

Lecture 01 (Abridged)

Today:

- ① plan of the class / logistics
- ② functions of one variable
- ③ functions of several variables.

Welcome to

Math 203 - Multivariate Calculus

[Plan of the class]

Textbooks:

(1) Thomas' Calculus (11th Edition)

(Chapters 14, 15, 16 and
appendix of 13)

- Weier, Hass and Giordano

(2) Calculus Volume 2 (2nd Edition)

- Tom M. Apostol.

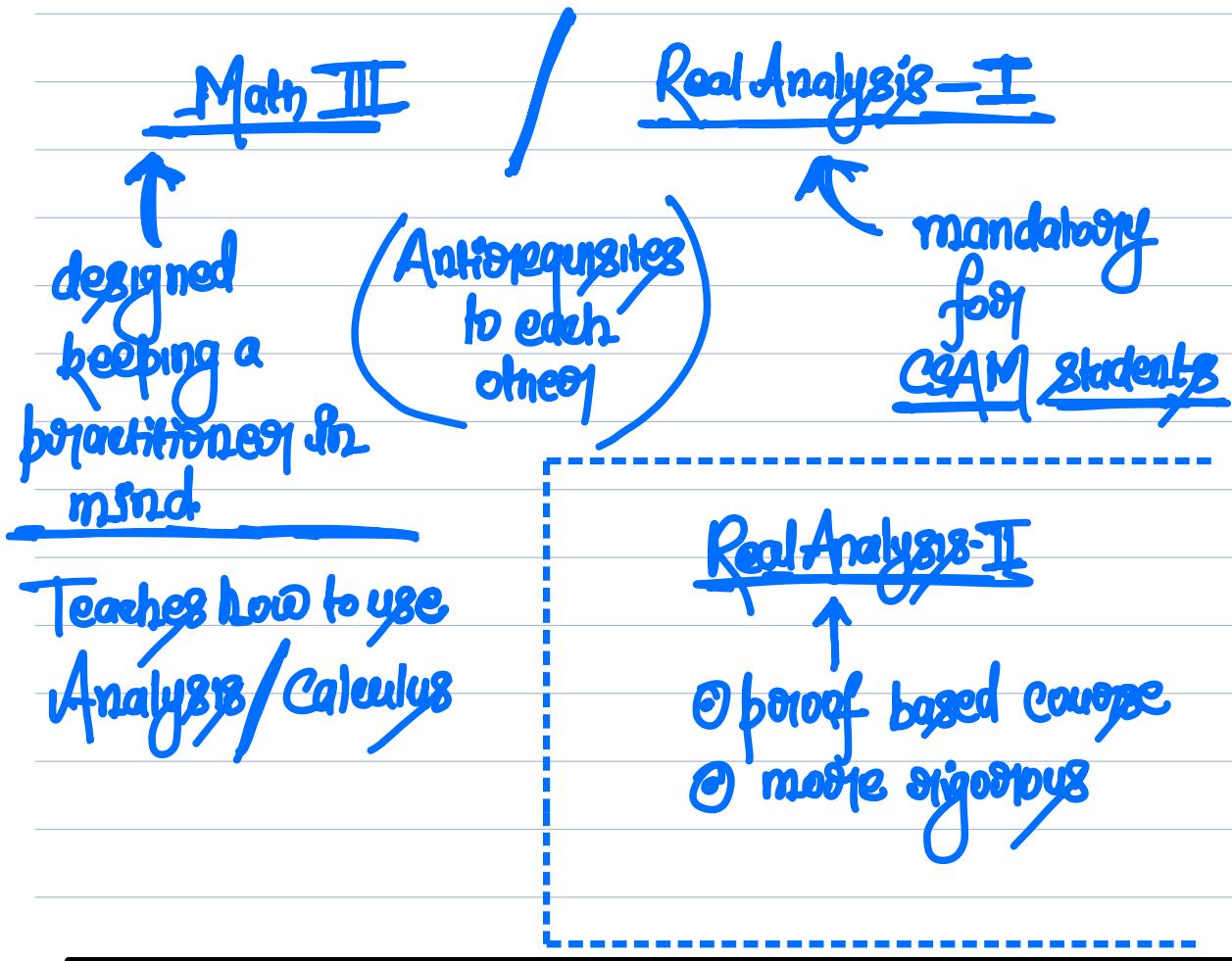
(3) Advanced Engineering Mathematics
(9th Edition)
- Efendin Kereyazig

Evaluation Criteria :

- ① Mid-sem examination — 30%.
- ② End-sem exam — 40%.
- ③ Two Quizzes — 10%.
- ④ 12 Worksheets — 20%
(5 best will be evaluated) 100/-

⑤ Tutorials — Thursdays 1:30-3:00

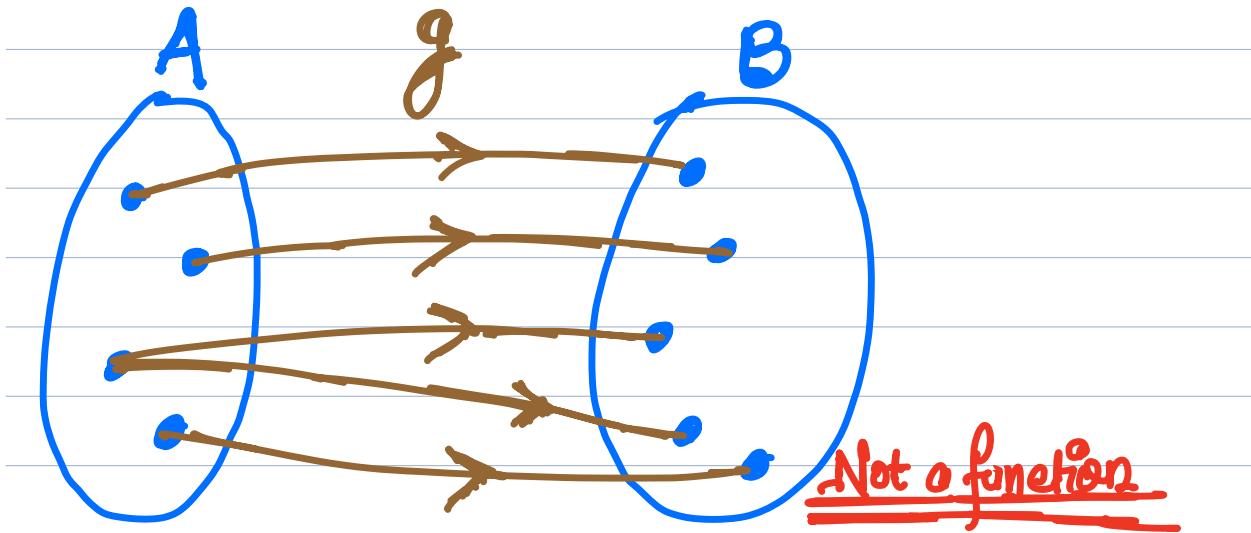
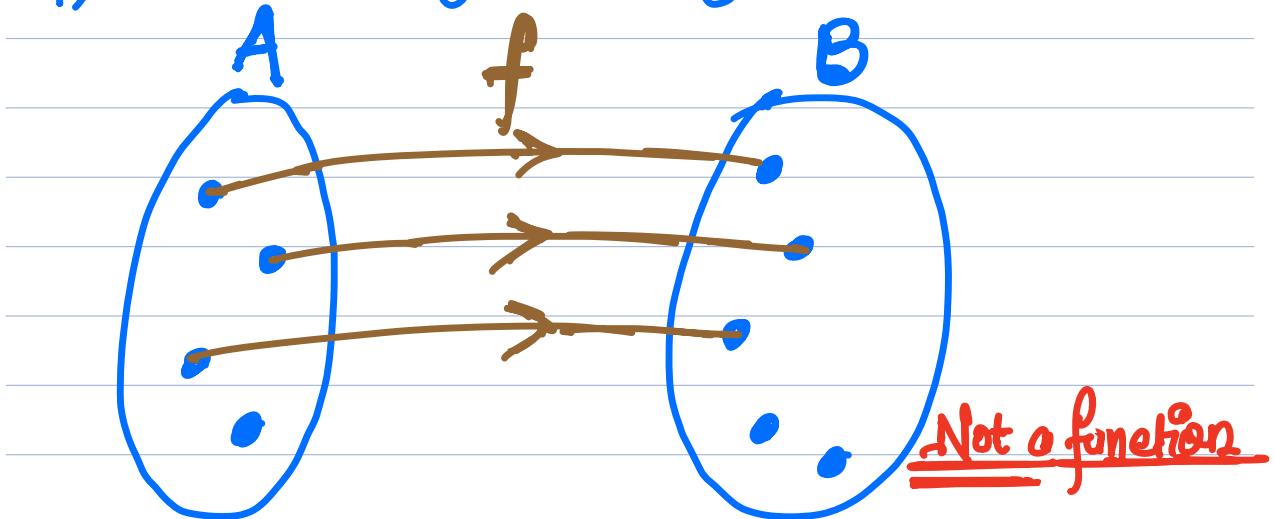
Plagiarism: — Do not do it!



Functions In almost every branch of Mathematics, functions are the central objects of interest / investigation.

Let's recall the defining properties of a function

A, B — arbitrarily nonempty sets.



Let $D \subseteq R$. Let $f: D \rightarrow R$, be a function from the set D to the set R . This statement means

$[f$ is a rule which assigns to each element $x \in D$, a unique element $y \in R]$

④ In this case, we write $f(x) = y$.

○ f \equiv The rule

○ $y = f(x) \leftarrow$ The value of f at x , or
The image of x under f .

○ $x \leftarrow$ the independent variable.

○ $y \leftarrow$ the dependent variable.

○ domain of f ?

Codomain of f ?

Range of f ?

○ for each element $x \in D$, there exists a unique element $y \in R$, such that $f(x) = y$; or,
fixed, $\exists! y \in R$, such that $f(x) = y$.

○ real-valued functions: - When the value of f at every point of the domain of f is a real number or, equivalently, $\text{ran}(f) \subseteq R$.

Q Clearly, the functions in discussion so far, are functions of single (independent) variable.

What about the case when the function in question is of two, or three, or, in general, several variables?

Let f be a "function" given by

$$f(x_1, x_2, x_3) = x_1 x_2 + x_3^2.$$

Then, clearly, the elements on which f is acting are coming from \mathbb{R}^3 . So, the domain must be a subset of \mathbb{R}^3 .

Similarly, if g be another "function" defined by

$$g(x_1, x_2) = x_1^2 + x_2^2 + x_1 \cdot x_2.$$

Then, the domain of g must be contained in \mathbb{R}^2 .

This compels us to understand the space \mathbb{R}^n .

\mathbb{R}^n dimensional Euclidean space

||

Euclidean n -space

|||

\mathbb{R}^n

Let n be a fixed, but arbitrarily chosen, positive integer.

\mathbb{R}^n as a set: It is the set of all ordered n -tuples of real numbers. That is,

$$\mathbb{R}^n := \{x = (x_1, \dots, x_n) : \forall j \in \{1, \dots, n\}, x_j \in \mathbb{R}\}$$

\mathbb{R}^n as a Vector Space:

a linear space (over the field \mathbb{R}).

|||
Vector space

[How? $\begin{cases} \rightarrow \text{Vector addition} \\ \rightarrow \text{Scalar multiplication} \end{cases}$]

\mathbb{R}^n is an inner product space:

(Vector space equipped with an inner product)

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are arbitrary elements of \mathbb{R}^n , then define an inner product of x and y , denoted by $\langle x, y \rangle$, by

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^n x_i y_i.$$

Alternatively, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
is an inner product on \mathbb{R}^n , defined via
$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i.$$

Properties: let $x, x_1, x_2, y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Then,

(1) $\langle x, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$

(2) $\langle x, x \rangle = 0 \iff x = 0$

(3) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

(4) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(5) $\langle x, y \rangle = \langle y, x \rangle$

Cauchy-Schwarz Inequality:

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle},$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and
for all $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Or, alternatively,

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2} \quad \forall x, y \in \mathbb{R}^n.$$

Moreover, the equality holds $\Leftrightarrow x$ and y are linearly dependent.

(Euclidean) norm induced by the inner product.

[If $x \in \mathbb{R}^n$, then define

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

In essence, $\|\cdot\| : \mathbb{R}^n \rightarrow [0, \infty)$

"Norm" (induced by $\langle \cdot, \cdot \rangle$)

A function from \mathbb{R}^n to $[0, \infty)$

Properties of norm $\|\cdot\|$ on \mathbb{R}^n

Let $x, y \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$;

(1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$

(2) $\|x\| = 0 \iff x = 0$

(3) $\|\alpha x\| = |\alpha| \|x\|$

(4) $\|x+y\| \leq \|x\| + \|y\|.$

↑ This is called the triangle inequality.
Qn: When does the equality hold?

Now, for any $x, y \in \mathbb{R}^n$, we can define

the distance between x and y by

$$\|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

$$= \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle}$$

$$= \sqrt{\sum_{i=1}^n x_i^2 + \sum_{j=1}^n y_j^2 - 2 \sum_{k=1}^n x_k y_k}$$

$$= \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2 - 2x_j y_j)}$$

$$= \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$$

In particular, $\|x\|^2 = \sum_{j=1}^n x_j^2$ for every $x \in \mathbb{R}^n$.

This norm is referred to as the Euclidean norm.

The real linear space \mathbb{R}^n , equipped with the inner product defined above (which induces the Euclidean norm) is referred to as the

n -dimensional Euclidean space, or

Euclidean n -space:

Function of n Variables

Defn: Let n be a positive integer.

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a rule which associates to each point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, a unique point $w \in \mathbb{R}$, and we write

$$w = f(x_1, \dots, x_n).$$

The value of
 f at the
point
 $x = (x_1, \dots, x_n)$

The rule

A function of
 n (independent)
variables x_1 to x_n

The image of x
under f

The dependent
variable of f

→ Example: Find the domain and range
of the following functions:

$$(1) Z = f(x,y) = \sin(y-x)$$

$$(2) Z = f(x,y) = \sqrt{9-x^2-y^2}$$

$$(3) W = f(x,y,z) = xy \ln z$$

$$(4) Z = f(x,y) = \sqrt{y-x^2}$$



Lecture 02

Warm-up problem:

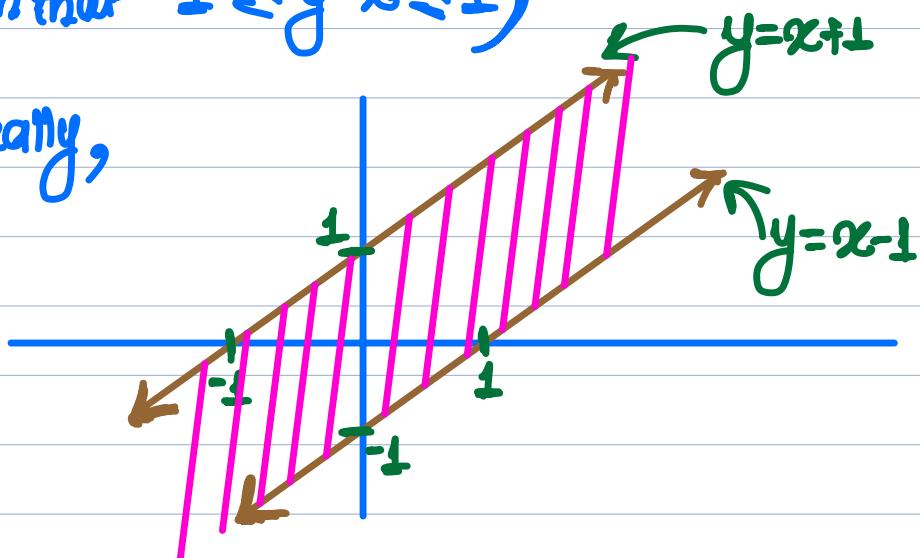
Find the domain and the range of the following functions.

