Lecture-03

Elements of point-set topology continued,

In the last lecture, we introduced:

1. open ball in $\mathbb{R}^n$ of radius $r>0$ centered at x_0
2. Interior point of a given subset of $\mathbb{R}^n$
3. boundary point of a given subset of $\mathbb{R}^n$
4. interior of a set
5. boundary of a set
6. open sets in $\mathbb{R}^n$
7. closed sets in $\mathbb{R}^n$

Let's introduce:

8. bounded subsets of $\mathbb{R}^n$:

Let $A \subseteq \mathbb{R}^n$. A is called a bounded subset of $\mathbb{R}^n$ if $\exists\, r>0$ such that $A \subseteq B_r(\vec{0})$, where $\vec{0} = (0, \dots, 0)$.

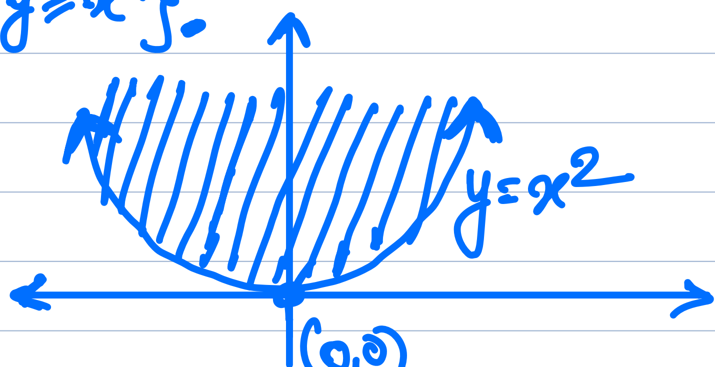
(9) unbounded subsets of $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. A is said to be unbounded if it is not bounded.

Example:- Let $f(x,y) = \sqrt{y - x^2}$.

What is the domain of f?

$dom(f) = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$.



- Is $dom(f)$ open? No! (why?)

$int(dom(f)) = \{(x,y) \in \mathbb{R}^2 : y > x^2\}$

$\therefore dom(f) \neq int(dom(f))$. $\therefore dom(f)$ NOT open.

- Is $dom(f)$ closed? Yes.
- bounded? No!

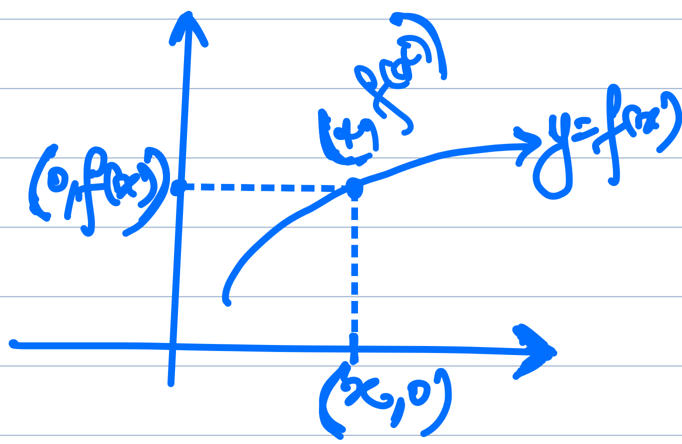
⊙ $\mathrm{Ran}(f) = [0,\infty) \subseteq \mathbb{R}$.

This is <u>closed</u>,
<u>not open</u>,
and <u>unbounded</u> in $\mathbb{R}$.

