Lecture 01 (Abridged)

Today:

1. plan of the class/logistics
2. functions of one variable
3. functions of several variables.

Welcome to

Math 203 - Multivariate Calculus

[Plan of the class]

Textbooks:

(1) Thomas' Calculus (11^{th} Edition)
(Chapters 14, 15, 16 and a part of 13)
- Weir, Hass and Giordano

(2) Calculus Volume 2 (2nd Edition)
- Tom M. Apostol.

(3) Advanced Engineering Mathematics
(9th Edition)
— Erwin Kreyszig

Evaluation Criteria :

1. Mid-sem examination — 30%
2. End-sem exam — 40%
3. Two Quizzes — 10%
4. 12 Worksheets — 20%
(10 best will be evaluated)

100%

5. Tutorials — Thursdays 1:30–3:00

Plagiarism :— Do not do it !

Math III / Real Analysis-I

designed keeping a practitioner in mind

Antirequisites to each other

mandatory for CSAM students

Teaches how to use Analysis/Calculus

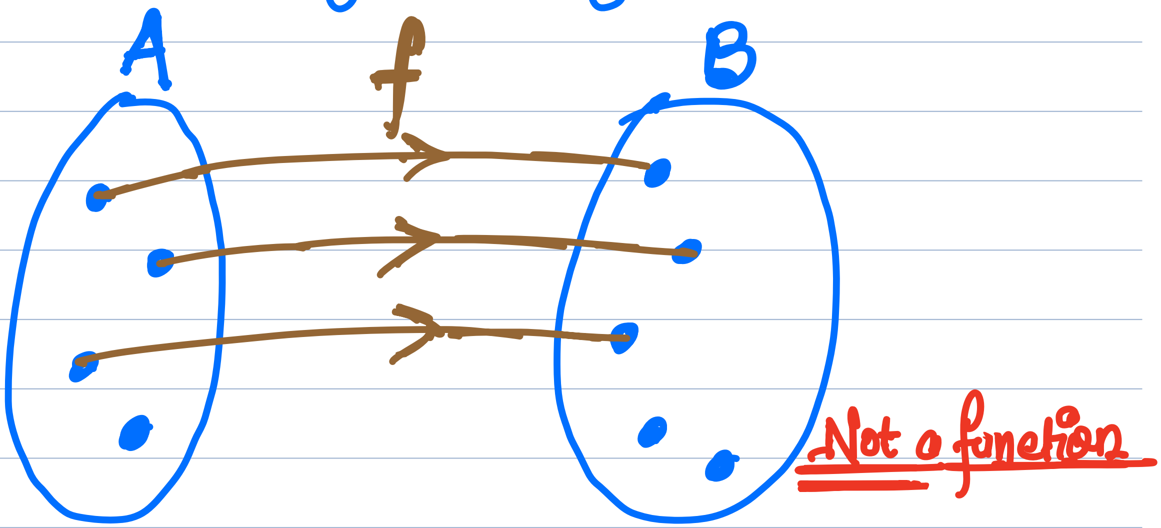
Real Analysis-II

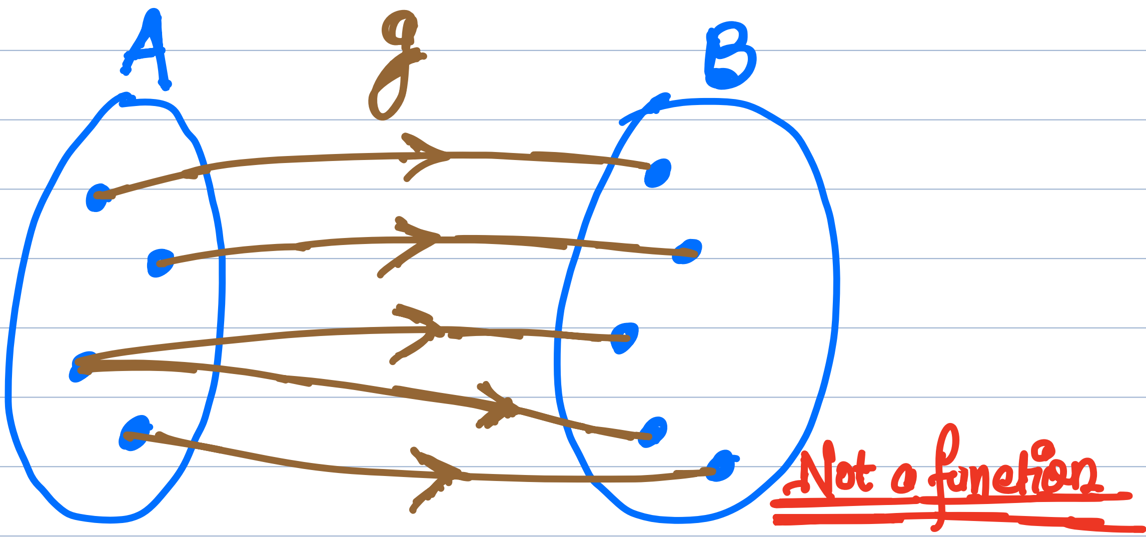
- proof based course
- more rigorous

Functions In almost every branch of Mathematics, functions are the central objects of interest/investigation.

Let's recall the defining properties of a function

A, B — arbitrary nonempty sets.





Let $D \subseteq \mathbb{R}$. Let $f: D \to \mathbb{R}$ be a function from the set D to the set $\mathbb{R}$. This statement means

[f is a rule which assigns to each element $x \in D$, a unique element $y \in \mathbb{R}$.]

- In this case, we write $f(x) = y$.

  - $f \equiv$ The rule
  - $y = f(x) \leftarrow$ The value of f at x, or
    The image of x under f.
  - $x \leftarrow$ the independent variable.
  - $y \leftarrow$ the dependent variable.

  - domain of f?
    Codomain of f?
    range of f?

- [for each element $x \in D$, there exists a unique element $y \in \mathbb{R}$, such that $f(x) = y$; or,
  $\forall x \in D, \exists! y \in \mathbb{R}$, such that $f(x) = y$.]

- Real-valued functions: — When the value of f at every point of the domain of f is a real number or, equivalently, $\operatorname{ran}(f) \subseteq \mathbb{R}$.

② Clearly, the functions in discussion so far, are functions of single (independent) variable.

What about the case when the function in question is of two, or three, or, in general, several variables?

Let f be a "function" given by

$$f((x_1, x_2, x_3)) = x_1 x_2 + x_3^2.$$

Then, clearly, the elements on which f is acting are coming from $\mathbb{R}^3$. So, the domain must be a subset of $\mathbb{R}^3$.

Similarly, if g be another "function" defined by

$$g((x_1, x_2)) = x_1^2 + x_2^2 + x_1 \cdot x_2.$$

Then, the domain of g must be contained in $\mathbb{R}^2$.

This compels us to understand the space $\mathbb{R}^n$.

n-dimensional Euclidean Space

$\cong$

Euclidean n-space

$\cong$

$\mathbb{R}^n$

Let n be a fixed, but arbitrarily chosen, positive integer.

$\mathbb{R}^n$ as a set: It is the set of all ordered n-tuples of real numbers. That is,

$$\mathbb{R}^n := \{x = (x_1, \dots, x_n) : \forall j \in \{1, \dots, n\}, x_j \in \mathbb{R}\}$$

$\mathbb{R}^n$ as a Vector space:

a linear space (over the field $\mathbb{R}$).

$\cong$

Vector space

How? $\rightarrow$ Vector addition

$\rightarrow$ Scalar multiplication

$\mathbb{R}^n$ as an inner product space:

(Vector space equipped with an inner product)

If $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ are arbitrary elements of $\mathbb{R}^n$, then define an inner product of x and y, denoted by $\langle x,y\rangle$, by

$$\langle x,y\rangle := x_1y_1+x_2y_2+\cdots+x_ny_n=\sum_{i=1}^{n} x_iy_i.$$

Alternatively, $\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is an inner product on $\mathbb{R}^n$, defined via

$$\langle x,y\rangle := \sum_{i=1}^{n} x_iy_i .$$

Properties: Let $x, x_1, x_2, y \in \mathbb{R}^n$, and $\alpha\in\mathbb{R}$. Then,

(1) $\langle x,x\rangle \geq 0$ for every $x\in\mathbb{R}^n$

(2) $\langle x,x\rangle = 0 \iff x=0$

(3) $\langle x_1+x_2,y\rangle = \langle x_1,y\rangle+\langle x_2,y\rangle$

(4) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$

(5) $\langle x,y\rangle = \langle y,x\rangle$

Cauchy-Schwarz Inequality:

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle},$$

for all $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, and for all $y=(y_1,\dots,y_n)\in\mathbb{R}^n$.

Or, alternatively,

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2}\left(\sum_{i=1}^n y_i^2\right)^{1/2} \quad \forall x,y\in\mathbb{R}^n.$$

Moreover, the equality holds $\Leftrightarrow$ x and y are linearly dependent.

(Euclidean) norm induced by the inner product.

If $x\in\mathbb{R}^n$, then define

$$\|x\| := \sqrt{\langle x,x\rangle}.$$

In essence, $\|\cdot\| : \mathbb{R}^n \longrightarrow [0,\infty)$

"Norm" (induced by $\langle\cdot,\cdot\rangle$)

$|||$

A function from $\mathbb{R}^n$ to $[0,\infty)$

Properties of norm $\|\cdot\|$ on $\mathbb{R}^n$

Let $x, y \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$;

(1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$

(2) $\|x\| = 0 \iff x = 0$

(3) $\|\alpha x\| = |\alpha| \, \|x\|$

(4) $\boxed{\|x+y\| \leq \|x\| + \|y\|.}$

This is called the triangle inequality.

Qn: When does the equality hold?

Now, for any $x, y \in \mathbb{R}^n$, we can define the distance between x and y by

$$\|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

$$= \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle}$$

$$= \sqrt{\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{n} y_j^2 - 2\sum_{k=1}^{n} x_k y_k}$$

$$= \sqrt{\sum_{j=1}^{n} (x_j^2 + y_j^2 - 2x_j y_j)}$$

$$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

In particular, $\|x\|^2 = \sum_{j=1}^{n} x_j^2$ for every $x \in \mathbb{R}^n$.
This norm is referred to as the Euclidean norm.

The real linear space $\mathbb{R}^n$, equipped with the inner product defined above (which induces the Euclidean norm) is referred to as the

n-dimensional Euclidean space, or

Euclidean n-space.

Function of n Variables

Defn: Let n be a positive integer.

A function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$, is a rule which associates to each point $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, a unique point $w\in\mathbb{R}$, and we write

$$w=f\big((x_1,\dots,x_n)\big).$$

The value of f at the point $x=(x_1,\dots,x_n)$

$\equiv$

The image of x under f

$\equiv$

The dependent variable of f

The rule

$\equiv$

A function of n (independent) variables x_1 to x_n

→ **Example:** Find the domain and range of the following functions:

(1) $z = f(x,y) = \sin^{-1}(y-x)$.

(2) $z = f(x,y) = \sqrt{9-x^2-y^2}$.

(3) $w = f(x,y,z) = xy \ln z$.

(4) $z = f(x,y) = \sqrt{y-x^2}$.

