Linear Algebra: Review

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Linear Algebra: Review

(Vector Spaces)

Learning Objectives

- In this section, we will study the concepts related to the vector spaces. In particular, we will study
    - Group
    - Field
    - Vector space
    - Metric
    - Norm
    - Inner product
    - Basis vectors
    - Linear independence of basis vectors
    - Span
    - Basis

Notations

- Scalars will be written as small case letters in italics: x
- Vectors will be written as small case letters in bold: $\mathbf{x}$
- Matrices will be written as upper case letters in bold: $\mathbf{X}$
- The $(i,k)^{\text{th}}$ component of a matrix $\mathbf{X}$ will be written as X_{ik}
- Single random variable will be written as upper case letter in italics: X
- Vector random variable will be written as upper case, italics, and bold: $\boldsymbol{X}$
- Complex conjugation of a variable x will be denoted as: $\bar{x}$
- Transpose of a vector will be represented using superscript T: $\mathbf{x}^{\mathrm{T}}$
- Complex conjugate transpose of a vector will be represented using superscript H: $\mathbf{x}^{\mathrm{H}}$

Group

A group $(G, *)$ consists of the following:

(i) A set G

(ii) A rule or binary operation '$*$' (set G is closed under this operation) which associates with each pair of elements x and y in G, an element $(x * y)$ in G such that

a. This binary operation is associative:

i.e., $x *(y * z)=(x * y) * z \quad \forall\, x, y, z \in G$.

b. There exists an element 'e' called identity of group G s.t. $e * x = x * e = x \quad \forall\, x \in G$

c. For every $x \in G$, there exists an element x^{-1} such that

$x * x^{-1} = e = x^{-1} * x \quad \forall\, x \in G$

d. If $x * y = y * x \quad \forall\, x, y \in G$

$\Longrightarrow$ Commutative group or Abelian group

Examples

1. Set of n x n invertible matrices with matrix multiplication
2. Set of continuous time periodic signals with time period T under binary operation '$*$' = '+'

Field

A field $(F, +, \cdot)$ consists of the following:

(i) A set F

(ii) Two binary operations '+' and '·' such that

a. $(F, +)$ is an Abelian group

b. Define $F^*=F-\{0\}$. $(F^*, \cdot)$ is an Abelian group

c. The multiplication operation distributes over addition:

- Left distributive:

$$x\cdot(y+z)=x\cdot y + x\cdot z \qquad \forall\, x, y, z \in F$$

- Right distributive:

$$(x+y)\cdot z=x\cdot z + y\cdot z \qquad \forall\, x, y, z \in F$$

Examples: Check for F = R; F = C; F= Z; F=Q

Vector Space

A set V together with a map

$$\text{'+'}: V \times V \rightarrow V$$

$$(\mathbf{v}_1, \mathbf{v}_2) \rightarrow (\mathbf{v}_1 + \mathbf{v}_2) \text{ called vector addition}$$

And

$$\text{'}\cdot\text{'}: F \times V \rightarrow V$$

$$(\alpha, \mathbf{v}) \rightarrow (\alpha \cdot \mathbf{v}) \text{ called scalar multiplication}$$

is called a F-vector space or vector space over the field F if the following are satisfied:

(i) $(V, +)$ is an Abelian group

(ii) $\alpha \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \alpha \cdot \mathbf{v}_1 + \alpha \cdot \mathbf{v}_2$ $\quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$ and $\forall\, \alpha \in F$

(iii) $(\alpha_1 + \alpha_2) \cdot \mathbf{v} = \alpha_1 \cdot \mathbf{v} + \alpha_2 \cdot \mathbf{v}$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha_1, \alpha_2 \in F$

(iv) $(\alpha\beta) \cdot \mathbf{v} = \alpha(\beta\mathbf{v}) = \beta(\alpha\mathbf{v})$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha, \beta \in F$

(v) $1 \cdot \mathbf{v} = \mathbf{v}$ $\quad \forall\, \mathbf{v} \in V$

(vi) $\alpha \cdot \mathbf{0} = \mathbf{0}$ $\quad \forall\, \alpha \in F$

(vii) $0 \cdot \mathbf{v} = \mathbf{0}$ $\quad \forall\, \mathbf{v} \in V$

(viii) If $\mathbf{v} \neq \mathbf{0}$, then $\alpha \cdot \mathbf{v} = \mathbf{0}$ implies that $\alpha = 0$.

(ix) If V is a vector space over a field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V.

Vector Space

Examples:

1. Set of continuous time periodic signals with the same time period T over the field $F = R$ form a vector space
2. Set of finite energy signals over the field $F = R$ form a vector space
3. $V_n(F)$ = n-dimensional vector space over the field $F = R$ or C

Metric Space

Metric is a map

$$d: X \times X \rightarrow R$$

that satisfies the following:

(i) $d(x, y) \geq 0$ and $d(x, y)=0$ iff $x = y$ $\forall\, x, y \in X$

(ii) $d(x, y) = d(y, x)$ $\forall\, x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ $\forall\, x, y, z \in X$

This map is called a **metric** and a set equipped with this map is called a **metric space** and is denoted by (X, d).

Note that metric is a generalization of the notion of distance. An example is Euclidean distance.

Norm

Let V be a F-vector space. A map

$\|\cdot\|: \mathrm{V} \rightarrow R$ is called a **norm** if it satisfies the following:

(i) $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ iff $\mathbf{v}=0$ $\quad \forall\, \mathbf{v} \in V$

(ii) $\|\alpha\mathbf{v}\| = |\alpha|\, \|\mathbf{v}\|$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha \in F$

(iii) $\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$ $\quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$

A vector space equipped with a norm is called a **normed vector space**.

Example: Let V be a F-vector space equipped with a norm.

Prove that $d(\mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{v}_1 - \mathbf{v}_2\|$ is a proper metric.

Inner Product

Let V be a F-vector space.

A map

$$<\,,\,>: V \times V \rightarrow F$$

is called an inner product if it satisfies the following:

(i) $<\mathbf{v}, \mathbf{v}> \;\geq\; 0$ and $<\mathbf{v}, \mathbf{v}> \;= 0$ iff $\mathbf{v} = 0$ $\qquad \forall\, \mathbf{v} \in V$

(ii) $<\mathbf{v}_1, \mathbf{v}_2> = \overline{<\mathbf{v}_2, \mathbf{v}_1>}$ $\qquad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$

'‾' denotes the complex conjugate operation.

(iii) It is linear in the first coordinate.

$$<\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2, \mathbf{w}> = \alpha_1 <\mathbf{v}_1, \mathbf{w}> + \alpha_2 <\mathbf{v}_2, \mathbf{w}> \quad \forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w} \in V \text{ and } \forall\, \alpha_1, \alpha_2 \in F$$

(iv) It is conjugate linear in the second coordinate.

$$<\mathbf{v}, \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2> = \overline{\alpha_1} <\mathbf{v}, \mathbf{w}_1> + \overline{\alpha_2} <\mathbf{v}, \mathbf{w}_2>$$

$$\forall\, \mathbf{w}_1, \mathbf{w}_2, \mathbf{v} \in V \text{ and } \forall\, \alpha_1, \alpha_2 \in F$$

Inner Product

Example 1: Define $\|\mathbf{v}\| = <\mathbf{v}, \mathbf{v}>^{1/2}$. Show that it is a proper norm.

Thus, if a space is an inner product space, we can define norm and then define metric.

Example 2: Consider $V_n(F)$, where vectors are n-tuples of scalars, i.e., $\mathbf{x} \in V$ and $x_i \in F$

i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

Define inner product as: $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i \bar{y}_i$

Show that it satisfies all the properties of an inner product.

Linear Independence of Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots.., \mathbf{v}_n\}$ in $V_n(F)$ is called linearly independent (LI) if $\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.

Definition: The number of maximal LI vectors in a vector space V is called the **dimension** of the vector space and the maximal LI vectors is called a **basis** for V.

If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots.., \mathbf{e}_n\}$ is a basis for $V_n(F)$, then for any $\mathbf{v} \in V$ and for some $\alpha_i \in F$

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$$

Orthogonal and Orthonormal Basis

Definition: A set of basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots.., \mathbf{v}_n\}$ spanning an inner product space V is called **orthogonal** basis if

(i) $\mathbf{v}_i \neq 0 \quad \forall\ i$

(ii) $<\mathbf{v}_i, \mathbf{v}_j> = 0 \quad \forall\ i \neq j$

If additionally, the below condition is satisfied:

(iii) $<\mathbf{v}_i, \mathbf{v}_j> = 1 \quad \forall\ i = j$

Then, the set of basis vectors is called as the orthonormal basis. In this case, we can also write the above conditions as $<\mathbf{v}_i\ , \mathbf{v}_j> = \delta_{ij}$.

Summary

- In this section, we studied the concepts of
    - Group
    - Field
    - Vector space
    - Metric
    - Norm
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    - Basis vectors
    - Linear independence of basis vectors
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    - Basis

with examples.

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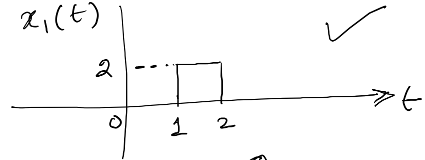
21/8/2024

C.T. Signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

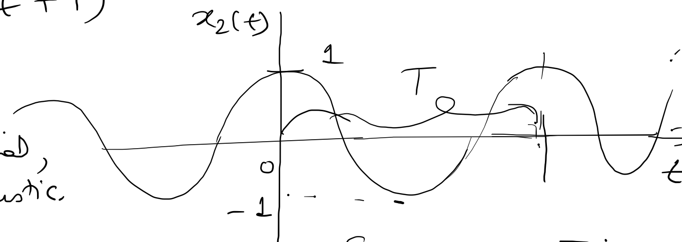


$$= \int_{1}^{2} |2|^2 dt = 4\int_{1}^{2} dt = 4 < \infty$$

Energy Signal

$x_1(t) \equiv$ energy signal, C.T. Signal, aperiodic, deterministic

$x_1(t) \neq x_1(t+T) \longrightarrow$ aperiodic signal

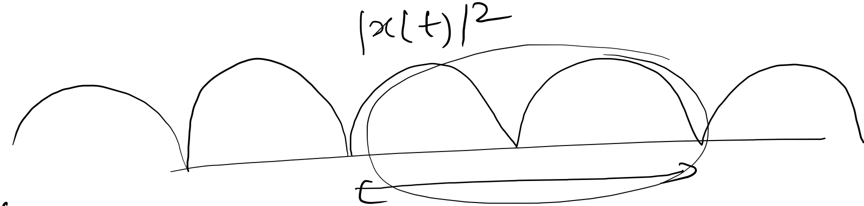


$x_2(t) \equiv$ C.T. Signal, periodic signal, deterministic

$x_2(t) = \cos \omega_0 t \quad \forall t$

fundamental Time period $\equiv$ smallest positive T

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



$$= \lim_{T_0 \to \infty} \underbrace{\int_{-T_0}^{T_0} |x(t)|^2 \, dt}$$

Not an energy signal

$$= \infty$$

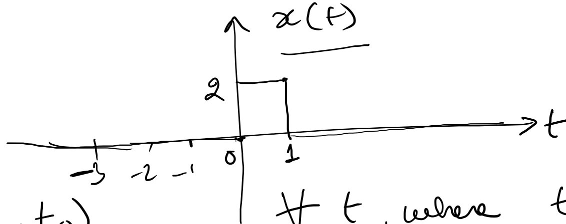
$$P = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} |x(t)|^2 \, dt = \text{finite no.}$$

$$\boxed{0 < P < \infty}$$

Power signal

Transformations of Signals

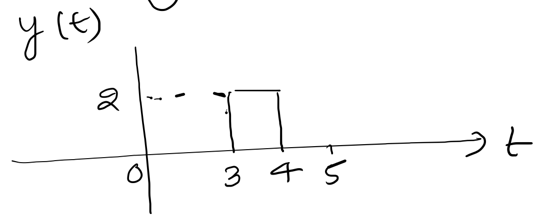
(1) Translate / Shift a signal



(a) Delay $y(t) = x(t - t_0)$ $\forall t$, where t_0 = constant

Delay

Let $t_0 = 3$



$y(t) = x(t-3)$

$y(0) = x(-3) = 0$

$y(3) = x(3-3) = x(0) = 2$

$y(4) = x(4-3) = x(1) = 2$

$y(5) = x(5-3) = x(2) = 0$

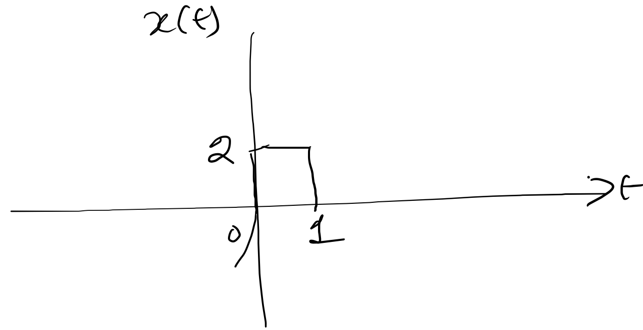
(b) Advance a signal —

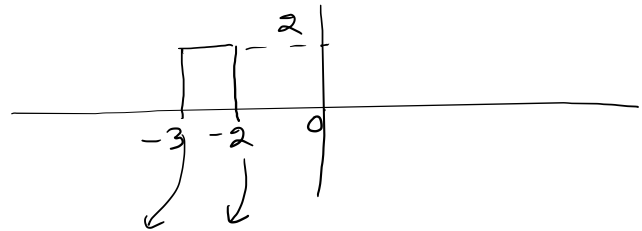
$$y(t) = x(t + t_0)$$

$\underline{t_0 = 3}$

$$y(-3) = x(-3 + 3) = x(0)$$

$$y(-2) = x(-2 + 3) = x(1)$$



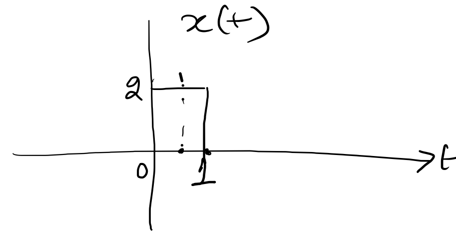


2

Scaling of a signal

$$y(t) = x(at)$$

(a) $a > 1$ $\longrightarrow$ compression of a signal on the t-axis

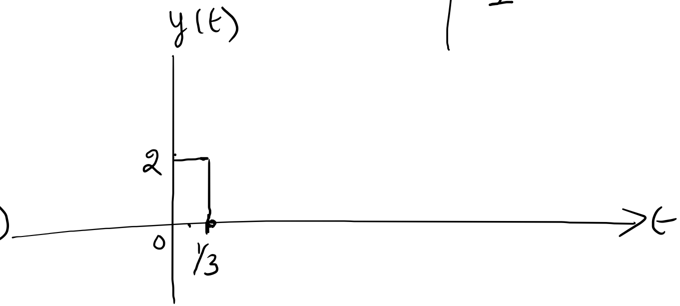


Let $a = 3$

$$y(t) = x(3t)$$

$$y(0) = x(0) = 2$$

$$y\left(\frac{1}{3}\right) = x\left(3 \cdot \frac{1}{3}\right) = x(1) = 2$$



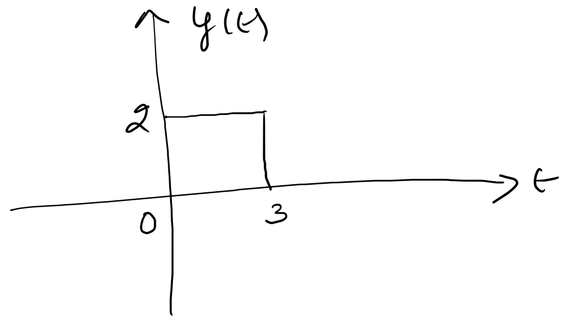
$y(t) = 0 \quad t > \frac{1}{3}$

& $t < 0$

$0 < a < 1$ — Dilation on the t-axis

(b)