(i) $Z = f(x, y) = \sin^{-1}(y-x)$.

Solution: $\text{dom}(f) = \{(x, y) \in \mathbb{R}^2 : -1 \leq y-x \leq 1\}$

(This is the set of all ordered pairs in \mathbb{R}^2
such that $-1 \leq y-x \leq 1$)

graphically,



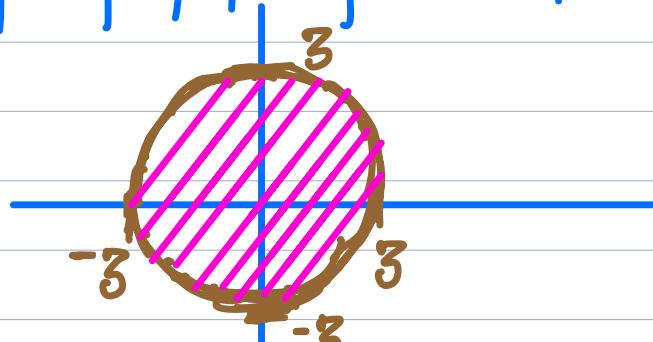
$$\text{ran}(f) = [-\pi/2, \pi/2].$$

$$\text{(ii)} \quad z = g(x, y) = \sqrt{9 - x^2 - y^2}$$

Solution: $\text{dom}(g) = \{(x, y) \in \mathbb{R}^2 : 9 - x^2 - y^2 \geq 0\}$

$$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$$

Graphically, the domain is the shaded region including the circumference of the circle.



$$\text{dom}(f) = [0, 3] \subseteq \mathbb{R},$$

$$\text{(iii)} \quad \omega = xy \ln z$$

Soln: Clearly ω is a function of three variables

\therefore domain must be contained in \mathbb{R}^3 .

Let $w = h(x, y, z) = xy \ln z$. Then,

$$\text{dom}(h) = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

[Geometrically, it is the half-space where the z-coordinate is strictly positive (and thus the x-y plane is excluded).]

$$\text{Span}(f) = \mathbb{R}_+ \cdot (\text{How?})$$

[Let $\alpha \in \mathbb{R}$ be arbitrary. Then one can choose $x=\alpha, y=1$ and $z=e$ so that $xy\ln(z) = \alpha$.]

Exercises

(iv) $\omega = \sqrt{y-x^2} ;$

(v) $f(x,y) = \sqrt{y-x} ;$

(vi) $f(x,y) = \frac{1}{\sqrt{16-x^2-y^2}} ;$

(vii) $g(x,y) = y/x^2 ;$

(viii) $h(x,y) = \frac{1}{x-y} .$

Elements of point-set Topology on \mathbb{R}^n

Let $n \in \mathbb{N}$ be fixed but arbitrarily chosen.

① Open ball in \mathbb{R}^n of radius $\delta > 0$
Centered at $x_0 \in \mathbb{R}^n$

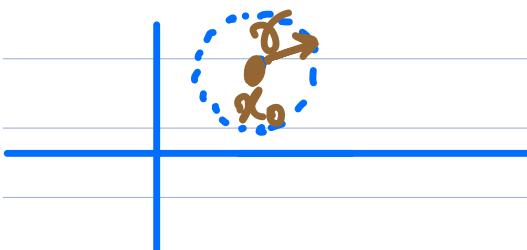
→ If $n=2$, then we are in \mathbb{R}^2 .

Suppose $x_0 \in \mathbb{R}^2$ is given and $\delta > 0$ is also given.
Then the open ball in \mathbb{R}^2 of radius δ centered
at x_0 is

the open disc of
radius δ centered
at x_0

$$= B_\delta(x_0) = \{x \in \mathbb{R}^2 : \|x - x_0\| < \delta\}$$

$$\subseteq \mathbb{R}^2$$



↑
Notation

→ If $\eta=1$, the space is \mathbb{R} .

$$\begin{aligned}B_\delta(x_0) &= \text{open interval } (x_0-\delta, x_0+\delta) \\&= \{x \in \mathbb{R} : |x-x_0| < \delta\}\end{aligned}$$

$$\therefore B_2(5) = \{x \in \mathbb{R} : |x-5| < 2\} = (3, 7)$$

→ \mathbb{R}^n : The open ball (\mathbb{R}^n) of radius δ
centered at $x_0 \in \mathbb{R}^n$ is given by

$$B_\delta(x_0) = \{x \in \mathbb{R}^n : \|x-x_0\| < \delta\}$$

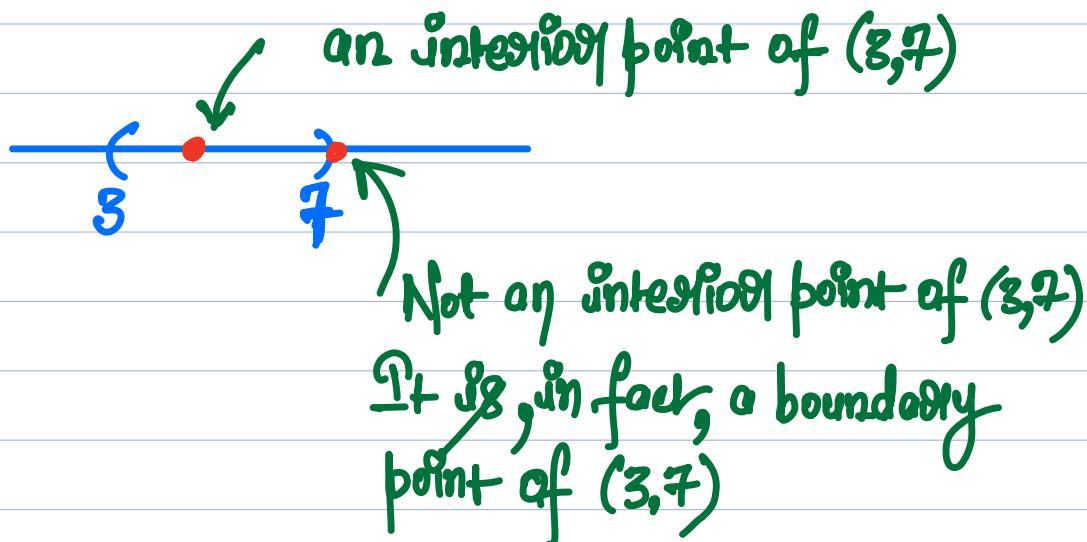
Question: Find $B_2(4)$;

$$B_3((2, 3));$$

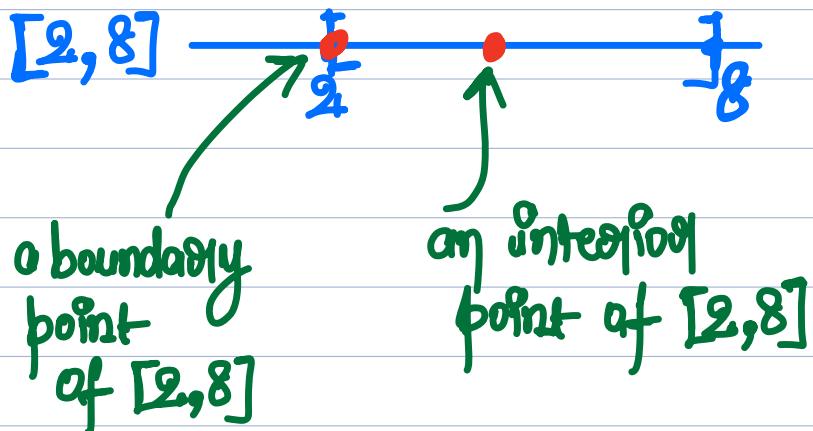
$$B_1((4, 0, 0)).$$

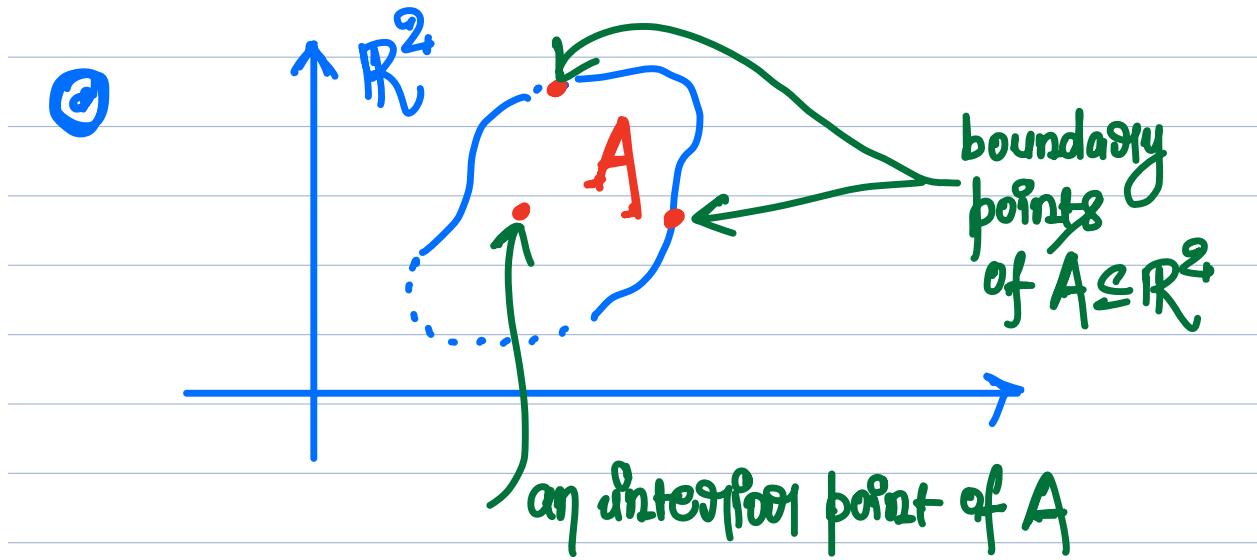
2. Interior point of a given subset of \mathbb{R}^n

Q Consider the open interval $(3, 7) \subseteq \mathbb{R}$,



Q Consider $[2, 8]$





definition Let $A \subseteq \mathbb{R}^n$. A point $x \in A$ is said to be an interior point of A if $\exists \delta > 0$ such that $B_\delta(x) \subseteq A$

[Symbol for "there exist(s)"]

[In definition "if" is essentially "if and only if"]

Alternatively, A point $x \in A$ is called an interior point of A if it is the centre of some open ball contained in A .

3. Boundary point of a given subset of \mathbb{R}^n

definition: Let $A \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a boundary point of A if for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap A^c \neq \emptyset$

[Symbol for "for every"]

[Again, this means "if and only if"]

Alternatively, a point $x \in \mathbb{R}^n$ is called a boundary point of A if every open ball centered at x contains points of A as well as points of A^c

4.

Interior of a set

Let $A \subseteq \mathbb{R}^n$. The interior of A (denoted by $\text{int}(A)$) is the set of all interior points of A.

5.

Boundary of a set

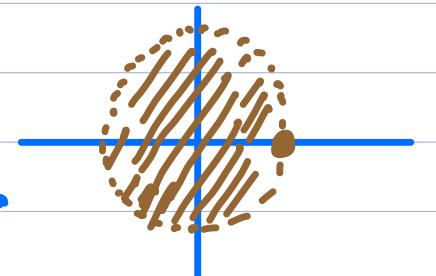
Let $A \subseteq \mathbb{R}^n$. The boundary of A (denoted by ∂A) is the set of all boundary points of A.

Exercise. Let $A \subseteq \mathbb{R}^n$. Prove that $\text{int}(A) \subseteq A$.

Example. Let $A = B_1((0,0)) \cup \{(1,0)\} \subseteq \mathbb{R}^2$.

Then, $\text{int}(A) = B_1((0,0))$,

$\partial A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.



⑥ Open set in \mathbb{R}^n

Let $A \subseteq \mathbb{R}^n$. We say that A is open in \mathbb{R}^n

or, A is an open subset of \mathbb{R}^n if $A = \text{int}(A)$.

① $[2, 3]$ is Not open in \mathbb{R} .

② $(2, 5)$ is open in \mathbb{R} .

③ $B_1((0,0)) \cup \{(1,0)\}$ is not open in \mathbb{R}^2 .



⑦ Closed set in \mathbb{R}^n

Let $A \subseteq \mathbb{R}^n$. We say that A is closed in \mathbb{R}^n

or, A is a closed subset of \mathbb{R}^n if $\partial A \subseteq A$.

① $(2, 3]$ is not closed in \mathbb{R} .

② $[5, 7)$ is not closed in \mathbb{R} .

③ $[a, b]$, where $a, b \in \mathbb{R}$, is closed in \mathbb{R} .

④ Let $A = (3, \infty) \subseteq \mathbb{R}$. Compute ∂A .

⑤ Give an example of a subset of \mathbb{R} that is neither open nor closed in \mathbb{R} .

⑥ Is \mathbb{R}^n an open subset of \mathbb{R}^m ?

⑦ Is \mathbb{R}^n a closed subset of \mathbb{R}^m ?

⑧ Let $A \subseteq \mathbb{R}^n$. Can we assert that $A = \text{int}(A) \cup \partial A$? If yes, prove it. If no, give a counterexample.

Proposition. Let $A \subseteq \mathbb{R}^n$:

If $x \in \mathbb{R}^n$, then one, and only one, of the following three possibilities holds.

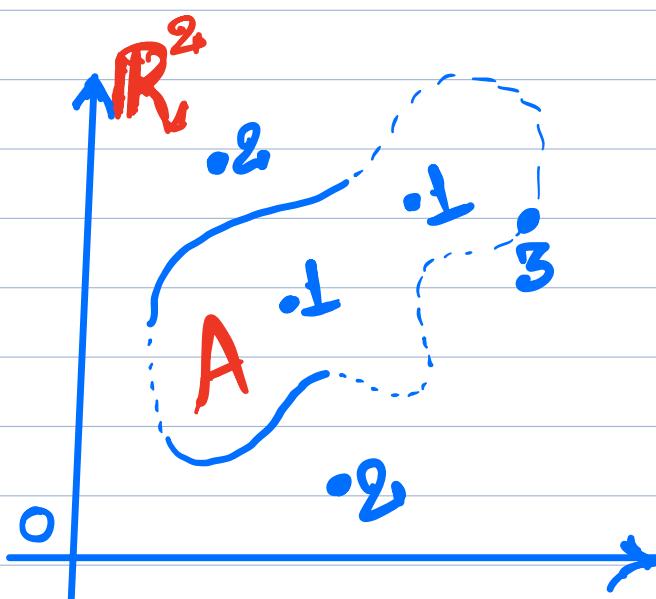
(1) $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq A$. [Interior pt. of A]

(2) $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq A^c$.

$\nearrow x$ is an exterior point of A

(3) $\forall \delta > 0$, $B_\delta(x) \cap A \neq \emptyset$ and $B_\delta(x) \cap A^c \neq \emptyset$.

boundary point
of A



Exercise: Find the exterior, interior and the boundary of the following sets.

$$\textcircled{1} \quad \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

$$\textcircled{2} \quad \{x \in \mathbb{R}^n : \|x\| = 1\}$$

$$\textcircled{3} \quad \{x \in \mathbb{R}^n : \|x\| < 1\}$$

$$\textcircled{4} \quad \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \forall j \in \{1, \dots, n\} \quad x_j \in \mathcal{O}_j\}$$

Lecture-03

Elements of point-set topology, Continued,

In the last lecture, we introduced:

- ① open ball in \mathbb{R}^n of radius $r > 0$ centered at x_0
- ② interior point of a given subset of \mathbb{R}^n
- ③ boundary point of a given subset of \mathbb{R}^n
- ④ interior of a set
- ⑤ boundary of a set
- ⑥ open sets in \mathbb{R}^n
- ⑦ closed sets in \mathbb{R}^n

Let's introduce:

- ⑧ bounded subsets of \mathbb{R}^n

Let $A \subseteq \mathbb{R}^n$. A is called a bounded subset of \mathbb{R}^n if
 $\exists r > 0$ such that $A \subseteq B_r(\vec{0})$, where $\vec{0} = (0, \dots, 0)$.