Lecture 02

Warm-up problem:

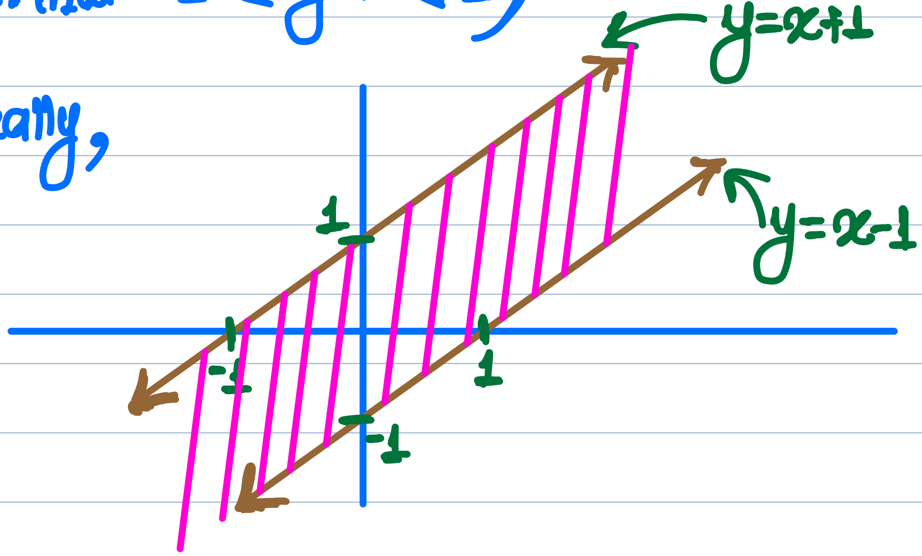
Find the domain and the range of the following functions.

(i) $z = f(x,y) = \sin^{-1}(y-x)$.

Solution: $\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 : -1 \leq y-x \leq 1\}$

(This is the set of all ordered pairs in $\mathbb{R}^2$ such that $-1 \leq y-x \leq 1$)

graphically,



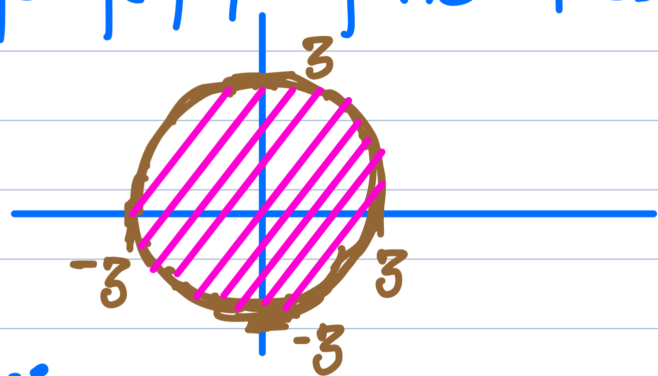
$\text{ran}(f) = [-\pi/2, \pi/2]$.

(ii) $z = g(x,y) = \sqrt{9-x^2-y^2}$

Solution: $\text{dom}(g) = \{(x,y) \in \mathbb{R}^2 : 9-x^2-y^2 \geq 0\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 9\}$

Graphically, the domain is the shaded region including the circumference of the circle.



$\text{ran}(f) = [0,3] \subseteq \mathbb{R}$.

(iii) $\omega = xy \ln z$

Soln: Clearly ω is a function of three variables

$\therefore$ domain must be contained in $\mathbb{R}^3$.

Let $w = h(x,y,z) = xy\ln z$. Then,

$\text{dom}(h) = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$

[Graphically, it is the half-space where the z-coordinate is strictly positive (and thus the x-y plane is excluded).]

$\mathrm{Ran}(f) = \mathbb{R}$. (How?)

[Let $\alpha \in \mathbb{R}$ be arbitrary. Then one can choose $x = \alpha$, $y = 1$ and $z = e$ so that $xy\ln(z) = \alpha$.]

Exercises

(iv) $\omega = \sqrt{y - x^2}$;

(v) $f(x,y) = \sqrt{y - x}$;

(vi) $f(x,y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$;

(vii) $g(x,y) = y/x^2$;

(viii) $h(x,y) = \dfrac{1}{x - y}$.

Elements of point-set Topology on $\mathbb{R}^n$

Let $n \in \mathbb{N}$ be fixed but arbitrarily chosen.

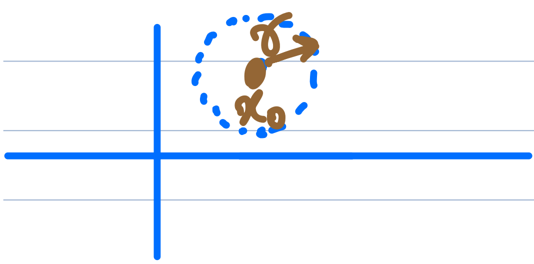
1. Open ball in $\mathbb{R}^n$ of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$

→ If $n = 2$, then we are in $\mathbb{R}^2$:

Suppose $x_0 \in \mathbb{R}^2$ is given and $r > 0$ is also given. Then the open ball in $\mathbb{R}^2$ of radius r centered at x_0 is

the open disc of radius r centered at x_0

$$= B_r(x_0) = \{x \in \mathbb{R}^2 : \|x - x_0\| < r\} \subseteq \mathbb{R}^2$$



Notation

→ If $n=1$, the space is $\mathbb{R}$.

$B_r(x_0) =$ open interval (x_0-r, x_0+r)

$= \{x \in \mathbb{R} : |x-x_0| < r\}$

$\therefore B_2(5) = \{x \in \mathbb{R} : |x-5| < 2\} = (3,7)$

→ **In $\mathbb{R}^n$**: The open ball (in $\mathbb{R}^n$) of radius r centered at $x_0 \in \mathbb{R}^n$ is given by

$$B_r(x_0) = \{x \in \mathbb{R}^n : \|x-x_0\| < r\}$$

Question: Find $B_2(4)$;

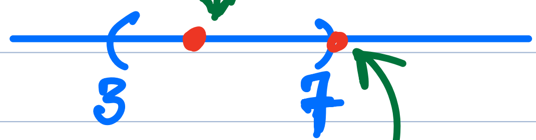
$B_3((2,3))$;

$B_1((1,0,0))$.

2. Interior point of a given subset of $\mathbb{R}^n$

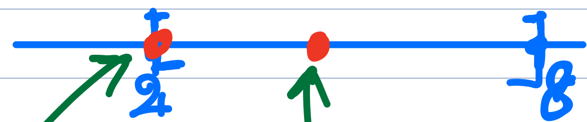
- Consider the open interval $(3,7) \subseteq \mathbb{R}$.

an interior point of $(3,7)$



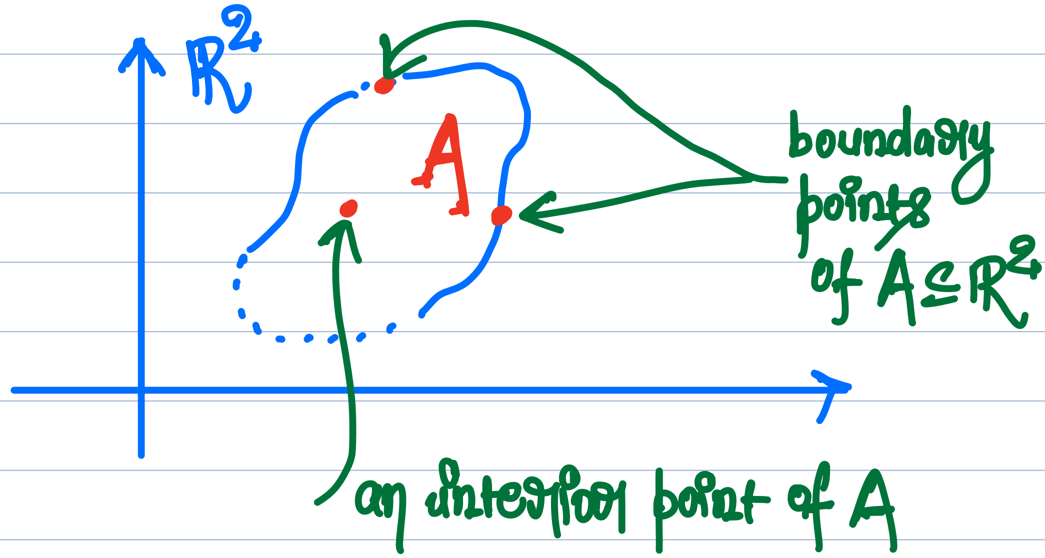
Not an interior point of $(3,7)$.
It is, in fact, a boundary point of $(3,7)$

- Consider $[2,8]$



a boundary point of $[2,8]$

an interior point of $[2,8]$



definition Let $A \subseteq \mathbb{R}^n$. A point $x \in A$ is said to be an interior point of A if $\exists\, r > 0$ such that $B_r(x) \subseteq A$

[symbol for "there exist(s)"]

[In definition "if" is essentially "if and only if"]

Alternatively, A point $x \in A$ is called an interior point of A if it is the centre of some open ball contained in A.

3. Boundary point of a given subset of $\mathbb{R}^n$

definition: Let $A \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a boundary point of A if

$\forall \delta > 0$, $B_\delta(x) \cap A \neq \phi$ and $B_\delta(x) \cap A^c \neq \phi$

[Symbol for "For every"]

[Again, this means "if and only if"]

Alternatively, a point $x \in \mathbb{R}^n$ is called a boundary point of A if every open ball centered at x contains points of A as well as points of A^c

4. Interior of a set

Let $A \subseteq \mathbb{R}^n$. The interior of A (denoted by $\mathrm{int}(A)$) is the set of all interior points of A.

5. Boundary of a set

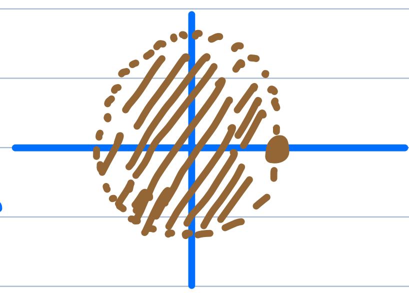
Let $A \subseteq \mathbb{R}^n$. The boundary of A (denoted by ∂A) is the set of all boundary points of A.

Exercise. Let $A \subseteq \mathbb{R}^n$. Prove that $\mathrm{int}(A) \subseteq A$.

Example Let $A = B_1((0,0)) \cup \{(1,0)\} \subseteq \mathbb{R}^2$.

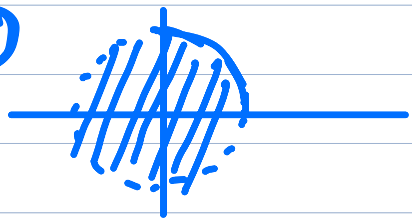
Then, $\mathrm{int}(A) = B_1((0,0))$,

$\partial A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$.



6. Open set in $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. We say that A is open in $\mathbb{R}^n$ or, A is an open subset of $\mathbb{R}^n$ if $A = \text{int}(A)$.

- $[2,3)$ is NOT open in $\mathbb{R}$.
- $(2,5)$ is open in $\mathbb{R}$.
- $B_1((0,0)) \cup \{(1,0)\}$ is not open in $\mathbb{R}^2$.
-  Is this open in $\mathbb{R}^2$?

7. Closed set in $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. We say that A is closed in $\mathbb{R}^n$ or, A is a closed subset of $\mathbb{R}^n$ if $\partial A \subseteq A$.

