._first
    curr_index = 0
    while curr is not None:
        if curr_index == i:
            return curr.item
        curr = curr.next
        curr_index += 1
```

```
raise IndexError

# Version 2: not using an early return, but using a "compound loop
# condition"
# HOMEWORK: read about this in 13.2
# curr = self._first
# curr_index = 0 # the index of the current node
#
# # Idea: modify the loop condition so that we stop when EITHER:
# # 1. we reach the end of the list (curr is None)
# # 2. we reach the right index (curr_index == i)
# while not (... or ...):
#     curr = curr.next
#     curr_index = curr_index + 1
#
# # Now, detect which of the two cases we're in (1 or 2)
# # and handle each case separately.
# if ...:
#     ...
# else:
#     ...

# version 2

def __getitem__(self, i: int) -> Any:
    """Return the item stored at index i in this linked list.
    Raise an IndexError if index i is out of bounds.
    Preconditions:
        - i >= 0
    """
    curr = self._first
    curr_index = 0
    while not (curr is not None or curr_index == i):
        curr = curr.next
        curr_index += 1

    if curr is None:
        raise IndexError
    else:
        return curr.item
```

13.3 Mutating Linked List

- "off-by-one" error - loop iterates one too many times

```
class LinkedList:

    def __init__(self, items: Iterable) -> None:
        """Initialize a new linked list containing the given items.
        """
        self._first = None
        for item in items:
            self.append(item)

    def append(self, item: Any) -> None:
        """...
        new_node = _Node(item)

        if self._first is None:
            self._first = new_node
        else:
            curr = self._first
            while curr.next is not None:
                curr = curr.next

        # After the loop, curr is the last node in the LinkedList.
        assert curr is not None and curr.next is None
        curr.next = new_node
```

- in `__init__`, removing the line `self._first = None`
→ error `AttributeError: 'LinkedList' object has no attribute '_first' in append`
- `append` running time analysis

```
assume self.size = n
if branch → 3
else branch:
while loop → n - 1
```

$$\text{total} \rightarrow 1 + 1 + 1 + (n - 1) + 1 = n + 3 \rightarrow \Theta(n)$$

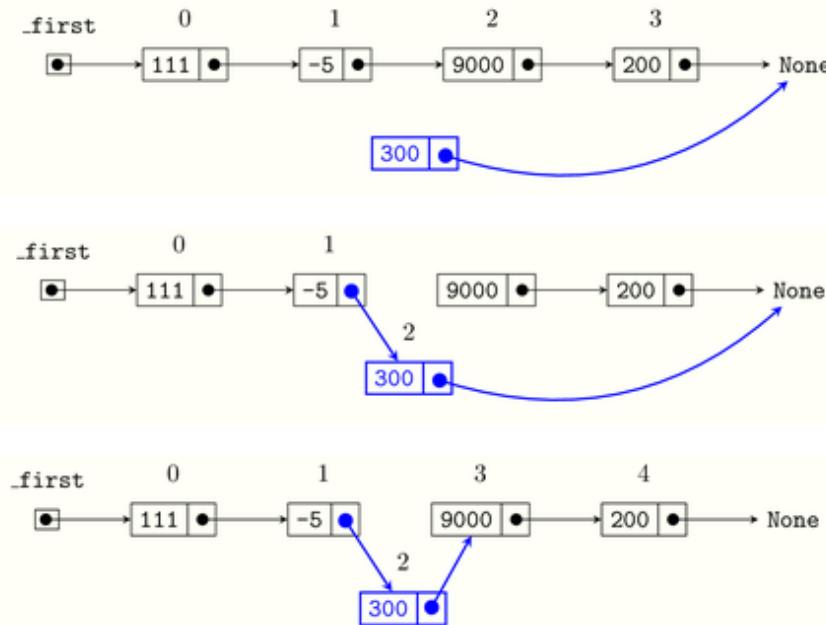
- `_init_` running time analysis

assume the size of items is n

total:

$$1 + 3 + (1 + 3) + (2 + 3) + \dots + (n - 1 + 3) = 4 + \sum_{i=1}^{n-1} (i + 3) = 4 + \frac{(n+6)(n-1)}{2} \rightarrow \Theta(n^2)$$

13.4 Index-Based Mutation



```
class LinkedList:
    def insert(self, i: int, item: Any) -> None:
        """
        ...
        new_node = _Node(item)

        curr = self._first
        curr_index = 0

        while not (curr is None or curr_index == i - 1):
            curr = curr.next
            curr_index = curr_index + 1
```

