I. MODEL HAMILTONIAN IN MOMENTUM SPACE

Let us consider our three-dimensional bosonic Hamiltonian. Here, the bosonic tight-binding Hamiltonian reads

$$
\begin{aligned}
\hat{\mathcal{H}} &= \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{int}, \\
\hat{\mathcal{H}}_0 &= \sum_l \sum_{\boldsymbol{k}} \left(f_{\boldsymbol{k}}^{A_{1g}} - \mu_0 \right) \hat{a}^\dagger_{\boldsymbol{k},l} \hat{a}_{\boldsymbol{k},l}, \\
f_{\boldsymbol{k}}^{A_{1g}} &= \mathsf{d}_\parallel \left(2 - \cos(k_x a) - \cos(k_y a) \right) + \mathsf{d}_z \left(1 - \cos(k_z c) \right),
\end{aligned}
$$

where $\hat{a}^\dagger_{\boldsymbol{k},l}$ ($\hat{a}_{\boldsymbol{k},l}$) creates (annihilates) a boson with momentum $\boldsymbol{k} = (k_x, k_y, k_z)$ of the degenerate component $l \in \{1, 2\}$. The interaction term is on-site only, and in real space it reads

$$
\hat{\mathcal{H}}_{int}^{\mathsf{D_{4h}}} = \frac{(u+w)}{2} \sum_{\boldsymbol{r}} \sum_l \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}_{\boldsymbol{r},l} \hat{a}_{\boldsymbol{r},l} - \frac{v}{2} \sum_{\boldsymbol{r}} \sum_l \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}_{\boldsymbol{r},\bar{l}} \hat{a}_{\boldsymbol{r},\bar{l}} + \frac{u+v-w}{2} \sum_{\boldsymbol{r}} \sum_l \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}^\dagger_{\boldsymbol{r},\bar{l}} \hat{a}_{\boldsymbol{r},l} \hat{a}_{\boldsymbol{r},\bar{l}}.
$$

We aim to decouple it in terms of the density channels

$$
n_{l,l'}(\boldsymbol{r}) = \langle \hat{a}^\dagger_{\boldsymbol{r},l} \hat{a}_{\boldsymbol{r},l'} \rangle, \qquad n_{l,l'}(\boldsymbol{q}) = \frac{1}{N_x N_y N_z} \sum_{\boldsymbol{k}} \langle \hat{a}^\dagger_{\boldsymbol{k}-\boldsymbol{q},l} \hat{a}_{\boldsymbol{k},l'} \rangle, \qquad \text{with} : n^*_{l,l'}(\boldsymbol{q}) = n_{l',l}(-\boldsymbol{q}),
$$

where we employed the Fourier transformation

$$
\hat{a}_{\boldsymbol{r},l} = \frac{1}{\sqrt{N_x N_y N_z}} \sum_{\boldsymbol{k}} e^{-\mathrm{i}\boldsymbol{r}\boldsymbol{k}} \hat{a}_{\boldsymbol{k},l}.
$$

A. Decoupling

For the mean-field decoupling we replace

$$
\hat{a}^\dagger_{\boldsymbol{k},l} \hat{a}_{\boldsymbol{k}',l'} \longrightarrow \langle \hat{a}^\dagger_{\boldsymbol{k},l} \hat{a}_{\boldsymbol{k}',l'} \rangle + \underbrace{\left(\hat{a}^\dagger_{\boldsymbol{k},l} \hat{a}_{\boldsymbol{k}',l'} - \langle \hat{a}^\dagger_{\boldsymbol{k},l} \hat{a}_{\boldsymbol{k}',l'} \rangle \right)}_{=\delta},
$$

and we strictly neglect the terms of the order $O(\delta^2)$. Then, we rewrite an interaction term $I_0 = \sum_{\boldsymbol{r}} \hat{a}^\dagger_{\boldsymbol{r},l_1} \hat{a}^\dagger_{\boldsymbol{r},l_2} \hat{a}_{\boldsymbol{r},l_3} \hat{a}_{\boldsymbol{r},l_4}$ in momentum space