- $(2,3]$ is not closed in $\mathbb{R}$.
- $[5,7)$ is not closed in $\mathbb{R}$.
- $[a,b]$, where $a,b \in \mathbb{R}$, is closed in $\mathbb{R}$.
- Let $A = (3,\infty) \subseteq \mathbb{R}$. Compute ∂A.
- Give an example of a subset of $\mathbb{R}$, that is neither open nor closed in $\mathbb{R}$.
- Is $\mathbb{R}^n$ an open subset of $\mathbb{R}^n$?
- Is $\mathbb{R}^n$ a closed subset of $\mathbb{R}^n$?
- Let $A \subseteq \mathbb{R}^n$. Can we assert that $A = \text{int}(A) \cup \partial A$? If yes, prove it. If no, give a counterexample.

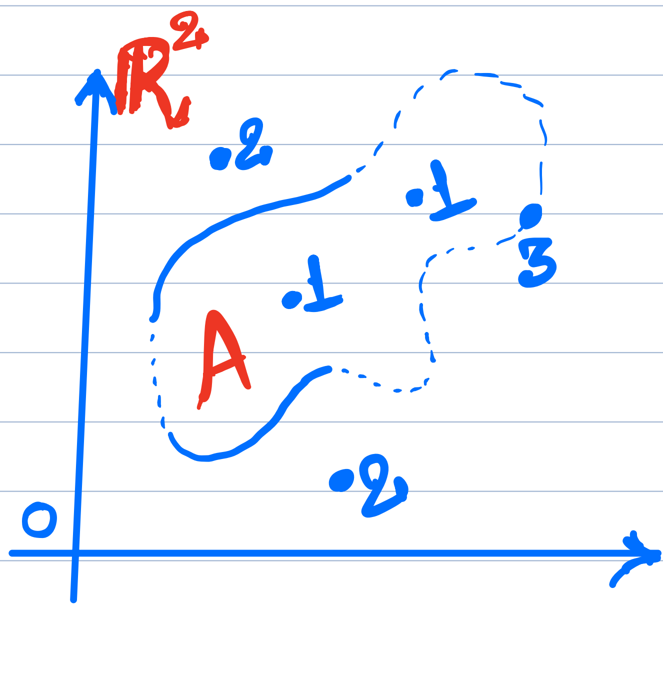
Proposition. Let $A \subseteq \mathbb{R}^n$.

If $x \in \mathbb{R}^n$, then one, and only one, of the following three possibilities holds.

(1) $\exists\, r > 0$ s.t. $B_r(x) \subseteq A$. ← interior pt. of A

(2) $\underbrace{\exists\, r > 0 \text{ s.t. } B_r(x) \subseteq A^c.}_{x \text{ is an exterior point of } A}$

(3) $\underbrace{\forall\, r > 0,\ B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap A^c \neq \emptyset.}_{\text{boundary point of } A}$



Exercise: Find the exterior, interior and the boundary of the following sets.

- $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$
- $\{x \in \mathbb{R}^n : \|x\| = 1\}$
- $\{x \in \mathbb{R}^n : \|x\| < 1\}$
- $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \forall j \in \{1, \dots, n\} \; x_j \in \mathbb{Q}\}$

Lecture-03

Elements of point-set topology, continued.

In the last lecture, we introduced:

1. open ball in $\mathbb{R}^n$ of radius $r>0$ centered at x_0
2. Interior point of a given subset of $\mathbb{R}^n$
3. boundary point of a given subset of $\mathbb{R}^n$
4. interior of a set
5. boundary of a set
6. open sets in $\mathbb{R}^n$
7. closed sets in $\mathbb{R}^n$

Let's introduce:

8. bounded subsets of $\mathbb{R}^n$:

Let $A \subseteq \mathbb{R}^n$. A is called a bounded subset of $\mathbb{R}^n$ if $\exists\, r>0$ such that $A \subseteq B_r(\vec{0})$, where $\vec{0} = (0, \ldots, 0)$.

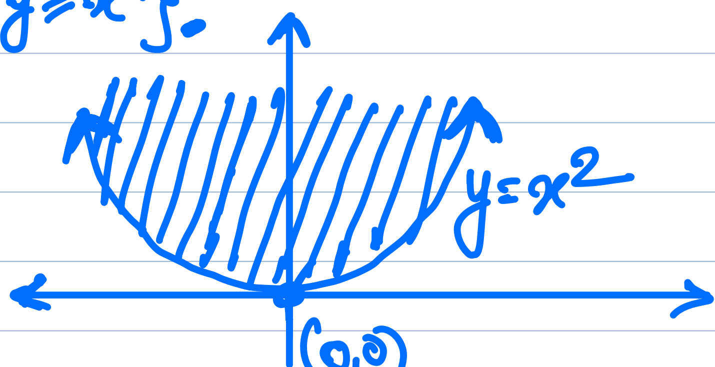
(9) unbounded subsets of $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. A is said to be unbounded if it is not bounded.

Example:- Let $f(x,y) = \sqrt{y-x^2}$.

What is the domain of f?

$\mathrm{dom}(f) = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$.



- Is dom(f) open?

No! (why?)

$\mathrm{int}(\mathrm{dom}(f)) = \{(x,y) \in \mathbb{R}^2 : y > x^2\}$

$\therefore \mathrm{dom}(f) \neq \mathrm{int}(\mathrm{dom}(f))$. $\therefore$ dom(f) NOT open.

- Is dom(f) closed? Yes.
- bounded? No!

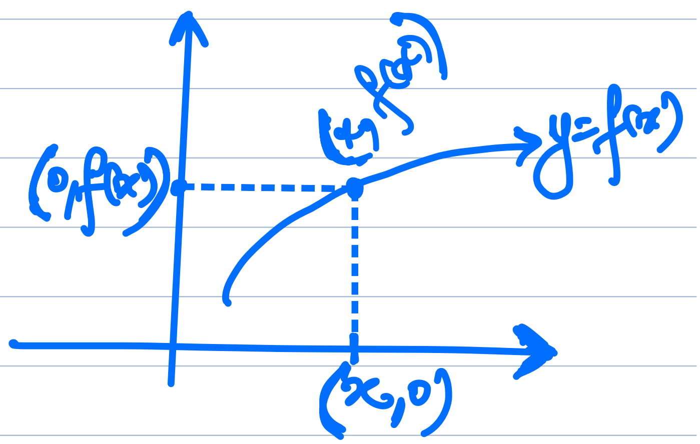
(3) $\operatorname{Ran}(f) = [0,\infty) \subseteq \mathbb{R}$.

This is **closed**,
not open,
and **unbounded** in $\mathbb{R}$.

Graph of a function

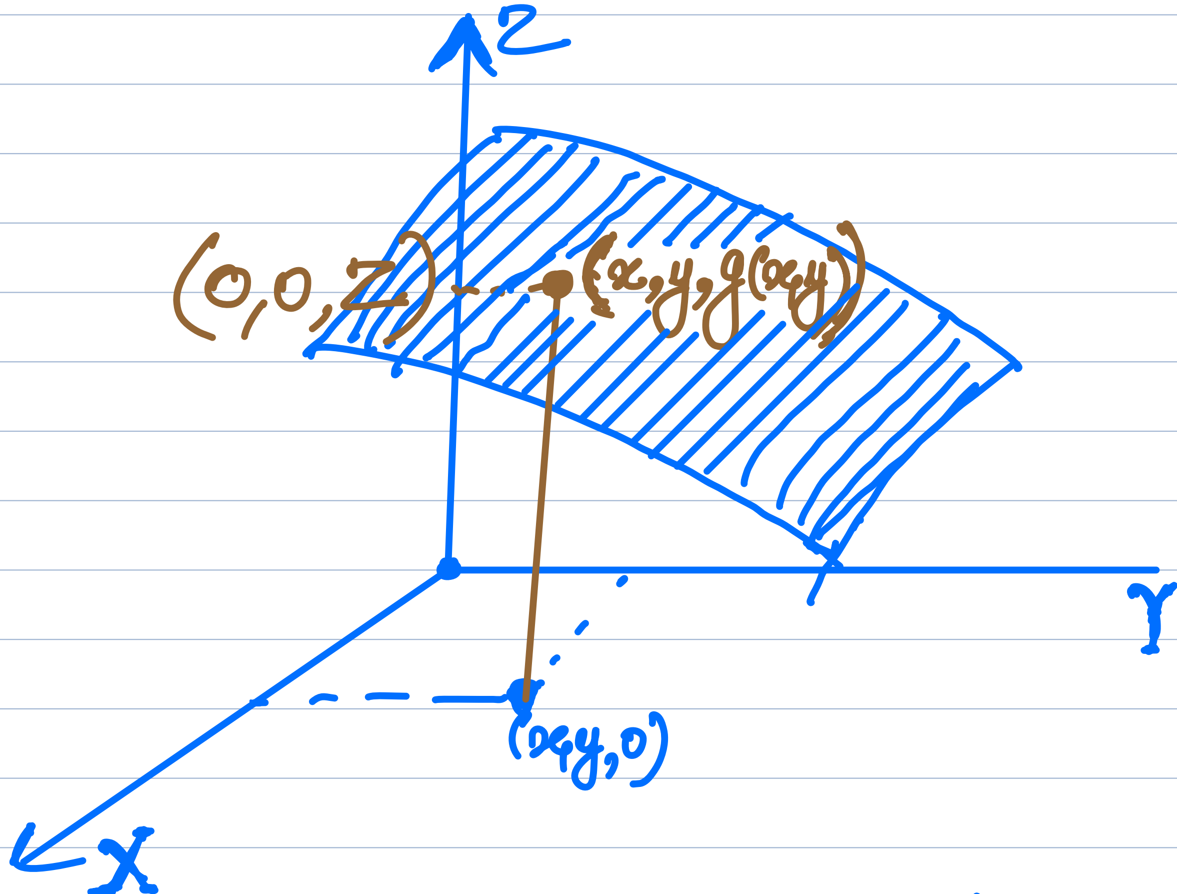
Case I: function of one variable: $y=f(x)$.

[The Graph of f is a "curve" (in $\mathbb{R}^2$)]



Case II: function of two variables

$z=g(x,y)$



Graph of $Z=g(x,y)$ is a "surface" (in $\mathbb{R}^3$)

Case III: function of "n" independent variables

Graph of a function:

Let n be any positive integer, let $D \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$, and let $f: D \longrightarrow \mathbb{R}$ be a function from D to $\mathbb{R}$. The graph of the function f, denoted by $G(f)$, is defined to be the set

$$G(f) = \{(\underbrace{x}_{n\text{-tuple}}, \underbrace{f(x)}_{\text{real number}}) : x \in D \subseteq \mathbb{R}^n\} \ (\subseteq \mathbb{R}^{n+1}).$$

n+1 coordinates.

