Harvard University Extension School
Computer Science E-121

Problem Set 8

Due November 20, 2015 at 11:59 PM.
Submit your solutions electronically on the course website, located at `https://canvas.harvard.edu/courses/4896/assignments`. On the site, use the assignments tab to find the correct problem set, then with the "submit assignment" button, upload the PDF file of your solution.

LATE PROBLEM SETS WILL NOT BE ACCEPTED.

See the syllabus for the collaboration policy.

PROBLEM 1 (2+2+2+2+2+2 points, suggested length of 2/3 page)

For each part (A) through (F), answer TRUE or FALSE and briefly justify your answers.

(A) $n^6 - n^4 = \Theta(n^6 - 5n^3)$.

(B) $\sqrt{n} = \Omega(n^{\sin n})$.

(C) $n! = O(2^n)$.

(D) $n^{100001} = o(1.00001^n)$. (Hint: L'Hôpital's rule)

(E) $3^n = 5^{O(n)}$.

(F) Suppose that $f(n) = g(O(n))$. Give a counterexample to prove that this does not necessarily imply that $f(n) = O(g(n))$, and show why your counterexample works.

PROBLEM 2 (6 points, suggested length of 1/3 page)

Two languages H and K are *recursively separable* if there exists some recursive language R such that $H \subseteq R$ and $K \subseteq \overline{R}$. Prove that the languages $L_1 = \{\langle M \rangle : M \text{ accepts } \langle M \rangle\}$ and $L_2 = \{\langle M \rangle : M \text{ rejects } \langle M \rangle\}$ are not recursively separable.

PROBLEM 3 (6 points, suggested length of 1/3 page)

A function $f : \Sigma^* \to \Sigma^*$ is *computable* if there exists a Turing Machine M that, when given $s \in \Sigma^*$ as input, halts with $f(s)$ written on the tape. The *range* of f is $\{f(x) : x \in \Sigma^*\}$. Prove that a nonempty language is r.e. if and only if it is the range of a computable function.

PROBLEM 4 (CHALLENGE! (3 extra credit points) points, suggested length of 1/2 page)

In this problem you will define a function that grows faster than any computable function. Thus you will prove the existence of an uncomputable function directly, without relying on Turing's diagonalization argument.

The **busy-beaver function** $\beta(n)$ is the largest number of a's that can be printed by any n-state, two-symbol Turing machine that eventually halts when started from the empty tape.

Show that β is not computable.

PROBLEM 5 (6+6 points, suggested length of 1/2+2/3 pages)

Prove that the class $\mathcal{P}$ is closed under:

(A) Concatenation.

(B) Kleene star. (*Hint:* Look at the algorithm we gave in class for recognizing context-free languages via Chomsky Normal Form.)

PROBLEM 6 (8 points, suggested length of .5 pages)

Prove that $\mathrm{ALL}_{\mathrm{DFA}} = \{\langle D \rangle : D \text{ is a DFA and } L(D) = \Sigma^*\}$ is in $\mathcal{P}$.