$$
\begin{aligned}
I_0 &= \frac{1}{\sqrt{N_x N_y N_z}^4} \sum_{\boldsymbol{r}} \sum_{\boldsymbol{k}_1,..,\boldsymbol{k}_4} e^{\mathrm{i}\boldsymbol{r}(\boldsymbol{k}_4 + \boldsymbol{k}_3 - \boldsymbol{k}_2 - \boldsymbol{k}_1)} \hat{a}^\dagger_{\boldsymbol{k}_4,l_1} \hat{a}^\dagger_{k_3,l_2} \hat{a}_{\boldsymbol{k}_2,l_3} \hat{a}_{\boldsymbol{k}_1,l_4} \\
&= \frac{1}{2 N_x N_y N_z} \left(\sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2} \hat{a}^\dagger_{\boldsymbol{k}_1,l_1} \hat{a}^\dagger_{\boldsymbol{k}_2,l_2} \hat{a}_{\boldsymbol{k}_1-\boldsymbol{q},l_3} \hat{a}_{\boldsymbol{k}_2+\boldsymbol{q},l_4} + \sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2} \hat{a}^\dagger_{\boldsymbol{k}_1,l_1} \hat{a}^\dagger_{\boldsymbol{k}_2,l_2} \hat{a}_{\boldsymbol{k}_2-\boldsymbol{q},l_3} \hat{a}_{\boldsymbol{k}_1+\boldsymbol{q},l_4} \right) \\
&= I_0^r + \frac{1}{2 N_x N_y N_z} \left(\sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2} \hat{a}^\dagger_{\boldsymbol{k}_1,l_1} \hat{a}_{\boldsymbol{k}_1-\boldsymbol{q},l_3} \hat{a}^\dagger_{\boldsymbol{k}_2,l_2} \hat{a}_{\boldsymbol{k}_2+\boldsymbol{q},l_4} + \sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2} \hat{a}^\dagger_{\boldsymbol{k}_1,l_1} \hat{a}_{\boldsymbol{k}_1+\boldsymbol{q},l_4} \hat{a}^\dagger_{\boldsymbol{k}_2,l_2} \hat{a}_{\boldsymbol{k}_2-\boldsymbol{q},l_3} \right), \\
I_0^r &= -\frac{1}{4} \left(\delta_{l_3,l_2} \sum_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k},l_1} \hat{a}_{\boldsymbol{k},l_4} + \delta_{l_3,l_1} \sum_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k},l_2} \hat{a}_{\boldsymbol{k},l_4} + \delta_{l_4,l_2} \sum_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k},l_1} \hat{a}_{\boldsymbol{k},l_3} + \delta_{l_4,l_1} \sum_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k},l_2} \hat{a}_{\boldsymbol{k},l_3} \right),
\end{aligned}
$$

and then, we decouple it according to

$$I_0 \to I_0^r + \frac{1}{2N_xN_yN_z}\sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2}\Big(-\langle\hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1-\boldsymbol{q},l_3}\rangle\langle\hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2+\boldsymbol{q},l_4}\rangle + \hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1-\boldsymbol{q},l_3}\langle\hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2+\boldsymbol{q},l_4}\rangle + \hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2+\boldsymbol{q},l_4}\langle\hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1-\boldsymbol{q},l_3}\rangle\Big)$$
$$+\frac{1}{2N_xN_yN_z}\sum_{\boldsymbol{q},\boldsymbol{k}_1,\boldsymbol{k}_2}\Big(-\langle\hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1+\boldsymbol{q},l_4}\rangle\langle\hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2-\boldsymbol{q},l_3}\rangle + \langle\hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2-\boldsymbol{q},l_3}\rangle\hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1+\boldsymbol{q},l_4} + \langle\hat{a}^\dagger_{\boldsymbol{k}_1,l_1}\hat{a}_{\boldsymbol{k}_1+\boldsymbol{q},l_4}\rangle\hat{a}^\dagger_{\boldsymbol{k}_2,l_2}\hat{a}_{\boldsymbol{k}_2-\boldsymbol{q},l_3}\Big)$$
$$= I_0^r + \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\Big(n_{l_2,l_4}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l_1}\hat{a}_{\boldsymbol{k}',l_3} + n_{l_1,l_3}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l_2}\hat{a}_{\boldsymbol{k}',l_4} + n_{l_2,l_3}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l_1}\hat{a}_{\boldsymbol{k}',l_4} + n_{l_1,l_4}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l_2}\hat{a}_{\boldsymbol{k}',l_3}\Big)$$
$$-\frac{N_xN_yN_z}{2}\sum_{\boldsymbol{q}}\big(n_{l_1,l_3}(-\boldsymbol{q})n_{l_2,l_4}(\boldsymbol{q}) + n_{l_1,l_4}(\boldsymbol{q})n_{l_2,l_3}(-\boldsymbol{q})\big).$$