Let $a = 1/3$



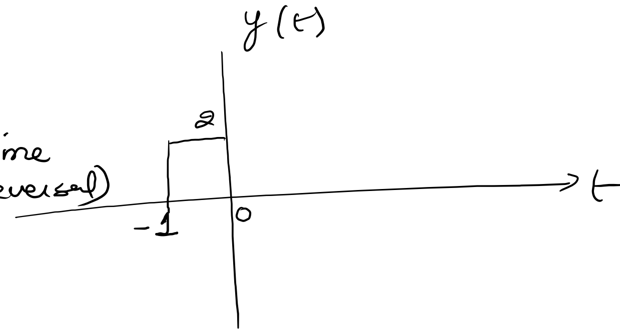
$$y(t) = x(at) = x\left(\frac{t}{3}\right)$$

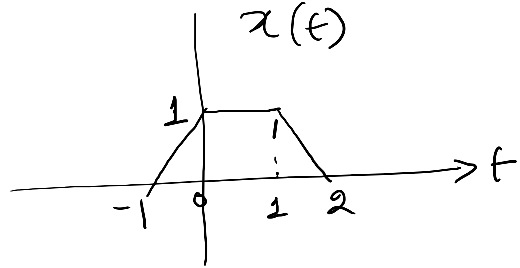
$$y(0) = x(0) = 2$$

$$y(3) = x\left(\frac{3}{3}\right) = x(1) = 2$$

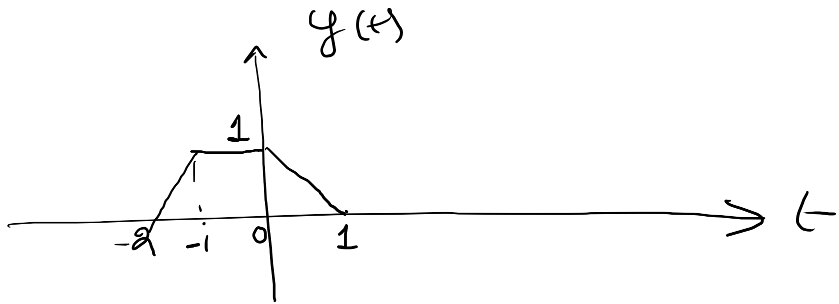
(c) Special case $a = -1$ (time-reversal)

$$y(t) = x(at) = x(-t)$$

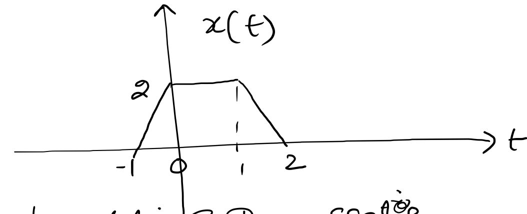




$y(t) = x(-t)$

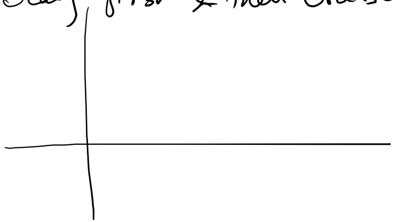


Example $\quad y(t) = x(2t+3)$



Scaling first & Then translation

translation & Then scaling



ⓐ

$x(t)$

$\downarrow$ scale

$x(2t)$

$\downarrow$ Translate/shift

$x(2(t+3))$

$= x(2t+6)$

ⓑ

$x(t)$

$\downarrow$ Translate (Shift)

$x(t+3)$

$\downarrow$ Scale

$x(2t+3)$

Fundamental Signals —

C.T. Signals

$$x(t) = Ce^{at}$$

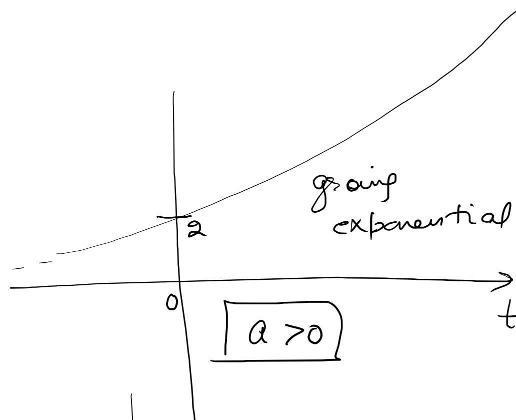
Case - 1

C = Real Constant $= 2$

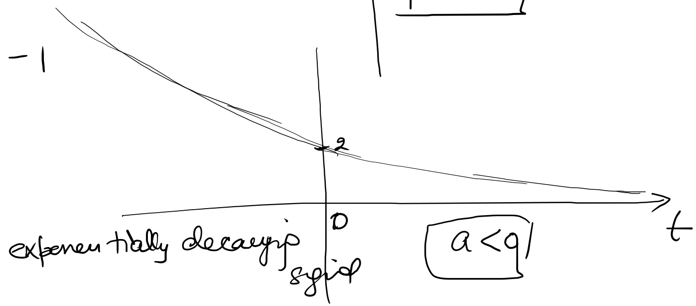
a = real constant

$$x(t) = 2e^{at}$$

(i) Let $a = 1$.



(ii) Let $a = -1$



(iii) $a = 0$

constant signal

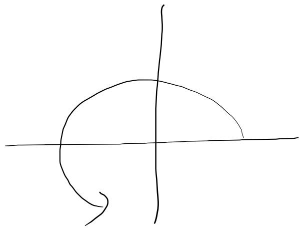
$x(t) = 2 \quad \forall t$

C.T. Signal

Case-2

C = real constant

a = purely imaginary $= j\omega$



$$\boxed{x(t) = e^{j\omega t}} \quad \forall\, t$$

periodic if $x(t) = x(t+T) \quad \forall\, t$

$$x(t+T) = e^{j\omega(t+T)} = e^{j\omega t} \cdot e^{j\omega T} = e^{j\omega t} \text{ if } e^{j\omega T} = 1$$

$$= x(t) \quad \forall\, t$$

$$E = \int_{-\infty}^{\infty} \left| e^{j\omega t} \right|^2 dt = \int_{-\infty}^{\infty} 1 \, dt = \infty$$

$e^{j\omega T} = 1$ when $\omega T = 2\pi$

$$\boxed{T = \frac{2\pi}{\omega}}$$ fundamental time period

ω = Units ~~of~~ rad/sec.

$C \equiv$ Complex no. $= |c| e^{j\theta} = |c| \cos\theta + j\, |c| \sin\theta$

Case -3

$a =$ real no.

$$x(t) = |c| e^{j\theta} \cdot e^{at} = |c| \cdot e^{at + j\theta}$$

$$\text{Re}\{x(t)\} = |c| \cos\theta \cdot e^{at}$$

$$\text{Im}\{x(t)\} = |c| \sin\theta \cdot e^{at}$$

Case 4

$C =$ Complex no. $\equiv |c| e^{j\theta}$

$a =$ Complex no. $\equiv r + j\omega$

$$x(t) = c\, e^{at} = |c| e^{j\theta} \cdot e^{(r + j\omega)t}$$

$$= |c| e^{rt} \cdot e^{j(\omega t + \theta)}$$

$$\Downarrow$$

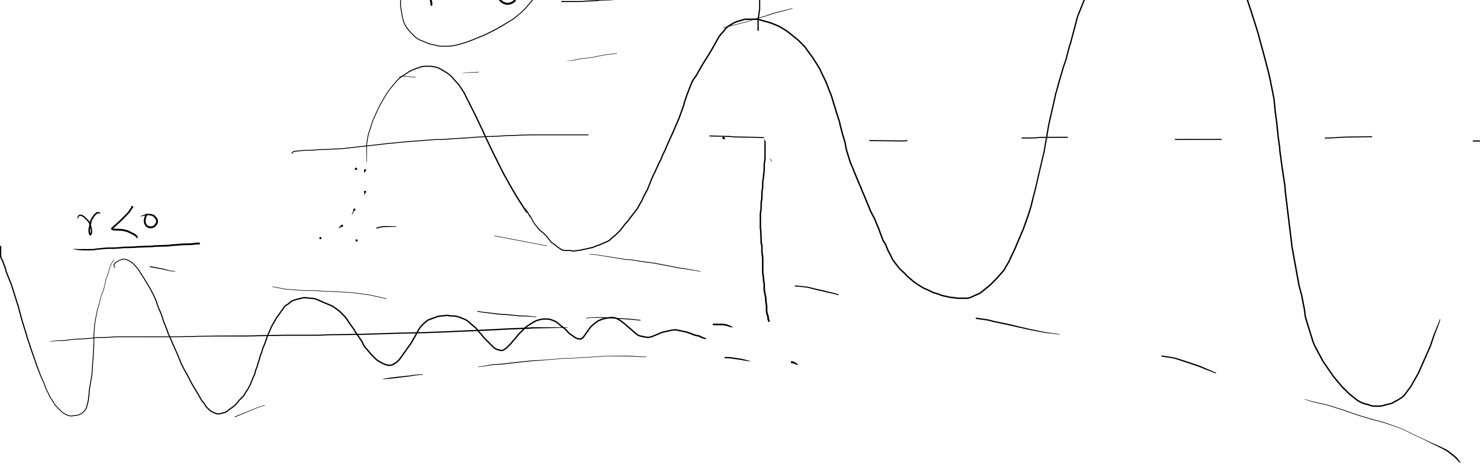
$$\cos(\omega t + \theta) + j \sin(\omega t + \theta)$$