Graph of a function

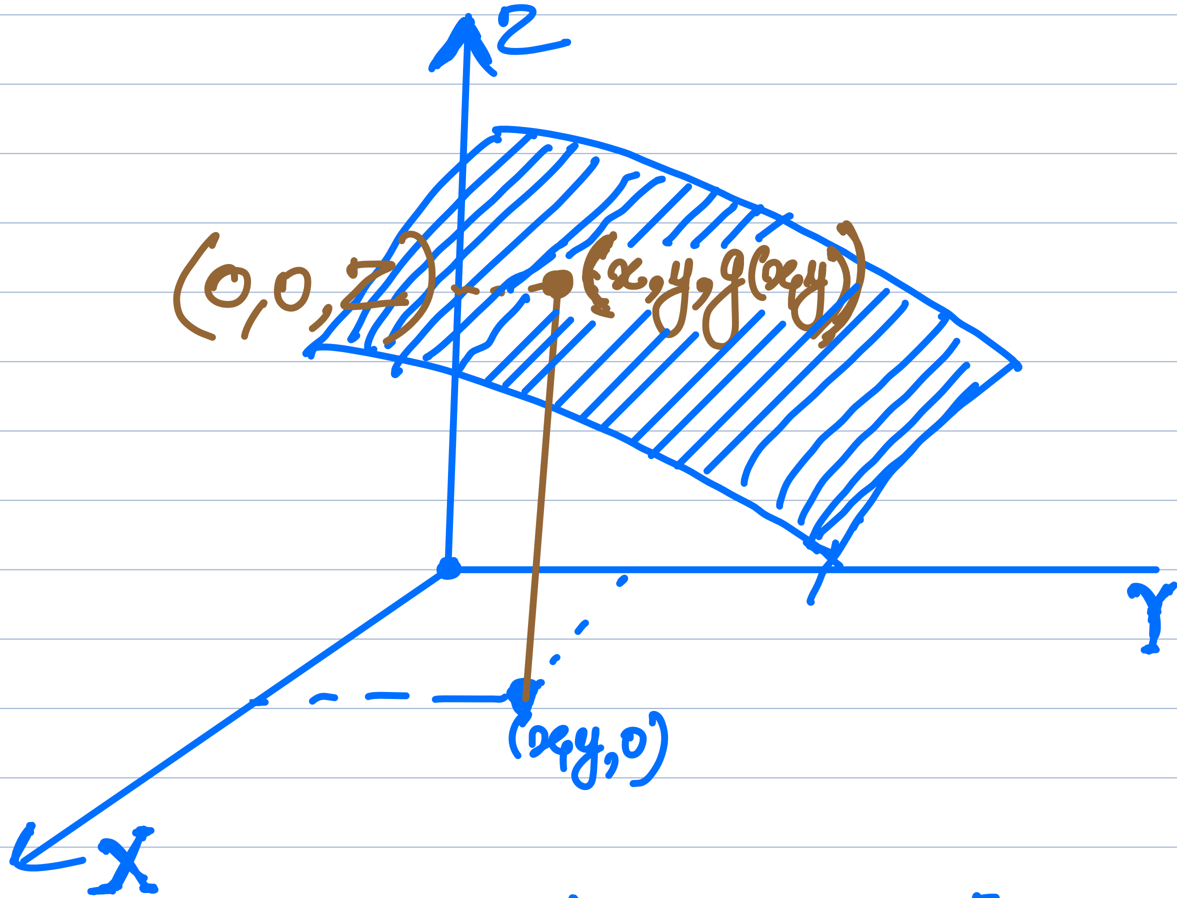
Case I: function of one variable: $y=f(x)$.

[The Graph of f is a "curve" (in $\mathbb{R}^2$)]



Case II: function of two variables

$z=g(x,y)$



Graph of $z=g(x,y)$ is a "surface" (in $\mathbb{R}^3$)

Case III: function of "n" independent variables

Graph of a function:

Let n be any positive integer, let $D \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$, and let $f: D \longrightarrow \mathbb{R}$ be a function from D to $\mathbb{R}$. The graph of the function f, denoted by $G(f)$, is defined to be the set

$$G(f) = \{ (x, f(x)) : x \in D \subseteq \mathbb{R}^n \} \quad (\subseteq \mathbb{R}^{n+1}).$$

x: n-tuple

$f(x)$: real number

$(x, f(x))$: n+1 coordinates.