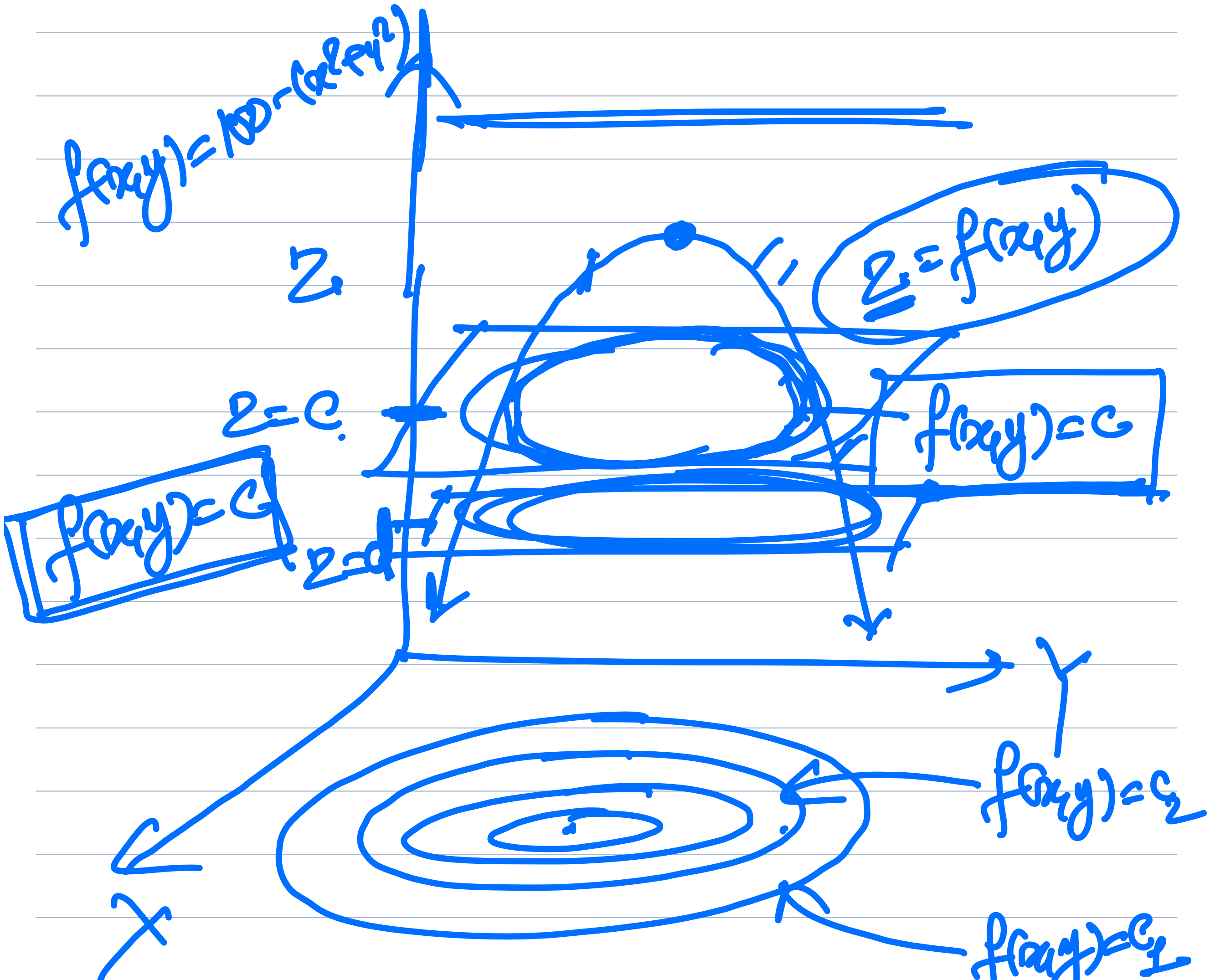
Level Sets

Defn: Let n be any positive integer, let $D \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a function from D to $\mathbb{R}$. Given a constant $c \in \operatorname{ran}(f)$, the Level set of f at the point c is defined to be the set

$$\{x \in D \subseteq \mathbb{R}^n : f(x) = c\} \ (\subseteq \mathbb{R}^n).$$

Remarks:

- Every Level set of the function f lies in the domain of the function f.
- On each level set, the value of the function is a constant.
- When $n=2$ (i.e. when we consider $D \subseteq \mathbb{R}^2$, so that f is a function of two variables), we call it a **Level Curve**.
- When $n=3$ (i.e. when we consider $D \subseteq \mathbb{R}^3$, so that f is a function of three variables), we call it a **Level Surface**.



f(x,y) = 100 − (x² + y²)
z
z = f(x,y)
z = c
f(x,y) = c
f(x,y) = c
z = d
Y
X
f(x,y) = c₂
f(x,y) = c₁

Example: Let $f(x,y) = 4 - x^2 - y^2$. Find the level curves.

Solution: We have $z = f(x,y)$. We replace z by some admissible constant c, and we get

$$f(x,y) = c$$

i.e., $4 - x^2 - y^2 = c$

or, $\left[x^2 + y^2 = 4 - c\right]$

Clearly, c must be either less than or equal to 4, i.e., $c \leq 4$.

So,

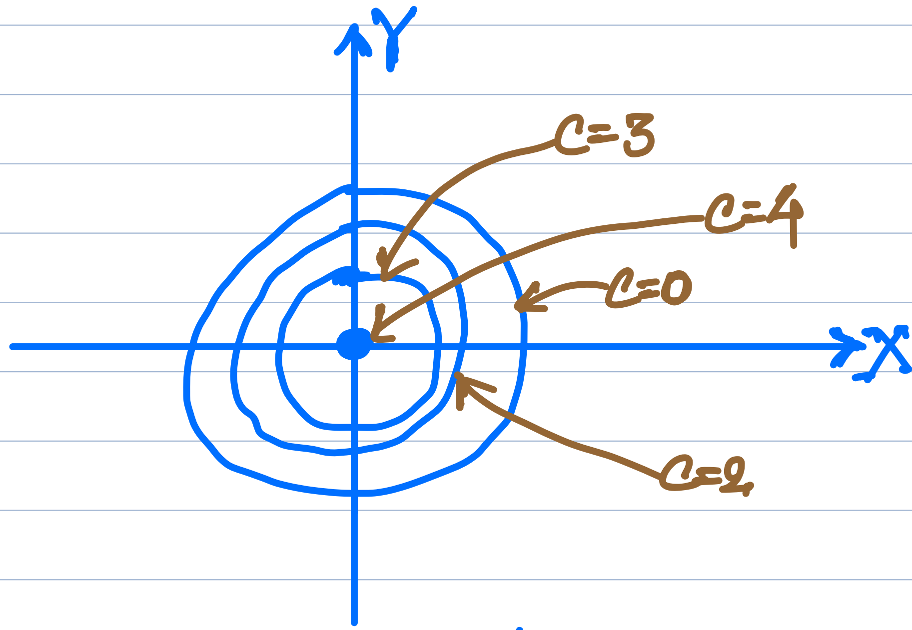
If $c = 4$, we get $x^2 + y^2 = 0$

(a circle of radius 0. ∴ merely the origin)

If $c=3$, we get $x^2+y^2=1$
(a circle of radius 1 centered at the origin)

If $c=0$, we get $x^2+y^2=4$ (a circle of radius 2)

If $c<0$, we get circle of radius >2.



__Exercise__ Let $f(x,y)=\frac{1}{\sqrt{16-x^2-y^2}}$. Find the level curves.

__Exercise__ Let $g(x,y,z)=x^2+y^2+z^2$. Find the level surfaces.

Lecture 04

Recall:

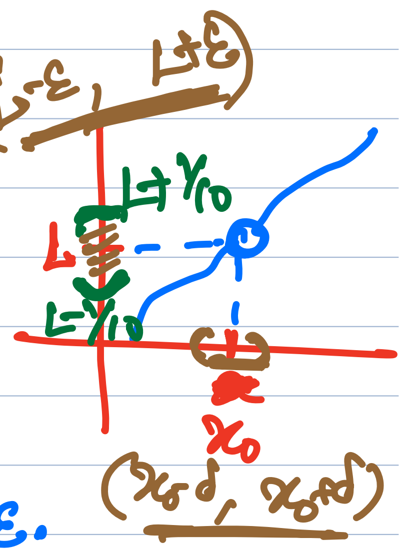
Let $f: \mathbb{R} \to \mathbb{R}$, be a function.

$$\lim_{x \to a} f(x) = L$$

(read as, "the limit of f at a")

We can get $f(x)$ as close to the real number L as desired, by choosing x sufficiently close to (but not equal to) a

$\Updownarrow$



Given any $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $x \in \mathrm{dom}(f)$ that satisfies $\underbrace{0 < |x-a| < \delta}_{[a-\delta < x < a+\delta \text{ and } x \neq x_0]}$, we have $\underbrace{|f(x) - L| < \varepsilon}_{[L-\varepsilon < f(x) < L+\varepsilon]}$.

definition: Let $S \subseteq \mathbb{R}$, and $f: S \to \mathbb{R}$ be a function. If $a \in \mathbb{R}$ be a limit point of S, ??

then a point $L \in \mathbb{R}$, is the limit of f at a if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $x \in \text{dom}(f) = S$ and $0 < |x - a| < \delta$,

and we write
$$\lim_{x \to a} f(x) = L.$$

Note: (1) $a \in \mathbb{R}$, but a need not necessarily belong to dom(f).

(2) Even if $a \in \text{dom}(f)$, it may be the case that
$$f(a) \neq \lim_{x \to a} f(x) = L$$

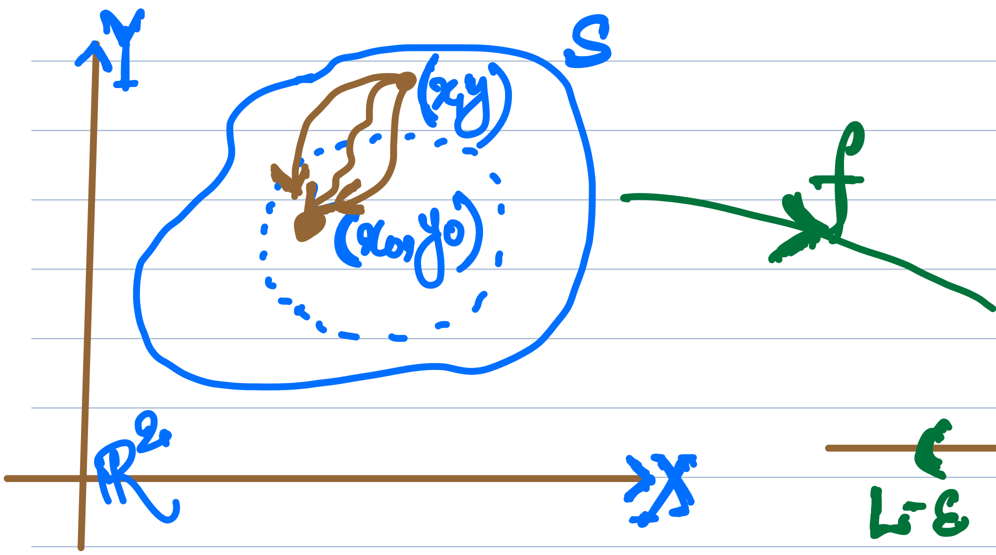
(3) If $a \in \text{dom}(f)$ and if $f(a) = \lim_{x \to a} f(x)$, then f is continuous at the point a.