Q.1

unbounded subsets of \mathbb{R}^n

Let $A \subseteq \mathbb{R}^n$. A is said to be unbounded if it is not bounded.

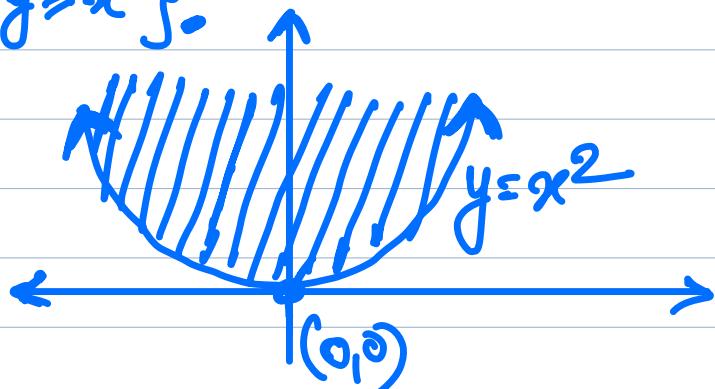
Example:- Let $f(x,y) = \sqrt{y-x^2}$.

What is the domain of f ?

$$\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}.$$

① Is $\text{dom}(f)$ open?

No! (why?)



$$\text{int}(\text{dom}(f)) = \{(x,y) \in \mathbb{R}^2 : y > x^2\}$$

$\therefore \text{dom}(f) \neq \text{int}(\text{dom}(f))$. $\therefore \text{dom}(f)$ Not open.

② Is $\text{dom}(f)$ closed? Yes.

③ bounded? No!

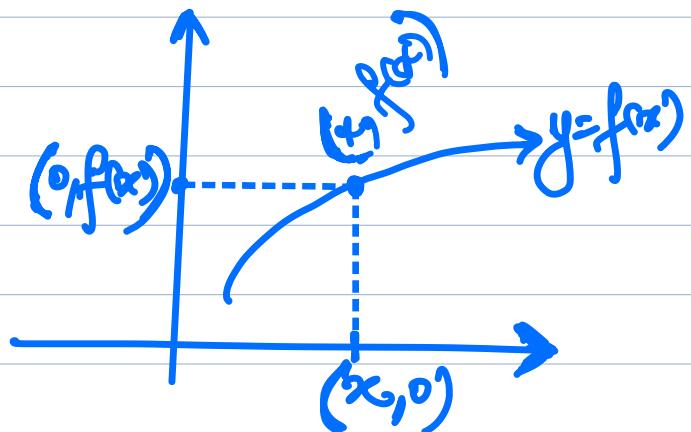
$\text{③ } \text{Dom}(f) = [0, \infty) \subseteq \mathbb{R}.$

This is closed,
not open,
and unbounded in \mathbb{R} .

Graph of a function

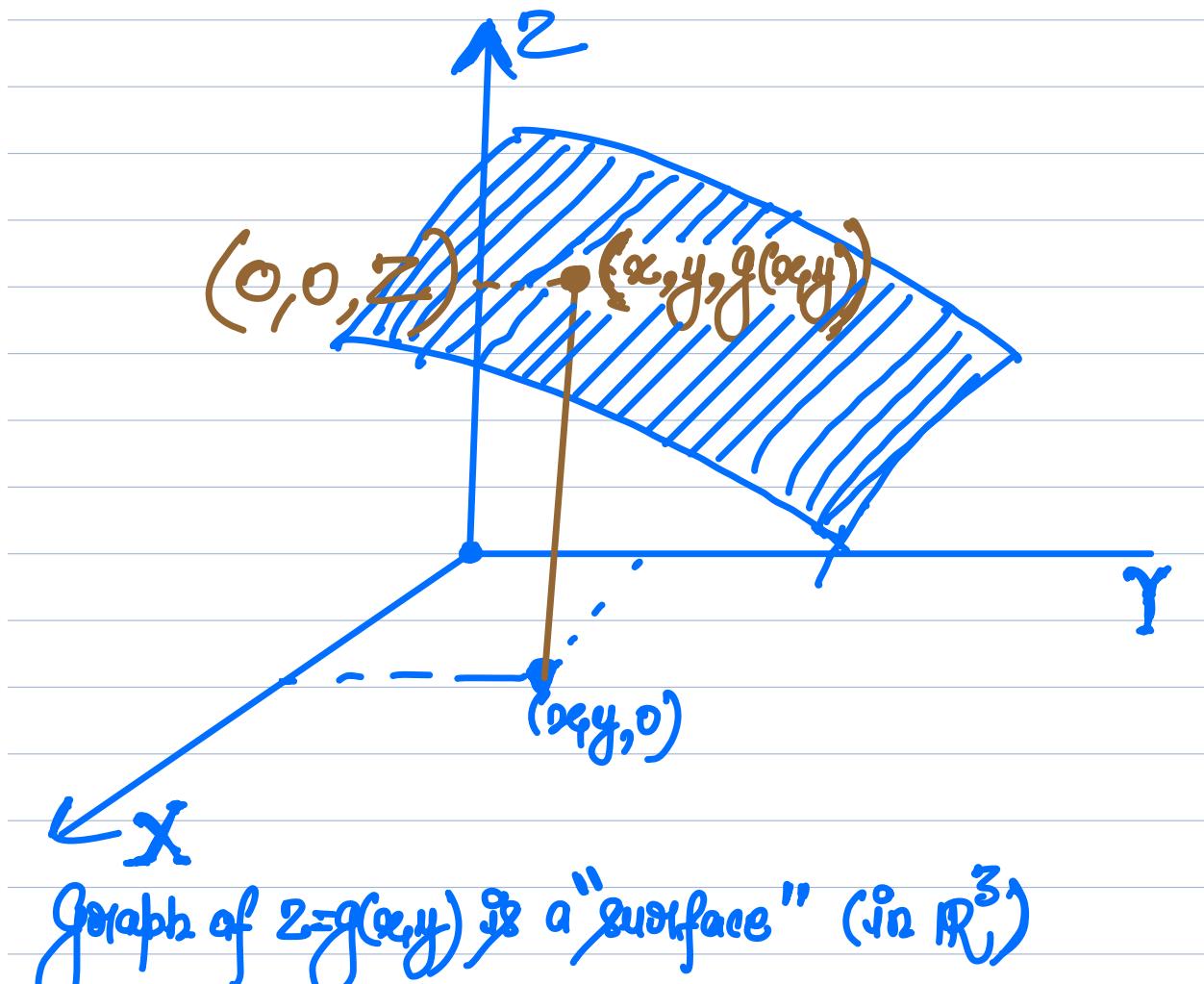
Case I: function of one variable: $y=f(x)$.

The Graph
of f is a
"curve" (in \mathbb{R}^2)



Case II: function of two variables

$$z = g(x, y)$$



Case III: function of " n " independent variables

Graph of a function:

Let n be any positive integer, let $D \subseteq \mathbb{R}^n$ be a subset of \mathbb{R}^n , and let $f: D \rightarrow \mathbb{R}$ be a function

from D to \mathbb{R} . The graph of the function f , denoted by $G(f)$, is defined to be the set

$$G(f) = \{ (\underline{x}, f(\underline{x})) : \underline{x} \in D \subseteq \mathbb{R}^n \} \quad (\subseteq \mathbb{R}^{n+1}).$$

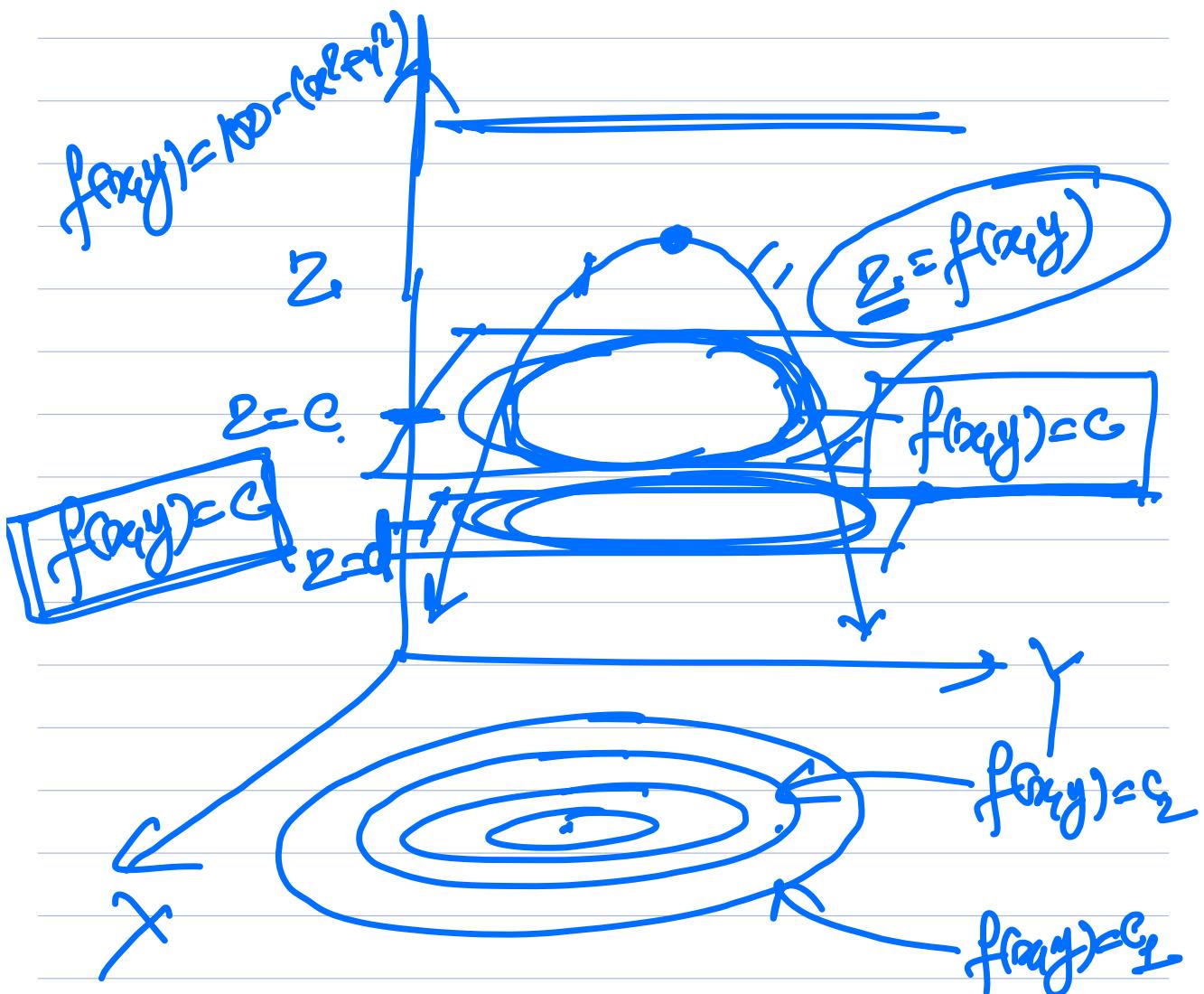
↓
 n-tuple
 ↓
 real number
 ↓
 n+1 coordinates

Level Sets

Defn: Let n be any positive integer,
 let $D \subseteq \mathbb{R}^n$ be a subset of \mathbb{R}^n , and
 let $f: D \rightarrow \mathbb{R}$ be a function from D to \mathbb{R} .
 Given a constant $c \in \text{span}(f)$,
 the level set of f at the point c
is defined to be the set
 $\{ \underline{x} \in D \subseteq \mathbb{R}^n : f(\underline{x}) = c \} \quad (\subseteq \mathbb{R}^n).$

Remarks:

- ① Every Level set of the function f lies in the domain of the function f .
- ② On each Level set, the value of the function is a constant.
- ③ When $n=2$, (i.e. When we consider $D \subseteq \mathbb{R}^2$, so that f is a function of two variables), we call it a Level Curve.
- ④ When $n=3$ (i.e. When we consider $D \subseteq \mathbb{R}^3$, so that f is a function of three variables), we call it a Level Surface.



Example: Let $f(x, y) = 4 - x^2 - y^2$. Find the level curves.

Solution: We have $Z = f(x, y)$. We replace Z by some admissible constant C , and we get

$$f(x, y) = C$$

i.e., $4 - x^2 - y^2 = C$

or, $[x^2 + y^2 = 4 - C]$

clearly, C must be either less than or equal to 4, i.e., $C \leq 4$.

So,

If $C=4$, we get $x^2 + y^2 = 0$

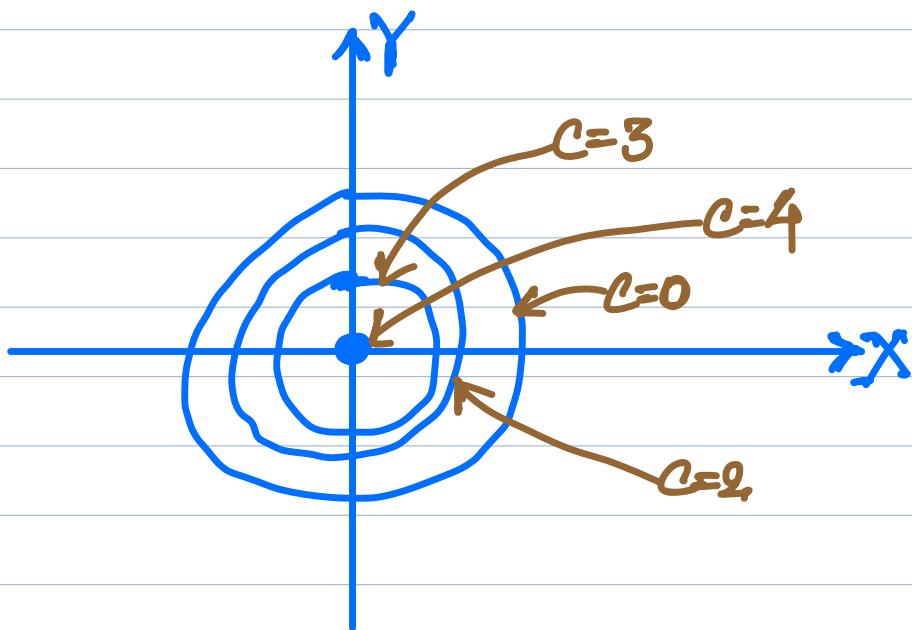
(a circle of radius 0.)
 \therefore merely the origin

If $c=3$, we get $x^2+y^2=1$

(a circle of radius 1)
centered at the origin)

If $c=0$, we get $x^2+y^2=4$ (a circle of radius 2)

If $c<0$, we get circle of radius >2 .



Exercise Let $f(x,y)=\frac{1}{\sqrt{16-x^2-y^2}}$. Find the level curves.

Exercise Let $g(x,y,z)=x^2+y^2+z^2$. Find the level surfaces.