$$\mathrm{Re}\{x(t)\} = |c|\, e^{rt} \cos(\omega t + \theta)$$

$$\mathrm{Im}\{x(t)\} = |c|\, e^{rt} \sin(\omega t + \theta)$$

Let $r = 2 \quad |c| = 1.5$

$$1.5\, e^{2t} \cos(\omega t + \theta)$$



D.T. Signal

$$x[n] = C\alpha^n$$

(1) Case -1

C = real constant

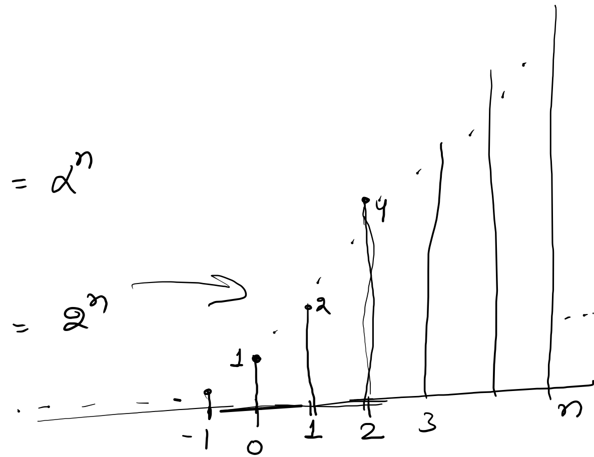
α = real

(i) $\alpha > 1$, $C = 1$

$x[n] = \alpha^n$

Let $\alpha = 2$

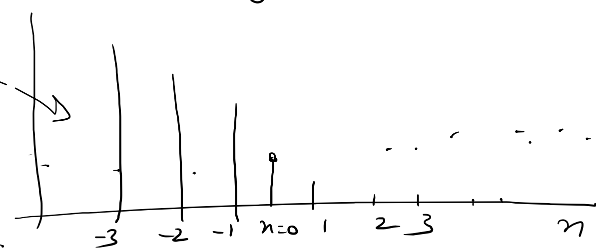
$x[n] = 2^n$



(ii) $0 < \alpha < 1$

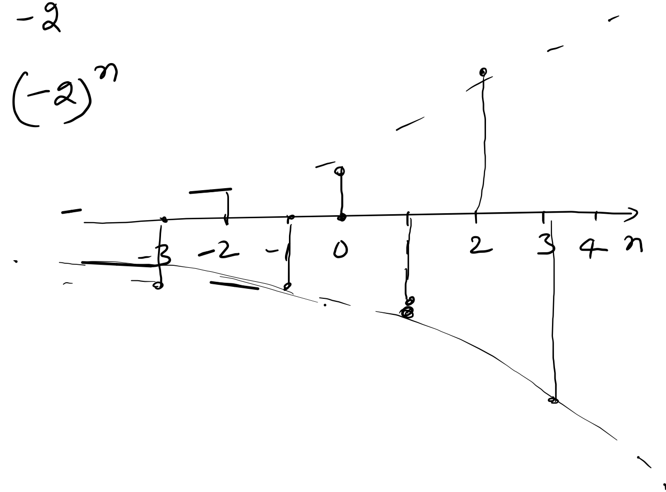
Let $\alpha = 0.5$

$x[n] = (0.5)^n = \left(\frac{1}{2}\right)^n$



(iii) $\underline{\alpha < -1}$ Let $\alpha = -2$

$x[n] = (-2)^n$

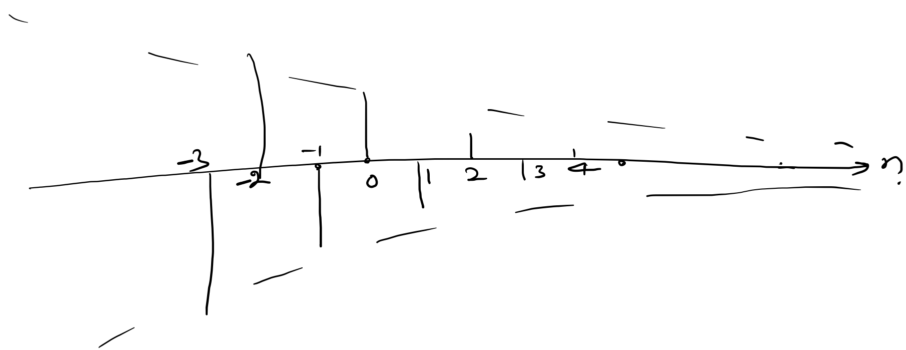


(iv) $\underline{-1 < \alpha < 0}$

Let $\alpha = -\frac{1}{2}$

$x[n] = \left(-\frac{1}{2}\right)^n$

$= (-1)^n \left(\frac{1}{2}\right)^n$



Case -2

C = real

$\alpha = e^{j\omega}$

Let $C = 1$

$$\boxed{x[n] = C\alpha^n = e^{j\omega n}}$$

if $x[n] = x[n+N] \quad \forall n$

then $x[n]$ is periodic

$$x[n+N] = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N}$$

$$= e^{j\omega n} \quad \text{if} \quad e^{j\omega N} = 1$$

$$e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi K$$

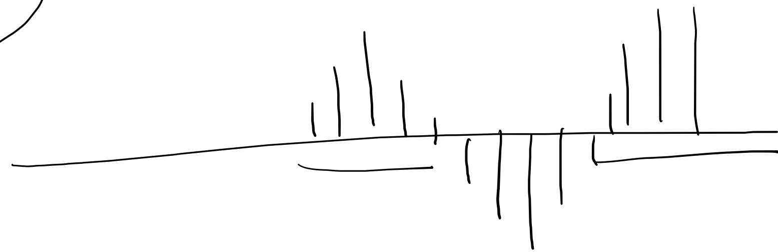
$$N = \frac{2\pi k}{\omega} \equiv \text{has to be an integer}$$

$$\frac{N}{k} \equiv \text{rational no.} = \frac{2\pi}{\omega}$$

$$x[n] = e^{j\frac{n}{6}} = e^{j\omega n}$$

$$\omega = \frac{1}{6}$$

$$\cos\frac{n}{6} + j\sin\frac{n}{6}$$



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C.T. Signals — frequencies are unique

$0 \longrightarrow \infty$

D.T. Signals — $[0, 2\pi]$ unique frequencies

$$x[n] = e^{j\omega_0 n}$$

Let $\omega_0 = 0$ $= e^{j0n} = e^{j0} = \cancel{\cos 0}^{=1} + j\cancel{\sin 0}^{=0}$

$= 1$

$\omega_0 = 2\pi$ $= e^{j2\pi . n} = \cos 2\pi n + j \sin 2\pi n$ $\forall\, n$

$= 1$

4π

6π

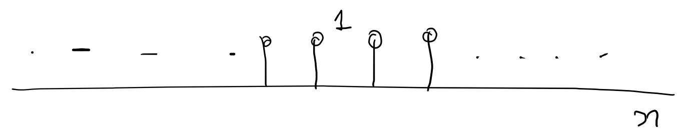
$\vdots$

-2π

$-4\pi \ldots$

$\omega_0 = 2\pi K$

$x[n]$



$w_0 = \pi$

$$x[n] = e^{jw_0 n} = e^{j\pi n}$$

$$= \cos \pi n + j \sin \pi n$$

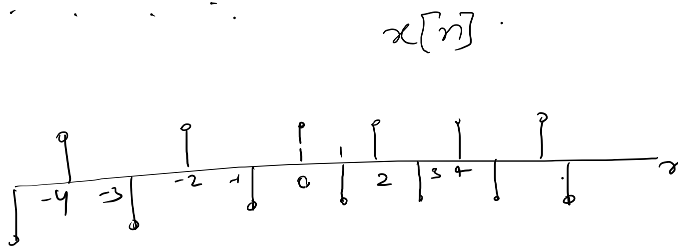
$$= (-1)^n$$

$w_0 = (2k+1)\pi$

$[-\pi, \pi]$

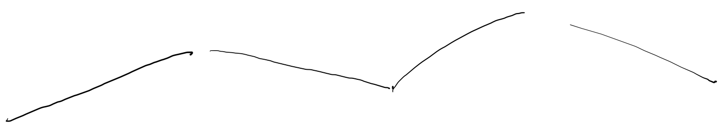
$[0, 2\pi]$

$\ldots -3\pi = -\pi = \pi = 3\pi = 5\pi \ldots$



$[0 \text{ — } 2\pi]$

$[0 \text{ — } \pi \text{ — } 2\pi]$

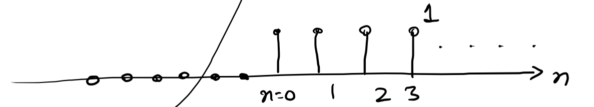


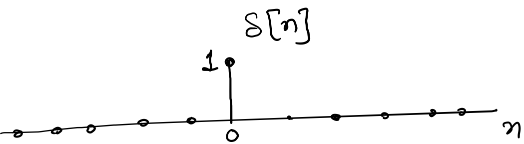
D.T. Signals

Unit Step Signal $u[n]$ | unit Impulse fn. $\delta[n]$

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{elsewhere} \end{cases}$$

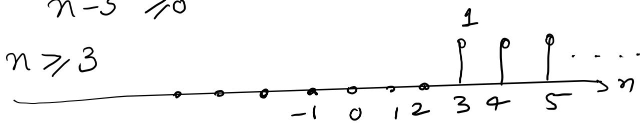
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$





$u[n-3]$ $\quad n-3 \geq 0$

$n \geq 3$



$\delta[n-3]$, $\delta[n-5]$, $\delta[n+1]$

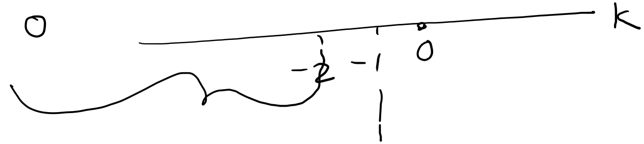
Prove ~~that~~ $\boxed{\delta[n] = u[n] - u[n-1]}$

first order difference equation

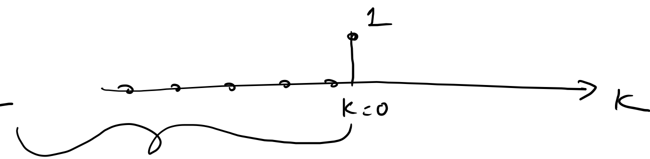
$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$u[n] = 0 \quad \text{for all } n < 0$$

$$u[-2] = \sum_{k=-\infty}^{-2} \delta[k] = 0$$

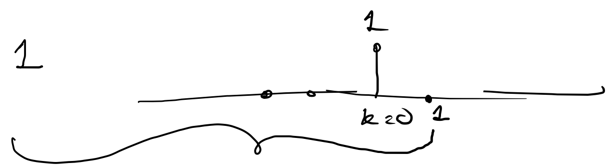


$$u[-1] = \sum_{k=-\infty}^{-1} \delta[k] = 0$$



$$u[0] = \sum_{k=-\infty}^{0} \delta[k] = 1$$

$$u[1] = \sum_{k=-\infty}^{1} \delta[k] = 1$$



$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \qquad \delta[0] = 1$$

$$= 0$$

$$u[-2] = \sum_{k=0}^{\infty} \delta[-2-k] \qquad -2-k=0$$

$$k = -2$$

$$= 0 + 0 + \cdots$$

$$= \delta[-2] + \delta[-3] + \delta[-4] + \cdots$$

$$= 0 + 0 + \cdots$$

$$= 0$$

$$u[1] = \sum_{k=0}^{\infty} \delta[1-k] = \underbrace{\delta[1]}_{0} + \overbrace{\delta[0]}^{=1} + \underbrace{\delta[-1]}_{0} + \underbrace{\delta[-2]}_{0} + \cdots$$

$$= 1$$

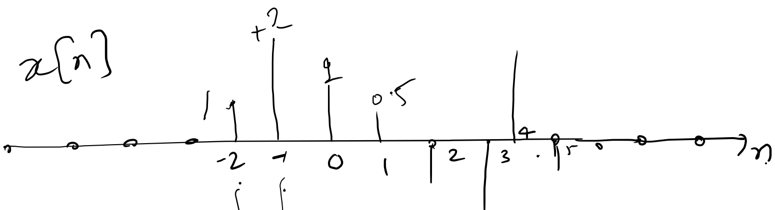
$$x[n]\,\delta[n] = \, ? \, = x[0]\,\delta[n] \neq x[0]$$

L.H.S $\Rightarrow$ δn

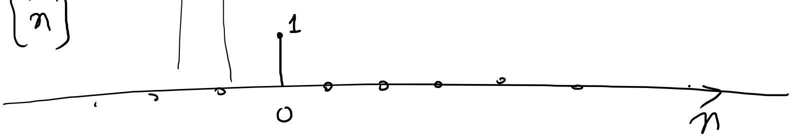
R.H.S $\Rightarrow$ Scalar

$$x[n]\,\delta[n-n_0] = \, ?$$

$x[n]$



$\delta[n]$



$$x[n]\,\delta[n] = \begin{cases} x[0] & n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

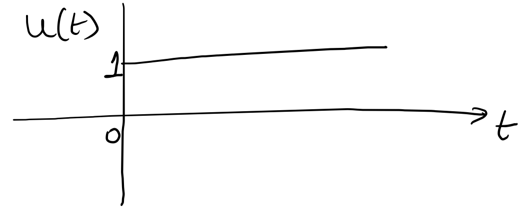
Sifting

Sampling

Unit Step fn —

C.T

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\delta(t) = \frac{d}{dt} u(t)$$

another variation

$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(t) = 0 \quad t \neq 0$$

Unit Impulse fn.

⇓

Singularity fn.

or

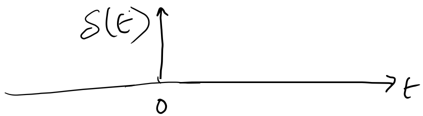
Generalized fn

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Operational definition

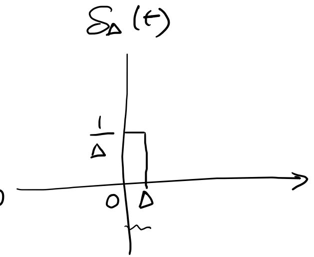
$$\delta(t) = 0 \qquad t \neq 0$$

& $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



$\delta_\Delta(t)$

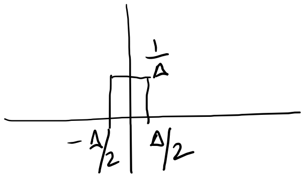
$$\delta_\Delta(t) = \begin{cases} 0 & t \geq \Delta \ \& \ t < 0 \\ \frac{1}{\Delta} & \text{otherwi} \end{cases}$$

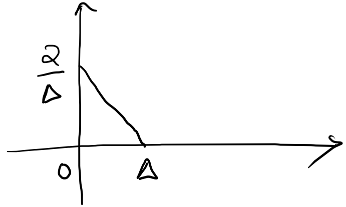


infinitesimally small duration Δ

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

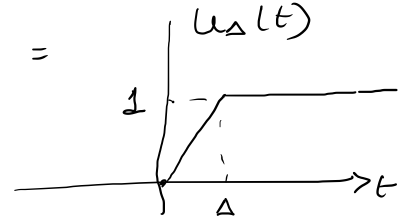
$$\int_{-\infty}^{\infty} \delta_\Delta(t)\,dt = \int_{0}^{\Delta} \delta_\Delta(t)\,dt = \frac{1}{\Delta}\int_{0}^{\Delta} dt = \frac{1}{\Delta}\, t\Big|_{0}^{\Delta} = 1$$





$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$u_\Delta(t) =$$

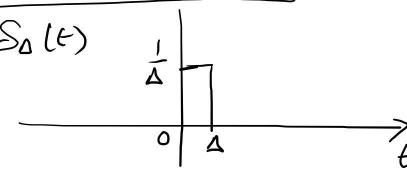


$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

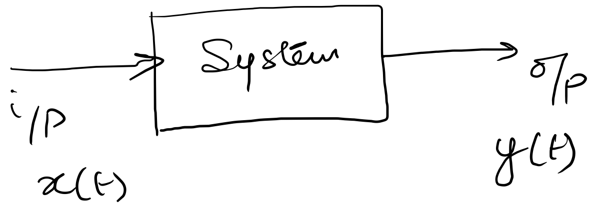
$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)}$$

$\delta_\Delta(t)$



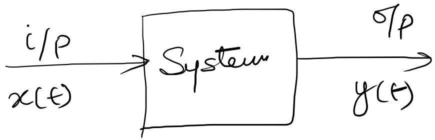
$$u(t) = \int_{0}^{\infty} \delta(t-\tau)\, d\tau$$

System



30/8/2024

Systems & Their properties



$x[n] \longrightarrow y[n]$

① System : with memory or without memory

$S_1 : y(t) = x(t)$

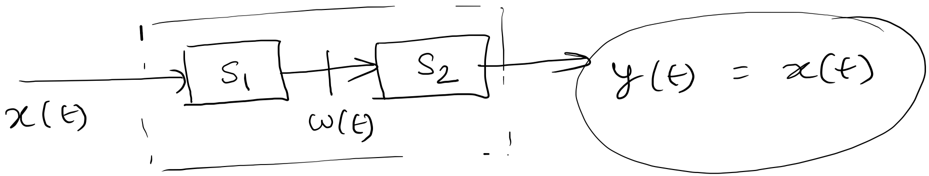
$S_2 : y(t) = x^2(t)$

$S_3 : y(t) = x(t-2)$

$S_4 : y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$

$S_5 : y(t) = x(t+2)$

② Invertible / Non-invertible Systems



if $S_2 = \text{inv}(S_1)$

ⓐ $S_1 : w(t) = 2x(t)$

$S_2 : y(t) = \frac{1}{2} w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible & the inverse system is S_2:

ⓑ $S_1 : w[n] = \sum_{k=-\infty}^{n} x[k]$

$S_2 : y[n] = w[n] - w[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$

$= \cancel{\sum_{k=-\infty}^{n-1} x[k]} + x[n] - \cancel{\sum_{k=-\infty}^{n-1} x[k]}$

$= x[n]$

(c)

$$x[n] \longrightarrow \boxed{S_1} \xrightarrow{w[n]} \boxed{S_2} \longrightarrow y[n]$$

$S_1: \quad w[n] = x[n] - x[n-1]$

$S_2: \quad y[n] = \sum_{k=-\infty}^{n} w[k]$

Invertible

(d)

$S_1: \quad w(t) = x^2(t)$

$S_2: \quad y(t) = \sqrt{w(t)}$

S_1 is not invertible

If distinct i/ps lead to distinct o/ps, the system is invertible, else not.