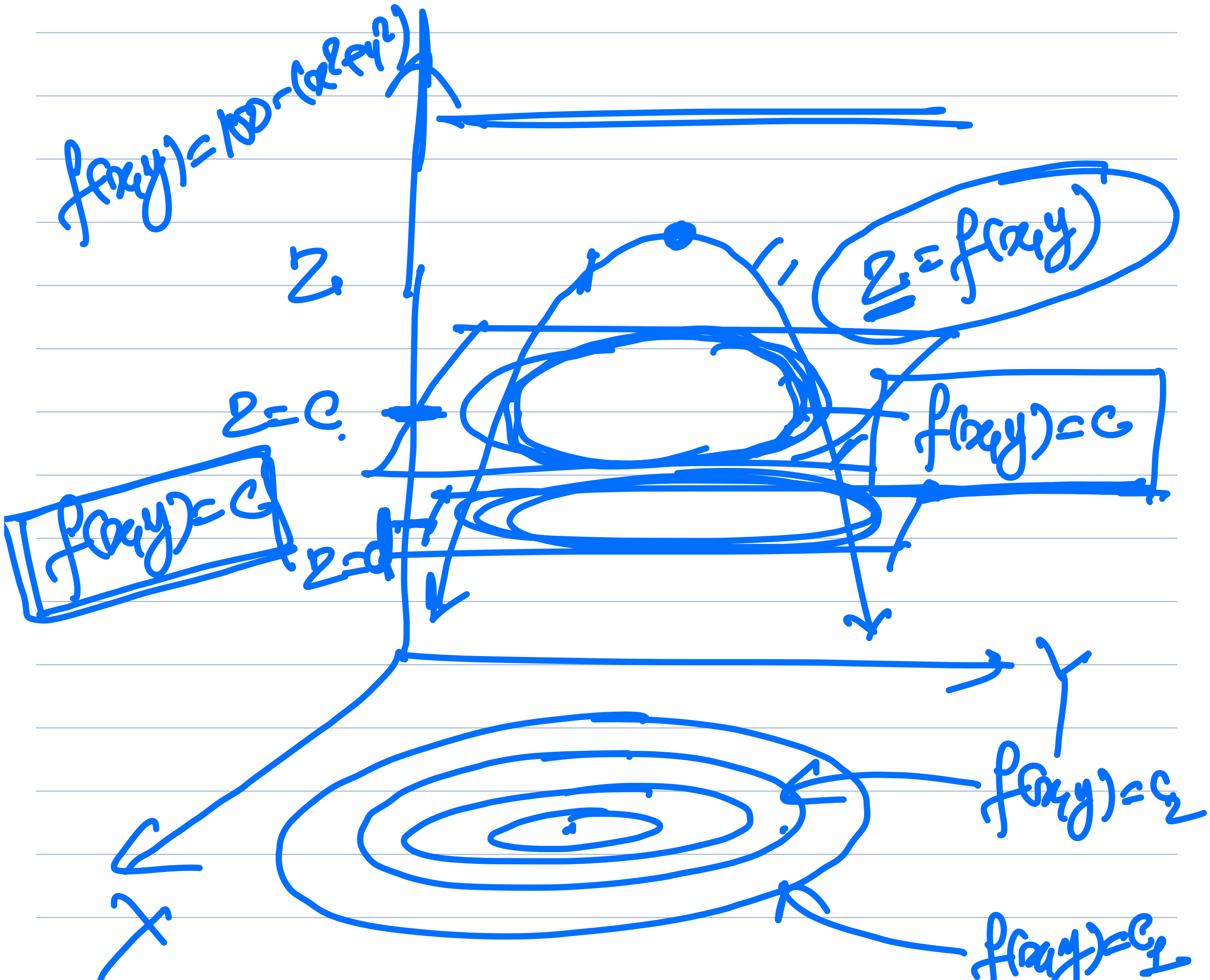
Level Sets

Defn: Let n be any positive integer, let $D \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$, and let $f : D \longrightarrow \mathbb{R}$ be a function from D to $\mathbb{R}$. Given a constant $c \in \operatorname{ran}(f)$, the Level set of f at the point c is defined to be the set

$$\{ x \in D \subseteq \mathbb{R}^n : f(x) = c \} \quad (\subseteq \mathbb{R}^n).$$