Lecture 04

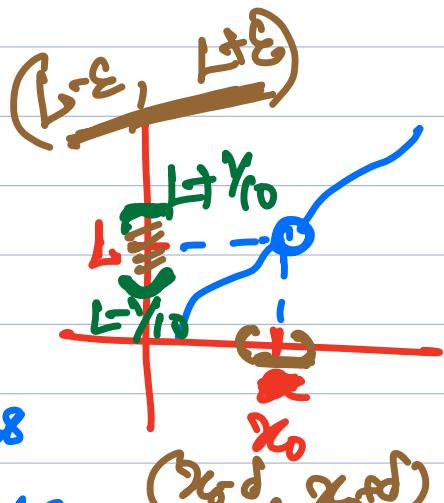
Recall:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$, be a function.

$$\lim_{x \rightarrow a} f(x) = L$$

(read as, "the limit of f at a ")

We can get $f(x)$ as close to the real number L as desired, by choosing x sufficiently close to (but not equal to) a



Given any $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $x \in \text{dom}(f)$ that satisfies $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$.

$[a - \delta < x < a + \delta \text{ and } x \neq x_0]$

$[L - \epsilon < f(x) < L + \epsilon]$

definition: Let $S \subseteq \mathbb{R}$, and $f: S \rightarrow \mathbb{R}$ be a function. If $a \in \mathbb{R}$ be a limit point of S , ^{??} then a point $L \in \mathbb{R}$, is the limit of f at a . If for every $\epsilon > 0$, there exists a $\delta > 0$ such that

If $|f(x) - L| < \epsilon$ whenever $x \in \text{dom}(f) = S$ and $0 < |x - a| < \delta$,

and we write

$$\lim_{x \rightarrow a} f(x) = L.$$

Note:

(1) $a \in \mathbb{R}$, but

a need not necessarily belong to $\text{dom}(f)$.

(2) Even if $a \in \text{dom}(f)$, it may be the case that $f(a) \neq \lim_{x \rightarrow a} f(x) = L$

(3) If $a \in \text{dom}(f)$ and if $f(a) = \lim_{x \rightarrow a} f(x)$, then f is continuous at the point a .

Let's move on to functions of several variables.

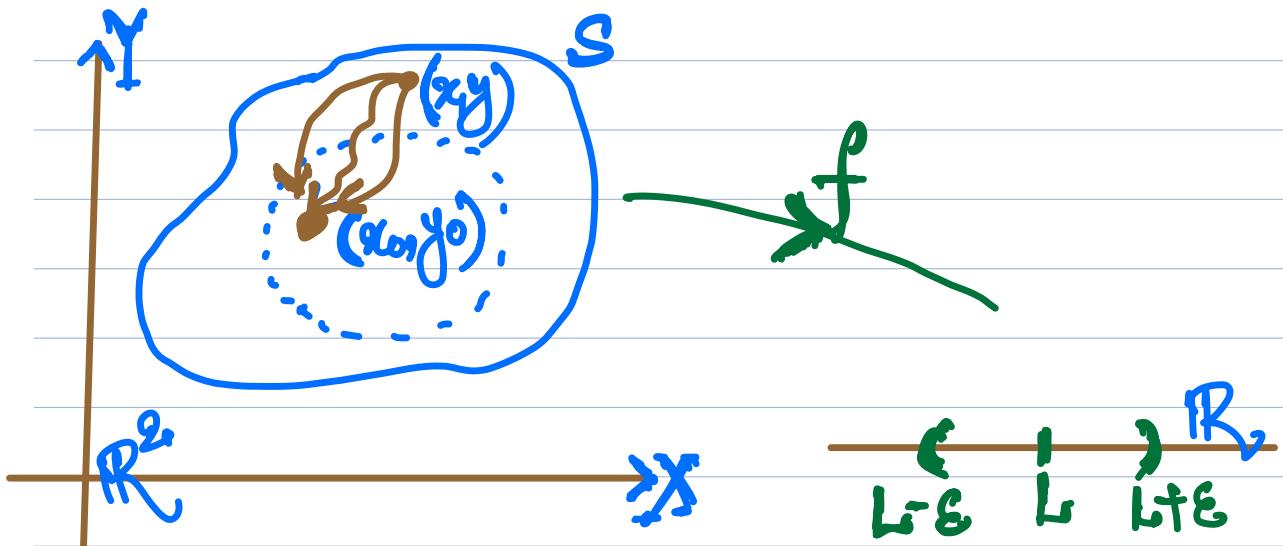
definition: Let $S \subseteq \mathbb{R}^n$ and $f: S \rightarrow \mathbb{R}$ be a function from S to \mathbb{R} . If $a \in \mathbb{R}^n$ is a limit point of S , then a point $L \in \mathbb{R}$ is called the limit of f at a , if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that $x \in S$ and $0 < \|x - a\| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$.

If $x = (x_1, x_2, \dots, x_n)$ and $a = (a_1, a_2, \dots, a_n)$, then

$$\|x - a\| = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}$$

or

$$\begin{cases} x \in B_\delta(a) \text{ and} \\ x \neq a \end{cases}$$



Example 1: $f(x,y) = x$.

$$\lim_{\substack{(x,y) \rightarrow (x_0, y_0)}} f(x,y) = x_0.$$

Example 2: $f(x,y) = c$

$$\lim_{\substack{(x,y) \rightarrow (x_0, y_0)}} f(x,y) = c.$$

Example 3: $\lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$

Solution: Note that

$$\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 : x \neq y+1\}.$$

\therefore Clearly $(4,3) \notin \text{dom}(f)$

$$\Rightarrow \lim_{(x,y) \rightarrow (4,3)} \frac{(\sqrt{x} - \sqrt{y+1})(\sqrt{x} + \sqrt{y+1})}{(x-y-1)(\sqrt{x} + \sqrt{y+1})}$$

$$= \lim_{(x,y) \rightarrow (4,3)} \frac{(x-y-1)}{(\cancel{x-y-1})(\sqrt{x} + \sqrt{y+1})}$$

$$= \lim_{(x,y) \rightarrow (4,3)} \frac{1}{(\sqrt{x} + \sqrt{y+1})} = \frac{1}{4} \text{ ans.}$$

Properties of Limits of fns. of several variables

Let f and g be functions from \mathbb{R}^2 to \mathbb{R} .

If $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = L$ and

$\lim_{(x,y) \rightarrow (x_0, y_0)} g(x,y) = M$, then

(1) $\lim_{(x,y) \rightarrow (x_0, y_0)} [f(x,y) \pm g(x,y)] = L \pm M$

(2) $\lim_{(x,y) \rightarrow (x_0, y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$

(3) $\lim_{(x,y) \rightarrow (x_0, y_0)} [c f(x,y)] = cL$

$$\underline{(4)} \lim_{(x,y) \rightarrow (x_0, y_0)} \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \frac{L}{M}$$

provided that $M \neq 0$.

(5) If $\alpha, \beta \in \mathbb{Z}$ such that there are no common factors of α and β (except 1) and if $\beta \neq 0$, then

$$\lim_{(x,y) \rightarrow (x_0, y_0)} [f(x,y)]^{\alpha/\beta} = L^{\alpha/\beta}$$

provided $L^{\alpha/\beta}$ is a real number.

Example 4: $\lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$

Solution: (use quotient rule)

Example 5: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

Solution: (Can't use quotient rule right away!)

Why? : $\sqrt{x} - \sqrt{y} \rightarrow 0$ as $(x,y) \rightarrow (0,0)$)

Remark: Can you see that the definition of the limit of a function does not help you "find" the limit; it only helps you verify whether or not L is the limit of f at a.

How to show that

(i) L is not the limit of f at a

(ii) the limit of f at a does not exist

For (ii) you need to "negate" the implication in the definition of the limit

Hint: How to negate $A \Rightarrow B$?

Negate " $\neg A \vee B$ ", and you shall get
" $A \wedge \neg B$ ".

" L is not the limit of f at a "

\Updownarrow
" $\lim_{\substack{(x_1, x_2) \rightarrow (a_1, a_2)}} f(x_1, x_2) \neq L$ "

\Updownarrow
" $\exists \varepsilon > 0$ such that $\forall \delta > 0$, we have

① $x = (x_1, x_2) \in \text{dom}(f)$,

② $0 < \|x - a\| < \delta$, and

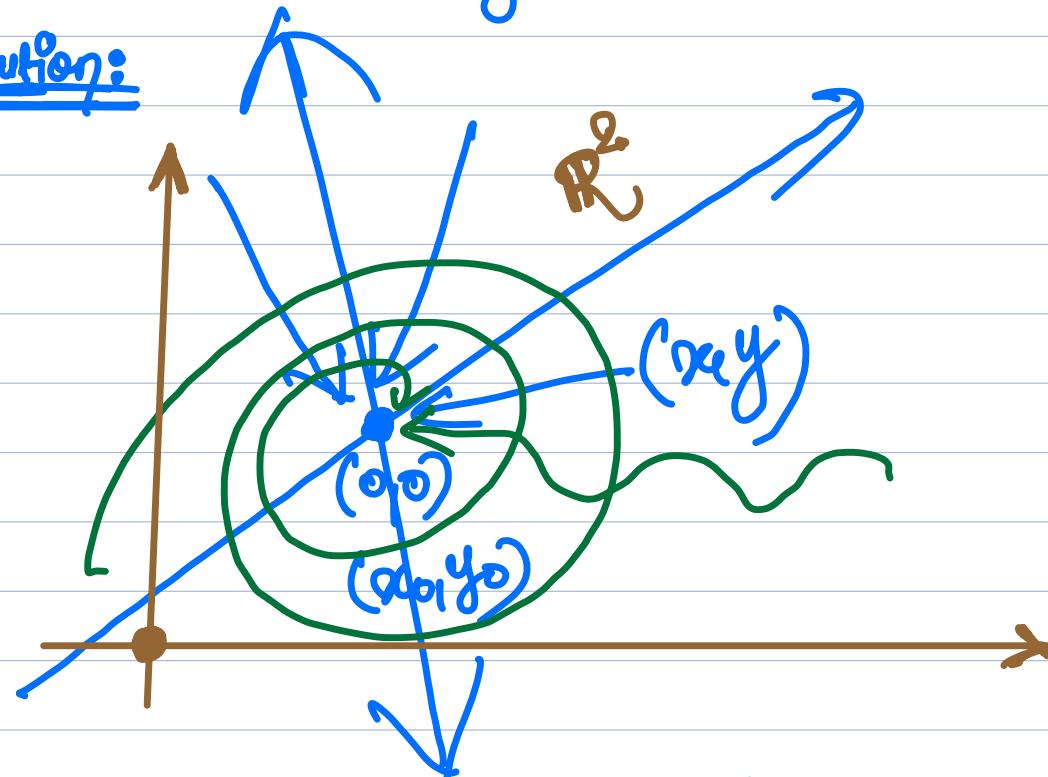
③ $|f(x) - L| \geq \varepsilon$

For (ii), we have, what we call, two-path test!

Example 6: Compute

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2+y^2} \text{ if it exists.}$$

Solution:



→ Let us approach $(0,0)$ along the line $y=2x$.

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=2x}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x \rightarrow 0} \frac{2x(2x)}{x^2 + (2x)^2}$$

$$= \lim_{x \rightarrow 0} \frac{4x^2}{5x^2} = 4/5$$

→ Let us approach $(0,0)$ along the line $y=5x$

$$\lim_{\substack{(xy) \rightarrow (0,0) \\ \text{along } y=5x}} \frac{2xy}{x^2 + y^2}$$

$$= \lim_{x \rightarrow 0} \frac{2x(5x)}{x^2 + (5x)^2}$$

$$= \lim_{x \rightarrow 0} \frac{10x^2}{26x^2} = 10/26 = 5/13.$$

So, it is evident that the limit must not exist!!

Reason: Different paths of approach to the origin $(0,0)$ can lead to different results!

In general,

→ Let us approach $(0,0)$ along the line $y=m x$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x \rightarrow 0} \frac{2x(mx)}{x^2 + (mx)^2}$$

$$= \lim_{x \rightarrow 0} \frac{2mx^2}{(1+m^2)x^2}$$

$= \frac{2m}{1+m^2}$ which depends on "m",
that is, it depends on the path!!!

Example 7: $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^4+y^4} = ?$

Hint: Try the path $y=cx^2$.

(By the two-path test, if has no limit
as (x,y) approaches $(0,0)$.)

Two-path test (for the nonexistence of limit)

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \rightarrow (x_0, y_0)$, then

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) \text{ does not exist}$$

Example 8: Compute

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+y^2} \text{ if it exists.}$$

Remark: (1) Let $f(x,y) = \frac{4xy^2}{x^2+y^2}$.

Then, $\text{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$. f is not defined at $(0,0)$. However, it makes sense to discuss the limit of the function f as (x,y) approaches $(0,0)$.

(Why? Since f is defined at every point around some "neighbourhood" of (x_0) .)

(2) quotient rule does not apply! (why?)

(3) Can you come up with some simplification? The denominator is not an additional function.

(4) What if we try 2-path test?

Path 1: (along x-axis)

$$\lim_{(x,0) \rightarrow (0,0)} \frac{4x \cdot 0^2}{x^2 + 0^2} = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0$$

Path 2: (along the y-axis)

$$\lim_{(0,y) \rightarrow (0,0)} \frac{4 \cdot 0 \cdot y^2}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

Claim: The limit is indeed "0". Prove it!!!

Classroom Proof:

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} \\ (x, y) \in \text{dom}(f) = \mathbb{R}^2 \setminus \{(0, 0)\} \\ \text{and} \\ 0 < \| (x, y) - (x_0, y_0) \| < \delta \end{array} \right] \rightarrow |f(x, y) - L| < \epsilon.$$

EquivAlently,

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} \\ (x, y) \neq (0, 0) \text{ and} \\ \sqrt{x^2 + y^2} < \delta \end{array} \right] \Rightarrow \left| \frac{4xy^2}{x^2 + y^2} \right| < \epsilon$$

Now, let $\epsilon > 0$ be given.

We are required to find some $\delta > 0$, depending only on $\epsilon > 0$, such that

$$\left| \frac{4xy^2}{x^2 + y^2} \right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

To find some $\delta > 0$, we need to estimate $\left| \frac{4xy^2}{x^2+y^2} \right|$.

In particular, we want

$$\left| \frac{4xy^2}{x^2+y^2} \right| < \epsilon.$$

Note that, we don't know for which points (x,y) the above inequality is satisfied.

We want this inequality to be satisfied by $(x,y) \neq (0,0)$ whenever $\sqrt{x^2+y^2} < \delta$ for a chosen δ .

Observe that

$$\left| \frac{4xy^2}{x^2+y^2} \right| = \frac{4|x|y^2}{x^2+y^2}$$

and since $y^2 \leq x^2+y^2$, we get

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x|.$$

So, it suffices to estimate $4|x|$, (that is,
if we establish that $4|x| < \epsilon$, we are through.)

Let's not forget that we want $4|x| < \varepsilon$; we don't already know it. We want $4|x| < \varepsilon$ whenever $0 < \sqrt{x^2 + y^2} < \delta$ (the delta that we are looking for!).

So, can we estimate $4|x|$ in terms of $\sqrt{x^2 + y^2}$?

Of course, $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$, so that $4|x| \leq 4\sqrt{x^2 + y^2}$. So, it is sufficient to make $4\sqrt{x^2 + y^2} < \varepsilon$. (Why?)

But expecting $4\sqrt{x^2 + y^2} < \varepsilon$ is same as expecting $\sqrt{x^2 + y^2} < \varepsilon/4$. This suggests us to choose $\delta = \varepsilon/4$.

Now, clearly, for any given $\varepsilon > 0$, we choose $\delta = \varepsilon/4$ to obtain the following implication:

$$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow \left| \frac{4xy^2}{x^2 + y^2} \right| = \frac{4|x|y^2}{x^2 + y^2}$$

$$\leq 4|x|$$

$$\leq 4\sqrt{x^2 + y^2}$$

$$< 4\delta$$

$$= 4\delta/4$$

$$=\epsilon.$$

Textbook proof:

Let $\epsilon > 0$ be given. We are required to find $\delta > 0$, which depends on the given $\epsilon > 0$, such that $0 < \|(\mathbf{x}, \mathbf{y}) - (\mathbf{x}_0, \mathbf{y}_0)\| < \delta \Rightarrow |f(\mathbf{x}, \mathbf{y}) - L| < \epsilon$.