(3) Causal Systems —

$S_1 : \quad y(t) = x(t - t_0)$ $\quad$ t_0 is +ve real no.

$S_2 : \quad y[n] = x[n+3]$

$S_1 \equiv$ causal system

$S_2 \equiv$ non-causal system

Causality: Unless there is a cause, there should be no effect.

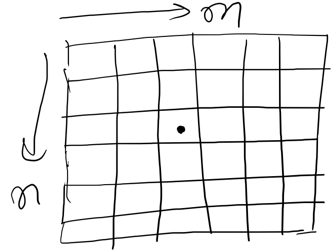
Unless there is a non-zero i/p, there should not be a

Or unless there is a non-zero i/p, there should not be a non-zero o/p.

Or a causal system depends on current & past i/p.

Accumulator: $\quad S_3 : \quad y[n] = \sum_{k=-\infty}^{n} x[k]$ causal.

S_4: $y[n,m] = \frac{1}{5}\{x[n,m] + x[n-1,m] + x[n+1,m] + x[n,m-1] + x[n,m+1]\}$



Non-causal System

Stability —

4

BIBO Stability

Stable System : A Bounded i/p should lead to a bounded o/p.

is on amplitude

Bounded i/p $\equiv$ $|x(t)| < M_x < \infty$ $\forall\, t$

$M_x \to$ finite value

Or $|x[n]| < M_x < \infty$ for all n.

$M_x \to$ finite value

this bounded i/p is fed to a system.

If the system generates an o/p, that is also ~~bounded~~

Then, we call it a stable system —

Examples of bounded i/ps.

$x(t) = u(t)$; $x[n] = u[n]$

unbounded signals

$x(t) = tu(t)$ or $x[n] = tu[n]$

if $|y(t)| \leq M_y < \infty \quad \forall t$

$\hookrightarrow$ finite no.

for bounded i/p, then the System is stable.

S_1: $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n+1]\}$

$|y[n]| = \left|\frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n+1]\right|$

$\leq \frac{1}{3}|x[n]| + \frac{1}{3}|x[n-1]| + \frac{1}{3}|x[n+1]| \quad \forall n$

if $|x[n]| < M_x \quad \forall n$

$|y[n]|$ = ~~bounded~~ for all n.

$S_2 : \; y[n] = \frac{1}{3}\{x[n] - x[n-1] - x[n-2]\}$

$|y[n]| = \frac{1}{3}|x[n] - x[n-1] - x[n-2]|$

Since $|x[n]| < M_x < \infty \; \forall n$

Sum & difference of 3 finite nos. will again be a finite no.

So $|y[n]| < M_y < \infty \quad \forall n$

Stable system.

S_3 : $y[n] = n\,x[n]$

$|x[n]| < M_x < \infty \quad \forall n$

bounded i/p

$|y[n]| = |n\,x[n]| \longrightarrow \infty$

as $n \longrightarrow \infty$

Unstable system

S_4 : $y(t) = x(2t)$ Stable system

(5)

Time - Invariant Systems

$$x(t) \xrightarrow{S} y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

$$x(t+t_0) \longrightarrow y(t+t_0)$$

$$x_1(t) \longrightarrow \boxed{} \longrightarrow y_1(t)$$

If $\quad x_2(t) \longrightarrow y_2(t)$

if $\quad x_2(t) = x_1(t-t_0) \longrightarrow$ Is $y_2(t) = y_1(t-t_0)$?

$y_2(t) =$

If yes, it is a time-invariant system

else, not.

$S: \; y(t) = x(2t)$

$x_1(t) \longrightarrow y_1(t) \boxed{= x_1(2t)}$

$x_2(t) \longrightarrow y_2(t) \boxed{= x_2(2t)}$

$\boxed{y_1(t - t_0) = x_1(2t - 2t_0) \longrightarrow \text{ⓐ} \quad \text{I o/p}}$

$x_2(t) = x_1(t - t_0) \Rightarrow y_2(t) = x_1(2t - t_0) \Rightarrow \text{ⓑ} \quad \text{II o/p}$

ⓐ $\neq$ ⓑ

$y_2(t) \neq y_1(t - t_0)$

So, "S" is a time-varying system

$S_2 : y[n] = n\,x[n]$

$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$

$\downarrow$ delayed o/p

$y_1[n-n_0] = (n-n_0)\,x_1[n-n_0]$ — (a)

$x_2[n] \longrightarrow y_2[n] = n\,x_2[n]$

Let $x_2[n] = x_1[n-n_0]$

delayed i/p

$y_2[n] = n\,x_1[n-n_0]$ — (b)

But $y_2[n] \neq y_1[n-n_0]$

Time-varying

(5) Linearity —

A system is linear if the properties of additivity & homogeneity holds true.

$$x_1(t) \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow y_2(t)$$

$$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$$

$\longrightarrow$ System is Linear

$\forall\, a, b \in \mathbb{R}$

Additivity : $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$

homogeneity : $a x_1(t) \longrightarrow a y_1(t)$

$S: \quad y(t) = 2x(t) + 3 = 2(i/p) + 3$

$x_1(t) \longrightarrow y_1(t) = 2x_1(t) + 3$

$x_2(t) \longrightarrow y_2(t) = 2x_2(t) + 3$

$\boxed{a x_1(t) + b x_2(t)}$ (i/p) $\longrightarrow 2(\ i/p\ ) + 3$

$= 2\left(a x_1(t) + b x_2(t)\right) + 3$ —— (a)

$= 2a x_1(t) + 2b x_2(t) + 3$

$a y_1(t) + b y_2(t) = a\left(2x_1(t) + 3\right) + b\left(2x_2(t) + 3\right)$

$= 2a x_1(t) + 3a + 2b x_2(t) + 3b$ (b)

Not Linear

A system that is both linear & time-invariant is called an LTI system

LTI $\equiv$ Linear & Time Invariant

4/9/2024

$$x[n] = \cos\left(\frac{\pi}{8} n^2\right)$$

Periodic if $x[n] = x[n+N]$ $\forall n$ for some positive N.

Smallest +ve N that satisfies this eqn, is called the fundamental time period.

$$x[n+N] = \cos\left(\frac{\pi}{8}(n+N)^2\right) = \cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{2\pi}{8}nN\right)$$

$$\cos(2\pi k + \theta) = \cos\theta \quad \forall k \in \mathbb{Z}$$

$\frac{\pi}{8}N^2 \longrightarrow$ Can it be a multiple of 2π

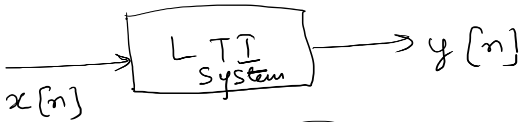
If $N = 4$ $\quad \frac{\pi}{\cancel{8}_{\cancel{2}}} \times \cancel{4} \times \cancel{4}^2 = 2\pi$

If $N = 8$ $\quad 2\pi \cdot 2$

$$\frac{\cancel{2}\pi}{\cancel{8}} \cancel{4}\, n = \pi n$$

$$\frac{2\pi}{\cancel{8}} \cdot \cancel{8}\, n = 2\pi n$$

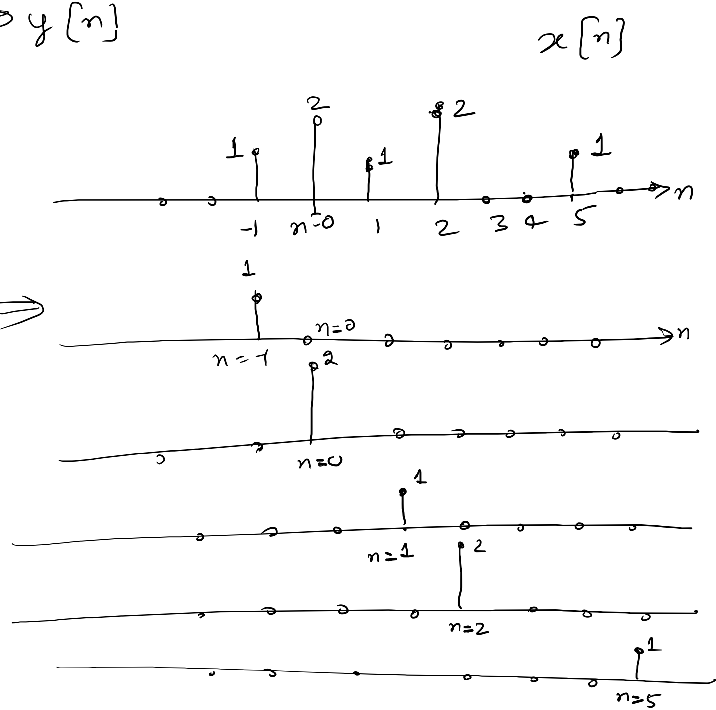
Discrete-time Convolution –



$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

$$\begin{aligned}
x[n] = \; & x[-1]\,\delta[n+1] = \delta[n+1] \\
+ \; & x[0]\,\delta[n] = 2\,\delta[n] \\
+ \; & x[1]\,\delta[n-1] = \delta[n-1] \\
+ \; & x[2]\,\delta[n-2] = 2\,\delta[n-2] \\
+ \; & x[5]\,\delta[n-5] = \delta[n-5]
\end{aligned}$$

$\Rightarrow$



Time-varying

$h[n, m]$

i/p variable

where the impulse was located at i/p

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$\delta[n] \longrightarrow y[n] = h[n]$$

= Impulse response of the LTI System

$$\delta[n-1] \longrightarrow h[n-1]$$

$$x[1]\,\delta[n-1] \longrightarrow x[1]\,h[n-1]$$

$$x[-1]\,\delta[n+1] \longrightarrow x[-1]\,h[n+1]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = y[n]}$$

D.T. convolution of

$$x[n] * h[n] = y[n]$$

"Convolution" $\longrightarrow$ linear, commutative

$$x[n] * h[n] = h[n] * x[n]$$

$$\big(x_1[n] + x_2[n]\big) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

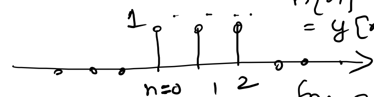
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Ex

Given an LTI System whose impulse response is

$$h[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$



$h[n] = y[n]$ for $\delta[n]$ as i/p.

Given $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$

Compute $y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1] = \frac{3}{5}\big[\delta[n] + \delta[n-1] + \delta[n-2]\big]$

$$+ \frac{1}{2}\big[\delta[n+1] + \delta[n] + \delta[n-1]\big]$$

Ex-2

LTI System $h[n] = u[n]$

Given $x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$h[n] = 0 \quad n < 0$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=0}^{\infty} \underbrace{h[k]}\, x[n-k] = \sum_{k=0}^{\infty} x[n-k]$$

$$= \sum_{k=0}^{\infty} \alpha^{n-k} u[n-k]$$

or

$u[n-k] = 0$

$n-k < 0$

$-n < k$

or $k > n$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

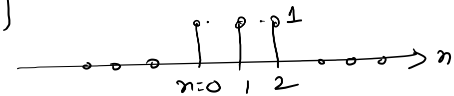
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$x[n]$ $\quad$ $y[n]$