```

# After the loop is over, either we've reached the end of the list
# or curr is the (i - 1)-th node in the list.
assert curr is None or curr_index == i - 1

if curr is None:
    # i - 1 is out of bounds. The item cannot be inserted.
    raise IndexError
else: # curr_index == i - 1
    # i - 1 is in bounds. Insert the new item.
    new_node.next = curr.next
    curr.next = new_node

```

- **Common Error**

- the **order** in which we update the links **really matters**

```

curr.next = new_node
new_node.next = curr.next

# equivalent to new_node.next = new_node

```

- use parallel assignments to avoid

```
curr.next, new_node.next = new_node, curr.next
```

- corner case $i == 0$

```

...
new_node = _Node(item)

if i == 0:
    # Insert the new node at the start of the linked list.
    self._first, new_node.next = new_node, self._first
else:
    ...
        curr.next, new_node.next = new_node, curr.next

```

- index-based deletion

```

class LinkedList:
    def pop(self, i: int) -> Any:
        """Remove and return item at index i.

        Preconditions:
            - i >= 0

        Raise IndexError if i >= the length of self.

        >>> linky = LinkedList([1, 2, 10, 200])
        >>> linky.pop(2)
        10
        >>> linky.pop(0)
        1
        >>> linky.to_list()
        [2, 200]
        """

        # 1. If the list is empty, you know for sure that index is out of
        bounds...
        curr = self._first
        if curr is None:
            raise IndexError

        # 2. Else if i is 0, remove the first node and return its item.
        else if i == 0:
            self._first = None

```

```

        return curr.item

    # 3. Else iterate to the (i-1)-th node and update links to remove
    # the node at position index. But don't forget to return the item!
    else:
        curr_index = 0
        while not (curr is None or curr_index == i - 1):
            curr = curr.next
            curr_index = curr_index + 1

        assert curr is None or curr_index == i - 1

        if curr is None:
            raise IndexError
        else:
            item = curr.next.item
            curr.next.next, curr.next = None, curr.next.next
            return item

```

13.5 Linked List Running-Time Analysis

- `LinkedList.insert` running time analysis
 - Let n be the length of `self`
 - **Case 1: Assume $i==0$.**
 - the if branch executes, which takes constant time, so we'll count it as one step
 - **Case 2: Assume $i>0$**
 - The first two statements in the else branch take constant time, so we'll count them as 1 step.
 - The statements after the while loop all take constant time, so we'll count them as one step
 - The while loop iterates until either it reaches the end of the list or until it reaches the correct index
 - so happens after n iterations or $i - 1$ iterations
 - the number of iterations taken is $\min(n, i - 1)$
 - each iteration takes 1 step, for a total of $\min(n, i - 1)$ steps

- total running time of $1 + \min(n, i - 1) + 1 = \min(n, i - 1) + 2$ steps
- In the first case, we have a running time of $\Theta(1)$. In the second case, we have a running time of $\Theta(\min(n, i))$. The second expression also becomes $\Theta(1)$ when $i = 0$, and the overall running time is $\Theta(\min(n, i))$
- note: $\min(n, i - 1) \in \Theta(\min(n, i))$ since $i - 1 \in \Theta(i)$
- assume that $0 \leq i < n$ in which case $\min(i, n) = i$, and we get that the running time is $\Theta(i)$. That is a simplification under an additional assumption that $i < n$.
- essentially, we say that if we treat i as small with respect to the size of the list, then the running time of the algorithm does not depend on the size of the list.
 - the most extreme case of this is when $i == 0$, so we're inserting into the front of the linked list. This takes constant time, meaning it does not depend on the length of the linked list.

Operation (assuming $0 \leq i < n$)	Running time (list)	Running time (LinkedList)
Indexing(<code>lst[i]</code>)	$\Theta(1)$	$\Theta(i)$
Insert into index i	$\Theta(n - i)$	$\Theta(i)$
Remove item at index i	$\Theta(n - i)$	$\Theta(i)$

14. Induction and Recursion

14.1 Proof by Induction

- the principle of induction applies to universal statements over the natural numbers $\forall x \in \mathbb{N}, P(n)$.
 - **base case:** prove that $P(0)$ holds
 - **inductive step:** prove that $\forall k \in \mathbb{N}, P(k) \implies P(k + 1)$

Given statement to prove: $\forall n \in \mathbb{N}, P(n)$.

Base case: Let $n = 0$.

[prove that $P(0)$ is true]

Inductive step: Let $k \in \mathbb{N}$, and assume that $P(k)$ is true. (induction hypothesis)

[prove that $P(k + 1)$ is true]

14.2 Recursively-Defined Functions

- recursion definition - f is defined in terms of itself
- f is a **recursively-defined function** when it contains a call to itself in its body
- **recursive call** - inner $f(n-1)$ call
- **recursion** - the programming technique of defining recursive functions to perform computations and solve problems
 - **base case:** does not require any additional "breaking down" of the problem
 - **recursive step:** require the problem to be broken down into an instance of a smaller size
- Python: "When we call $f(_)$, the recursive call $f(_)...$ " → extremely time-consuming and error-prone
- **inductive approach (partial tracing)** - assume that the recursive call returns the correct result, based on the function's specification, and without relying on having explicitly traced that call.

1. when we call $f(100)$, then the recursive call $f(100-1)=f(99)$ is made.

Assuming this call is correct, it returns 4950

2. Then $4950+100=5050$ is returned.