Since $(x_0, y_0) = (0, 0)$ and $L=0$, we are required to find $\delta > 0$, which depends on $\epsilon > 0$, such that

$$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow \left| \frac{4xy^2}{x^2 + y^2} \right| < \epsilon.$$

To this end, let us estimate $\left| \frac{4xy^2}{x^2 + y^2} \right|$. Now observe that,

$$\left| \frac{4xy^2}{x^2 + y^2} \right| = \frac{4|x|y^2}{x^2 + y^2} \leq 4|x| \leq 4\sqrt{x^2 + y^2}.$$

This suggests that we should choose δ to be $\epsilon/4$, for then,

$$\sqrt{x^2 + y^2} < \delta = \epsilon/4 \Rightarrow 4\sqrt{x^2 + y^2} < \epsilon$$

and since

$$\left| \frac{4xy^2}{x^2 + y^2} \right| \leq 4\sqrt{x^2 + y^2},$$

it follows that

$$\left| \frac{4xy^2}{x^2+y^2} \right| < \varepsilon.$$

This completes the proof.



Now, please go through the ~~classroom~~ proof
Once again!

Lecture - 05

Continuity of a function at a point

Let $S \subseteq \mathbb{R}$ and $f: S \rightarrow \mathbb{R}$, be a function.

We say that f is continuous at $a \in S$, if

(i) $a \in \text{dom}(f) = S$;

(ii) $\lim_{x \rightarrow a} f(x)$ exists;

(iii) $\lim_{x \rightarrow a} f(x) = f(a)$.

\Updownarrow (Equivalently,

Let $S \subseteq \mathbb{R}$, $a \in S$ and $f: S \rightarrow \mathbb{R}$, be a function.

We say that f is continuous at $a \in S$ if

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that for all $x \in \text{dom}(f) = S$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Analogously,

Let $S \subseteq \mathbb{R}^2$, $a = (a_1, a_2) \in S$ and $f: S \rightarrow \mathbb{R}$ be a function. We say that f is continuous at $a \in S$ if for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $(x_1, x_2) \in \text{dom}(f) = S$

$$\|(x_1, x_2) - (a_1, a_2)\| < \delta \Rightarrow |f(x_1, x_2) - f(a_1, a_2)| < \epsilon$$

$$\| \cdot \| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$$

Note: f is called "continuous" if it is continuous at each point of its domain.

Example 1: $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Is f continuous at $(0, 0)$?

Soln: (See Example 6 from the previous lecture)

Example 2: Let $f(x,y) = \frac{4xy^2}{x^2+y^2}$.

→ Prove/Verify that

(a) $(0,0) \notin \text{dom}(f)$

(b) $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ exists

Hint: See Example 8 of the previous lecture

→ Define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Example 3: Let $g(x,y) = \frac{3xy}{x^2+y^2}$.

Define $g(0,0)$ in such a way that extends g to be continuous at the origin.

Example 4: Let $f(x,y) = \ln\left(\frac{3x^2 - xy^2 + 3y^2}{x^2 + y^2}\right)$.

Define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Homework: (use polar coordinates to find the limit of f at the point $(0,0)$)

Example 5: Let $f(x,y) = \begin{cases} \frac{x^2}{x^2+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Show that f is continuous at every point except $(0,0)$.

Solution: Hint: Try two cases:

Case I: When $(x_0, y_0) \neq (0,0)$. Now, use $\epsilon-\delta$ argument to show that f is continuous at (x_0, y_0) .

Case II: Show that f is not continuous at $(0,0)$.

Question(8): At what points in the plane/
space are the following functions continuous?

$$(1) g(x,y) = \sin\left(\frac{1}{xy}\right)$$

$$(2) h(x,y) = \frac{x^2+y^2}{x^2-3x+2}$$

$$(3) g(x,y) = \frac{x+y}{2+\cos x}$$

$$(4) f(x,y,z) = \frac{1}{x^2+z^2-1}$$

$$(5) h(x,y,z) = \frac{1}{|y|+|z|}$$

Partial derivatives

Let us first recall the following definition
In the case of functions of one variable.

defn: Let $D \subseteq \mathbb{R}$, let $x_0 \in D$ be an interior point and let $f: D \rightarrow \mathbb{R}$ be a function.

We say that f is differentiable at x_0 , or, f has a derivative at x_0 , provided that

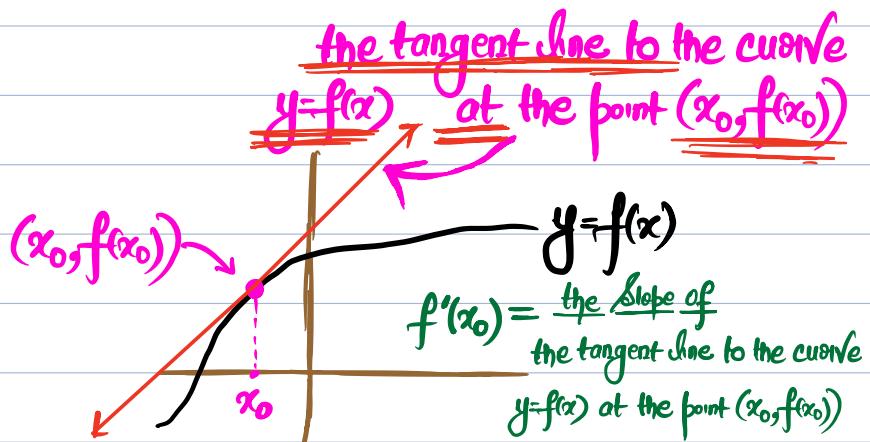
the following limit exists,

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}).$$

When this limit exists, we denote it by

$$f'(x_0) \text{ or } \left. \frac{df}{dx} \right|_{x=x_0}$$

Question: What is the geometrical interpretation of $f'(x_0)$?



On the language you may encounter

Let $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function.
Then the following statements essentially mean the same thing.

- f is differentiable at a point $x_0 \in D$
- f has a derivative at a point $x_0 \in D$

Question

What is the analogous notion of the "derivative"
in case of $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, or $f: \mathbb{R}^n \rightarrow \mathbb{R}$, or
 $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$? [\rightarrow Later!]

First, let us talk about partial derivatives.

defn: Let $D \subseteq \mathbb{R}^2$, let $(x_0, y_0) \in D$ be an interior point of D and let $f: D \rightarrow \mathbb{R}$ be a real-valued function of two variables.

We say that f has a partial derivative w.r.t x at the point (x_0, y_0) provided that the following limit exists,

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

When this limit exists, we denote it by

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \text{ or } f_x(x_0, y_0) \text{ or } \left. \frac{d}{dx} f(x, y_0) \right|_{x=x_0}$$

and call it "the partial derivative of f w.r.t. x at the point (x_0, y_0) ".

Example: $f(x,y) = xy + 3x^2 + y$

Compute $\frac{\partial f}{\partial x}$ at $(1,2)$.

$$\frac{\partial f}{\partial x} = y + 6x$$

$$\frac{\partial f}{\partial x}(1,2) = 2+6=8.$$

Example: Calculate the partial derivative

$\frac{\partial f}{\partial x}$ of the function

$$f(x,y) = x^3 - 3x^2y^3 + y^2.$$

Soln: $\frac{\partial f}{\partial x} = 3x^2 - 6xy^3.$

defn: Let $D \subseteq \mathbb{R}^2$, $\text{Int}(x_0, y_0) \in D$ be an interior point of D and let $f: D \rightarrow \mathbb{R}$ be a real-valued function of two variables.

We say that f has a partial derivative w.r.t. y at the point (x_0, y_0) provided that the following limit exists,

$$\lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

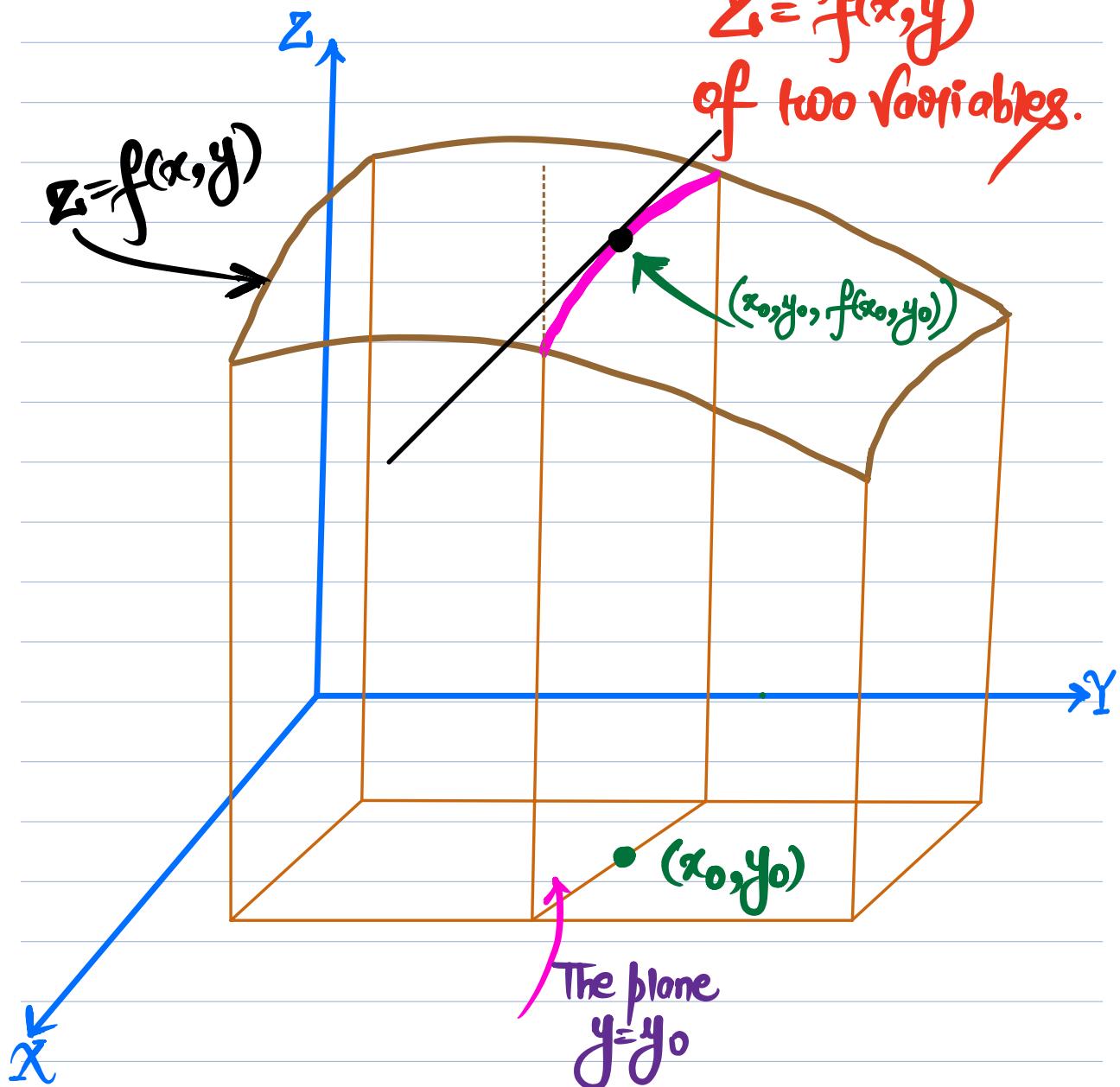
When this limit exists, we denote it by

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} \text{ or } f_y(x_0, y_0) \text{ or } \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0}$$

and call it "the partial derivative of f w.r.t. y at the point (x_0, y_0) ".

Geometric interpretation of partial derivatives —

for a function
 $Z = f(x, y)$
of two variables.



① The graph of $z = f(x, y)$ is a surface
— the one sketched above.

② (x_0, y_0) is a given point in the interior
of the domain of the function (this is a point in
the $X-Y$ plane, such that $(x_0, y_0, f(x_0, y_0))$
is a point on the surface).

③ We wish to interpret $\frac{\partial f}{\partial x}$ at (x_0, y_0)
④ To treat the variable "y" as constant,
and the constant must of
course be y_0 , i.e., $y = y_0$.

⑤ To hold y fixed at the value y_0 means
To intersect the surface $z = f(x, y)$
with the plane $y = y_0$.

① The intersection of
the surface $\Sigma = f(x, y)$
and the plane $y = y_0$
is the curve $z = f(x, y_0)$.

(this curve is on the plane $y = y_0$)

② What does then the value
 $\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)} = f_x(x_0, y_0)$ denote?

It is the slope of the tangent line
to the curve $z = f(x, y_0)$
at the point $x = x_0$.

Lecture 06

Relationship between Continuity and existence of partial derivatives

Example 1: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

$\rightarrow \frac{\partial f}{\partial x} \Big|_{(0,0)} = 0$ and $\frac{\partial f}{\partial y} \Big|_{(0,0)} = 0$

$\rightarrow f$ is not continuous at $(0,0)$

[use 2-path test to show that the limit of f at $(0,0)$ does not exist, and hence f is not continuous.
Compute the limit along $y=ex^2$.]

Example 2.: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

(1) f is not cont. at $(0,0)$

(2)

$$\frac{\partial f}{\partial x}$$

partial derivative of f

W.r.t. x

partial derivative of f

W.r.t. ~~the other ind. variable~~

Now, $\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0$??
(How)

$$\lim_{h \rightarrow 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\underline{1} - \underline{1}}{h} = 0.$$

Next, $\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\underline{1} - \underline{1}}{h}$$

$$= 0.$$

∴ The partial derivatives exist at $(0,0)$

(3) However, f is not continuous at $(0,0)$.
(Verify!!!)

Thus, unlike in one-variable case, where existence of derivatives guarantees continuity, existence of (all) partial derivatives at a point does not even guarantee that the function is continuous at that point!

Need a new notion of derivative!

[Total Derivative]

Differentiability (single-variable case)

Let us recall the following definition from the last lecture.

Definition 1: Let $D \subseteq \mathbb{R}$, $x_0 \in \text{int}(D)$ and $f: D \rightarrow \mathbb{R}$ be a (real-valued) function (of one variable). We say that f is differentiable at $x_0 \in \text{int}(D)$



the following limit exists, $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

When this limit exists, we denote it by

$f'(x_0)$, and call it "the derivative of f at x_0 ".

(End of the definition)

 (convince yourself)

\exists a real number, denoted by $f'(x_0)$,

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0).$$

or, equivalently,

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - h f'(x_0)}{h} = 0$$

[Definition 2.: (Reformulation of the first defn)]

Let $D \subseteq \mathbb{R}$, $x_0 \in \text{int}(D)$ and $f: D \rightarrow \mathbb{R}$

be a (real-valued) function (of one variable).

We say that f is differentiable at $x_0 \in \text{int}(D)$



\exists a real number, denoted by $f'(x_0)$,
such that

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0$$

Notation:

Let $f'(x_0)$ exists. Define a function E_{x_0} as follows:

(*) $E_{x_0}(h) := f(x_0+h) - f(x_0) - f'(x_0)h.$

(*)
$$f(x_0+h) - f(x_0) = f'(x_0)h + E_{x_0}(h)$$

Approximation of
 $f(x_0+h) - f(x_0)$

Observations:

① $f'(x_0)$ is a linear transformation from \mathbb{R} to \mathbb{R} ,

Proof: define $T: \mathbb{R} \rightarrow \mathbb{R}$ by

$$T(h) = f'(x_0)h. \quad (\text{for every } h \in \mathbb{R})$$

Check: ① $T(h_1 + h_2) = T(h_1) + T(h_2)$
② $T(\lambda h) = \lambda T(h)$

Note: In fact, every real number λ
can be thought of as a linear mapping of
 \mathbb{R} into \mathbb{R} . (How?)

