Lecture 01 (Abridged)

Today:

1. plan of the class/logistics
2. functions of one variable
3. functions of several variables.

Welcome to

Math 203 - Multivariate Calculus

[Plan of the class]

Textbooks:

(1) Thomas' Calculus (11th Edition)
(Chapters 14, 15, 16 and a part of 13)
- Weir, Hass and Giordano

(2) Calculus Volume 2 (2nd Edition)
- Tom M. Apostol.

(3) Advanced Engineering Mathematics
(9th Edition)
— Erwin Kreyszig

Evaluation Criteria :

1. Mid-sem examination — 30%
2. End-sem exam — 40%
3. Two Quizzes — 10%
4. 12 Worksheets — 20%
(10 best will be evaluated)

100%

5. Tutorials — Thursdays 1:30–3:00

Plagiarism :— Do not do it!

Math III / Real Analysis - I

Antirequisites to each other

designed keeping a practitioner in mind

Teaches how to use Analysis/Calculus

mandatory for CSAM students

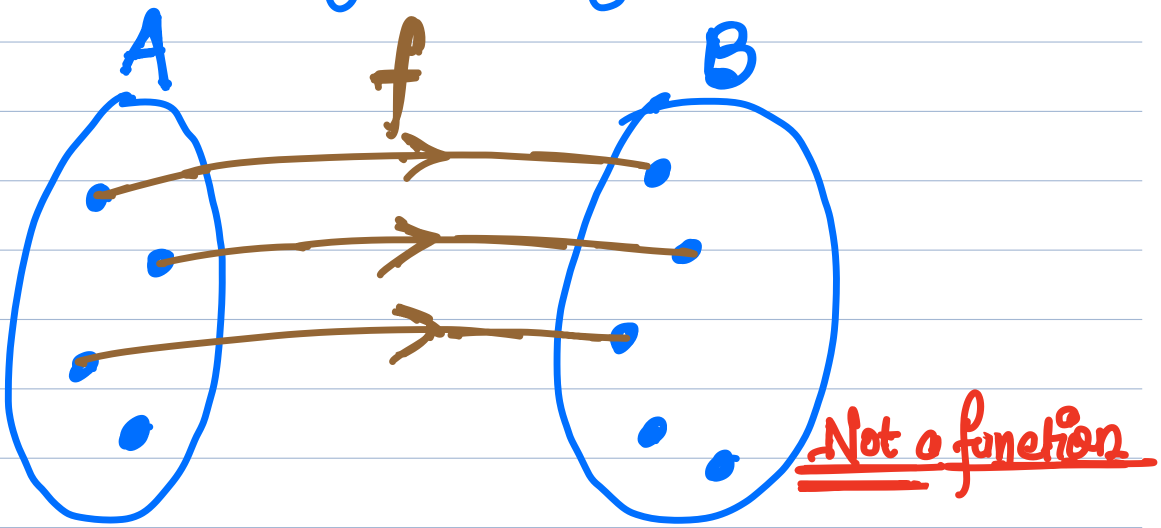
Real Analysis - II

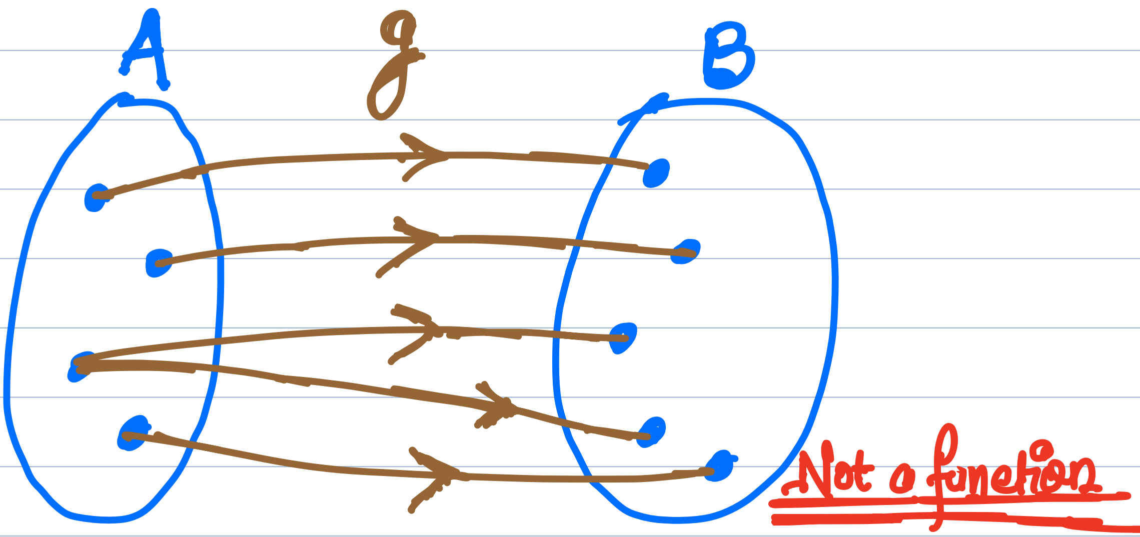
- proof based course
- more rigorous

Functions In almost every branch of Mathematics, functions are the central objects of interest / investigation.

Let's recall the defining properties of a function

A, B — arbitrary nonempty sets.





Let $D \subseteq \mathbb{R}$. Let $f: D \to \mathbb{R}$ be a function from the set D to the set $\mathbb{R}$. This statement means

[f is a rule which assigns to each element $x \in D$, a unique element $y \in \mathbb{R}$.]