$h[n]$

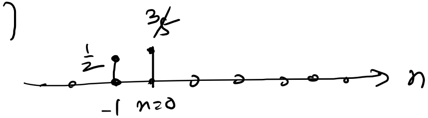
$$h[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$h[n]$



$$x[n] = \frac{3}{5}\,\delta[n] + \frac{1}{2}\,\delta[n+1]$$

$x[n]$

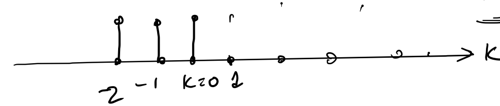


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$

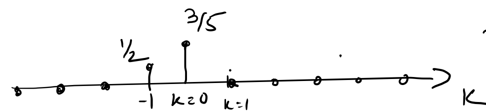
$$= 0 + \frac{1}{2} \times 1 + \frac{3}{5} \times 1 + 0 + 0 \ldots$$

$$= \frac{1}{2} + \frac{3}{5}$$

$h[-k]$



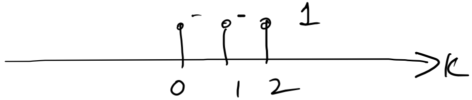
$x[k]$



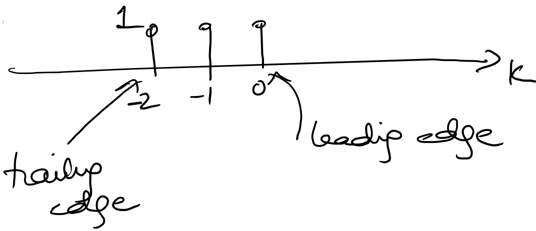
$h[1-k]$

$= h[-k+1]$

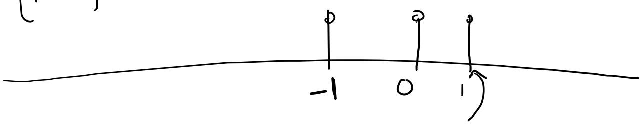
$h[k]$



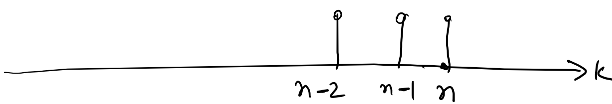
$h[-k]$



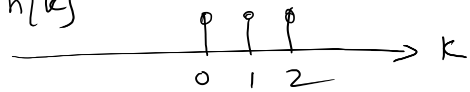
$h[1-k]$



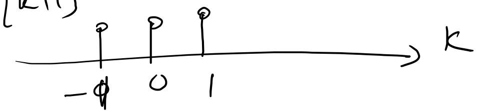
$h[n-k]$ & $n>0$



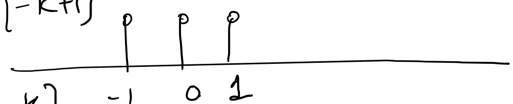
$h[k]$



$h[k+1]$

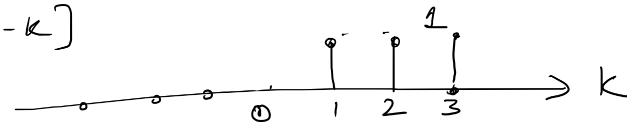


$h[-k+1]$



$= h[1-k]$

$h[3-k]$



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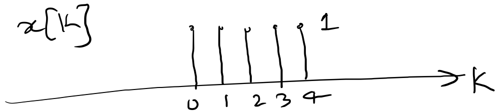
Discrete-time convolution

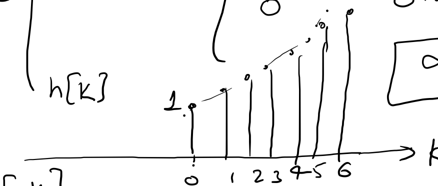
<u>Example</u>:

$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

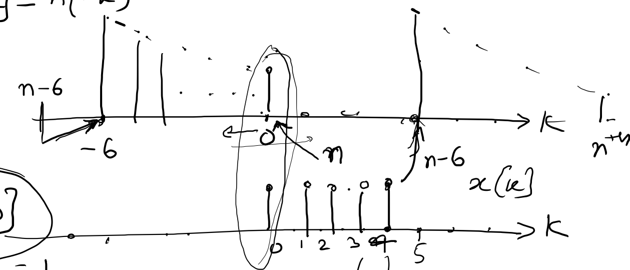
$$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\alpha > 1}$





$g[k] = h[-k]$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, g[k]$$

$$= + \dots x[-1]\,g[-1] + x[0]\,g[0] \dots = 1$$

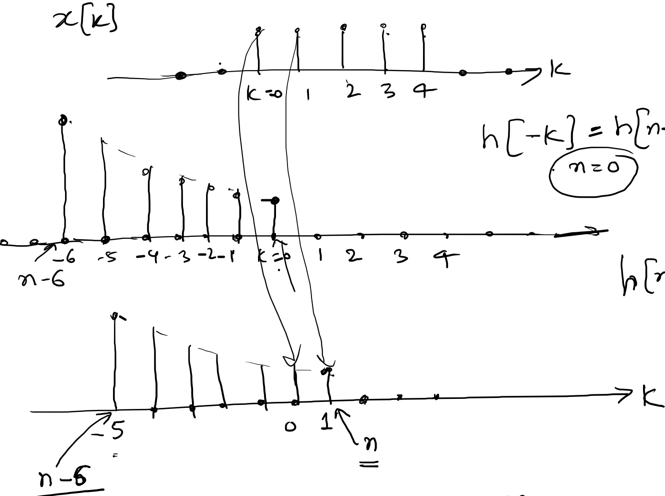
$$+ x[1]\,g[1] + \dots$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k]$$

$n < 0$

$n = -1, -2 \dots$

$n - 6$



$h[-k] = h[n-k]$

$n = 0$

$h[n-k]\big|_{n=1} = h[1-k]$

$k = 1$

$h[0]$

$h[n-k] \neq 0$ $\qquad n-6 \longrightarrow n$

$-1-6=-7$ to -1 ← range of argument for which $h[n-k]$ is non-zero.

$n = -1$

$n = 1$ $\qquad -5$ to 1

$n = -2$ $\qquad -8$ to -2

$$y[n] = \sum_k x[k]\,h[n-k]$$

overlap of the regions where

I $\quad y[n] = 0$, $\quad n < 0$ $\qquad n = -1$

or say $n = -1$

$h[n-k] \equiv -7$ to -1 on k-axis

$x[k] \equiv 0$ to 4 on k-axis

$n = 11$

$h[n-k] \neq 0$ $\quad 5$ to 11

$x[k] \neq 0$ $\quad 0$ to 4

$n > 10$ & $n < 0$

$y[n] = 0$

② $0 \le n < 4$

$$\boxed{y[n] = \sum_{k=0}^{n} x[k]\,h[n-k]}$$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^k$$

$$= \alpha^n \frac{\left(1 - (\alpha^{-1})^{n+1}\right)}{1 - (\alpha^{-1})}$$

$k = 0$ | $n = 0$

$k = 0, 1$ | $n = 1$

$k = 0, 1, 2$ | $n = 2$

$k = 0, 1, 2, 3$ | $n = 3$

$k = 0, \ldots 4$ | $n = 4$

$h[n-k]$

-6 to 0

-5 to 1

-4 to 2

-3 to 3

-2 to 4

$x[k]$

0 to 4

③ $4 \le n \le 6$

$$y[n] = \sum_{k=0}^{4} x[k]\,h[n-k]$$

④ $6 < n \le 10$

$$y[n] = \sum_{k=n-6}^{4} x[k]\,h[n-k]$$

C.T.

$$x(t) \rightarrow \boxed{LTI} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

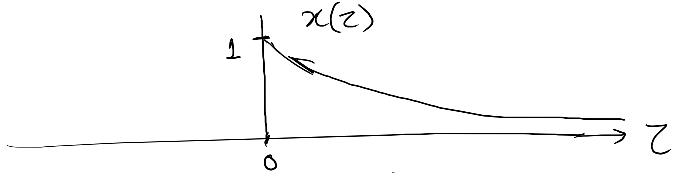
$$\boxed{y(t) = x(t) * h(t)}$$

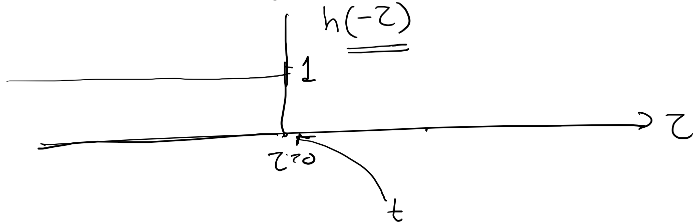
" convolution " operator

Ex

$x(t) = e^{-at}u(t) \quad a>0$

$h(t) = u(t)$





$h(t-\tau)$

when $t=0$ $\quad h(-\tau)$

① $t<0$

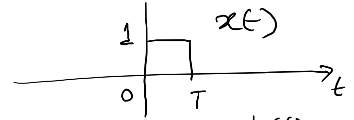
$y(t)=0$

② $t>0$

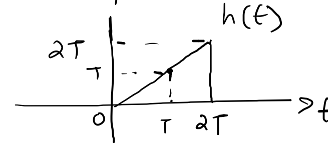
$$y(t) = \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

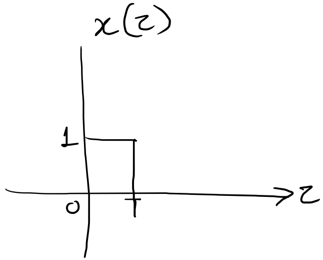
Ex

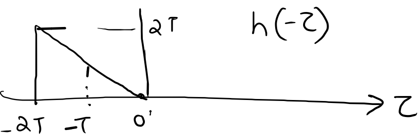
$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$



$$h(t) = \begin{cases} t & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}$$







① $y(t) = 0$ $\quad t < 0$ & $t > 3T$

② $0 \leq t \leq T$

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

③ $T \leq t \leq 2T$

$$y(t) = \int_0^T x(\tau)\, h(t-\tau)\, d\tau$$

④ $2T \leq t \leq 3T$

$$y(t) = \int_{t-2T}^{T} x(\tau)\, h(t-\tau)\, d\tau$$

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Convolution

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

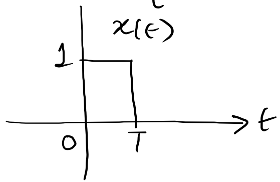
$0 < t < T$

$0 \le t \le T$

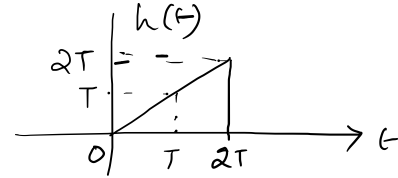
otherwise

$0 \le t < T$

$0 < t \le T$



$$h(t) = \begin{cases} t & 0 \le t \le 2T \\ 0 & \text{otherwise} \end{cases}$$

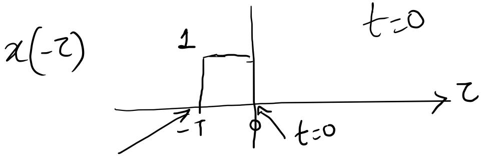


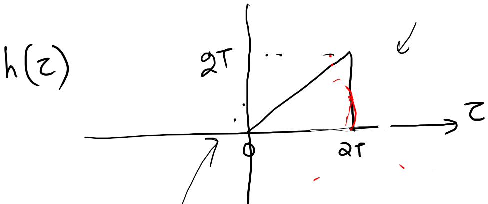
Find $y(t)$?

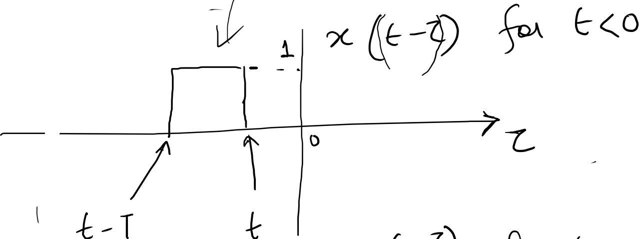
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

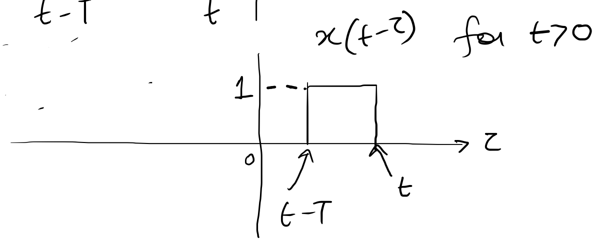
$$= x(t) * h(t)$$

$$= h(t) * x(t)$$









$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(0-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(-\tau)\, d\tau$$

① Case - 1 $t < 0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= 0$$

② Case - 2 $t > 3T$

$$y(t) = 0$$

Let $t = \frac{5T}{2}$

Case-3 : $0 < t < T$

$$y(t) = \int_0^t h(\tau)\, d\tau = \int_0^t \tau\, d\tau = \left.\frac{\tau^2}{2}\right|_0^t = \frac{t^2}{2}$$

Case-4 : $T \leq t < 2T$

$$y(t) = \int_{t-T}^{t} h(\tau)\, d\tau = \left.\frac{\tau^2}{2}\right|_{t-T}^{t} = \frac{t^2}{2} - \frac{(t-T)^2}{2}$$

$$= \frac{-T^2 + 2tT}{2} = -\frac{T^2}{2} + tT$$

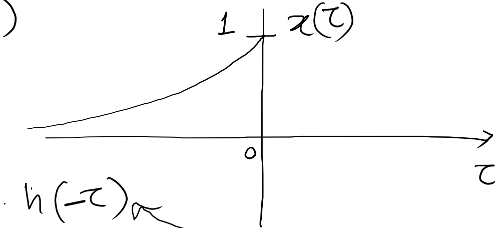
Case-5 $2T \leq t \leq 3T$

$$y(t) = \int_{t-T}^{2T} h(\tau)\, d\tau = \left.\frac{\tau^2}{2}\right|_{t-T}^{2T}$$

Ex

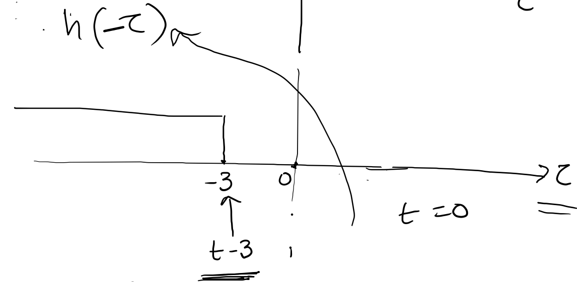
$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$



Case -1 $\quad t \leq 3$

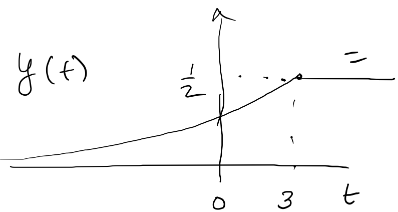
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

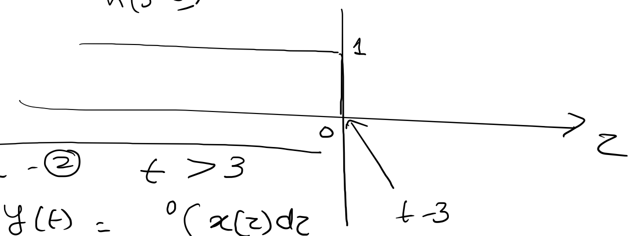


$$y(t) = \int_{-\infty}^{t-3} x(\tau)\, d\tau$$

$$= \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \left.\frac{e^{2\tau}}{2}\right|_{-\infty}^{t-3}$$

$$= \frac{e^{2t-6}}{2}$$





Case -2 $\quad t > 3$

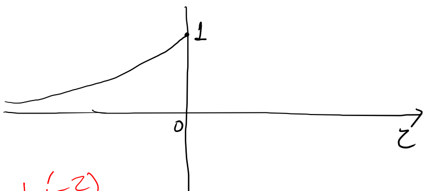
$$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \frac{1}{2}$$