② If f is differentiable at x_0 , then

$$\frac{E_{x_0}(h)}{h} \rightarrow 0 \text{ as } h \rightarrow 0$$

i.e., not only the error $E_{x_0}(h)$ tends to 0 as h tends to 0, but it does so rapidly that it still tends to 0 when divided by h !!!

(The error $E_{x_0}(h)$ is of smaller order than h (when h is small))

✳ The "total derivative" of a function from \mathbb{R}^n to \mathbb{R}^m will now be defined in such a way that it preserves these two properties.

But before that, let us re-write the definition of differentiability of a function at a point, in the form which will be generalized to functions from \mathbb{R}^n to \mathbb{R}^m .

[Definition 3 (Yet another reformulation)]

Let $D \subseteq \mathbb{R}$, $x_0 \in \text{int}(D)$ and $f: D \rightarrow \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in \text{int}(D)$



there exists a linear transformation

$$T_{x_0}: \mathbb{R} \rightarrow \mathbb{R} \quad \text{such that}$$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - T_{x_0}(h)}{h} = 0$$

When f is differentiable at x_0 , we set

$$f'(x_0) = T_{x_0} \quad \text{and call it}$$

"the derivative of f at x_0 "

Note, let us generalize this version of the definition to the case of a real-valued function in several variables.

Differentiability of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$,

at a point $\vec{x}_0 \in \mathbb{R}^n$ (The Total Derivative)

defn: Let $D \subseteq \mathbb{R}^2$, $\vec{x}_0 \in \text{int}(D)$,
let $f: D \rightarrow \mathbb{R}$. We say that f is
differentiable at $\vec{x}_0 \in \text{int}(D) \subseteq \mathbb{R}^2$, if
 f has a total derivative at $\vec{x}_0 \in \mathbb{R}^2$

there exists a linear transformation

$T_{\vec{x}_0}: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - T_{\vec{x}_0}(\vec{h})}{\|\vec{h}\|} = 0$$

$\vec{h} = (h_1, h_2) \rightarrow (0, 0)$

When the total derivative exists,
 we say $f'(\vec{x}_0) = \vec{T}_{\vec{x}_0}$ and
 call it "the total derivative of f at \vec{x}_0 "

(End of the definition)

Remarks:

$$\textcircled{1} \quad \lim_{\vec{h} \rightarrow \vec{0}} \frac{E_{\vec{x}_0}(\vec{h})}{\|\vec{h}\|_2} = 0 \quad (E_{\vec{x}_0}: \mathbb{R}^q \rightarrow \mathbb{R})$$

2: Reformulation of the above defn.

Let $D \subseteq \mathbb{R}^n$, der $\vec{x}_0 \in \text{int}(D)$, der
 $f: D \rightarrow \mathbb{R}$. Then f is differentiable
 at \vec{x}_0 if
 $\forall \varepsilon > 0 \ (\exists \delta > 0$

$$(0 < \|h\| < \delta \Rightarrow \left| \frac{E_{x_0}(h)}{\|h\|} \right| < \varepsilon).$$

3. Reformulation := (Yet again)

Let $D \subseteq \mathbb{R}^n$, der $\vec{x}_0 \in \text{int}(D)$, der
 $f: D \rightarrow \mathbb{R}$. Then f is differentiable
 at \vec{x}_0 if $\forall \varepsilon > 0 \ (\exists \delta > 0 \ ($

$$\|\vec{h}\| < \delta \Rightarrow |E_{x_0}(\vec{h})| < \|\vec{h}\| \varepsilon$$

• □

Jhm 1: If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $\vec{x}_0 \in \mathbb{R}^n$, then \exists a unique linear transformation $T_{x_0}: \mathbb{R}^n \rightarrow \mathbb{R}$ s.t.

$$\lim_{\substack{\rightarrow \\ \vec{h} \rightarrow 0}} \frac{|E_{x_0}(\vec{h})|}{\|\vec{h}\|_2} = 0$$

Jhm 2: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be
differentiable at $\vec{x}_0 \in \mathbb{R}^2$ with
the total derivative

$T_{\vec{x}_0}: \mathbb{R}^2 \rightarrow \mathbb{R}$, then .

① both the partial derivatives
of f at \vec{x}_0 exist,

and we have

$$\textcircled{1} \quad T_{\vec{x}_0} = \left[\frac{\partial f(\vec{x}_0)}{\partial x}, \frac{\partial f(\vec{x}_0)}{\partial y} \right].$$

*analogous
to $f'(\vec{x}_0)$*

So that,

$$T_{x_0}(h) = \left[\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0) \right] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}.$$

analogous
to $f'(\vec{x}_0)h$

End of Theorem 2.



Theorem 3: (Differentiability \Rightarrow continuity)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$ and $f: D \rightarrow \mathbb{R}$,
be a function.

If f is differentiable at \vec{x}_0 , then f is continuous at \vec{x}_0 .

~~Proof:-~~

(Beyond the scope of this course, but I would be more than happy to walk you through it if you are really interested!)

Generalization

Recall the definition of a vector-valued function of several variables (or, a vector field).

Let $D \subseteq \mathbb{R}^n$ and let $\vec{f}: D \rightarrow \mathbb{R}^m$ be a (vector-valued) function (of n variables).

The function \vec{f} (in fact, any such function) has the form

$$\vec{f}(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$$

where we call the functions

$f_i : D \rightarrow \mathbb{R}$, the "component functions".

Differentiability (The Total Derivative)

defn: Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \text{int}(D)$ and let $\vec{f}: D \rightarrow \mathbb{R}^m$ be a (vector-valued) function (of n variables).

We say that \vec{f} is differentiable at \vec{x}_0
(or, \vec{f} has a total derivative at \vec{x}_0) if

there exists a linear transformation

$$T_{\vec{x}_0} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ such that}$$

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{\|\vec{f}(\vec{x}_0 + \vec{h}) - \vec{f}(\vec{x}_0) - T_{\vec{x}_0}(\vec{h})\|}{\|\vec{h}\|} = 0.$$

When the total derivative exists, we get

$$\vec{f}'(\vec{x}_0) = T_{\vec{x}_0}, \text{ and call it}$$

"the total derivative of \vec{f} at \vec{x}_0 "

The Total derivative expressed
in terms of partial derivatives

Defn: Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \text{Int}(D)$, let
 $\vec{f}: D \rightarrow \mathbb{R}^m$ be a (vector-valued) function
 (of n independent variables x_1, x_2, \dots, x_n)
 and let $f_i: D \rightarrow \mathbb{R}$, $1 \leq i \leq m$, denote the
 component functions of \vec{f} .

If \vec{f} is differentiable at \vec{x}_0 , then for
 all $i \in \{1, \dots, m\}$ and for all $j \in \{1, \dots, n\}$

$\frac{\partial f_i}{\partial x_j}(\vec{x}_0)$ exists and

$$\vec{f}'(\vec{x}_0) = T_{\vec{x}_0} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right].$$

Proof: Trivial.

A clever use of matrix multiplication)))

Lecture 07

A word on notation:

① We use \vec{x} to denote an element of \mathbb{R}^n where $n \geq 2$;

The "arrow" is used to indicate that the element is an n-tuple!

② \vec{f} is used when the range of the function is \mathbb{R}^n with $n \geq 2$;
when $n=1$, we simply use f .

③ x, \vec{x}, f, \vec{f}

The Jacobian matrix

Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \text{Int}(D)$, let $\vec{f}: D \rightarrow \mathbb{R}^m$ be a (vector-valued) function (of n independent variables x_1, x_2, \dots, x_n) and let $f_i: D \rightarrow \mathbb{R}$, $1 \leq i \leq m$, denote the component functions of \vec{f} .

Now, suppose that \vec{f} is differentiable at \vec{x}_0 . Let $T_{\vec{x}_0} = \vec{f}'(\vec{x}_0)$ be the total derivative of \vec{f} at \vec{x}_0 . Of course, $T_{\vec{x}_0}$ is a linear transformation from \mathbb{R}^n to \mathbb{R}^m .

definition:

The $m \times n$ matrix of the linear transformation $T_{\vec{x}_0}$, with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m , is called the Jacobian matrix of the function \vec{f} at the point \vec{x}_0 .

Example 1: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x,y) = 8\sin x -$$

Compute $f'((2,3))$

Ans: $f'((2,3)) = [\cos 2, 0]$

Example 2: Let $\vec{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$\vec{f}(x,y) = (\sin x \cos y, \sin x, \sin y, \cos x \cos y).$$

Determine the Jacobian matrix of \vec{f} at an arbitrary point $(x,y) \in \mathbb{R}^2$.

Solution: (HW) $\vec{f}'(x,y) = [\quad]_{3 \times 2}$

$$f_1(x,y) = \sin x \cos y$$

$$f_2(x,y) = \sin x \sin y$$

$$f_3(x,y) = \cos x \cos y$$

Thus,

$$\vec{f}'(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix}$$

■

Example 3: (when range is 1-dimensional)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$, let $f: D \rightarrow \mathbb{R}$ be a function that is differentiable

at \vec{x}_0 . Then $f'(\vec{x}_0)$ is a $1 \times n$ matrix given by

$$\left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \frac{\partial f}{\partial x_2}(\vec{x}_0), \dots, \frac{\partial f}{\partial x_n}(\vec{x}_0) \right].$$

In this case, the Jacobian matrix of f at \vec{x}_0 is a row-matrix, which is also called "the gradient of f at \vec{x}_0 ", and is denoted by $\nabla f|_{\vec{x}_0}$ or $\nabla f(\vec{x}_0)$.

Remarks:

① The symbol ∇ is read as "del".

② $\nabla f(\vec{x}_0) = \left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \dots, \frac{\partial f}{\partial x_n}(\vec{x}_0) \right]$

is a linear transformation from \mathbb{R}^n to \mathbb{R} ,
and is called "the gradient of f at \vec{x}_0 ".

③ $\nabla f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right]$ is called
"the gradient of f ", or simply, "the del f ".

Note that, the gradient of f — ∇f , is
NOT a linear transformation from \mathbb{R}^n to \mathbb{R} .

④ Instead, $\nabla f: E \subseteq \mathbb{R}^n \rightarrow M_{1 \times n}(\mathbb{R})$

where $E \subseteq D$ is the set of all points in D
where the partial derivatives

$\frac{\partial f}{\partial x_j}$ exists for every $j \in \{1, \dots, n\}$.

⑤ Consider the function $f(x,y) = x^2y$.

Then,

$$\text{(i)} \quad \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \end{bmatrix}$$

(This is a function from \mathbb{R}^2 to $M_{1,2}(\mathbb{R})$)

$$\text{(ii)} \quad \nabla f \Big|_{(1,2)} = \begin{bmatrix} \frac{\partial f}{\partial x}(1,2) & \frac{\partial f}{\partial y}(1,2) \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix}$$

(This is a linear transformation from \mathbb{R}^2 to \mathbb{R})

$$\text{(iii)} \quad \underline{\text{the gradient of } f} = \underline{\begin{bmatrix} 2xy & x^2 \end{bmatrix}};$$

$$\underline{\text{the gradient of } f \text{ at } (1,2)} = \underline{\begin{bmatrix} 4 & 4 \end{bmatrix}}.$$



Example 4: (When the domain is 1-dimensional)

Let $D \subseteq \mathbb{R}$,

$x_0 \in \text{int}(D)$,

$\vec{f}: D \rightarrow \mathbb{R}^m$ be a function

that is differentiable at x_0 .

Then $\vec{f}'(x_0)$ is the $m \times 1$ matrix given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) \end{bmatrix} = \left[\frac{\partial f_i}{\partial x_1}(x_0) \right].$$

In this case,

- (a) We omit the ∂ notation and
(b) We use the ordinary derivative sign (Why?)

and the Jacobian matrix of \vec{f} at x_0 is
a column matrix, which in Calculus,
is denoted by $\frac{d\vec{f}}{dx}(x_0)$.

That is,

$$\vec{f}'(x_0) = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \vdots \\ \frac{df_m}{dx}(x_0) \end{bmatrix} = \frac{d\vec{f}}{dx}(x_0).$$

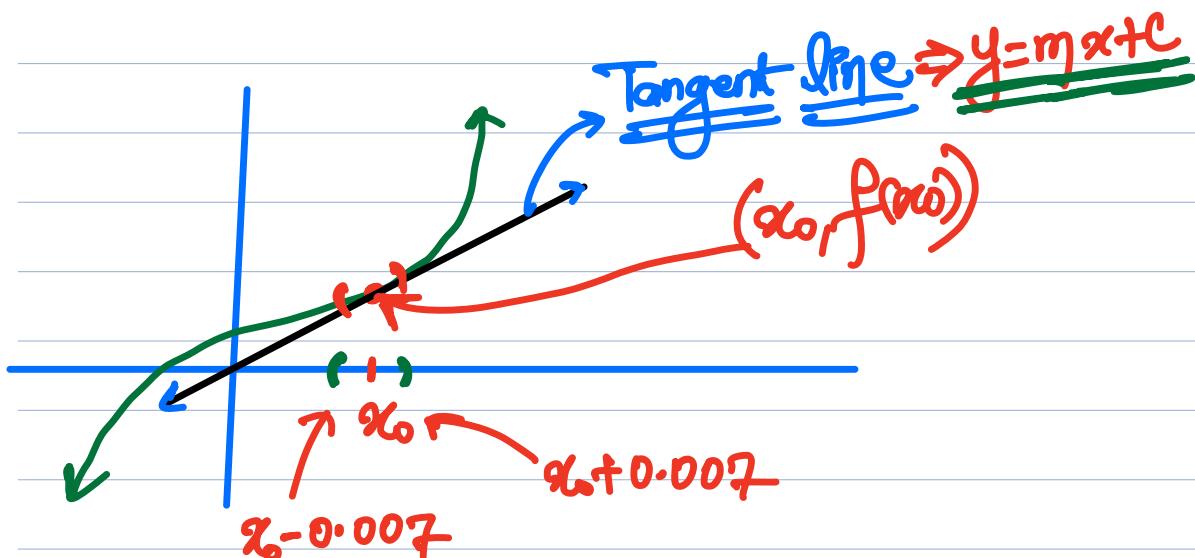
Standard linear approximation

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

\Downarrow
 x_0

f is differentiable at x_0 .

$$\text{Let } f(x) = 99x^{87} + 6x^{85} + 90x^{23} + 2023.$$



$$f(x_0 + 0.003) = ?$$

$$L(x) = mx + c$$

↑
standard linear approximation
of $f(x)$ by $L(x)$

In Calculus, linear function refers to a function whose graph is line.