Now, we apply the decoupling on the interaction Hamiltonian to obtain

$$\hat{\mathcal{H}}^{\mathrm{D}_{4h}}_{int} \to \frac{(u+w)}{4}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{l}\Big(4n_{l,l}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k}',l}\Big) - \frac{v}{4}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{l}\Big(4n_{l,\bar{l}}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k}',\bar{l}}\Big)$$
$$+\frac{u+v-w}{4}\sum_{l}\Big(2n_{l,l}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},\bar{l}}\hat{a}_{\boldsymbol{k}',\bar{l}} + 2n_{l,\bar{l}}(\boldsymbol{k}-\boldsymbol{k}')\hat{a}^\dagger_{\boldsymbol{k},\bar{l}}\hat{a}_{\boldsymbol{k}',l}\Big)$$
$$-N_xN_yN_z\sum_{l}\sum_{\boldsymbol{q}}\left(\frac{u+v-w}{4}\Big(n_{l,l}(-\boldsymbol{q})n_{\bar{l},\bar{l}}(\boldsymbol{q}) + n_{l,\bar{l}}(\boldsymbol{q})n_{\bar{l},l}(-\boldsymbol{q})\Big) + \frac{(u+w)}{2}n_{l,l}(-\boldsymbol{q})n_{l,l}(\boldsymbol{q}) - \frac{v}{2}n_{l,\bar{l}}(-\boldsymbol{q})n_{l,\bar{l}}(\boldsymbol{q})\right)$$
$$-\frac{(u+w)}{2}\sum_{l}\sum_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k},l} - \frac{u+v-w}{4}\sum_{l}\sum_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k},l}.$$

B. Mean-field Hamiltonian

Eventually, one obtains the mean–field Hamiltonian

$$\hat{\mathcal{H}} = \mathcal{H}_c + \sum_{\boldsymbol{k},\boldsymbol{k}'}\hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}}M_{\boldsymbol{k},\boldsymbol{k}'}\hat{\boldsymbol{a}}_{\boldsymbol{k}'},$$
$$\mathcal{H}_c = -\frac{N_xN_yN_z}{2}\sum_{l}\sum_{\boldsymbol{q}}\left(\frac{u+v-w}{2}\Big(n_{l,l}(-\boldsymbol{q})n_{\bar{l},\bar{l}}(\boldsymbol{q}) + n_{l,\bar{l}}(\boldsymbol{q})n_{\bar{l},l}(-\boldsymbol{q})\Big) + (u+w)n_{l,l}(-\boldsymbol{q})n_{l,l}(\boldsymbol{q}) - vn_{l,\bar{l}}(-\boldsymbol{q})n_{l,\bar{l}}(\boldsymbol{q})\right),$$
$$\tilde{\mu}_0 = \mu_0 + \frac{(u+w)}{2} + \frac{u+v-w}{4} = \mu_0 + \frac{3u+v+w}{4},$$

in the basis

$$\hat{\boldsymbol{a}}_{\boldsymbol{k}'} = \begin{pmatrix}\hat{a}_{\boldsymbol{k},1}\\ \hat{a}_{\boldsymbol{k},2}\end{pmatrix},$$