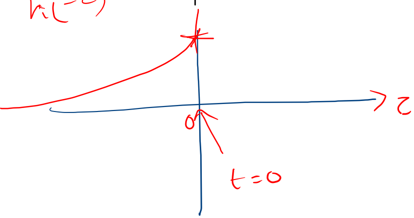
Ex

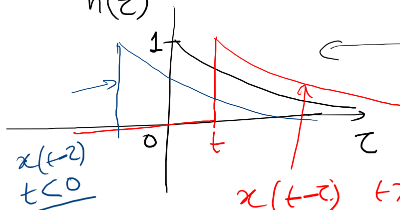
$$x(t) = e^{at}u(-t) \qquad a>0$$

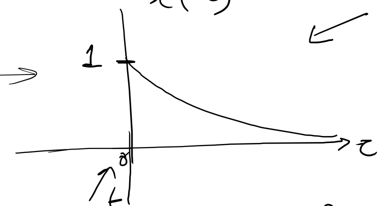
$$h(t) = e^{-at}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

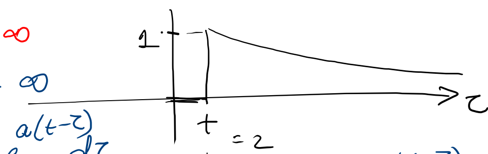








let $t=2$



Case-1 $\quad t \geq 0 \qquad t \text{ — } \infty$

Case-2 $\quad t<0 \qquad 0 \text{ — } \infty$

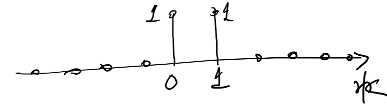
Case 1 $\quad y(t) = \int_{t}^{\infty} e^{-a\tau} e^{a(t-\tau)}\,d\tau$

Case-2 $\quad y(t) = \int_{0}^{\infty} e^{-a\tau} e^{a(t-\tau)}\,d\tau$

System Properties

there is an LTI System

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$



S_1: $y[n] = x[n] + x[n-1]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

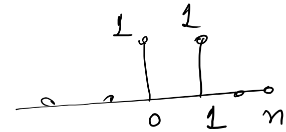
$$h[n] = \delta[n] + \delta[n-1] \implies h[k] = \delta[k] + \delta[k-1]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\delta[k] + \delta[k-1]\right) x[n-k]$$

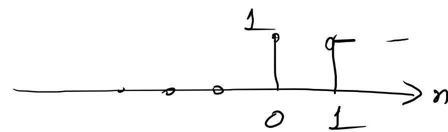
$$= \sum_{k=-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{k=-\infty}^{\infty} \delta[k-1]\, x[n-k]$$

$$= x[n] + x[n-1]$$

$$S_2 : \quad y[n] = \left(x[n] + x[n-1] \right)^2$$



$$S_3 : \quad y[n] = \max\left\{ x[n],\ x[n-1] \right\}$$



System Properties [holds true for ~~both~~ C.T. & D.T. Systems]

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② (i) Commutativity —

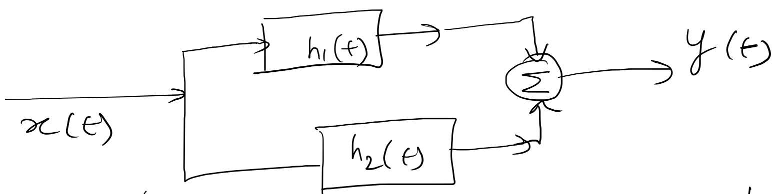
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

Prove that

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

(ii) Convolution distributes over addition

$$y(t) = x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$



$$y(t) = h(t) * \left(x_1(t) + x_2(t)\right) = h(t) * x_1(t) + h(t) * x_2(t)$$

Ex

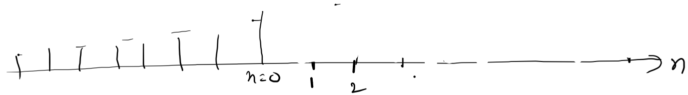
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n] = x_1[n] + x_2[n]$$

$$y[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$+$

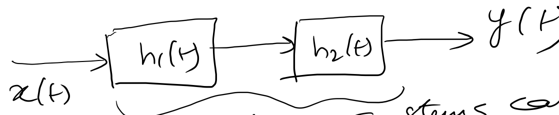
$$x_2[n] = 2^n u[-n]$$



(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)]$$

$$= [x(t) * h_1(t)] * h_2(t)$$



order of these systems can also be changed for LTI systems

Invertibility —

$$x(t) \rightarrow \boxed{h_1(t)} \xrightarrow{w(t)} \boxed{h_2(t)} \rightarrow y(t)$$

If $h_1(t) * h_2(t) = \delta(t)$

$$y(t) = x(t) * \left(h_1(t) * h_2(t)\right)$$
$$= x(t) * \delta(t)$$
$$= x(t)$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

Operational definition of $\delta(t)$

$\delta(t) = 0 \quad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Non-LTI

$h(t, \tau)$

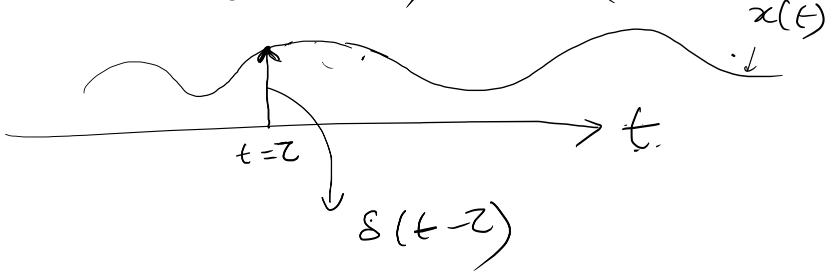
o/p at t

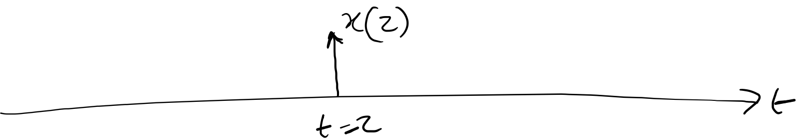
Unit impulse i/p applied at τ

D.T.

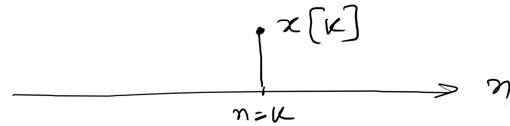
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$
$$= x[n] * \delta[n]$$

$$x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$$





$$\underline{x[k]\,\delta[n-k]}$$



Causality –

Claim: ~~If~~ If a system is causal, $h(t) = 0$ for $t < 0$ (C.T)

or $h[n] = 0$ for $n < 0$ (D.T)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k].$$

for causal

C.T.

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

D.T

$$\boxed{y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]} \Leftarrow \text{if } h[n] = 0 \text{ for } n < 0$$

$$\boxed{y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]} \Leftarrow$$

Memory or memoryless systems

$S_1: \ y(t) = k\,x(t) \implies h(t) = k\,\delta(t)$

$S_2: \ y[n] = k\,x[n] \implies h[n] = k\,\delta[n]$

Stability —

BIBO Stability condition

For every bounded i/p signal, if we get bounded o/p signal system is stable.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

Let i/n signal is bounded $\Rightarrow \ |x(t)| < M_x \ \forall\, t$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)\,x(t-\tau)|\,d\tau$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau$$

$$|y(t)| \leq M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \leq M_x \cdot \text{some finite no.} < \infty \quad \forall t$$

$h(t) = u(t)$

LTI, causal, with memory, unstable

If the system is absolutely integrable —

C.T. Systems

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

absolute Integrability

D. T. System

$$|y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

absolute summability

↓ System is stable

$h[n] = u[n]$

LTI, unstable, causal, with memory

Constant Coefficient difference equation for Causal LTI Systems

$$\sum_{k=0}^{N} b_k y[n-k] = \sum_{k=0}^{M} a_k x[n-k]$$

① N = order of the system

② The system is at rest - auxilliary conditions

If $x[n] = 0 \quad n < n_0$

$y[n] = 0 \quad n < n_0$

$y[n-1] = 0 \quad n < n_0$

$\vdots$

$y[n-N] = 0 \quad n < n_0$

if $N > 0$

(MA system)

moving average system

Case I: $N = 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots + a_M x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \dots + \frac{a_M}{b_0} x[n-M]$$

System is called Finite Impulse response FIR System

Case-2 Let $N = 1$, $M = 0$

a_0, b_0, b_1 are constants

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$\alpha_0 = a_0/b_0$

$\alpha_1 = b_1/b_0$

$$\boxed{y[n] = \alpha_0 x[n] - \alpha_1 y[n-1]}$$

Is it FIR?

If $x[n] = \delta[n]$

$y[n] = h[n]$

$h[0] = \alpha_0$

$h[1] = -\alpha_1 \alpha_0$

$h[2] = +\alpha_1^2 \alpha_0$

$h[3] = \vdots$

IIR System
Infinite Impulse response system

Autoregressive System
(AR) system

Case 3

$N > 0 \; , \; M > 0$

ARMA Systems

18/9/2024 $\boxed{\text{C.T.}}$ Constant coefficient differential equ. representation of Causal LTI system

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)$$

Condition of initial rest is satisfied

if $x(t) = 0$ for $t < t_0$

$\frac{d^k}{dt^k} y(t) = 0$ for $t < t_0$

$\forall\ k = 0, 1, \ldots N$

$N \equiv$ order of the system

Let $N = 2$ & $M = 3$

$$b_2 \frac{d^2}{dt^2} y(t) + b_1 \frac{d}{dt} y(t) + b_0 y(t) = a_0 x(t) + a_1 \frac{d}{dt} x(t) + a_2 \frac{d^2}{dt^2} x(t) + a_3 \frac{d^3}{dt^3} x(t)$$

Block Diagram representation

D.T.

$a_0 x[n] \equiv$ Scalar multiplier

$x[n-1] \equiv$ Delay element

$S_1: \; y[n] = a_0 x[n] + a_1 x[n-1]$

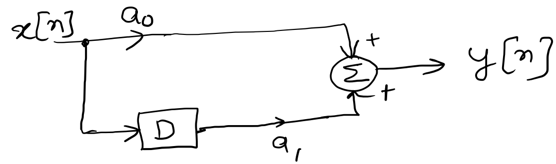


Fig.: S_1

$S_2 : \; b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

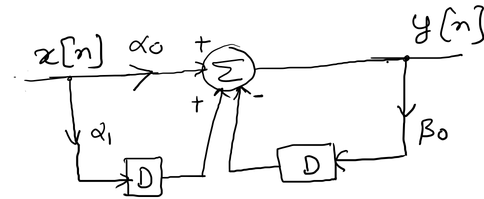
$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$

$$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$$

where $\alpha_0 = \frac{a_0}{b_0}$

⋮

constants



Fourier Series

C.T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \forall t$$

for Periodic Signal $x(t)$

Synthesis equation $\Rightarrow$

vector

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)}$$

$\rightarrow$ basis fns.

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$$

$$\boxed{\omega_0 = \frac{2\pi}{T}}$$

$T \equiv$ fundamental time period of $x(t)$

$$\phi_1(t) = e^{j\omega_0 t} \qquad \phi_2(t) = e^{2j\omega_0 t}$$

$$\phi_{-1}(t) = e^{-j\omega_0 t} \qquad \phi_{-2}(t) = e^{-j2\omega_0 t}$$

$\cdots$

$$\overbrace{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

$$= \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\phi_0(t) = e^{j0} = \cos 0 + j \sin 0$$

$$= 1$$

$$\boxed{\langle \phi_k(t), \phi_l(t) \rangle_T \triangleq \frac{1}{T}\int_T \phi_k(t)\,\phi_l^*(t)\,dt}$$

$\equiv$ Defining inner product

$$= \frac{1}{T}\int_T e^{jk\omega_0 t}\, e^{-jl\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{j(k-l)\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\, dt = \left.\frac{e^{jm\omega_0 t}}{jm\omega_0 T}\right|_0^T = \frac{e^{jm2\pi} - e^{j0}}{jm\omega_0 T} = 0$$

$= 1$ if $m = 0$

$= 0$ if $m \neq 0$ if $k \neq l$

$= 1$ if $k = l$

Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{\phi_k(t)\}_{k\in\mathbb{Z},\,T}$ = Span a vector space

form an orthonormal basis (O.N.B)

Analysis : $\boxed{a_k = \langle x(t), \phi_k(t) \rangle_T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \right\rangle_T = \langle \dots + a_{-10}\phi_{-10}(t) + \dots + a_0\phi_0(t) + \dots \; \dots a_9\phi_9(t) + \dots, \phi_l(t) \rangle_T$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \langle \phi_k(t), \phi_l(t) \rangle_T \right)$$

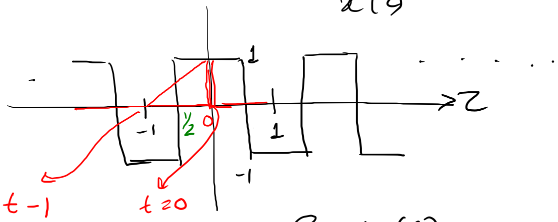
$$= \dots \langle a_{-10}\phi_{-10}(t), \phi_l(t) \rangle_T + \dots + \langle a_0\phi_0(t), \phi_l(t) \rangle_T + \dots + \langle a_9\phi_9(t), \phi_l(t) \rangle_T + \dots$$