Remarks:

- Every level set of the function f lies in the domain of the function f.
- On each level set, the value of the function is a constant.
- When $n=2$ (i.e. when we consider $D \subseteq \mathbb{R}^2$, so that f is a function of two variables), we call it a **Level Curve**.
- When $n=3$ (i.e. when we consider $D \subseteq \mathbb{R}^3$, so that f is a function of three variables), we call it a **Level Surface**.



f(x,y)=100-(x²+y²)
z
z=f(x,y)
z=c
f(x,y)=c
f(x,y)=c
z=d
y
x
f(x,y)=c2
f(x,y)=c1

Example: Let $f(x,y) = 4 - x^2 - y^2$. Find the level curves.

Solution: We have $z = f(x,y)$. We replace z by some admissible constant c, and we get

$$f(x,y) = c$$

i.e., $4 - x^2 - y^2 = c$

or, $\boxed{x^2 + y^2 = 4 - c}$

Clearly, c must be either less than or equal to 4, i.e., $c \leq 4$.

So,

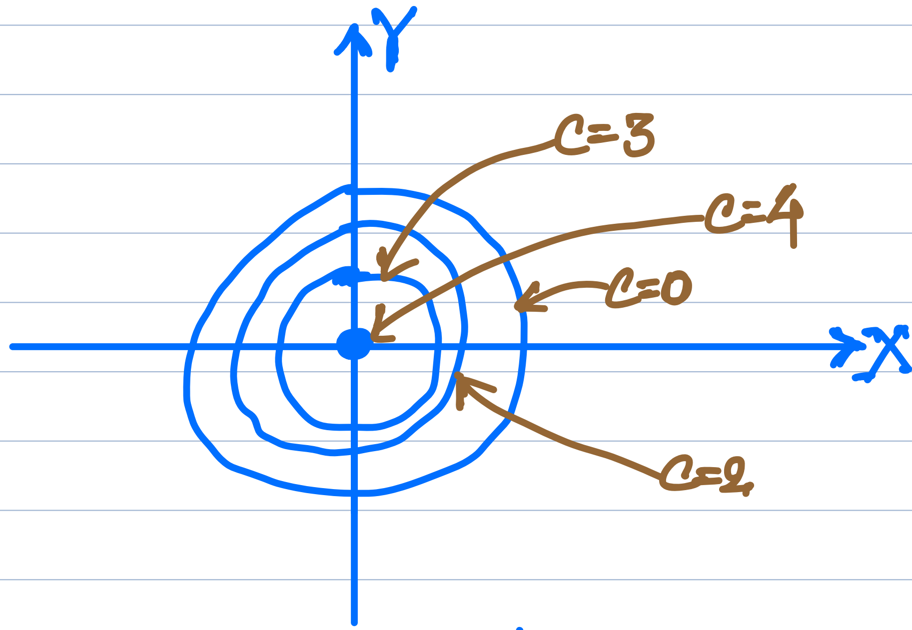
If $c = 4$, we get $x^2 + y^2 = 0$

(a circle of radius 0. ∴ merely the origin)

If $c=3$, we get $x^2+y^2=1$
(a circle of radius 1 centered at the origin)

If $c=0$, we get $x^2+y^2=4$ (a circle of radius 2)

If $c<0$, we get circle of radius >2.



Exercise Let $f(x,y)=\dfrac{1}{\sqrt{16-x^2-y^2}}$. Find the level curves.

Exercise Let $g(x,y,z)=x^2+y^2+z^2$. Find the level surfaces.