Exercise Class 11

Modes of Convergence

If one has a sequence of complex numbers $(x_n)_{n\in\mathbb{N}}$, it is unambiguous what it means for that sequence to converge to a limit $x \in \mathbb{R}$. More generally, if we have a sequence $(v_n)_{n\in\mathbb{N}}$ of d-dimensional vectors in a real vector space $\mathbb{R}^n$, it is clear what it means for a sequence to converge to a limit. We usually consider convergence with respect to the Euclidean norm, but for the purposes of convergence, these norms are all equivalent[1].

If, however, one has a sequence of real-valued functions $(f_n)_{n\in\mathbb{N}}$ on a common domain Ω and a perceived limit f, there can now be many different ways how f_n may or may not converge to f. Since the function spaces we consider are infinite dimensional, the functions f_n have an infinite number of degrees of freedom, and this allows them to approach f in any number of inequivalent ways.

We now introduce different convergence concepts for sequences of measurable functions and then compare them to each other.

Definition 1: Modes of Convergence

Let $(f_n)_{n\in\mathbb{N}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable functions. Then we say that (f_n) converges to f

– *μ-almost everywhere* (μ-a.e.) if there is a measurable set N with $\mu(N) = 0$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in N^c.$$

We write $f_n \to f$ μ-a.e.

– *in measure* μ if for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in \Omega : \big|f(x) - f_n(x)\big| > \varepsilon\}) = 0.$$

We write $f_n \xrightarrow{\mu} f$.

– *in* $L^1(\Omega, \mu)$ if

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\Omega,\mu)} := \lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0.$$

We write $f_n \xrightarrow{L^1} f$.

If there is no confusion as to what the space (Ω, μ) is, one often just speaks of L^1 convergence (instead of $L^1(\Omega, \mu)$ convergence).

Remark. The L^1 mode of convergence is a special case of the L^p mode of convergence, which is just convergence with respect to the L^p norm.

One particular advantage of L^1 convergence is that, in the case when the f_n are μ-summable, it implies convergence of the integrals

$$\int_\Omega f_n d\mu \to \int_\Omega f d\mu.$$

[1] In fact, on a finite dimensional vector space, any two norms are equivalent.

This follows directly by the triangle inequality, i.e.

$$\left|\int_\Omega f_n d\mu - \int_\Omega f d\mu\right| \leq \int_\Omega |f_n - f|\, d\mu.$$

Proposition 2: Simple Implications

Convergence in $L^1(\Omega,\mu)$ both implies convergence in measure μ. Moreover, if $\mu(\Omega) < \infty$, then convergence μ-a.e. implies convergence in measure μ too.

Proof. By replacing f_n with $f_n - f$, we can assume that $f \equiv 0$ without loss of generality.

1. Recall Chebyshev's inequality which states that for every μ-summable $f : \Omega \to \overline{\mathbb{R}}$, we have
$$\mu(x \in \Omega : \left|f(x)\right| > a) \leq \frac{1}{a}\int_\Omega |f|\, d\mu \quad \forall a > 0.$$
It immediately follows that for all $\varepsilon > 0$
$$\mu(\{x \in \Omega : |f_n| > \varepsilon\}) \leq \frac{1}{\varepsilon}\int_\Omega |f_n|\, d\mu = \frac{1}{\varepsilon}\|f_n\|_{L^1}\,.$$
So $L^1(\Omega,\mu)$-convergence implies convergence in measure.

2. From Egorov's theorem, it follows that for every $\delta > 0$, there exists $F_\delta \subset \Omega$ μ-measurable with $\mu(\Omega \setminus F_\delta) < \delta$ such that $(f_n)_n$ converges uniformly to f on F_δ. In other words,
$$\forall \varepsilon > 0\ \exists N \geq 0 : \sup_{x \in F_\delta}\left|f_n(x) - f(x)\right| < \varepsilon \quad \forall n \geq N.$$
Therefore, for $n \geq N$
$$\{x \in \Omega : \left|f_n(x) - f(x)\right| > \varepsilon\} \subset \Omega \setminus F_\delta.$$
Hence
$$\mu(\{x \in \Omega : \left|f_n(x) - f(x)\right| > \varepsilon\}) \leq \mu(\Omega \setminus F_\delta) < \delta.$$
Since $\delta > 0$ was arbitrary, we can conclude.

Alternatively, the latter can be proven by applying the dominated convergence theorem to the integral of $\mathbb{1}_{\{|f_n - f| > \varepsilon\}}$, which is dominated by 1 on a finite measure space. □

All other implications between different convergence concepts are not true in general.

Example 1. As the sequence $f_n = \mathbb{1}_{[n,n+1]}$ shows, the finiteness assumption in the second implication is necessary. The sequence f_n converges to 0 pointwise (and thus μ-a.e.), however it does not converge in measure.

Example 2. Let $\Omega = [0,1]$ and λ be the Lebesgue measure.

– (Convergence μ-a.e. $\not\Rightarrow$ L^1 convergence)
The sequence $f_n := n\mathbb{1}_{(0,\frac{1}{n})}$, $n \in \mathbb{N}$, converges to 0 pointwise, hence also λ-a.e. and hence in measure (since we are dealing with a finite measure space). But $\int f_n d\lambda = 1$ and thus (f_n) does not converge to 0 in $L^1([0,1],\lambda)$.