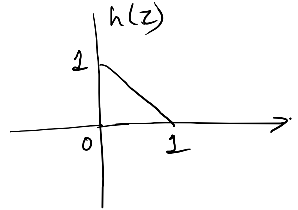
$$= \sum_{k=-\infty}^{\infty} a_k \, \delta[k-l]$$

$$\langle x(t), \phi_l(t) \rangle_T = a_l$$

=

Ex

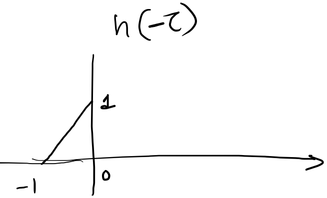




$h(-\tau) = 1+\tau$

$h(-\tau+t) = 1+\tau-t$

Convolve $x(t)$ & $h(t)$



$$y(0) = \int_{-1}^{-1/2} x(\tau)\,h(t-\tau)\,d\tau + \int_{-1/2}^{0} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-1}^{-1/2} (-1)(1+\tau)\,d\tau + \int_{-1/2}^{0} (1)(1+\tau)\,d\tau$$

$$= -\left(\tau + \frac{\tau^2}{2}\right)\Bigg|_{-1}^{-1/2} + \left(\tau + \frac{\tau^2}{2}\right)\Bigg|_{-1/2}^{0}$$

<u>Case 1</u>: $0 \le t \le \frac{1}{2}$

$$y(t) = -\int_{t-1}^{-1/2} (1+\tau-t)\,d\tau + \int_{-1/2}^{t} (1+\tau-t)\,d\tau$$

C.T. Fourier Series

20/9/2024

periodic signals

Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$

$\downarrow$ rad/s

$f_0 = \frac{1}{T}$ $\equiv$ per sec or Hz

$e^{jk\omega_0 t} = \phi_k(t)$ = pure imaginary exponential (in power)

$T \equiv$ fundamental time period

$$x(t) = 3 + 2e^{j\omega_0 t} + e^{j2\omega_0 t}$$

$$= 3 + 2(\cos\omega_0 t + j\sin\omega_0 t) + \cos 2\omega_0 t + j\sin 2\omega_0 t$$

$$= \sum_{k=0}^{2} a_k e^{jk\omega_0 t}$$

$a_0 = 3$

$a_1 = 2$

$a_2 = 1$

all other a_k's for $k \neq 0,1,2$ are zero

Analysis equation $\quad a_k = \langle x(t), \phi_k(t) \rangle_T = \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt$

$$a_k = \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$\{\phi_k(t)\}_{k\in\mathbb{Z}} \equiv \text{orthogonal basis} \equiv \int_T \phi_m(t)\,\phi_k^*(t)\,dt = T\,\delta[m-k]$$

$$\text{R.H.S} = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m\left(\int_T \phi_m(t)\,\phi_k^*(t)\right)dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \underbrace{\int_T e^{j(m-k)\omega_0 t}\,dt}\right]$$

$$= \cancel{\frac{1}{T}}\sum_{m=-\infty}^{\infty} a_m \cdot \cancel{T}\,\delta[m-k] = a_k = \text{L.H.S.}$$

$\phi_1(t)$, $\phi_{-1}(t)$ $\downarrow$ I harmonic

$\phi_2(t)$, $\phi_{-2}(t)$ $\downarrow$ 2nd harmonics

&

so on

Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ | Analysis : $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$

$e^{st} \equiv$ act as eigenfunctions of an LTI system

where $H(S) \equiv$ corresponding eigen value

$$x(t) \rightarrow \boxed{LTI} \rightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Let us assume $\boxed{x(t) = e^{st}}$ $\boxed{\text{where } s = j\omega_0}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = \int_{-\infty}^{\infty} h(\tau) e^{st} \cdot e^{-s\tau} d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

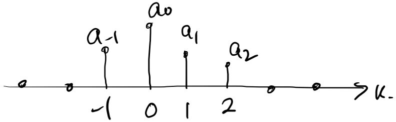
$$\boxed{y(t) = e^{st} H(s)}$$

$$e^{st} \rightarrow \boxed{LTI} \rightarrow e^{st} \cdot H(S)$$

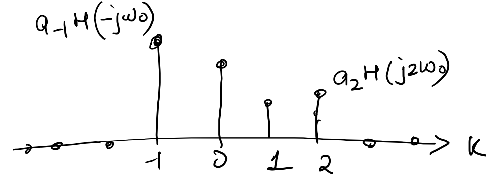
$$A\underline{v} = \lambda \underline{v}$$

Given an Line Spectrum $x(t)$

frequency domain representation

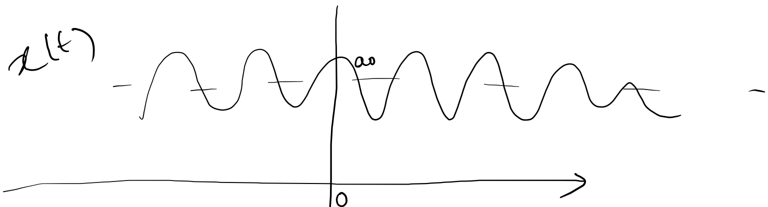


$\xrightarrow[\text{System}]{\text{LTI}}$



Line Spectrum of $y(t)$

$x(t)$



Given $$\boxed{x(t) = a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}}$$

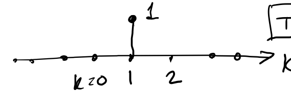
if $H(-j\omega_0) = 0$

$a_{-1}H(-j\omega_0) = 0$

& this freq. will disappear in o/p

Ex

S: $y(t) = x(t-2)$ let $x(t) = e^{j2t}$ $\omega_0 = 2\,\text{rad/s} = \frac{2\pi}{T}$ $\boxed{T = \pi}$



$$y(t) = e^{j2(t-2)}$$

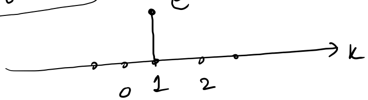
$$= e^{j2t} \cdot e^{-j4}$$

$$= e^{j2t} H(j2)$$

$$\boxed{H(j2) = e^{-j4}} \qquad \boxed{s = j2}$$

Line spectrum of $y(t)$

e^{-j4}



Ex

What is $h(t)$? $h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = \int_{-\infty}^{\infty} \delta(\tau-2)\, e^{-j2\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau-2) \cdot e^{-j2.2} d\tau$$

$$= e^{-j4} \boxed{\int_{-\infty}^{\infty} \delta(\tau-2)\, d\tau}$$

$$= e^{-j4}$$

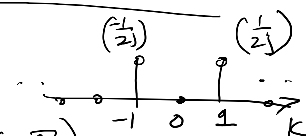
$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Ex

$$x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \left(\frac{1}{2j}\right) e^{j\omega_0 t} + \left(\frac{-1}{2j}\right) e^{-j\omega_0 t}$$

Find and plot the line spectrum of $x(t)$



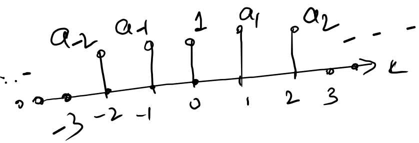
Ex

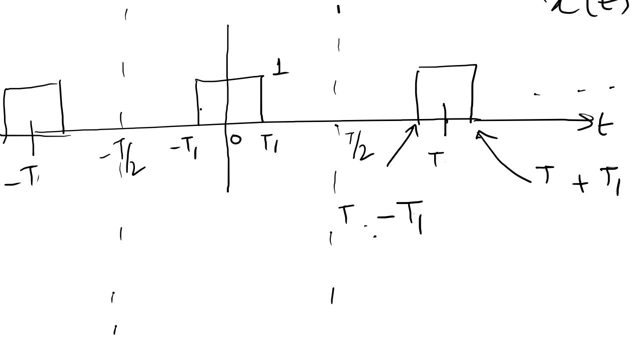
$$x(t) = 1 + \sin \omega_0 t + \cos\left(2\omega_0 t + \pi/4\right)$$

$$= 1 + \frac{e^{j\omega_0 t}}{2j} + \left(\frac{-1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$= 1 + \left(\frac{1}{2j}\right)\phi_1(t) + \left(\frac{-1}{2j}\right)\phi_{-1}(t) + \left(\frac{1}{2}e^{j\pi/4}\right)\phi_2(t) + \left(\frac{1}{2}e^{-j\pi/4}\right)\phi_{-2}(t)$$

$$= a_0 + a_1\phi_1(t) + a_{-1}\phi_{-1}(t) + a_2\phi_2(t) + a_{-2}\phi_{-2}(t)$$





$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

periodic

$$\omega_0 = \frac{2\pi}{T}$$

fundamental freq.

① if $T \longrightarrow \infty$, signal will become aperiodic

② there should be a dc bias term, so $a_0 \neq 0$

③

Solve it, plot the line spectrum.

25/9

C.T. F.S

① Linearity

$x(t) \xleftrightarrow{F.S} a_k$

$y(t) \xleftrightarrow{F.S.} b_k$

$x(t)$ & $y(t)$ are periodic with fundamental time period $= T$.

$z(t) = \boxed{\alpha x(t) + \beta y(t)} \longleftrightarrow \alpha a_k + \beta b_k$

& $z(t) \equiv$ Same fundamental time period T.

$$\boxed{z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

$\omega_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \left(\alpha x(t) + \beta y(t)\right) e^{-jk\omega_0 t} dt = \alpha \cdot \underbrace{\frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt}_{a_k} + \beta \cdot \underbrace{\frac{1}{T}\int_T y(t) e^{-jk\omega_0 t} dt}_{b_k}$$

$$= \alpha a_k + \beta b_k \longrightarrow \text{Proved.}$$

② Time Shifting

$$x(t) \xrightarrow[T]{F.S.} a_k \qquad \omega_0 = \frac{2\pi}{T}$$

Find F.S. coeff. of $x(t-t_0)$, where t_0 = constant.

$$z(t) = x(t-t_0) \xrightarrow[T]{} ? = C_k = a_k e^{-jk\omega_0 t_0}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} z(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

Let $t - t_0 = \tau$

$dt = d\tau$

$$= \frac{1}{T}\int_{-T/2-t_0}^{T/2-t_0} x(\tau)\, e^{-jk\omega_0 (t_0+\tau)}\, d\tau$$

$$C_k = e^{-jk\omega_0 t_0} \underbrace{\left(\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau\right)}_{} = a_k e^{-jk\omega_0 t_0}$$

(3) Time - reversal

$$x(t) \xrightarrow[T]{F.S.} a_k$$

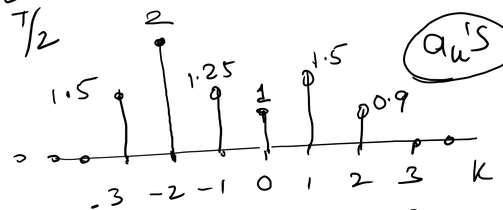
$$\boxed{\begin{aligned} \tau &= -t \\ d\tau &= -dt \end{aligned}}$$

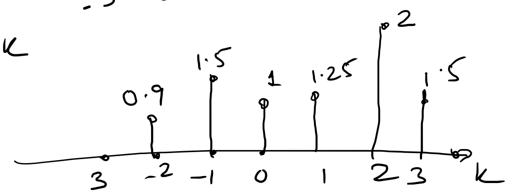
$$z(t) = x(-t) \xrightarrow[T]{F.S.} c_k = a_{-k}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(-t)\, e^{-jk\omega_0 t}\, dt = -\frac{1}{T}\int_{T/2}^{-T/2} x(\tau)\, e^{jk\omega_0 \tau}\, d\tau$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{jk\omega_0 \tau}\, d\tau$$

$$c_{-k} = \frac{1}{T}\int_{T} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau = a_k$$





$$\boxed{c_k = a_{-k}}$$

$c_0 = a_0$

$c_1 = a_{-1}$

(4)

$x(t) \xrightarrow[T]{F.S.} a_k$ Time period T

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

$z(t) = x^*(t) \longrightarrow a^*_{-k}$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Case-1 Let $x(t)$ is real.

$x(t) = x^*(t)$

$$\boxed{a_k = a^*_{-k}}$$

$$\boxed{a_k^* = a_{-k}}$$

$x(t) = x(-t)$

Case-2 Let $x(t)$ is real & even.

$a_k^* = a_{-k}$ $\quad$ $\boxed{a_k = a_{-k}}$

$$\boxed{a_k^* = a_k}$$

Conclusion: a_k's are also real & even.

Case 3

$x(t)$ is real & odd

$x(t) = x^*(t) \qquad x(-t) = -x(t)$

$a_k^* = a_{-k} \qquad -a_k = +a_{-k}$

$$a_k^* = -a_k$$

coeff. are purely imaginary.