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)(x - x_0) = 0$$

$$f(x) \approx f(x_0) + f'(x_0)[x - x_0]$$

$L(x)$
[a polynomial of degree 1]

[Linearization in multivariable setting]

Let $D \subseteq \mathbb{R}^n$ and $\vec{x}_0 \in \text{int}(D)$. Let $f: D \rightarrow \mathbb{R}$ be a function. Then

f is differentiable at \vec{x}_0

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - f'(\vec{x}_0)\vec{h}}{\|\vec{h}\|} = 0$$

or, equivalently, by replacing $\vec{x}_0 + \vec{h}$ by \vec{x} ,

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0) - f'(\vec{x}_0)(\vec{x} - \vec{x}_0)}{\|\vec{x} - \vec{x}_0\|} = 0$$



$$f(\vec{x}) \approx f(\vec{x}_0) - f'(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

If \vec{x} is "very" close to \vec{x}_0

$$\boxed{L(\vec{x}) := f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0)}$$

Linearization of f at \vec{x}_0

Standard linear approximation of f at \vec{x}_0

Remark:

(1) $f'(\vec{x}_0)$ is $1 \times n$ matrix given by

$$\left[\frac{\partial f(\vec{x}_0)}{\partial x_1} \quad \frac{\partial f(\vec{x}_0)}{\partial x_2} \quad \dots \quad \frac{\partial f(\vec{x}_0)}{\partial x_n} \right]$$

(2) $\vec{x} - \vec{x}_0$ is an $n \times 1$ column matrix,
given by

$$\begin{bmatrix} x_1 - x_{0,1} \\ x_2 - x_{0,2} \\ \vdots \\ x_n - x_{0,n} \end{bmatrix}$$

(3) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\rightarrow f(x, y)$$

Let $\vec{x}_0 = (a, b) \in \mathbb{R}^2$ and $\vec{x} = (x, y)$

Now,

$$\begin{aligned} & f(\vec{x}) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0) \\ &= f(a, b) + [f_x(a, b) \quad f_y(a, b)] \begin{bmatrix} x-a \\ x-b \end{bmatrix} \\ &= f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b). \end{aligned}$$

$$\begin{aligned} \therefore L(x, y) &= f(a, b) \\ &+ f_x(a, b)(x-a) \\ &+ f_y(a, b)(y-b) \end{aligned}$$

~~The usual
version
found in Calculus
textbooks~~

(4) Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$

Let $\vec{a}_0 = (a, b, c) \in \mathbb{R}^3$ and $\vec{x} = (x, y, z)$

Then show that

$$L(x, y, z) = f(a, b, c) + f_{ax}(a, b, c)(x-a) + f_{ay}(a, b, c)(y-b) + f_{az}(a, b, c)(z-c)$$

Example: Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point $(3,2)$.

$$\text{Solln: } L(x,y) = f(3,2) + \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix} \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$$

To be done later...

(Qn:-)

Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point (3,2).

Find an upper bound for the error incurred in replacing f by L on the rectangle R : $|x-3| \leq 0.1$

$$|y-2| \leq 0.1$$



$$\left| f(x,y) - L(x,y) \right|$$

Error

~~Later~~

(Once 2nd derivative test,
extreme points, etc.
are done!)

Qn.: find the linearization of

$$f(x,y,z) = x^2 - xy + 3xyz \text{ at}$$

the point $(x_0, y_0, z_0) = (2, 1, 0)$.

Find an upper bound for the error incurred
in replacing f by L on the rectangle

$$R: |x-2| \leq 0.01, |y-1| \leq 0.02, |z| \leq 0.01.$$

Lecture 08

Recapitulation of a few old concepts

(1) Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$, $\vec{f}: D \rightarrow \mathbb{R}^m$ be a function, $f_i: D \rightarrow \mathbb{R}$, $1 \leq i \leq m$, be the components of the function \vec{f} .

Suppose \vec{f} is differentiable at \vec{x}_0 .

Then $T_{\vec{x}_0} = \vec{f}'(\vec{x}_0) = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right]_{m \times n}$.

Now, let's observe an arbitrary column of the above matrix, say the j -th column.

$$\begin{array}{c|cccc}
 & \frac{\partial f_1}{\partial x_j}(\vec{x}_0) & & & \\
 & \frac{\partial f_2}{\partial x_j}(\vec{x}_0) & & & \\
 * & & & & \\
 & \frac{\partial f_3}{\partial x_j}(\vec{x}_0) & * & * & * \\
 & \vdots & & & \\
 & \frac{\partial f_m}{\partial x_j}(\vec{x}_0) & & & \\
 \hline
 \frac{\partial f}{\partial x_j}(\vec{x}_0) & & \xrightarrow{\text{fth column } \in \mathbb{R}^m} & &
 \end{array}$$

Let us agree to use
 this notation; it
 does make sense!

Q2: What about the others?

The j -th row = the total derivative of f_i at \vec{x}_0 .
 Needless to say, $f'_i(\vec{x}_0)$ is a $1 \times n$ matrix!

(3) Recall: [composition of functions g and f]

Let g and f be two functions such that $\text{ran}(f) \subseteq \text{dom}(g)$, then we can define a new function $g \circ f: \text{dom}(f) \rightarrow \text{ran}(g)$, and we call it " g composed with f "

(4) Suppose $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $S: \mathbb{R}^m \rightarrow \mathbb{R}^p$ are linear transformations. Then,

$$S \circ T: \mathbb{R}^n \rightarrow \mathbb{R}^p$$

is also a linear transformation.

If the matrix associated with T is $B_{m \times n}$ and the matrix associated with S is $A_{p \times m}$, then the matrix associated with $S \circ T$ is AB .

—X— End of speculation —X—

The Chain Rule [Section 14.4 / Thomas & Finney] 11th Edition

Single-Variable case

Suppose that

• $f: S \subseteq \mathbb{R} \rightarrow \mathbb{R}$, is a function defined on some nonempty subset S of \mathbb{R} ;

• $\text{ran}(f) \subseteq T \subseteq \mathbb{R}$;

• $g: T \subseteq \mathbb{R} \rightarrow \mathbb{R}$, is a function.

Also assume that

• $x_0 \in \text{int}(S)$ and

• $f(x_0) \in \text{int}(T)$.

If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composition of g and f , gof , is differentiable at x_0 and $(gof)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

Alternatively, $w = f(x)$, $x = g(t)$

$$\Rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt}$$

Multivariable setting

$$w = f(x, y)$$

$$\left. \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right\}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$w = f(x, y, z)$$

$$\left. \begin{array}{l} x = x(t) \\ y = y(t) \\ z = z(t) \end{array} \right\}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



Alternatively,

$$\frac{dw}{dt} = \left[\frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \quad \frac{\partial w}{\partial z} \right] \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$

What does this refer to?

Example: Use the chain rule to find the

derivative of $w = xy + z$
w.r.t. "t" along the path

of a Helix $x = \cos t$

$$y = \sin t$$

$$z = t.$$

What is the derivative's value at
 $t=0$ & $t=\pi/2$.

Qn:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{d\omega}{dt} = y \cdot (-g \sin t) + x \cdot (g \cos t) + 1 \cdot (1)$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\frac{du}{dt} \Big|_{t=0} = 2$$

A thick, dark blue brushstroke is drawn across the bottom of the page, starting from the left edge and ending near the right edge. The stroke is irregular and expressive, with varying thicknesses and some small loops.

$$\frac{d\omega}{de} \Big|_{t=\pi/2} = ?.$$

Cafe when

$$\omega = f(x, y, z)$$

$$x = g(\sigma \beta)$$

$$y = \text{h}(\sigma_8)$$

$$Z = R(\theta, \delta).$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

In the matrix notation,

$$\frac{\partial w}{\partial s} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{bmatrix}$$

What does this refer to?

Example: $w = x + 2y + z^2$ and

$$x = s/8$$

$$y = s^2 + \ln s$$

$$z = 2s$$

Express $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ in terms of s and t .

(HW)

Chain rule revisited

$$\omega = f(x, y) \text{ and } x = x(t), y = y(t)$$

$$\text{then, } \left. \frac{d\omega}{dt} \right|_{t=t_0} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

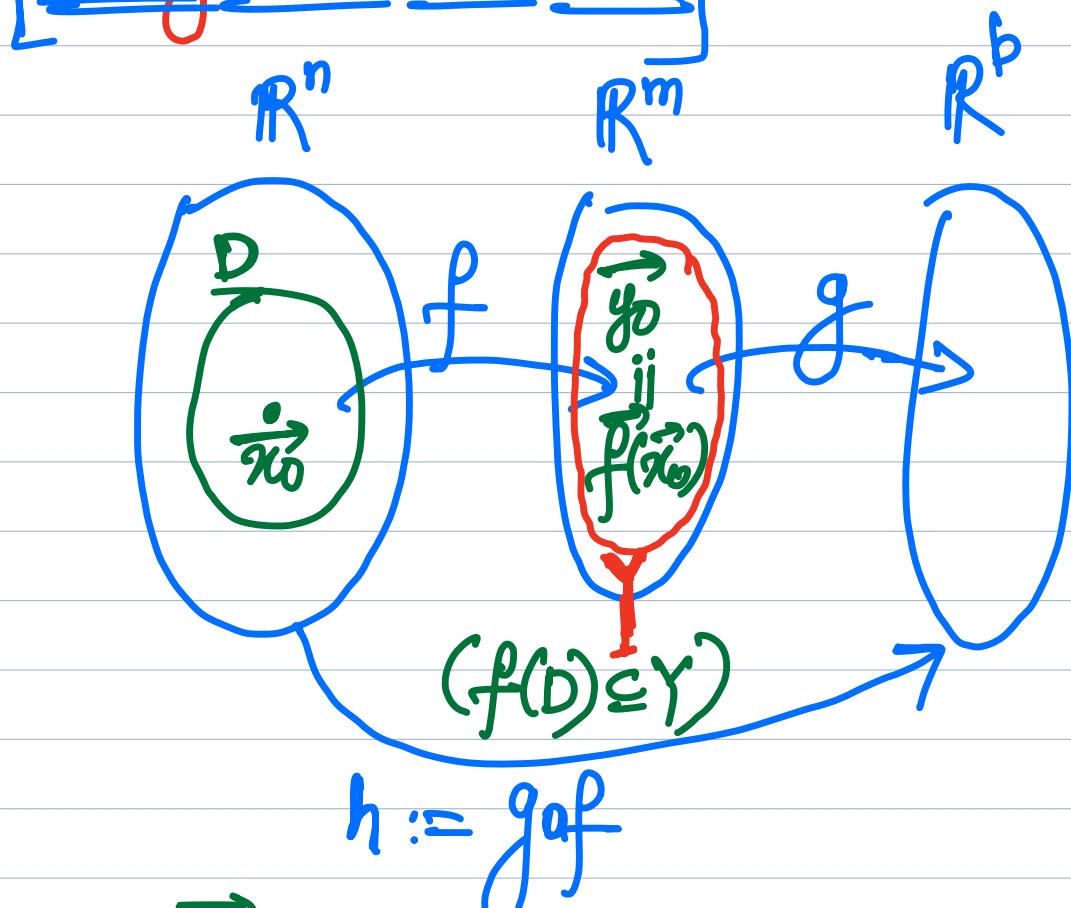
$$= \left[\frac{\partial f}{\partial x} \right]_{x=?} \left. \frac{\partial f}{\partial y} \right|_{y=?} \cdot \begin{bmatrix} \left. \frac{dx}{dt} \right|_{t=t_0} \\ \left. \frac{dy}{dt} \right|_{t=t_0} \end{bmatrix}$$

$$= \underbrace{f'(x(t_0), y(t_0))}_{\vec{x}(t_0)} \cdot \vec{x}'(t_0)$$

matrix multiplication

i.e., $f \circ \vec{x}: \underline{\mathbb{R}} \xrightarrow{\vec{x}} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$
 $t \mapsto (\vec{x}(t), y(t)) \mapsto f(\vec{x}(t), y(t)).$

The general chain rule



If \vec{f} is differentiable at \vec{x}_0 , and

\vec{g} is diffble at $\vec{y}_0 := \vec{f}(\vec{x}_0)$,

then \circ $\vec{g} \circ \vec{f}$ is diffble at \vec{x}_0

$$(\vec{g} \circ \vec{f})'(\vec{x}_0) = \vec{g}'(\vec{f}(\vec{x}_0)) \cdot \vec{f}'(\vec{x}_0)$$

$\uparrow p \times m$ $\uparrow m \times n$

matrix
multiplication.

If we use $B_{m \times n}$ to denote

the matrix $\vec{f}'(\vec{x}_0)$,

& use $A_{p \times m}$ to denote

the matrix $\vec{g}'(\vec{y}_0)$

then $(\vec{g} \circ \vec{f})(\vec{x}_0)$ is given by AB .

Let's write down the theorem in a rigorous manner.

The chain rule:

Suppose that

$\vec{f}: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function defined on some nonempty subset S of \mathbb{R}^n ;

① $\text{ran}(\vec{f}) \subseteq T \subseteq \mathbb{R}^m$;

② $\vec{g}: T \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a function.

Also assume that

③ $\vec{x}_0 \in \text{int}(S)$ and

④ $\vec{f}(\vec{x}_0) \in \text{int}(T)$.

If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composition of g and f , gof , is differentiable at x_0 and $(gof)'(\vec{x}_0) = \vec{g}'(\vec{f}(\vec{x}_0)) \cdot \vec{f}'(\vec{x}_0)$.

~~Remark:~~

① $(\vec{g} \circ \vec{f})'(\vec{x}_0)$ = the total derivative
of $\vec{g} \circ \vec{f}$ at the point \vec{x}_0
(A p -by- q matrix)

② $\vec{g}'(\vec{f}(\vec{x}_0))$ = the total derivative
of \vec{g}' at the point $\vec{f}(\vec{x}_0)$
(A p -by- m matrix)

③ $\vec{f}'(\vec{x}_0)$ = the total derivative of \vec{f}
at the point \vec{x}_0
(An m -by- q matrix)

④ $\vec{g}'(\vec{f}(\vec{x}_0)) \cdot \vec{f}'(\vec{x}_0)$.

$\underbrace{\quad}_{p \times m} \quad \underbrace{\quad}_{m \times n}$
(matrix multiplication)

Question: Let $W = f(x, y, z, \dots, v)$

and $\left\{ \begin{array}{l} x = x(p, q, r, \dots, t) \\ y = y(p, q, r, \dots, t) \\ z = z(p, q, r, \dots, t) \\ \vdots \\ v = v(p, q, r, \dots, t) \end{array} \right.$

What is $\frac{\partial W}{\partial r}$?

Ans: $\frac{\partial W}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial r} + \dots + \frac{\partial W}{\partial v} \cdot \frac{\partial v}{\partial r}$

(Qn: Explain in terms of matrices)

IHWI

(Implicit differentiation)

c) Find $\frac{dy}{dx}$ of $y^2 - x^2 - 8\sin xy = 0$.

Soln:

$$2y \cdot \frac{dy}{dx} - 2x - (\cos(xy)) \cdot (1y + x \cdot \frac{dy}{dx}) = 0$$

$$\frac{dy}{dx} (2y - x \cos(xy)) = 2x + y \cos(xy).$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y \cos(xy)}{2y - x \cos(xy)}.$$

If $F(xy) = y^2 - x^2 - 8\sin xy$.

Compute $\frac{dy}{dx}$.

(Jhm 8
Pg-1002)

Ans:-
$$\boxed{\frac{dy}{dx} = - \frac{F_x}{F_y} \quad (F_y \neq 0)}$$

Lecture - 09

Consider a function $f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and let $(x_0, y_0) \in \text{Int}(D)$.

Suppose f has a partial derivative w.r.t. the first variable (i.e., w.r.t. x) at the point (x_0, y_0) .

Then,

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{f((x_0, y_0) + h(\overrightarrow{1, 0})) - f(x_0, y_0)}{h}$$

$\overrightarrow{u} = (u_1, u_2)$

\equiv

Similarly, suppose that f has a partial derivative with respect to y at the point (x_0, y_0) .

Then, $\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{f((x_0, y_0) + h(0, 1)) - f(x_0, y_0)}{h}$$

=

$\rightarrow \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} =$ the directional derivative of \vec{f}
 in the direction $\vec{u} = (1, 0)$
 at the point (x_0, y_0) .

$\rightarrow \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} =$ the directional derivative of \vec{f}
 in the direction $\vec{v} = (0, 1)$
 at the point (x_0, y_0) .

///

Directional derivatives

defn: Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$,

$\vec{f}: D \rightarrow \mathbb{R}^m$ be a function,

($f_i: D \rightarrow \mathbb{R}$, $1 \leq i \leq m$, be the component functions of \vec{f}),

and let $\vec{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$.