and with the matrix defined as

$$M_{\boldsymbol{k},\boldsymbol{k}'} = \begin{pmatrix} (f^{A_{1g}}_{\boldsymbol{k}}-\tilde{\mu}_0)\,\delta_{\boldsymbol{k}\boldsymbol{k}'}+(u+w)n_{1,1}(\boldsymbol{k}-\boldsymbol{k}')+\frac{u+v-w}{2}n_{2,2}(\boldsymbol{k}-\boldsymbol{k}') & -vn_{1,2}(\boldsymbol{k}-\boldsymbol{k}')+\frac{u+v-w}{2}n_{2,1}(\boldsymbol{k}-\boldsymbol{k}') \\ -vn_{2,1}(\boldsymbol{k}-\boldsymbol{k}')+\frac{u+v-w}{2}n_{1,2}(\boldsymbol{k}-\boldsymbol{k}') & (f^{A_{1g}}_{\boldsymbol{k}}-\tilde{\mu}_0)\,\delta_{\boldsymbol{k}\boldsymbol{k}'}+(u+w)n_{2,2}(\boldsymbol{k}-\boldsymbol{k}')+\frac{u+v-w}{2}n_{1,1}(\boldsymbol{k}-\boldsymbol{k}')\end{pmatrix}.$$

C. Homogeneous densities

Now, let us significantly simplify the problem by only searching for homogeneous densities, i.e.

$$n_{l,l'}(\boldsymbol{k}-\boldsymbol{k}') = n_{l,l'}(0)\delta_{\boldsymbol{k}\boldsymbol{k}'}.$$

Then, we introduce the symmetry-classified combinations

$$\begin{aligned}
R_0 &= -\tilde{\mu}_0 + \mathsf{C}_{A_{1g}} \\
\mathsf{C}_{A_{1g}} &= \frac{3u+v+w}{4}\left(n_{1,1}(0)+n_{2,2}(0)\right), \\
\mathsf{C}_{B_{1g}} &= \frac{u-v+3w}{4}\left(n_{1,1}(0)-n_{2,2}(0)\right), \\
\mathsf{C}_{B_{2g}} &= \frac{u-v-w}{4}\left(2\Re n_{1,2}(0)\right), \\
\mathsf{C}_{A_{2g}} &= \frac{u+3v-w}{4}\left(2\Im n_{1,2}(0)\right),
\end{aligned}$$

and rewrite the above Hamiltonian accordingly,

$$\begin{aligned}
\hat{\mathcal{H}} &= \mathcal{H}_c + \sum_{\boldsymbol{k}} \hat{\boldsymbol{a}}_{\boldsymbol{k}}^{\dagger} M_{\boldsymbol{k}} \hat{\boldsymbol{a}}_{\boldsymbol{k}}, \qquad (1)\\
\mathcal{H}_c &= -\frac{N_x N_y N_z}{2}\left(\frac{4}{3u+v+w}\mathsf{C}_{A_{1g}}^2 + \frac{4}{u-v+3w}\mathsf{C}_{B_{1g}}^2 + \frac{4}{u-v-w}\mathsf{C}_{B_{2g}}^2 + \frac{4}{u+3v-w}\mathsf{C}_{A_{2g}}^2\right), \\
M_{\boldsymbol{k}} &= \left(f_{\boldsymbol{k}}^{A_{1g}} - \tilde{\mu}_0 + \mathsf{C}_{A_{1g}}\right)s^0 + \mathsf{C}_{B_{1g}}s^z + \mathsf{C}_{B_{2g}}s^x + \mathsf{C}_{A_{2g}}s^y,
\end{aligned}$$

The Hamiltonian (1) can be diagonalized by a simple unitary transformation. To keep things most simple, in the following, we do assume that only one of the three $\{\mathsf{C}_{B_{1g}}, \mathsf{C}_{B_{2g}}, \mathsf{C}_{A_{2g}}\}$ is finite. It has been checked that in all three cases, this leads to the same thermodynamic equations. For practical purposes, we only treat $\mathsf{C}_{B_{1g}}$ as in that case the Hamiltonian

$$\hat{\mathcal{H}} = \mathcal{H}_c + \sum_{\boldsymbol{k}}\sum_{l} \hat{a}_{\boldsymbol{k},l}^{\dagger}\hat{a}_{\boldsymbol{k},l}\omega_{\boldsymbol{k},l},$$

is already diagonal with

$$\omega_{\boldsymbol{k},1} = f_{\boldsymbol{k}}^{A_{1g}} - \tilde{\mu}_0 + \mathsf{C}_{A_{1g}} + \mathsf{C}_{B_{1g}}, \qquad \omega_{\boldsymbol{k},2} = f_{\boldsymbol{k}}^{A_{1g}} - \tilde{\mu}_0 + \mathsf{C}_{A_{1g}} - \mathsf{C}_{B_{1g}}.$$