```
def euclidean_gcd_rec(a: int, b: int) -> int:  
    """Return the gcd of a and b (using recursion!).  
  
    Preconditions:  
        - a >= 0 and b >= 0  
    """  
    if b == 0:  
        return a  
    else:  
        return euclidean_gcd_rec(b, a % b)
```

14.4 Nested Lists and Structural Recursion

- **nested list of integers** - as one of two types of values
 - For all $n \in \mathbb{Z}$, the single integer n is a nested list of integers.
 - For all $k \in \mathbb{N}$ and nested lists of integers a_0, a_1, \dots, a_{k-1} , the list $[a_0, a_1, \dots, a_{k-1}]$ is also a nested list of integers.
- $\text{nested_sum}(x) = \begin{cases} x, & \text{if } x \in \mathbb{Z} \\ \sum_{i=0}^{k-1} \text{nested_sum}(a_i), & \text{if } x = [a_0, a_1, \dots, a_{k-1}] \end{cases}$
 - The sum of a nested list that's an **integer** is simply the value of that integer itself.
 - The sum of a nested list of the form $[a_0, a_1, \dots, a_{k-1}]$ is equal to the sum of each of the a'_i 's added together.
- `int | list` means `int` or `list`

```
def sum_nested_v1(nested_list: int | list) -> int:  
    """Return the sum of the given nested list.
```

This version uses a loop to accumulate the sum of the sublists.

```
    """  
  
    if isinstance(nested_list, int):  
        return nested_list  
    else:  
        sum_so_far = 0  
        for sublist in nested_list:  
            sum_so_far += sum_nested_v1(sublist)  
        return sum_so_far
```

```
def sum_nested_v2(nested_list: int | list) -> int:  
    """Return the sum of the given nested list.
```

This version uses a comprehension and the built-in `sum` aggregation function.

```
    """  
  
    if isinstance(nested_list, int):  
        return nested_list  
    else:  
        return sum(sum_nested_v2(sublist) for sublist in nested_list)
```

- suppose we want to trace the call

```
>>> sum_nested_v1([1, [2, [3, 4], 5], [6, 7], 8])
```

36

Iteration	sublist	sum_nested_v1(sublist)	Accumulator sum_so_far
0	N/A	N/A	0
1	1		
2	[2,[3,4],5]		
3	[6,7]		
4	8		

- we will assume that each recursive call is correct (NOTE: this assumption depends only on the specification of `sum_nested_v1` written in its **docstring**, and not its implementation)

Iteration	sublist	sum_nested_v1(sublist)	Accumulator sum_so_far
0	N/A	N/A	0
1	1	1	1
2	[2,[3,4],5]	14(2+3+4+5)	15
3	[6,7]	13(6+7)	28
4	8	8	36

- recursive function design recipe for nested lists

1. Write a doctest example to illustrate the base case of the function, when the function is called on a single `int` value
2. Write a doctest example to illustrate the recursive step of the function

- pick a nested list with around 3 sublists, where at least one sublist is a single `int`, and another sublist is a `list` that **contains** other lists
 - your doctest should show the correct return value of the function for this input nested list
3. Use the following [nested list recursion code template](#) to follow the recursive structure of nested lists
4. **Implement the functions base case**, using your first doctest example to test. Most base cases are pretty straightforward to implement, though this depends on the exact function you're writing
5. Implement the function's recursive step by doing two things:
- Use your **second doctest example** to write down the relevant sublists and recursive function calls (these are the second and third columns of the loop accumulation table we showed above). Fill in the recursive call output based on the function specification, not any code you have written!
 - **Analyze the output of the recursive calls and determine how to combine them to return the correct value for the original call.** This will almost certainly involve some aggregation of the recursive call return values.

14.5 Recursive Lists