– (L^1 convergence $\not\Rightarrow$ convergence μ-a.e.)
For $n \in \mathbb{N}$ and $k = 1, \ldots, 2^n$, define $f_{nk} := \mathbb{1}_{[(k-1)2^{-n}, k2^{-n}]}$. We then renumber the double sequence $f_{11}, f_{12}, f_{21}, \ldots$ to a single sequence $(g_m)_{m \in \mathbb{N}}$. Then we have $\int f_{nk} d\lambda = 2^{-n}$ and hence $g_m \to 0$ in $L^1([0,1], \lambda)$ for $m \to \infty$. Thus, we also have $g_m \to 0$ in measure. However, $\limsup_{m\to\infty} g_m = 1$ and $\liminf_{m\to\infty} g_m = 0$ show that (g_m) does not converge to 0 λ-a.e. Intuitively, this is a sequence of indicator functions of intervals of decreasing length, marching across the unit interval over and over again. This sequence is also known as the *typewriter sequence*.

Example 2 shows, in particular, that convergence in measure is a strictly weaker notion than the other two, as convergence in measure implies neither of them.

Convergence μ-a.e. and convergence in $L^1(\Omega, \mu)$ do not seem to be related. However, if one imposes additional assumptions (such as $\mu(\Omega) < \infty$) to shut down these "escape to infinity" scenarios, then one can obtain some additional implications between the different concepts.

The dominated convergence theorem of Lebesgue states that μ-a.e. convergence together with the existence of a μ-summable bound for a sequence of measurable functions imply convergence in $L^1(\Omega, \mu)$. These conditions are only sufficient, but not necessary. Thus it is of interest to look for an even sharper result.

Example 3. Let $\Omega = [0,1]$ and consider the Lebesgue measure λ. We define the functions

$$f_n := \frac{n}{\log(n)} \mathbb{1}_{(0, \frac{1}{n}]} \quad \forall n \geq 1.$$

Then we have $f_n \to 0$ pointwise and hence also λ-a.e. Moreover

$$\int_{[0,1]} f_n d\lambda = \frac{1}{\log(n)} \to 0$$

so that $f_n \to 0$ in $L^1([0,1], \lambda)$ since $f_n \geq 0$. However, there exists no μ-summable function g with $g \geq f_n$ λ-a.e. for all n.

Indeed, such a g would have to satisfy $g \geq \frac{n}{\log(n)}$ λ-a.e. on $(0, \frac{1}{n}]$ for all n. But then

$$\int_{[0,1]} g \mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n}]} d\lambda \geq \frac{n}{\log(n)} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{(n+1)\log(n)}$$

and hence

$$\int_{[0,1]} g d\lambda = \sum_{n=1}^{\infty} \int_{[0,1]} g \mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n}]} d\lambda \geq \sum_{n=2}^{\infty} \frac{1}{n \log(n)} = \infty.$$

Definition 3: Uniform Summability

The family $(f_n)_{n \in \mathbb{N}}$ is called *uniformly μ-summable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and $A \subset \Omega$ μ-measurable with $\mu(A) < \delta$ it holds

$$\int_A |f_n| \, d\mu < \varepsilon.$$

This allows us to formulate a necessary and sufficient condition for L^1 convergence.

Theorem 4: Vitali Convergence Theorem

If $\mu(\Omega) < \infty$, the following conditions are equivalent:

i) $f_n \to f$ in $L^1(\Omega, \mu)$.

ii) $f_n \xrightarrow{\mu} f$ and $(f_n)_{n\in\mathbb{N}}$ is uniformly μ-summable.

As we hinted at in the remark, L^1-convergence is just one particular case of a more general concept called L^p-convergence. We will discuss this concept now without really introducing L^p-spaces.

Definition 5: L^p convergence

Let $p \in [1, \infty)$. For $f : \Omega \to \overline{\mathbb{R}}$, we define the $L^p(\Omega, \mu)$ norm by

$$\|f\|_{L^p(\Omega,\mu)} = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}} \leq \infty.$$

For $p = \infty$, we define the $L^\infty(\Omega, \mu)$ norm by

$$\|f\|_{L^\infty(\Omega,\mu)} := \mu\text{-ess}\sup_{x\in\Omega} \big| f(x) \big| = \inf\{C : |f| \leq C \ \mu\text{-a.e.}\}.$$

For $1 \leq p \leq \infty$, we say that a sequence of μ-measurable functions $(f_n)_{n\in\mathbb{N}}$ converges in $L^p(\Omega, \mu)$ to a measurable function f if

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega,\mu)} = 0.$$

For $1 \leq p < \infty$, $L^p(\Omega, \mu)$ convergence is the same as $|f_n - f|^p \to 0$ in $L^1(\Omega, \mu)$; for $p = \infty$, it means that there exists a set N with $\mu(N) = 0$ such that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on N^c.

Proposition 6

If $\mu(\Omega) < \infty$, then for $1 \leq r < s \leq \infty$, we have

$$L^s(\Omega, \mu) \subset L^r(\Omega, \mu).$$

In particular, we have that convergence in $L^s(\Omega, \mu)$ implies convergence in $L^r(\Omega, \mu)$.

Assume that $\mu(\Omega) < \infty$ and $1 \leq r \leq s \leq \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. To summarize, we have the following implications:

$$\begin{tikzcd}
L^s \arrow[r, Rightarrow, "r\leq s"] & L^r \arrow[r, Rightarrow] & L^1 \arrow[d, Rightarrow] \\
& \mu\text{-a.e.} \arrow[r, Rightarrow] & \text{in measure } \mu \arrow[u, Rightarrow, dashed, bend right, "+\text{ uniformly } \mu\text{-summable}"']
\end{tikzcd}$$