D. Thermodynamic relations

Next, we compute the resulting thermodynamic relations. For the grand-canonical partition function, one obtains

$$Z_G = e^{-\beta\mathcal{H}_c}\prod_{\boldsymbol{k}}\prod_{l=1}^{2}\frac{1}{1-e^{-\beta\omega_{l,\boldsymbol{k}}}},$$

and consequently, the potential and the number equation

$$\begin{aligned}\Omega &= -T\log Z_g & \bar{N} &= -\frac{\partial\Omega}{\partial\mu_0}\Big|_{V,T} \\ &= \mathcal{H}_c + T\sum_{\boldsymbol{k}}\sum_{l=1}^{2}\log\left(1-e^{-\beta\omega_{l,\boldsymbol{k}}}\right), & &= -\sum_{\boldsymbol{k}}\sum_{l=1}^{2}\frac{e^{-\beta\omega_{l,\boldsymbol{k}}}}{1-e^{-\beta\omega_{l,\boldsymbol{k}}}}\frac{\partial\omega_{l,\boldsymbol{k}}}{\partial\mu_0} \qquad (2)\\ &= \mathcal{H}_c - \sum_{\boldsymbol{k}}\sum_{l=1}^{2}\left(\omega_{l,\boldsymbol{k}} + T\log n_B(\omega_{l,\boldsymbol{k}})\right), & &= \sum_{\boldsymbol{k}}\sum_{l=1}^{2} n_B(\omega_{l,\boldsymbol{k}}) \qquad (3)\end{aligned}$$

with the Bose function

$$n_B(E) = \frac{1}{e^{\beta E}-1}.$$

In the present study, we want to keep the particle density fixed,

$$\bar{n} = \frac{1}{N_x N_y N_z}\bar{N} = \text{const.}$$

Note that it holds

$$\langle \hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k}',l'}\rangle = \frac{1}{Z_G}\mathrm{tr}\left(\hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k}',l'}e^{-\beta\hat{\mathcal{H}}}\right) = n_B(\omega_{\boldsymbol{k},l})\delta_{l,l'}\delta_{\boldsymbol{k},\boldsymbol{k}'},$$

and thus,

$$\begin{aligned}\mathsf{C}_{A_{1g}} &= \frac{3u+v+w}{4}\left(n_{1,1}(0)+n_{2,2}(0)\right) \\ &= \frac{1}{N_x N_y N_z}\frac{3u+v+w}{4}\sum_{\boldsymbol{k}}\sum_{l=1}^{2}\langle \hat{a}^\dagger_{\boldsymbol{k},l}\hat{a}_{\boldsymbol{k},l}\rangle \\ &= \frac{3u+v+w}{4}\bar{n}\end{aligned}$$

is constrained to be a constant. Eventually, we aim to minimize the free energy with respect to C_x [not $\mathsf{C}_{A_{1g}}$ since it is bound to be a constant], which reads

$$\begin{aligned}F[T,\bar{N},\mathsf{C}_x] &= \mu_0(T,\mathsf{C}_x,\bar{N})\bar{N} + \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})] \qquad (4)\\ &= \mu_0\bar{N} + \mathcal{H}_c - \sum_{\boldsymbol{k}}\sum_{l=1}^{2}\left(\omega_{l,\boldsymbol{k}} + T\log n_B(\omega_{l,\boldsymbol{k}})\right), \qquad (5)\end{aligned}$$

where we use C_x with $x \in \{B_{1g}, B_{2g}, A_{2g}\}$ to emphasize that the equation is the same in any of the cases—where only one of them is finite.