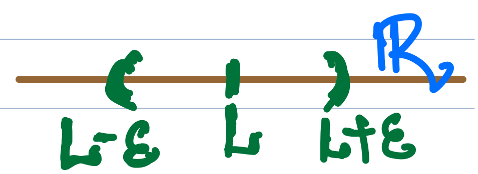
Let's move on to functions of several variables.

definition: Let $S \subseteq \mathbb{R}^n$ and $f: S \to \mathbb{R}$ be a function from S to $\mathbb{R}$. If $a \in \mathbb{R}^n$ is a limit point of S, then a point $L \in \mathbb{R}$ is called the limit of f at a, if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$x \in S \text{ and } 0 < \|x - a\| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

$\left[\begin{array}{l} \text{If } x = (x_1, x_2, \dots, x_n) \text{ and } \\ \quad a = (a_1, a_2, \dots, a_n), \text{ then} \\ \|x - a\| = \sqrt{\sum_{i=1}^n (x_i - a_i)^2} \end{array}\right]$ or $\left[\begin{array}{l} x \in B_\delta(a) \text{ and} \\ x \neq a \end{array}\right]$





Example 1: $f(x,y)=x$.

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0.$$

Example 2: $f(x,y)=c$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = c.$$

Example 3: $$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$$

Solution: Note that

$$\operatorname{dom}(f)=\{(x,y)\in\mathbb{R}^2 : x\neq y+1\}.$$

$\therefore$ Clearly $(4,3)\notin \operatorname{dom}(f)$

$$\Rightarrow \lim_{(x,y)\to(4,3)} \frac{(\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})}{(x-y-1)\ (\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{\cancel{(x-y-1)}}{\cancel{(x-y-1)}\ (\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{1}{(\sqrt{x}+\sqrt{y+1})} = 1/4 \text{ ans.}$$

Properties of Limits of fns. of Several Variables

Let f and g be functions from $\mathbb{R}^2$ to $\mathbb{R}$.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and

$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$, then

(1) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M$

(2) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$

(3) $\lim_{(x,y)\to(x_0,y_0)} [c f(x,y)] = cL$

(4) $$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$$

provided that $M \neq 0$.

(5) If $r, s \in \mathbb{Z}$ such that there are no common factors of r and s (except 1) and if $s \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number.

Example 4: $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$$

Solution: (use quotient rule)

Example 5: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Solution: (Can't use quotient rule right away!
Why? $\because \sqrt{x}-\sqrt{y} \to 0$ as $(x,y)\to(0,0)$)

Remark: Can you see that the definition of the limit of a function does not help you "find" the limit; it only helps you verify whether or not L is the limit of f at a.

How to show that

(i) L is not the limit of f at a

(ii) the limit of f at a does not exist

For $\underline{\underline{(i)}}$ you need to "negate" the implication in the definition of the limit

Hint: How to negate $A \Rightarrow B$?
Negate "$\neg A \vee B$," and you shall get "$A \wedge \neg B$."

"L is not the limit of f at a"

$$\Updownarrow$$

"$$\lim_{(x_1,x_2)\to(a_1,a_2)} f(x_1,x_2) \neq L$$"

$$\Updownarrow$$

$\exists\, \varepsilon>0$ such that $\forall\, \delta>0$, we have

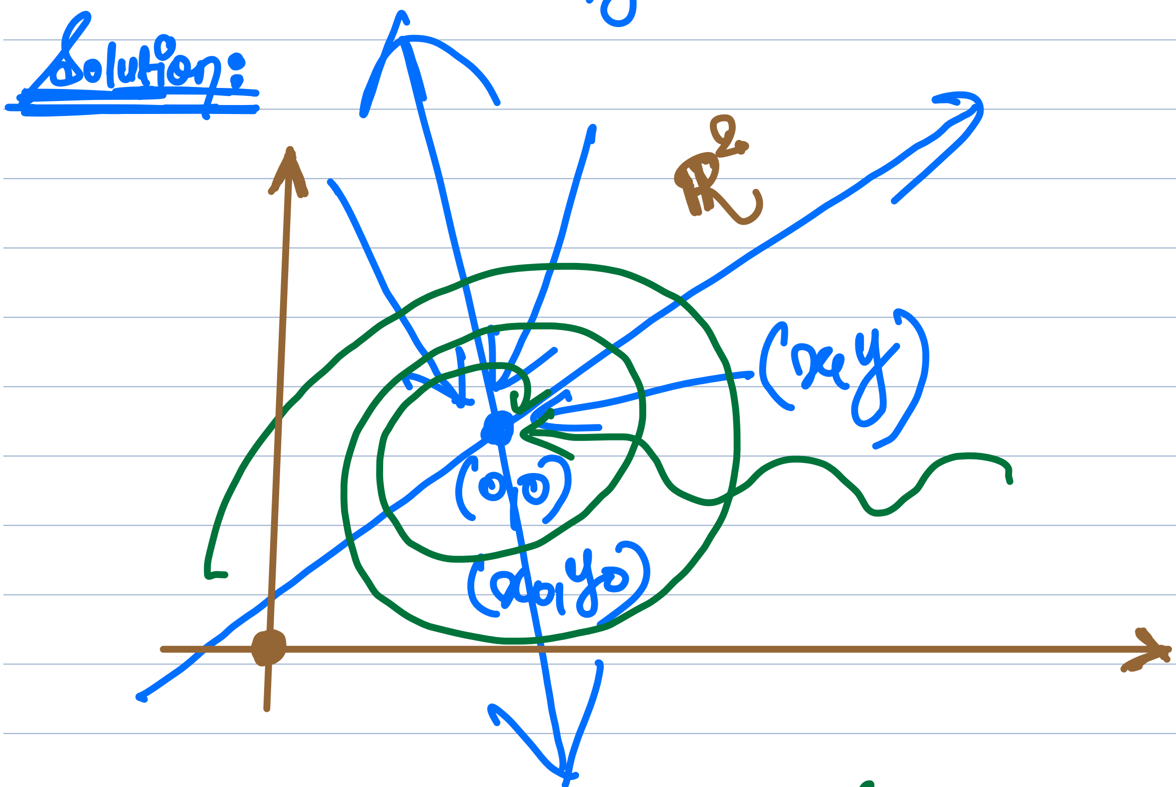
- $x=(x_1,x_2)\in \mathrm{dom}(f)$,
- $0<\|x-a\|<\delta$, and
- $|f(x)-L| \geq \varepsilon$

For (ii), we have, what we call, **two-path test!**

Example 6: Compute

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$ if it exists.

Solution:



→ **Let us approach (0,0) along the line $y=2x$.**

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{2x(2x)}{x^2+(2x)^2}$$

$$= \lim_{x\to 0} \frac{4x^2}{5x^2} = 4/5$$

→ **Let us approach (0,0) along the line $y=5x$**

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=5x}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{2x(5x)}{x^2+(5x)^2}$$

$$= \lim_{x\to 0} \frac{10x^2}{26x^2} = 10/26 = 5/13.$$

So, it is evident that the limit must not exist!!!

Reason: Different paths of approach to the origin (0,0) can lead to different results!

In general,

→ **Let us approach (0,0) along the line $y=mx$**

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{2x(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{2mx^2}{(1+m^2)x^2}$$

$$= \frac{2m}{1+m^2}$$ which depends on "m", that is, it depends on the path!!!

Example 7: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = ?$

Hint: Try the path $y=cx^2$.

(By the two-path test, f has no limit as (x,y) approaches $(0,0)$.)

Two-path test (for the non existence of limit)

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \to (x_0,y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ does not exist}$$

Example 8: Compute

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} \text{ if it exists.}$$

Remark: **(1)** Let $f(x,y) := \dfrac{4xy^2}{x^2+y^2}$.

Then, $\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$. f is not defined at $(0,0)$. However, it makes sense to discuss the limit of the function f as (x,y) approaches $(0,0)$.

(Why? Since f is defined at every point around some "neighborhood" of $(0,0)$.)

(2) quotient rule does not apply! (why?)

(3) Can you come up with some simplification? The denominator is not an irrational function.

(4) What if we try 2-path test?

Path 1: (along x-axis)

$$\lim_{(x,0)\to(0,0)} \frac{4x\cdot 0^2}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Path 2: (along the y-axis)

$$\lim_{(0,y)\to(0,0)} \frac{4\cdot 0\cdot y^2}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

Claim: The limit is indeed "0." Prove it!!!

Classroom proof:

$$\left.\begin{array}{l}\forall \varepsilon>0, \exists \delta>0 \text{ s.t.} \\ (x,y) \in \operatorname{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\} \\ \text{and} \\ 0<\|(x,y)-(x_0,y_0)\|<\delta\end{array}\right] \Rightarrow |f(x,y)-L|<\varepsilon.$$

Equivalently,

$$\left.\begin{array}{l}\forall \varepsilon>0, \exists \delta>0 \text{ s.t.} \\ (x,y) \neq (0,0) \text{ and} \\ \sqrt{x^2+y^2}<\delta\end{array}\right. \Rightarrow \left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon$$

Now, let $\varepsilon>0$ be given.
We are required to find some $\delta>0$, depending only on $\varepsilon>0$, such that

$$\left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon \quad \text{whenever} \quad 0<\sqrt{x^2+y^2}<\delta.$$

To find some $\delta>0$, we need to estimate $\left|\frac{4xy^2}{x^2+y^2}\right|$.
In particular, we want

$$\left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon.$$

[Note that, we don't know for which points (x,y) the above inequality is satisfied.

We want this inequality to be satisfied by $(x,y)\neq(0,0)$ whenever $\sqrt{x^2+y^2}<\delta$ for a chosen δ.]

Observe that

$$\left|\frac{4xy^2}{x^2+y^2}\right|=\frac{4|x|y^2}{x^2+y^2}$$

and since $y^2\leq x^2+y^2$, we get

$$\frac{4|x|y^2}{x^2+y^2}\leq 4|x|.$$

So, it suffices to estimate $4|x|$, (that is, if we establish that $4|x|<\varepsilon$, we are through.)

Let's not forget that we want $4|x| < \varepsilon$; we don't already know it. We want $4|x| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$ (the delta that we are looking for!)

So, can we estimate $4|x|$ in terms of $\sqrt{x^2+y^2}$?

Of course, $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$, so that $4|x| \leq 4\sqrt{x^2+y^2}$. So, it is sufficient to make $4\sqrt{x^2+y^2} < \varepsilon$. (why?)

But expecting $4\sqrt{x^2+y^2} < \varepsilon$ is same as expecting $\sqrt{x^2+y^2} < \varepsilon/4$. This suggests us to choose $\delta = \varepsilon/4$.

Now, clearly, for any given $\varepsilon > 0$, we choose $\delta = \varepsilon/4$ to obtain the following implication:

$$0<\sqrt{x^2+y^2}<\delta \Rightarrow \left|\frac{4xy^2}{x^2+y^2}\right| = \frac{4|x|y^2}{x^2+y^2}$$

$$\leq 4|x|$$

$$\leq 4\sqrt{x^2+y^2}$$

$$< 4\delta$$

$$= 4\,\varepsilon/4$$

$$= \varepsilon.$$

Textbook proof:

Let $\varepsilon>0$ be given. We are required to find $\delta>0$, which depends on the given $\varepsilon>0$, such that
$$0<\|(x,y)-(x_0,y_0)\|<\delta \Rightarrow |f(x,y)-L|<\varepsilon.$$

Since $(x_0, y_0) = (0,0)$ and $L = 0$, we are required to find $\delta > 0$, which depends on $\varepsilon > 0$, such that

$$0 < \sqrt{x^2+y^2} < \delta \Rightarrow \left| \frac{4xy^2}{x^2+y^2} \right| < \varepsilon.$$

To this end, let us estimate $\left| \dfrac{4xy^2}{x^2+y^2} \right|$.