- In this case, we write $f(x)=y$.

  - $f \equiv$ The rule
  - $y=f(x) \leftarrow$ The value of f at x, or The image of x under f.
- $x \leftarrow$ the independent variable.
- $y \leftarrow$ the dependent variable.
- domain of f?
  Codomain of f?
  range of f?

- [for each element $x \in D$, there exists a unique element $y \in \mathbb{R}$, such that $f(x)=y$; or,
  $\forall x \in D, \exists! y \in \mathbb{R}$, such that $f(x)=y$.]

- **real-valued functions:** — When the value of f at every point of the domain of f is a real number or, equivalently, $\text{ran}(f) \subseteq \mathbb{R}$.

② Clearly, the functions in discussion so far, are functions of single (independent) variable.

What about the case when the function in question is of two, or three, or, in general, several variables?

Let f be a "function" given by

$$f((x_1, x_2, x_3)) = x_1 x_2 + x_3^2.$$

Then, clearly, the elements on which f is acting are coming from $\mathbb{R}^3$. So, the domain must be a subset of $\mathbb{R}^3$.

Similarly, if g be another "function" defined by

$$g((x_1, x_2)) = x_1^2 + x_2^2 + x_1 \cdot x_2.$$

Then, the domain of g must be contained in $\mathbb{R}^2$.

This compels us to understand the space $\mathbb{R}^n$.

[n-dimensional Euclidean Space
$\parallel$
Euclidean n-space
$\parallel$
$\mathbb{R}^n$]

Let n be a fixed, but arbitrarily chosen, positive integer.

$\mathbb{R}^n$ as a set: It is the set of all ordered n-tuples of real numbers. That is,

$$\mathbb{R}^n := \{x = (x_1, \dots, x_n) : \forall j \in \{1, \dots, n\}, x_j \in \mathbb{R}\}$$

$\mathbb{R}^n$ as a Vector space:

a linear space (over the field $\mathbb{R}$).
$\parallel$
Vector space

[How? → Vector addition
→ Scalar multiplication]

$\mathbb{R}^n$ as an inner product space:

(Vector space equipped with an inner product)

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are arbitrary elements of $\mathbb{R}^n$, then define an inner product of x and y, denoted by $\langle x, y \rangle$, by

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i .$$

Alternatively, $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an inner product on $\mathbb{R}^n$, defined via

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i .$$

Properties: Let $x, x_1, x_2, y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Then,

(1) $\langle x, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$

(2) $\langle x, x \rangle = 0 \iff x = 0$

(3) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

(4) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(5) $\langle x, y \rangle = \langle y, x \rangle$

Cauchy-Schwarz Inequality:

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle},$$

for all $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, and for all $y=(y_1,\dots,y_n)\in\mathbb{R}^n$.

Or, alternatively,

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2}\left(\sum_{i=1}^n y_i^2\right)^{1/2} \;\forall x,y\in\mathbb{R}^n.$$

Moreover, the equality holds $\Leftrightarrow$ x and y are linearly dependent.

(Euclidean) norm induced by the inner product.

If $x\in\mathbb{R}^n$, then define

$$\|x\| := \sqrt{\langle x,x\rangle}.$$

In essence, $\|\cdot\| : \mathbb{R}^n \longrightarrow [0,\infty)$

"Norm" (induced by $\langle\cdot,\cdot\rangle$)

$|||$

A function from $\mathbb{R}^n$ to $[0,\infty)$

Properties of norm $\|\cdot\|$ on $\mathbb{R}^n$

Let $x, y \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$;

(1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$

(2) $\|x\| = 0 \iff x = 0$

(3) $\|\alpha x\| = |\alpha| \, \|x\|$

(4) $\boxed{\|x+y\| \leq \|x\| + \|y\|.}$

This is called the triangle inequality.

Qn: When does the equality hold?

Now, for any $x, y \in \mathbb{R}^n$, we can define the distance between x and y by

$$\|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

$$= \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle}$$

$$= \sqrt{\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{n} y_j^2 - 2\sum_{k=1}^{n} x_k y_k}$$

$$= \sqrt{\sum_{j=1}^{n} (x_j^2 + y_j^2 - 2x_j y_j)}$$

$$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

In particular, $\|x\|^2 = \sum_{j=1}^{n} x_j^2$ for every $x \in \mathbb{R}^n$.
This norm is referred to as the <u>Euclidean norm</u>.

The real linear space $\mathbb{R}^n$, equipped with the inner product defined above (which induces the Euclidean norm) is referred to as the

<u>n-dimensional Euclidean Space</u>, or

<u>Euclidean n-space</u>.

Function of n Variables

Defn: Let n be a positive integer.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a rule which associates to each point $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, a unique point $w\in\mathbb{R}$, and we write

$$w = f((x_1,\dots,x_n)).$$

The value of f at the point $x=(x_1,\dots,x_n)$

The image of x under f

The dependent variable of f

The rule

A function of n (independent) variables x_1 to x_n

→ **Example:** Find the domain and range of the following functions:

(1) $Z = f(x,y) = \sin^{-1}(y-x)$.

(2) $Z = f(x,y) = \sqrt{9-x^2-y^2}$.

(3) $W = f(x,y,z) = xy \ln z$.

(4) $Z = f(x,y) = \sqrt{y-x^2}$.