E. Instability with respect to C_x

Unfortunately, the free energy (5) seems to be unstable with respect to the possible C_x. We demonstrate this by showing that

$$\frac{\partial^2 F}{\partial \mathsf{C}_x^2}\Big|_{\mathsf{C}_x=0} < 0$$

for generic values of the interaction parameters. The second derivative of the free energy (4) becomes

$$
\begin{aligned}
\frac{\partial F[T,\bar{N},\mathsf{C}_x]}{\partial \mathsf{C}_x} &= \frac{\partial \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x}\Big|_{\mu_0} + \underbrace{\frac{\partial \Omega[\mathsf{C}_x,T,\mu_0]}{\partial \mu_0}}_{-\bar{N}} \frac{\partial \mu_0}{\partial \mathsf{C}_x} + \frac{\partial \mu_0}{\partial \mathsf{C}_x}\bar{N},\\
\frac{\partial^2 F[T,\bar{N},\mathsf{C}_x]}{\partial \mathsf{C}_x^2} &= \frac{\partial^2 \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x^2}\Big|_{\mu_0} + \frac{\partial}{\partial \mu_0}\left(\frac{\partial \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x}\Big|_{\mu_0}\right)\frac{\partial \mu_0}{\partial \mathsf{C}_x}.
\end{aligned}
\tag{6}
$$

Thus, we carry out the derivatives

$$
\begin{aligned}
\frac{\partial \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x}\Big|_{\mu_0} &= -N_xN_yN_z\frac{4}{u_x}\mathsf{C}_x + \sum_{\boldsymbol{k}}\left(\frac{1}{e^{\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}+\mathsf{C}_x\right)}-1} - \frac{1}{e^{\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}-\mathsf{C}_x\right)}-1}\right),\\
\frac{\partial^2 \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x^2}\Big|_{\mu_0} &= -\frac{1}{4}\beta\sum_{\boldsymbol{k}}\left(\frac{1}{\sinh^2\left(\frac{1}{2}\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}+\mathsf{C}_x\right)\right)} + \frac{1}{\sinh^2\left(\frac{1}{2}\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}-\mathsf{C}_x\right)\right)}\right)\\
&\quad - N_xN_yN_z\frac{4}{u_x},\\
\frac{\partial}{\partial \mu_0}\left(\frac{\partial \Omega[\mathsf{C}_x,T,\mu_0(T,\mathsf{C}_x,\bar{N})]}{\partial \mathsf{C}_x}\Big|_{\mu_0}\right) &= \frac{1}{4}\beta\sum_{\boldsymbol{k}}\left(\frac{1}{\sinh^2\left(\frac{1}{2}\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}+\mathsf{C}_x\right)\right)} - \frac{1}{\sinh^2\left(\frac{1}{2}\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}-\mathsf{C}_x\right)\right)}\right),
\end{aligned}
$$

and evaluate them at zero field, to obtain

$$
\begin{aligned}
f_1 &= \frac{\partial F}{\partial \mathsf{C}_x}\Big|_{\mathsf{C}_x=0} = 0,\\
f_2 &= \frac{\partial^2 F}{\partial \mathsf{C}_x^2}\Big|_{\mathsf{C}_x=0} = -N_xN_yN_z\frac{4}{u_x} - \frac{1}{2}\beta\sum_{\boldsymbol{k}}\frac{1}{\sinh^2\left(\frac{1}{2}\beta\left(f_{\boldsymbol{k}}^{A_{1g}}-\tilde{\mu}_0+\mathsf{C}_{A_{1g}}\right)\right)}.
\end{aligned}
\tag{7}
$$

Here, we have the problem that the second term in (7) is strictly negative, and only the first one can potentially be positive, if $u_x < 0$, i.e. the channel is attractive. However, keeping in mind, that the relation (7) is valid for any of the density channels, there will always be repulsive channels with $u_x > 0$ and consequently, a non-negotiable negative free energy curvature. As a result those densities would always be finite, and in the worst cases even $\mathsf{C}_x \to \pm\infty$. The relation (7) seems to be physical non-sense. The question remains if the model can be adjusted such that the results become physical.