Now observe that,

$$\left| \frac{4xy^2}{x^2+y^2} \right| = \frac{4|x|y^2}{x^2+y^2} \leqslant 4|x| \leq 4\sqrt{x^2+y^2}.$$

This suggests that we should choose δ to be $\varepsilon/4$, for then,

$$\sqrt{x^2+y^2} < \delta = \varepsilon/4 \Rightarrow 4\sqrt{x^2+y^2} < \varepsilon$$

and since

$$\left| \frac{4xy^2}{x^2+y^2} \right| \leqslant 4\sqrt{x^2+y^2},$$

it follows that

$$\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon .$$

This completes the proof. $\blacksquare$

Now, please go through the classroom proof once again!

Lecture - 05

Continuity of a function at a point

Let $S \subseteq \mathbb{R}$ and $f: S \to \mathbb{R}$ be a function.
We say that f is continuous at $a \in \mathbb{R}$ if

(i) $a \in dom(f) = S$;

(ii) $\lim_{x \to a} f(x)$ exists;

(iii) $\lim_{x \to a} f(x) = f(a)$.

$\Updownarrow$ (Equivalently,)

Let $S \subseteq \mathbb{R}$, $a \in S$ and $f: S \to \mathbb{R}$ be a function.
We say that f is continuous at $a \in S$ if
$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that for all
$x \in dom(f) = S$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Analogously,

Let $S \subseteq \mathbb{R}^2$, $a = (a_1, a_2) \in S$ and $f : S \to \mathbb{R}$, be a function. We say that **f is continuous at $a \in S$** if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $(x_1, x_2) \in \mathrm{dom}(f) = S$

$$\underbrace{\|(x_1, x_2) - (a_1, a_2)\|}_{= \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}} < \delta \Rightarrow |f(x_1, x_2) - f(a_1, a_2)| < \varepsilon$$

Note: f is called "continuous" if it is continuous at each point of its domain.

Example 1:
$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is f continuous at $(0,0)$?

Soln: (See Example 6 from the previous lecture)

Example 2: Let $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$.

→ Prove/Verify that

(a) $(0,0) \notin \mathrm{dom}(f)$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists

[Hint: See Example 8 of the previous lecture

→ define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Example 3: Let $g(x,y) = \dfrac{3x^2y}{x^2+y^2}$.

Define $g(0,0)$ in such a way that extends g to be continuous at the origin.

Example 4: Let $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$.

Define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Homework: (use polar coordinates to find the limit of f at the point $(0,0)$)

Example 5: Let $f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that f is continuous at every point except $(0,0)$.

Solution: **Hint:** Try two cases:

Case I: When $(x_0, y_0) \neq (0,0)$. Now, use $\varepsilon-\delta$ argument to show that f is continuous at (x_0, y_0).

Case II: Show that f is not continuous at $(0,0)$.

Question (8): At what points in the plane/space are the following functions continuous?

(1) $g(x,y) = \sin\left(\frac{1}{xy}\right)$

(2) $h(x,y) = \frac{x^2+y^2}{x^2-3x+2}$

(3) $g(x,y) = \frac{x+y}{2+\cos x}$

(4) $f(x,y,z) = \frac{1}{x^2+z^2-1}$

(5) $h(x,y,z) = \frac{1}{|y|+|z|}$

Partial derivatives

Let us first recall the following definition in the case of functions of one variable.

defn: Let $D \subseteq \mathbb{R}$, let $x_0 \in D$ be an interior point and let $f: D \to \mathbb{R}$ be a function.

We say that <u>f is differentiable at x_0</u>, or, <u>f has a derivative at x_0</u>, provided that the following limit exists,

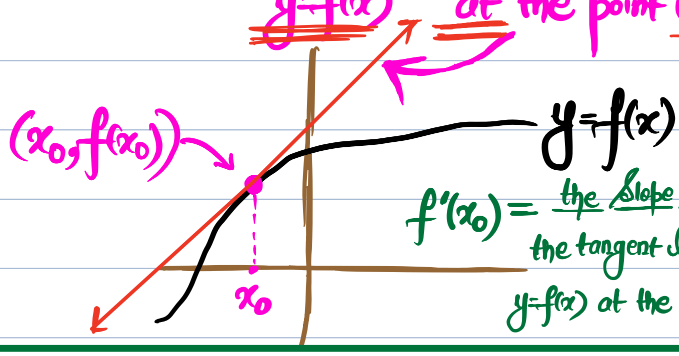
$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \left(= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right).$$

When this limit exists, we denote it by

$$f'(x_0) \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=x_0}$$

<u>**Question:**</u> <u>What is the geometrical interpretation of $f'(x_0)$?</u>

<u>the tangent line to the curve $y=f(x)$ at the point $(x_0, f(x_0))$</u>



$f'(x_0)$ = the slope of the tangent line to the curve $y=f(x)$ at the point $(x_0, f(x_0))$

On the language you may encounter

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function.
Then the following statements essentially mean the same thing.

- f is differentiable at a point $x_0 \in D$
- f has a derivative at a point $x_0 \in D$

Question

What is the analogous notion of the "derivative" in case of $f: \mathbb{R}^2 \to \mathbb{R}$, or $f: \mathbb{R}^n \to \mathbb{R}$, or $f: \mathbb{R}^n \to \mathbb{R}^m$? [$\longrightarrow$ Later!]

First, let us talk about partial derivatives.

defn: Let $D \subseteq \mathbb{R}^2$, let $(x_0, y_0) \in D$ be an interior point of D and let $f: D \to \mathbb{R}$ be a real-valued function of two variables.

We say that **f has a partial derivative w.r.t x at the point (x_0, y_0)** provided that the following limit exists,

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

When this limit exists, we denote it by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \text{ or } f_x(x_0, y_0) \text{ or } \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}$$

and call it "**the partial derivative of f w.r.t. x at the point (x_0, y_0).**"

Example: $f(x,y) = xy + 3x^2 + y$

Compute $\dfrac{\partial f}{\partial x}$ at $(1,2)$.

$$\frac{\partial f}{\partial x} = y + 6x$$

$$\frac{\partial f}{\partial x}(1,2) = 2+6 = 8.$$

Example: Calculate the partial derivative $\dfrac{\partial f}{\partial x}$ of the function

$$f(x,y) = x^3 - 3x^2y^3 + y^2.$$

Soln: $$\frac{\partial f}{\partial x} = 3x^2 - 6xy^3.$$

defn: Let $D \subseteq \mathbb{R}^2$, let $(x_0, y_0) \in D$ be an interior point of D and let $f : D \to \mathbb{R}$ be a real-valued function of two variables.

We say that f has a partial derivative w.r.t y at the point (x_0, y_0) provided that the following limit exists,

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.$$

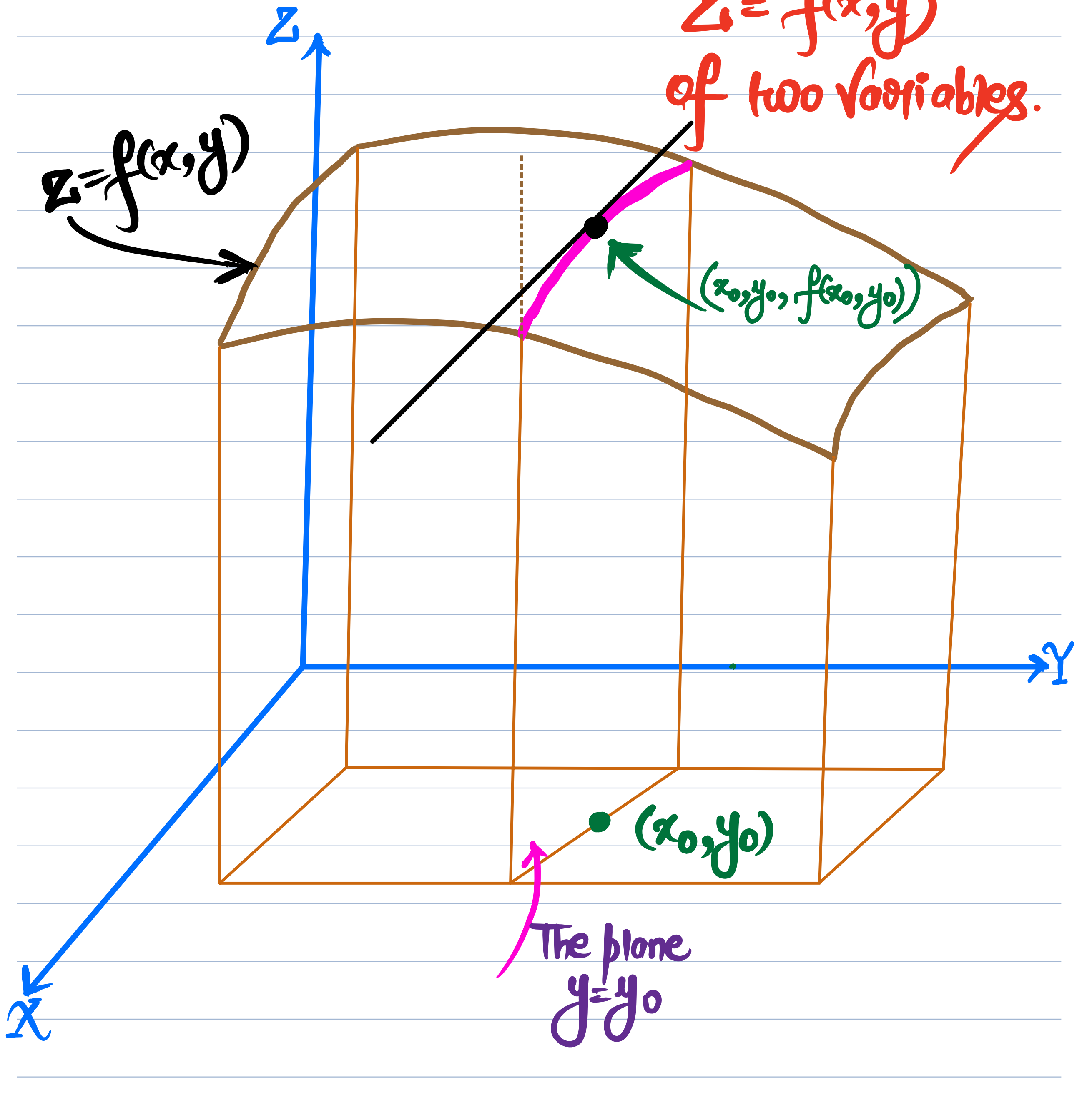
When this limit exists, we denote it by

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \text{ or } f_y(x_0, y_0) \text{ or } \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0}$$

and call it "the partial derivative of f w.r.t. y at the point (x_0, y_0).

Geometric interpretation of partial derivatives —

for a function $z = f(x,y)$ of two variables.



⊙ The graph of $z=f(x,y)$ is a surface — the one sketched above.

⊙ (x_0,y_0) is a given point in the interior of the domain of the function (this is a point in the X-Y plane, such that $(x_0,y_0,f(x_0,y_0))$ is a point on the surface.