(5) Time scaling

$x(t) \xrightarrow[T]{F.S.} a_k \qquad \alpha > 0 \qquad \alpha \neq -1$

$z(t) = x(\alpha t) \xrightarrow[T/\alpha \,,\, \alpha\omega_0]{F.S.} c_k = ?$

def $\alpha t = \tau$

$\alpha\, dt = d\tau$

$dt = \dfrac{d\tau}{\alpha}$

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(\alpha\omega_0)t}$$

$$c_k = \frac{1}{T/\alpha}\int_{-T/2\alpha}^{T/2\alpha} x(\alpha t)\, e^{-jk(\alpha\omega_0)t}\, dt$$

$$= \frac{1}{T/\cancel{\alpha}}\int_{-T/2}^{T/2} x(\tau)\, e^{-jk\omega_0\tau}\, \frac{d\tau}{\cancel{\alpha}} = a_k$$

6 Multiplication

$$x(t) \xrightarrow[T]{F.S.} a_k \qquad x(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}$$

$$y(t) \xrightarrow[T]{F.S.} b_k \qquad y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}$$

$$z(t) = x(t)y(t) \xrightarrow{F.S.} c_k = ?$$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \left(x(t)y(t)\right) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \sum_l \sum_m a_l b_m e^{j(l+m)\omega_0 t} e^{-jk\omega_0 t} dt$$

$$= \sum_l \sum_m a_l b_m \underbrace{\left(\frac{1}{T}\int_T e^{-j(k-l-m)\omega_0 t} dt\right)}_{=1}$$

$$k-l-m=0$$

$$l = k-m$$

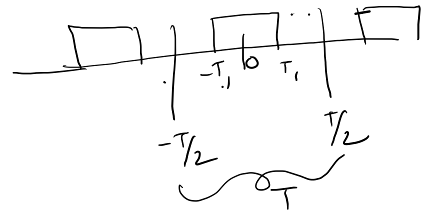
$$= \boxed{\sum_m a_{k-m} b_m = \sum_l a_l b_{k-l}}$$

multiplication in the time domain $\Longleftrightarrow$ Convolution in the frequency domain

Fix T_1

1. $T = 4T_1$
2. $T = 8T_1$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$



$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} dt = \boxed{\frac{2T_1}{T} = a_0} = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt \Longleftarrow k \neq 0$$

$$a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1)$$

$k \neq 0$

$$= \frac{1}{k\pi}\sin\left(k\frac{2\pi}{\cancel{4}T_1}\cancel{T_1}\right)$$

$$= \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

(7) Parseval's relation $x(t) \xrightarrow[T]{F.S} a_k$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |a_k|^2$$

D.T. F.S

Given a discrete-time periodic signal $x[n]$

N = Fundamental time period

Synthesis $\equiv$ $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

$0 \quad \omega_0, 2\omega_0, \ldots (N-1)\omega_0, N\omega_0$

$0, \frac{2\pi}{N}, \frac{2.2\pi}{N} \ldots (N-1)\cdot\frac{2\pi}{N}, 2\pi = 0$

$$= \sum_{k=\langle N\rangle} a_k \phi_k[n]$$

$$\phi_k[n] = e^{jk\omega_0 n}$$

$$\phi_{k+N}[n] = \phi_k[n] = e^{jk\omega_0 n}$$

$$\langle x[n], \phi_l[n] \rangle_N = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jl\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \left(\sum_{k=\langle N \rangle} a_k\, e^{jk\omega_0 n} \right) e^{-jl\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k\, e^{j(k-l)\omega_0 n}$$

$$= \sum_{k=\langle N \rangle} a_k \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} e^{j(k-l)\omega_0 n}}_{=1}$$

when $k = l$

$$= \boxed{a_l = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, \phi_l^*[n]}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\omega_0 n}$$

27/9/2024 Ex

$x[n] = \sin \omega_0 n$ $\quad \omega_0 = \frac{2\pi}{N} \longrightarrow$ periodic with period N.

$a_k = ?$

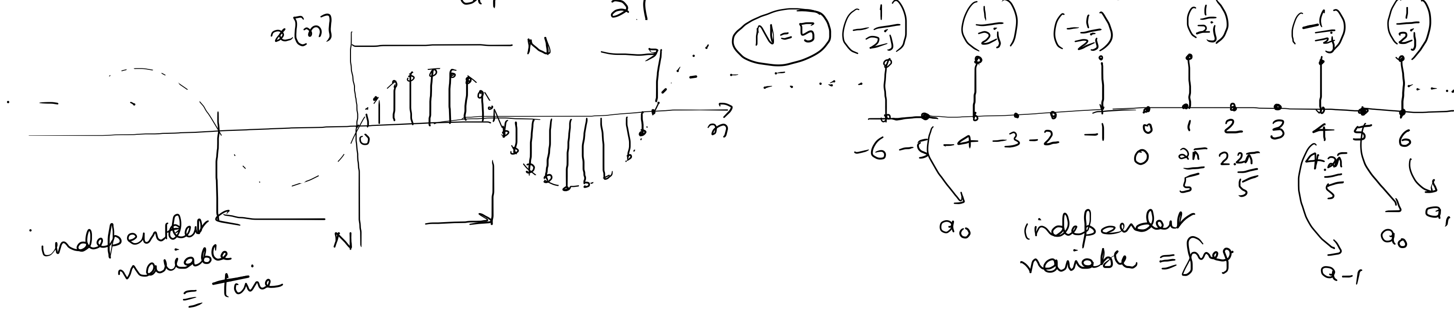
$$x[n] = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$$

$$= \left(\frac{1}{2j}\right)e^{j\omega_0 n} + \left(\frac{-1}{2j}\right)e^{-j\omega_0 n}$$

a_1, a_{-1} are non-zero, rest all are zero in $k = \langle N \rangle$.

$a_1 = \frac{1}{2j}$; $a_{-1} = \left(\frac{-1}{2j}\right)$

Line Spectrum



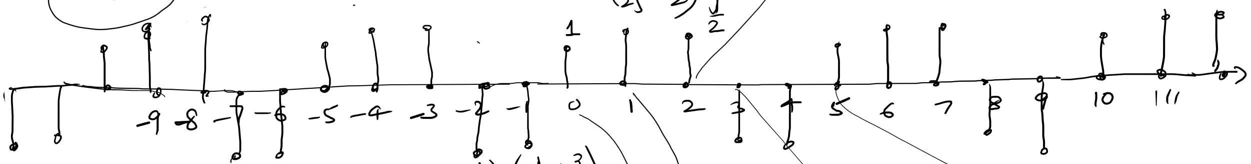
Ex

$$x[n] = 1 + \sin\frac{2\pi}{N}n + 3\cos\frac{2\pi}{N}n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$k = \langle N \rangle$

$N = 5$

$\left(\frac{1}{2j} + \frac{3}{2}\right)$ $\quad \frac{j}{2}$ $\quad 2\cdot\frac{2\pi}{5}$



$\left(-\frac{j}{2}\right)$ $\left(\frac{-1}{2j} + \frac{3}{2}\right)$

$0 \qquad \frac{2\pi}{5} \qquad 3\cdot\frac{2\pi}{5} \qquad 2\pi \equiv 0$

$N = 7$ 2 zero, amplitude pts. in one N

zero-amplitude pts. in one period in the

$N = 11$ 6 zero-amplitude pts. in one period in the frequency domain

$, \; -\frac{2\pi}{3}, \; 0., \; \frac{2\pi}{3}, \; 2\cdot\frac{2\pi}{3}, \; 2\pi \equiv 0$

$N = 3$

$$e^{j2\cdot\frac{2\pi}{3}n} = e^{j\left(2\pi - \frac{2\pi}{3}\right)n} = e^{j2\pi n}\, e^{-j\frac{2\pi}{3}n} = \left(e^{j2\pi}\right)^n \cdot e^{-j\frac{2\pi}{3}n} = e^{-j\frac{2\pi}{3}n}$$

$$x[n] = 1 + \sin\frac{2\pi}{3}n + 3\cos\frac{2\pi}{3}n + \cos\left(\frac{4\pi}{3}n + \frac{\pi}{2}\right)$$

$$x[n] = 1 + \frac{1}{2j}\left(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}\right) + \frac{3}{2}\left(e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}\right)$$

$$+ \cancel{\cos\frac{4\pi}{3}n} \times 0 - \sin\frac{4\pi}{3}n$$

$$= 1 + \left(\frac{1}{2j} + \frac{3}{2}\right)e^{j\frac{2\pi}{3}n} + \left(-\frac{1}{2j} + \frac{3}{2}\right)e^{-j\frac{2\pi}{3}n}$$

$$\left(+\frac{1}{2j}\right)e^{j\frac{2\pi}{3}n} + \left(-\frac{1}{2j}\right)e^{-j\frac{2\pi}{3}n}$$

$$= 1 + \left(\frac{1}{j} + \frac{3}{2}\right)e^{j\frac{2\pi}{3}n} + \left(-\frac{1}{j} + \frac{3}{2}\right)e^{-j\frac{2\pi}{3}n}$$

$$-\sin\frac{4\pi}{3}n$$

$$= -\left(\frac{e^{j\frac{4\pi}{3}n} - e^{-j\frac{4\pi}{3}n}}{2j}\right)$$

$$= \frac{-e^{-j\frac{2\pi}{3}n} + e^{j\frac{2\pi}{3}n}}{2j}$$

Ex $x[n] \equiv$ periodic with period N

Show that the F.S. coefficients ~~of~~ the periodic signal

$$g(t) = \sum_{k=-\infty}^{\infty} x[k]\,\delta(t-kT)$$

are periodic with period N.

Ex

1. $x(t)$ is a real signal $\Rightarrow a_k^* = a_{-k}$ $\Rightarrow$ $a_{-1} = a_1^*$
2. $x(t)$ is period with $T = 4$
3. $a_k = 0$ for $|k| > 1$ — only a_{-1}, a_0, a_1 have non-zero amplitude
4. A signal with F.S. coeff. $b_k = e^{-j\frac{\pi k}{2}} a_{-k}$ is odd
5. $\boxed{\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}}$ $\equiv$ average power in one time period

$= |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = \frac{1}{2}$

$$x(t) \xrightarrow[T]{F.S} a_k$$

$$z(t) = x(-t+1) \xrightarrow[T]{F.S} ?$$

$x(t)$ is real

$z(t) = x(-t+1)$

real & odd

odd & purely imaginary

$b_0 = 0$

$$|b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$$

$b_1 = \frac{j}{2}$, $\quad b_{-1} = -\frac{j}{2}$

$b_1 = ja$

$b_{-1} = -ja$

$$|ja|^2 + |-ja|^2 = \frac{1}{2}$$

$$a^2 + a^2 = \frac{1}{2}$$

$$a^2 = \frac{1}{4}$$

$$a = \pm\frac{1}{2}$$

① $x[n]$ is periodic with period $N=6$

② $$\sum_{n=0}^{5} x[n] = 2$$

③ $$\sum_{n=2}^{7} (-1)^n x[n] = 1$$

④ $x[n]$ has the min. power per period among the set of signals satisfying the preceding 3 conditions

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N}kn}$$

1/10/2024

C.T.

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = e^{st} H(s)$$

C.T.F.S.

$$s_k = jk\omega_0$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$$

$$= \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt}$$

D.T.

$$x[n] = z^n \longrightarrow \boxed{\text{LTI}}_{h[n]} \longrightarrow y[n] = ?$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$= z^n \boxed{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}$$

$$y[n] = z^n H(z)$$

Let $z = e^{j\omega_0}$

$$x[n] = e^{j\omega_0 n} \longrightarrow e^{j\omega_0 n} H(e^{j\omega_0})$$

D.T.F.S.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \xrightarrow[\substack{\text{System}\\ h[n]}]{\text{LTI}} y[n] = \sum_{k=\langle N\rangle} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$x[n] = \sum_{k=\langle 5\rangle} a_k e^{jk\frac{2\pi}{5}n}$$

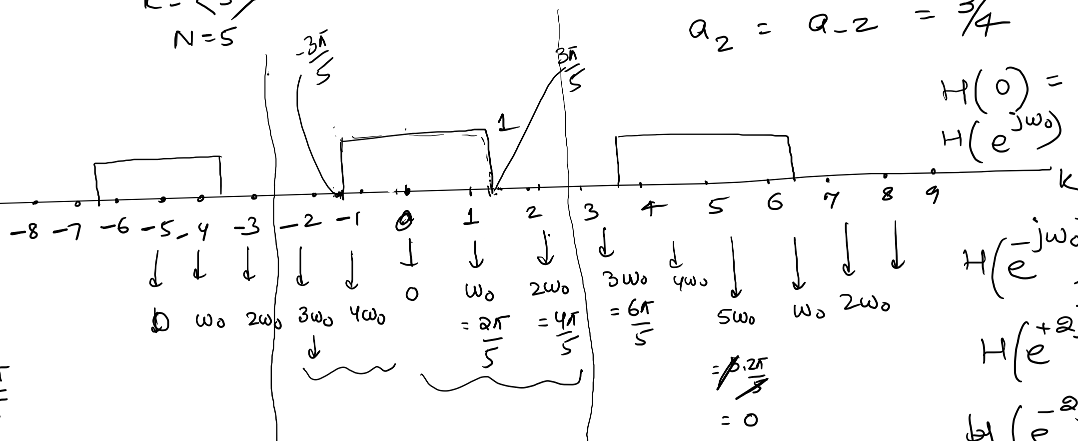
$N=5$

$a_0 = 1$

$a_1 = a_{-1} = 1/2$

$a_2 = a_{-2} = 3/4$

$= 1 + \cos\omega_0 n + \frac{3}{2}\cos 2\omega_0 n$



$H(0) = 1$

$H(e^{j\omega_0}) = 1$

$H(e^{-j\omega_0}) = 1$

$H(e^{+2j\omega_0}) = 0$

$H(e^{-2j\omega_0}) = 0$

$3\omega_0 = \frac{6\pi}{5} = 2\pi - \frac{4\pi}{5}$

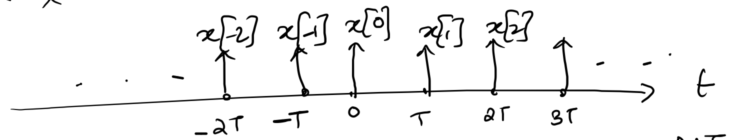
$= -\frac{4\pi}{5}$

Ex

$x[n]$ is a periodic signal, N $N=3$

Show that F.S. coeffi. of a periodic signal $g(t) = \sum_{k=-\infty}^{\infty} x[k]\,\delta(t-kT)$ are also periodic with period N.

$g(t)$



$$= x[0]\,\delta(t) + x[1]\,\delta(t-T) + x[2]\,\delta(t-2T) + \cdots + \cdots + x[-1]\,\delta(t+T).$$

$g(t)$ is periodic with NT.

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{NT}t}$$

$$a_0 = \frac{1}{NT}\int_{NT} g(t)\,dt = \frac{x[0]+x[1]+\cdots+x[N-1]}{NT}$$

$$a_l = \frac{1}{NT}\int_{NT} g(t)\, e^{-jl\frac{2\pi}{NT}t}\, dt$$

$$a_l = \frac{1}{NT}\int_{NT} \sum_{k=-\infty}^{\infty} x[k]\,\delta(t-kT)\, e^{-jl\frac{2\pi}{NT}t}\, dt$$

$$= \frac{1}{NT}\int_{0}^{NT} \sum_{k=0}^{N-1} x[k]\, e^{-jl\frac{2\pi}{NT}\cdot kT}\,\delta(t-kT)\, dt$$

$$= \frac{1}{NT}\sum_{k=0}^{N-1} x[k]\int_{0}^{NT} e^{-jlk\cdot\frac{2\pi}{N}}\,\delta(t-kT)\, dt$$

$$\boxed{a_l = \frac{1}{NT}\sum_{k=0}^{N-1} x[k]\, e^{-jk\cdot\frac{2\pi}{N}\cdot l}} = \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{x[n]}{T}\right) e^{-jl\frac{2\pi}{N}n}$$

$$=$$