Harvard University Extension School
Computer Science E-121

Problem Set 9

Due November 29, 2015 at 11:59 PM.
Submit your solutions electronically on the course website, located at `https://canvas.harvard.edu/courses/4896/assignments`. On the site, use the assignments tab to find the correct problem set, then with the "submit assignment" button, upload the PDF file of your solution.

LATE PROBLEM SETS WILL NOT BE ACCEPTED.

See the syllabus for the collaboration policy.

PROBLEM 1 (5 points, suggested length of 1/3 of a page)

Show that if $\mathcal{P} = \mathcal{NP}$ then every non-trivial language $L \in \mathcal{NP}$ is $\mathcal{NP}$-complete. (Non-trivial languages are languages other than $\Sigma^*, \emptyset$)

PROBLEM 2 (10 points, suggested length of 1 page)

A *pattern* is a string over the alphabet $\{0, 1, ?\}$. Say that a pattern π *covers* a string w over $\{0, 1\}$ if w can be obtained from π by replacing each occurrence of ? in π with either 0 or 1. (For example, 0??1 covers the four strings 0001, 0011, 0101, 0111.)

We define the problem Pattern as the set $\{\Pi : \Pi$ is a set of patterns, each of length n, such that there exists a string s of length n over $\{0, 1\}$ where no pattern in Π covers $s\}$.

Show that Pattern is $\mathcal{NP}$-complete. (*Hint*: reduce from n-variable 3SAT.)

PROBLEM 3 ((3)+5 points, suggested length of 1 page)

Note: Part A of this problem is optional extra credit; Part B is required.

If a divorce is friendly, the spouses often have the problem of trying to divide up the property evenly. If the divorce is hostile, instead of dividing the property two ways, the property is often divided three ways: between the first spouse, the second spouse, and the various lawyers. Define Amicable Divorce as follows: Given a set of numbers, is it possible to split the set into two subsets such that every element of the original set is in exactly one of the two subsets and the sum of each subset is the same? Define Hostile Divorce as follows: Given a set of numbers, is it possible to split the set into three subsets such that every element of the original set is in exactly one of the subsets and the sum of each subset is the same?

(A) (CHALLENGE! Extra Credit) Show that Amicable Divorce is $\mathcal{NP}$-complete.

(B) Assuming that Amicable Divorce is $\mathcal{NP}$-complete, show that Hostile Divorce is $\mathcal{NP}$-complete.

PROBLEM 4 (3+8+3 points, suggested length of 1 page)

A language is $\mathcal{NP}$*-hard* if all problems in $\mathcal{NP}$ $\leq_p$-reduce to it. (Recall that $\mathcal{NP}$-completeness also requires that the language is in $\mathcal{NP}$.) Let:

$$A = \{\langle M\rangle \,:\, M \text{ is a DFA and } M \text{ accepts some string.}\}$$

$$B = \{\langle M_1, M_2, ..., M_k\rangle \,:\, \text{Each } M_i \text{ is a DFA and all of the } M_i \text{ accept some common string.}\}$$

(A) Show that $A \in \mathcal{P}$.

(B) Show that B is $\mathcal{NP}$-hard by giving a reduction from some $\mathcal{NP}$-complete problem to B.

(C) We can convert an instance of B into an instance of A by applying the product construction (See p.45-46 of *Sipser*) $k-1$ times in succession. Does this show that $\mathcal{P} = \mathcal{NP}$? Why or why not?

PROBLEM 5 (6 points, suggested length of 1/2 page)

Let DoubleSAT $= \{\phi \,:\, \phi$ has at least two satisfying assignments$\}$. Show that DoubleSAT is $\mathcal{NP}$-complete.

PROBLEM 6 (8 points, suggested length of 1/2 page)

A strong nondeterministic TM is one that has three possible halt states: "yes", "no", or "maybe". We say such a machine decides L in polynomial time iff all computations run in polynomial time, and if the following holds: if $x \in L$, then all computations end up with "yes" or "maybe," and at least one ends up "yes". If $x \notin L$, then all computations end up in "no" or "maybe", but at least one ends up "no". Prove that L is decided by a strong nondeterministic TM in polynomial time if and only if $L \in \mathcal{NP} \cap \text{co-}\mathcal{NP}$.