⊙ We wish to interpret $\frac{\partial f}{\partial x}$ at (x_0,y_0)

⊙ Treat the variable "y" as constant, and the constant must of course be y_0, i.e., $y=y_0$.

⊙ <u>To hold y fixed at the value y_0 means</u> <u>To intersect the surface $z=f(x,y)$ with the plane $y=y_0$.</u>

- The intersection of

the surface $z = f(x,y)$

and

the plane $y = y_0$

is the curve $z = f(x, y_0)$.

(this curve is on the plane $y = y_0$)

- What does then the value

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) \text{ denote ?}$$

It is the slope of the tangent line to the curve $z = f(x, y_0)$ at the point $x = x_0$.

Lecture 06

Relationship between Continuity and existence of partial derivatives

Example 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

$\longrightarrow$ ⊙ $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = 0$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = 0$

$\longrightarrow$ ⊙ f is <u>not</u> continuous at $(0,0)$

[use 2-path test to show that the limit of f at $(0,0)$ does not exist, and hence f is not continuous. Compute the limit along $y = cx^2$.]

Example 2: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

(1) f is not cont. at $(0,0)$

(2) $\dfrac{\partial f}{\partial x}$

"

partial derivative of f w.r.t. x

"

partial derivative of f w.r.t. ~~x~~ the first ind. variable

Now, $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = 0$?? (How)

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0.$$

Next, $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{1-1}{h}$$

$$= 0.$$

$\therefore$ The partial derivatives exist at $(0,0)$

(3) However, f is not continuous at $(0,0)$.
(Verify!!!)

Thus, unlike in one-variable case, where existence of derivatives guarantees continuity, existence of (all) partial derivatives at a point does not even guarantee that the function is continuous at that point!

Need a new notion of derivative!

[Total Derivative]

Differentiability (single-variable case)

Let us recall the following definition from the last lecture.

Definition 1: Let $D \subseteq \mathbb{R}$, $x_0 \in \text{int}(D)$ and $f: D \to \mathbb{R}$ be a (real-valued) function (of one variable). We say that f is differentiable at $x_0 \in \text{int}(D)$

$\Updownarrow$

the following limit exists, $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

When this limit exists, we denote it by $f'(x_0)$, and call it "the derivative of f at x_0".

(End of the definition)

$\Updownarrow$ (Convince yourself)

$\exists$ a real number, denoted by $f'(x_0)$,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0).$$

or, equivalently,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-\overset{f'(x_0)h}{\cancel{h f'(x_0)}}}{h} = 0$$

[Definition 2: (reformulation of the first defn)]

Let $D \subseteq \mathbb{R}$, $x_0 \in \operatorname{int}(D)$ and $f : D \to \mathbb{R}$ be a (real-valued) function (of one variable). We say that f is differentiable at $x_0 \in \operatorname{int}(D)$

$\Updownarrow$

$\exists$ a real number, denoted by $f'(x_0)$, such that

$$\boxed{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-f'(x_0)h}{h} = 0}$$

Notation:

Let $f'(x_0)$ exists. Define a function E_{x_0} as follows:

(*) $E_{x_0}(h) := f(x_0+h)-f(x_0)-f'(x_0)h$.

(*) $$\boxed{f(x_0+h)-f(x_0) = \underbrace{f'(x_0)h}_{\substack{\text{Approximation of}\\ f(x_0+h)-f(x_0)}} + \underbrace{E_{x_0}(h)}_{\text{Error}}}$$

Observations:

① $f'(x_0)$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

proof: define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(h) = f'(x_0)h. \quad (\text{for every } h \in \mathbb{R})$$

Check: • $T(h_1 + h_2) = T(h_1) + T(h_2)$

• $T(\lambda h) = \lambda T(h)$

[Note: In fact, every real number λ can be thought of as a linear mapping of $\mathbb{R}$ into $\mathbb{R}$. (How?)

② If f is differentiable at x_0, then

$$\frac{E_{x_0}(h)}{h} \to 0 \quad \text{as } h \to 0$$

i.e., not only the error $E_{x_0}(h)$ tends to 0 as h tends to 0, but it does so rapidly that it still tends to 0 when divided by h !!!

(The error $E_{x_0}(h)$ is of smaller order than h (when h is small))

* The "**total derivative**" of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ will now be defined in such a way that it preserves these two properties.

But prior to that, let us re-write the definition of differentiability of a function at a point, in the form which will be generalized to functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 3 (Yet another reformulation)

Let $D \subseteq \mathbb{R}$, $x_0 \in \operatorname{int}(D)$ and $f : D \to \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in \operatorname{int}(D)$

$$\Updownarrow$$

there exists a linear transformation $T_{x_0} : \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - T_{x_0}(h)}{h} = 0$$

When f is differentiable at x_0, we set

$$f'(x_0) = T_{x_0}$$ and call it

"the derivative of f at x_0"

Now, let us generalize this version of the definition to the case of a real-valued function in several variables.

Differentiability of a function $f: \mathbb{R}^n \to \mathbb{R}$, at a point $\vec{x}_0 \in \mathbb{R}^n$ (The Total Derivative)

defn: Let $D \subseteq \mathbb{R}^2$, $\vec{x}_0 \in \text{int}(D)$, Let $f: D \to \mathbb{R}$. We say that f is differentiable at $\vec{x}_0 \in \text{int}(D) \subseteq \mathbb{R}^2$ (or, f has a total derivative at $\vec{x}_0 \in \mathbb{R}^2$)

$\Updownarrow$

there exists a linear transformation

$T_{\vec{x}_0}: \mathbb{R}^2 \to \mathbb{R}$, such that

$$\lim_{\vec{h}=(h_1,h_2)\to(0,0)} \frac{f(\vec{x}_0+\vec{h}) - f(\vec{x}_0) - T_{\vec{x}_0}(\vec{h})}{\|\vec{h}\|} = 0$$

When the total derivative exists, we set $f'(\vec{x_0}) = T_{x_0}$ and call it "the total derivative of f at $\vec{x_0}$"

(End of the definition)

Remarks:

① $$\lim_{\vec{h} \to \vec{0}} \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|_2} = 0$$ $\left(E_{x_0} : \mathbb{R}^2 \to \mathbb{R}\right)$

② Reformulation of the above defn.

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{int}(D)$, let $f: D \to \mathbb{R}$. Then f is differentiable at $\vec{x_0}$ iff

$$\forall \varepsilon > 0 \left(\exists r > 0 \left(0 < \|\vec{h}\| < r \Rightarrow \left| \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|} \right| < \varepsilon \right) \right).$$

∎

8. Reformulation :(Yet again)

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{int}(D)$, let $f: D \to \mathbb{R}$. Then f is differentiable at $\vec{x_0}$ iff $\forall \varepsilon > 0 \left(\exists r > 0 \right($

$$\|\vec{h}\| < r \Rightarrow |E_{x_0}(\vec{h})| < \|\vec{h}\| \varepsilon) . \blacksquare$$

Thm 1: If $f: \mathbb{R}^n \to \mathbb{R}$ is diff'ble at $\vec{x_0} \in \mathbb{R}^n$, then $\exists$ a unique linear transformation $T_{x_0}: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\lim_{\vec{h} \to 0} \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|_2} = 0$$

Thm 2: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be diff'ble at $\vec{x_0} \in \mathbb{R}^2$ with the total derivative

$T_{\vec{x_0}} : \mathbb{R}^2 \to \mathbb{R}$, then

① both the partial derivatives of f at $\vec{x_0}$ exist,

and we have

② $$T_{\vec{x_0}} = \left[\frac{\partial f}{\partial x}(\vec{x_0}), \frac{\partial f}{\partial y}(\vec{x_0}) \right].$$

analogous to $f'(\vec{x_0})$

So that,

$$T_{\vec{x}_0}(\vec{h}) = \left[\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0)\right]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}$$

analogous to $f'(\vec{x}_0)h$

End of Theorem 2.

Theorem 3: (Differentiability ⇒ continuity)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$ and $f: D \to \mathbb{R}$, be a function.

If f is differentiable at $\vec{x_0}$, then f is continuous at $\vec{x_0}$.

Proof: (Beyond the scope of this course, but I would be more than happy to walk you through it if you are really interested!)

Generalization

Recall the definition of a vector-valued function of several variables (or, a vector field).

Let $D \subseteq \mathbb{R}^n$ and let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n variables). The function $\vec{f}$ (in fact, any such function) has the form

$$\vec{f}(\vec{x}) = \Big(f_1(\vec{x}), \dots, f_m(\vec{x})\Big)$$

where we call the functions

$f_i : D \to \mathbb{R}$, the "<u>component functions</u>".

Differentiability (The Total Derivative)

<u>defn:</u> Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \text{int}(D)$ and let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n variables).

We say that $\vec{f}$ is differentiable at $\vec{x_0}$ (or, $\vec{f}$ has a total derivative at $\vec{x_0}$) if there exists a linear transformation

$$T_{\vec{x_0}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ such that}$$

$$\lim_{\vec{h} \to \vec{0}} \frac{\left\| \vec{f}(\vec{x_0}+\vec{h}) - \vec{f}(\vec{x_0}) - T_{\vec{x_0}}(\vec{h}) \right\|}{\|\vec{h}\|} = 0.$$

When the total derivative exists, we set

$$\vec{f}\,'(\vec{x_0}) = T_{\vec{x_0}}, \text{ and call it}$$

"the total derivative of $\vec{f}$ at $\vec{x_0}$."

The Total derivative expressed in terms of partial derivatives

Thm: Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \operatorname{Int}(D)$, let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n independent variables $x_1, x_2, \dots, x_n$) and let $f_i: D \to \mathbb{R}$, $1 \le i \le m$, denote the component functions of $\vec{f}$.

If $\vec{f}$ is differentiable at $\vec{x}_0$, then for all $i \in \{1, \dots, m\}$ and for all $j \in \{1, \dots, n\}$

$$\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \text{ exists and}$$

$$\vec{f}\,'(\vec{x}_0) = T_{\vec{x}_0} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right].$$

proof: Trivial.

A clever use of matrix multiplication)))!

∎

Lecture 07

A word on notation:

- We use $\vec{x}$ to denote an element of $\mathbb{R}^n$ where $n \geq 2$;

  The "arrow" is used to indicate that the element is an n-tuple!

- $\vec{f}$ is used when the range of the function is $\mathbb{R}^n$ with $n \geq 2$;

  when $n = 1$, we simply use f.

- $x, \vec{x}, f, \vec{f}$

The Jacobian matrix

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{Int}(D)$, let $\vec{f} : D \to \mathbb{R}^m$ be a (vector-valued) function (of n independent variables $x_1, x_2, \ldots, x_n$) and let $f_i : D \to \mathbb{R}$, $1 \leq i \leq m$, denote the component functions of $\vec{f}$.

Now, suppose that $\vec{f}$ is differentiable at $\vec{x_0}$. Let $T_{\vec{x_0}} = \vec{f}'(\vec{x_0})$ be the total derivative of $\vec{f}$ at $\vec{x_0}$. Of course, $T_{\vec{x_0}}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

definition:

The $m \times n$ matrix of the linear transformation $T_{\vec{x_0}}$, with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, is called **the Jacobian matrix of the function $\vec{f}$ at the point $\vec{x_0}$**.

Example 1: $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \sin x.$$

Compute $f'((2,3))$

Ans: $f'((2,3)) = [\cos 2, 0]$

Example 2: Let $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$\vec{f}(x,y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$.

Determine the Jacobian matrix of $\vec{f}$ at an arbitrary point $(x,y) \in \mathbb{R}^2$.

Solution: (HW) $\vec{f}'(x,y) = [\quad]_{3\times 2}$

$f_1(x,y) = \sin x \cos y$

$f_2(x,y) = \sin x \sin y$

$f_3(x,y) = \cos x \cos y$

Thus,

$$\vec{f}'((x,y)) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix} \blacksquare$$

Example 3: (when range is 1-dimensional)

Let $D \subseteq \mathbb{R}^n$, $\vec{x_0} \in \text{int}(D)$, let $f: D \to \mathbb{R}$ be a function that is differentiable at $\vec{x_0}$. Then $f'(\vec{x_0})$ is a $1 \times n$ matrix given by

$$\left[\frac{\partial f}{\partial x_1}(\vec{x_0}), \frac{\partial f}{\partial x_2}(\vec{x_0}), \ldots, \frac{\partial f}{\partial x_n}(\vec{x_0})\right].$$

In this case, the Jacobian matrix of f at $\vec{x_0}$ is a row-matrix, which is also called "<u>the gradient of f at $\vec{x_0}$</u>", and is denoted by $\nabla f\big|_{\vec{x_0}}$ or $\nabla f(\vec{x_0})$.

Remarks:

(1) The symbol ∇ is read as "del".

(2) $\nabla f(\vec{x}_0) = \left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0)\right]$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ and is called "the gradient of f at $\vec{x}_0$".

(3) $\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$ is called "the gradient of f", or simply, the del f.

Note that, the gradient of f — ∇f, is NOT a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$;

(4) Instead, $\nabla f : E \subseteq \mathbb{R}^n \longrightarrow M_{1\times n}(\mathbb{R})$

where $E \subseteq D$ is the set of all points in D where the partial derivatives

$\frac{\partial f}{\partial x_j}$ exists for every $j \in \{1, \ldots, n\}$.

5. Consider the function $f(x,y) = x^2 y$.

Then,

(i) $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \end{bmatrix}$

(This is a function from $\mathbb{R}^2$ to $M_{1,2}(\mathbb{R})$)

(ii) $\nabla f \Big|_{(1,2)} = \begin{bmatrix} \frac{\partial f}{\partial x}(1,2) & \frac{\partial f}{\partial y}(1,2) \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix}$

(This is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$)

(iii) the gradient of f $= \begin{bmatrix} 2xy & x^2 \end{bmatrix}$;

the gradient of f at $(1,2)$ $= \begin{bmatrix} 4 & 4 \end{bmatrix}$.

Example 4: (When the domain is 1-dimensional)

Let $D \subseteq \mathbb{R}$,

$x_0 \in \operatorname{int}(D)$,

$\vec{f} : D \longrightarrow \mathbb{R}^m$ be a function

that is differentiable at x_0.

Then $\vec{f}'(x_0)$ is the $m \times 1$ matrix given by

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x_0) \\ \dfrac{\partial f_2}{\partial x_1}(x_0) \\ \vdots \\ \dfrac{\partial f_m}{\partial x_1}(x_0) \end{bmatrix} = \left[\dfrac{\partial f_i}{\partial x_1}(x_0) \right].$$

In this case,

(a) We omit the ∂ notation and

(b) We use the ordinary derivative sign

(Why?)

and the Jacobian matrix of $\vec{f}$ at x_0 is a column matrix, which in Calculus, is denoted by $\dfrac{d\vec{f}}{dx}(x_0)$.

That is,

$$\vec{f}'(x_0) = \begin{bmatrix} \dfrac{df_1}{dx}(x_0) \\ \dfrac{df_2}{dx}(x_0) \\ \vdots \\ \dfrac{df_m}{dx}(x_0) \end{bmatrix} = \dfrac{d\vec{f}}{dx}(x_0).$$

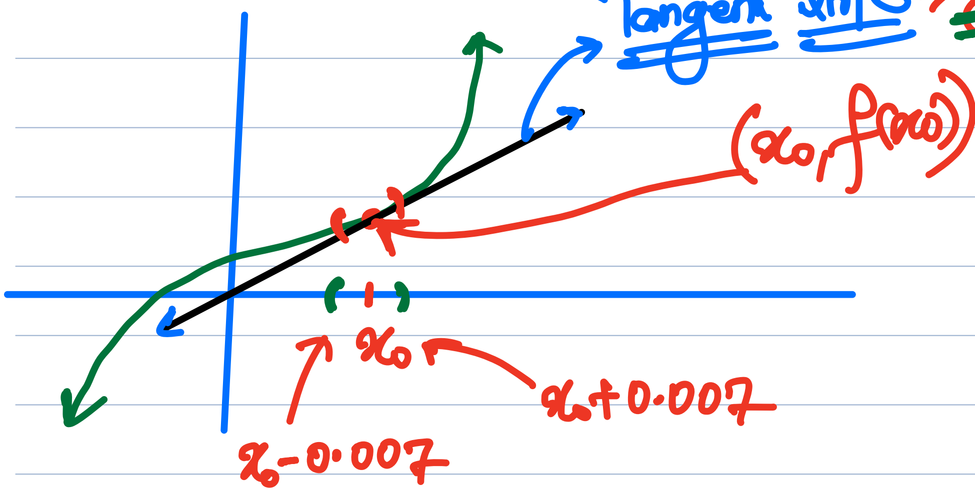
Standard linear approximation

$$f : \mathbb{R} \to \mathbb{R}$$
$$\Downarrow$$
$$x_0$$

f is differentiable at x_0.

$$\text{Let } f(x) = 99x^{87} + 6x^{85} + 90x^{23} + 2023.$$

Tangent line $\Rightarrow$ $y = mx + c$



$$f(x_0 + 0.003) = ?$$

$$\boxed{L(x) = mx + c}$$

Standard linear approximation of $f(x)$ by $L(x)$

[In Calculus, linear function refers to a function whose graph is line.]

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

$$\boxed{f(x) \approx f(x_0) + f'(x_0)[x - x_0]}$$

$L(x)$

[A polynomial of degree 1]

Linearization in multivariable setting

Let $D \subseteq \mathbb{R}^n$ and $\vec{x_0} \in \operatorname{int}(D)$. Let $f : D \to \mathbb{R}$ be a function. Then

f is differentiable at $\vec{x_0}$

$$\Updownarrow$$

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x_0}+\vec{h}) - f(\vec{x_0}) - f'(\vec{x_0})h}{\|h\|} = 0$$

or, equivalently, by replacing $\vec{x_0}+\vec{h}$ by $\vec{x}$,

$$\lim_{\vec{x} \to \vec{x_0}} \frac{f(\vec{x}) - f(\vec{x_0}) - f'(\vec{x_0})(\vec{x}-\vec{x_0})}{\|\vec{x}-\vec{x_0}\|} = 0$$

⊛ $\boxed{f(\vec{x}) \approx f(\vec{x_0}) - f'(\vec{x_0})(\vec{x}-\vec{x_0}) \text{ if } \vec{x} \text{ is "very" close to } \vec{x_0}}$

$$\therefore \boxed{L(\vec{x}) := f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0)}$$

Linearization of f at $\vec{x}_0$

or

Standard linear approximation of f at $\vec{x}_0$

Remark:

(1) $f'(\vec{x}_0)$ is $1 \times n$ matrix given by

$$\left[\frac{\partial f}{\partial x_1}(\vec{x}_0) \quad \frac{\partial f}{\partial x_2}(\vec{x}_0) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\vec{x}_0) \right]$$

(2) $\vec{x} - \vec{x}_0$ is an $n \times 1$ column matrix, given by

$$\begin{bmatrix} x_1 - x_{0,1} \\ x_2 - x_{0,2} \\ \vdots \\ x_n - x_{0,n} \end{bmatrix}$$

(3) Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$\qquad \hookrightarrow f(x,y)$

Let $\vec{x}_0 = (a,b) \in \mathbb{R}^2$ and $\vec{x} = (x,y)$

Now,

$$f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

$$= f((a,b)) + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ x-b \end{bmatrix}$$

$$= f((a,b)) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

$$\therefore L(x,y) = f((a,b)) + f_x((a,b))(x-a) + f_y((a,b))(y-b)$$

[The usual version found in Calculus text-books]

(4) Let $f : \mathbb{R}^3 \to \mathbb{R}$

$\quad \hookrightarrow f(x,y,z)$

Let $\vec{x_0} = (a,b,c) \in \mathbb{R}^3$ and $\vec{x} = (x,y,z)$

Then show that

$$\left[\begin{aligned} L(x,y,z) = f((a,b,c)) &+ f_x(a,b,c)(x-a) \\ &+ f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) \end{aligned} \right]$$

Example: Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point $(3,2)$.

Soln: $L(x,y) = f((3,2)) + \left[2x-y\Big|_{(3,2)} \quad -x+y\Big|_{(3,2)} \right] \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$

To be done later...

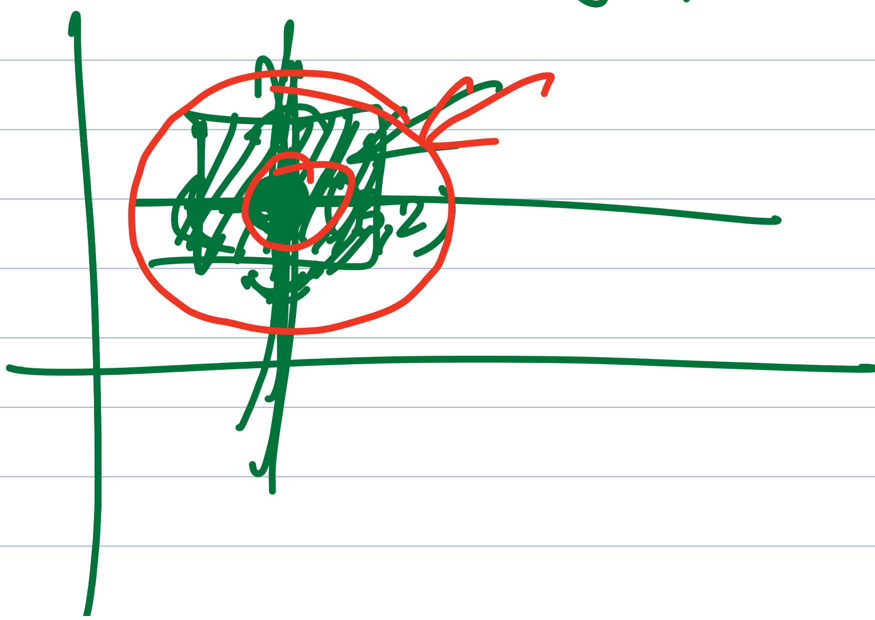
Qn:- Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point $(3,2)$.

Find an upper bound for the error incurred in replacing f by L on the rectangle $R: |x-3| \leq 0.1$

$|y-2| \leq 0.1$



Error

$$|f(x,y) - L(x,y)|$$

Later

(once 2^{nd} derivative test, extreme points, etc. are done!)

Qn: Find the linearization of $f(x,y,z) = x^2 - xy + 3\sin z$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$.

Find an upper bound for the error incurred in replacing f by L on the rectangle

$R: |x-2| \le 0.01, \ |y-1| \le 0.02, \ |z| \le 0.01.$