We say that \vec{f} has a directional

derivative in the direction \vec{u} at the

point \vec{x}_0 if

the following vector-valued limit exists,

$$\lim_{h \rightarrow 0} \frac{\vec{f}(\vec{x}_0 + h\vec{u}) - \vec{f}(\vec{x}_0)}{h}.$$

When this limit exists, we denote it

by $\vec{f}'_{\vec{u}}(\vec{x}_0)$,

(and call it "the directional derivative
of \vec{f} in the direction \vec{u} at the
point \vec{x}_0 ").

Remarks:

① We don't assume $\|\vec{u}\|_2 = 1$.

(Thomas & Finney's book does! Be cautious.)

② When $\vec{u} = \vec{0} \in \mathbb{R}^n$, then

$\vec{f}'_{\vec{u}}(\vec{x}_0) = \vec{0} \in \mathbb{R}^m$. This is
not an interesting case.

So, we will be interested in the case when $\vec{U} \neq \vec{0}$.

③ Qn: Let $\vec{0} \neq \vec{U} \in \mathbb{R}^n$ and let $\hat{\vec{u}} = \frac{\vec{U}}{\|\vec{U}\|_2}$:

Then find the relation between

$\vec{f}_{\vec{U}}'(\vec{x}_0)$ and $\vec{f}_{\hat{\vec{u}}}(\vec{x}_0)$.

④ If the above limit exists, then

$\vec{f}_{\vec{U}}'(\vec{x}_0) \in \mathbb{R}^m$ (Always express this as a column matrix).

⑤ The limit $\vec{f}_{\vec{U}}'(\vec{x}_0)$ exists

\updownarrow
if $\{1, \dots, m\}$, the open limit

$$\lim_{h \rightarrow 0} \frac{f_i(\vec{x}_0 + h\vec{u}) - f_i(\vec{x}_0)}{h} (= b_i)$$

exists. In this case

$$\vec{f}'_{\vec{u}}(\vec{x}_0) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

(6) Case: $m=1$ (that is, $f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$)

$$\text{Let } \vec{x}_0 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

$$\text{let } \vec{u} = \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad \text{jth entry}$$

Then, let's compute $\vec{f}'_{\vec{e}_j}(\vec{x}_0)$.

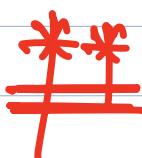
$$\vec{f}'_{\vec{e}_j}(\vec{x}_0) = \lim_{h \rightarrow 0} \frac{\vec{f}(\vec{x}_0 + h\vec{e}_j) - \vec{f}(\vec{x}_0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\vec{f}(a_1, \dots, a_{j-1}, a_j + h, a_{j+1}, \dots, a_n) - \vec{f}(a_1, \dots, a_n)}{h}$$

$$= \frac{\partial \vec{f}}{\partial x_j}(\vec{x}_0).$$

The partial derivative
of f w.r.t the
 j -th independent variable.

(7.)



This observation motivates us to keep using this notation even when $m > 1$!

That is, $\vec{f}: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, i.e.,

$$\vec{f}(\vec{x}_1, \dots, \vec{x}_n) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$$

Then,

$$\vec{f}'_{\vec{e}_j}(\vec{x}_0) = \frac{\partial \vec{f}}{\partial x_j}(\vec{x}_0)$$

[the j -th column of the Jacobian matrix of \vec{f} at \vec{x}_0 .]

(This is what we "Observed" at the beginning of today's lecture, and called it, "the partial derivative of \vec{f} at \vec{x}_0 in the j -th direction")

Let's recall once again:

① $\vec{f}'_{\vec{u}}(\vec{x}_0) = \lim_{h \rightarrow 0} \frac{\vec{f}(\vec{x}_0 + h\vec{u}) - \vec{f}(\vec{x}_0)}{h}$

the directional derivative of $\vec{f}: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$
in the direction $\vec{u} \in \mathbb{R}^n$ at the point $\vec{x}_0 \in \text{int}(D)$.

② Example 1: Find the dir. derivative of
 $f(x,y) = x^2 + xy$ at the point
(1,2) in the direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Soln: $\lim_{h \rightarrow 0} \frac{\vec{f}((1,2) + h(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})) - \vec{f}(1,2)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\vec{f}\left(1 + \frac{h}{\sqrt{2}}, 2 + \frac{h}{\sqrt{2}}\right) - \vec{f}(1,2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(1+h/\sqrt{2})^2 + (1+h/\sqrt{2})(2+h/\sqrt{2})] - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h^2/2 + \sqrt{2}h) + (2+h/\sqrt{2} + \frac{2h}{\sqrt{2}} + \frac{h^2}{2}) - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 2\sqrt{2}h + h/\sqrt{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2(h + 2\sqrt{2} + 1/\sqrt{2})}{h^2} = \underline{5/\sqrt{2}} \text{ ans.}$$

Example 2: $f(x,y,z) = x^2 - y + z^2$

$$\vec{x}_0 \equiv (1, 2, 1); \vec{u} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

Soln: $\vec{f}'_{\vec{u}}(\vec{x}_0) = 18$

Remark: Let $\hat{u} := \frac{\vec{u}}{\|\vec{u}\|}$.

Compute $f'_{\hat{u}}(\vec{x}_0)$. (Ans: = 3).

Example 3:

Consider a function $\vec{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$.
Observe that

(a) Partial derivatives are special cases of directional derivatives.

(b) If $\vec{f}'_{\hat{u}}(\vec{x}_0)$ exists for every \hat{u} ,
then all partial derivatives exist
(i.e., $\forall j \in \{1, \dots, n\} \frac{\partial \vec{f}}{\partial x_j}(\vec{x}_0)$ exists)

(c) The converse of (b) is not always true!
Consider the following function:

$$f(x,y) = \begin{cases} x+y, & \text{if } x=0 \text{ or } y=0 \\ 1, & \text{otherwise} \end{cases}$$

The word "or" is inclusive in Mathematics.

Then Show that

① $\frac{\partial f}{\partial x}(0,0) = 1$

② $\frac{\partial f}{\partial y}(0,0) = 1$

③ If \vec{u} is a vector in \mathbb{R}^2 different from the above two vectors, then $f_{\vec{u}}'(0,0)$ does not exist.



Example 4: Consider the following function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Given $\vec{u} = (u_1, u_2) \neq (0,0)$, Show that

(a) $f'_{\vec{u}}((0,0)) = \begin{cases} \frac{u_2^2}{u_1} & \text{if } u_1 \neq 0 \\ 0 & \text{if } u_1 = 0 \end{cases}$ and

(b) f is NOT continuous at $(0,0)$.

Inference: Existence of the directional derivatives
at a point in every direction
 $\not\Rightarrow$ Continuity at that point.

Theorem: let $D \subseteq \mathbb{R}^n$, $x_0 \in \text{Int}(D)$,

let $\vec{f}: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function,

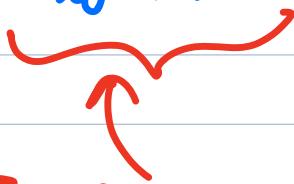
If \vec{f} is differentiable at \vec{x}_0 with

$$\vec{f}'(\vec{x}_0) = T_{\vec{x}_0},$$

then the directional derivatives of \vec{f} at the point \vec{x}_0 also exist in all directions,

and

$$\vec{f}'_{\vec{u}}(\vec{x}_0) = T_{\vec{x}_0}(\vec{u}).$$



This is matrix multiplication!

$$\left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right]_{m \times n} \left[\vec{u} \right]_{n \times 1}$$

Proof:- Not needed!

Verification:- (Not a proof)

$$\text{※ } \vec{f}_{\vec{e}_j}(\vec{x}_0) = j^{\text{th}} \text{ column of } T_{\vec{x}_0}$$

$$= j^{\text{th}} \text{ col. of } \left[\begin{array}{c} \vec{f}_1(\vec{x}_0) \\ \vdots \\ \vec{f}_m(\vec{x}_0) \end{array} \right]_{m \times n}$$

$$= \left[\begin{array}{c} T_{\vec{x}_0} \\ \vdots \\ T_{\vec{x}_0} \end{array} \right]_{m \times n} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right]_{n \times 1} \xrightarrow{\text{if entry}}$$
$$= T_{\vec{x}_0}(\vec{e}_j)$$

Lecture 10 (part-1)

Example 1: (reconsider Example 2)

$$f(x,y,z) = x^2 - y + z^2$$

$$\vec{x}_0 = (1, 2, 1)$$

$$\vec{u} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

Soln:

$$\vec{f}'_{\vec{u}}(\vec{x}_0) = [T_{\vec{x}_0}] [\vec{u}]$$

$$= \begin{bmatrix} 2x & -1 & 2z \\ (1, 2, 1) & (1, 2, 1) & (1, 2, 1) \end{bmatrix}_{1 \times 3} \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} = 8 + 2 + 8 = 18.$$

Remark:

$$[2 \ -1 \ 2] \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} \leftarrow \begin{array}{l} \text{This is, of course,} \\ \text{matrix multiplication} \end{array}$$

But if $m=1$ (i.e., $f: \mathbb{R}^n \rightarrow \mathbb{R}$)

then this matrix multiplication can also be viewed as the dot product of the two vectors $(2, -1, 2)$ and $(4, -2, 4)$

Do you remember gradient? (Lecture 07)

In this light, the total derivative of f at \vec{x}_0 .

$$[2 \ -1 \ 2] \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} = \vec{\nabla} f|_{\vec{x}_0} \cdot \vec{u}$$

Matrix multiplication
of ∇f and \vec{u}

This is how
we studied
in the previous
theorem.

The dot product
of ∇f and \vec{u}

(This is how Thomas &
Finney's book treats
 $f'_{\vec{u}}(\vec{x}_0)$)

What's the significance of expressing the
directional derivative of (real-valued) function
(of several variables) as the dot product ???

$$f'_{\vec{u}}(\vec{x}_0) = \left\| \nabla f \Big|_{\vec{x}_0} \right\|_2 \cdot \left\| \vec{u} \right\|_2 \cos \theta$$

If \vec{u} happens to be a unit vector,
then

$$f'_{\vec{u}}(\vec{x}) = \|\nabla f|_{\vec{x}}\|_2 \cos \theta$$

What happens if $\theta=0, \theta=\pi, \theta=\pi/2$?

====

A few remarks/insights on this
technique of computing "directional derivatives"
of $f: \mathbb{R}^n \rightarrow \mathbb{R}$ via "gradient of \vec{f} " are in
order:

(Let us choose $\eta=2$)

Suppose a function $f(x,y)$ is given, and an interpol point (x_0, y_0) in the domain of the function f is also given. Now, if we choose different directions (i.e., unit vectors \vec{u} in \mathbb{R}^2), then, of course, the directional derivatives $\vec{f}'_{\vec{u}}((x_0, y_0))$ would be different.

What can be said about these different directional derivatives of f at (x_0, y_0) by analysing $\frac{\vec{u}}{\|\vec{u}\|}$?

① When $\theta = 0$:

Then (and only then) \vec{u} is parallel to ∇f ,
and thus $f'_{\hat{u}}((x_0, y_0))$ assumes the
maximum value when we choose
 \hat{u} in the direction of ∇f .

In this case,

$$f'_{\hat{u}}((x_0, y_0)) = \left\| \nabla f \right\|_{(x_0, y_0)}^2.$$

This also means that f increases (why?)
most rapidly in the direction of the
gradient vector ∇f (why?)
 (x_0, y_0)

② When $\theta = \pi$:

- Then \vec{u} is in the direction of $-\nabla f$.
→ The value of $f'_{\hat{u}}((x_0, y_0))$ is $-\|\nabla f\|$.

→ Thus, f decreases (why?) most
rapidly in the direction of $-\nabla f$.

③ When $\theta = \pi/2$:

That is, \vec{u} is orthogonal
(perpendicular) to the gradient ∇f .

In this case $f'_{\hat{u}}((x_0, y_0)) = 0$.

This simply means that the function f does not change in a direction that is perpendicular to the gradient.

Alternatively, any direction (\vec{u}) orthogonal to that of ∇f is a "direction of zero change".

Example 2: Find the direction in which the

function $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

- (a) increases most rapidly at the point $(1,1)$;
- (b) decreases " " " " " " " .
- (c) what are the directions of zero change in f at $(1,1)$?

Solution:

(a) $\nabla f|_{(1,1)} = [1, 1] = \vec{i} + \vec{j}$.

(b) $-\vec{i} - \vec{j}$

(c) $(a\vec{i} + b\vec{j}) \cdot (\vec{i} + \vec{j}) = 0 \Rightarrow a+b=0$
 $\Rightarrow b=-a$. That is, $\vec{i} - \vec{j}$ or $-\vec{i} + \vec{j}$.

Homework: Geometrical interpretation
of the directional derivative
of a function $f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$,
at a point $\vec{x} \in \text{int}(D)$ in
a specified direction $\vec{v} \in \mathbb{R}^n$.

Homework: Go through Section 12.5
entirely (of Thomas' Calculus)
(Lines and planes in space)

Lecture 10 (part-2)

Digression (prerequisites)

Things you already know

Source: Thomas' Calculus (11th Edition)
Chapter 12 (section 12.5 mostly)

Throughout this section, we shall be working in
the realm of \mathbb{R}^3 .

① dist. between points (x_1, x_2, x_3) and (y_1, y_2, y_3)

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}$$

② Midpoint $\equiv \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \frac{x_3 + y_3}{2} \right)$

③ Let $\vec{u} = (u_1, u_2, u_3)$ and
 $\vec{v} = (v_1, v_2, v_3)$.

Then $\vec{u} \cdot \vec{v} = \sum_{j=1}^3 u_j v_j$.

④ The projection of \vec{u} on \vec{v}

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \quad \text{or} \quad \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

⑤ $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

⑥ Area of the parallelogram determined by vectors \vec{u} and \vec{v} = magnitude of $\vec{u} \times \vec{v}$.



Lines in \mathbb{R}^3

A line in \mathbb{R}^3 can be uniquely determined in the following three ways:

(a) passing through two points

(b) intersection of two planes



(c) passing through a point and parallel to a specified vector

Given $\Theta P_0 = (x_0, y_0, z_0)$ and

$$\Theta \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k},$$

Let $P = (x, y, z)$ be an arbitrary point in \mathbb{R}^3 .

Then, P lies on the line passing through P_0 .

$$\boxed{\vec{P}P_0 = t \vec{v} \quad t \in \mathbb{R}}$$

08,

$$(x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$$

$$= t(\vec{v}_1\vec{i} + \vec{v}_2\vec{j} + \vec{v}_3\vec{k}),$$

that is,

$$\left. \begin{aligned} x &= x_0 + v_1 t \\ y &= y_0 + v_2 t \\ z &= z_0 + v_3 t \end{aligned} \right\} \begin{array}{l} \text{parametric} \\ \text{equation} \\ \text{of the} \\ \text{line } L \end{array}$$

P_0

the vector \vec{v}

which can be rewritten as

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

symmetric equation
of the line L.

Qn: What happens when one of the λ_i 's is 0? Can we still use the above equation?

Qn: A line L goes through points

$$P_0 = (3, -2, 1) \text{ and } P_1 = (5, 1, 0).$$

Find the parametric equations and the symmetric equations of L. Also find the points at which this line pierces the three coordinate planes.

Soln: Consider P_0 to be the point and $\vec{P_0P_1} = (5-3)\vec{i} + (1+2)\vec{j} + (0-1)\vec{k}$ to be the vector \vec{v} . Solve.

question: does your answer change if we consider $\vec{P_1P_0}$ as our vector instead of $\vec{P_0P_1}$?

Ques: Given

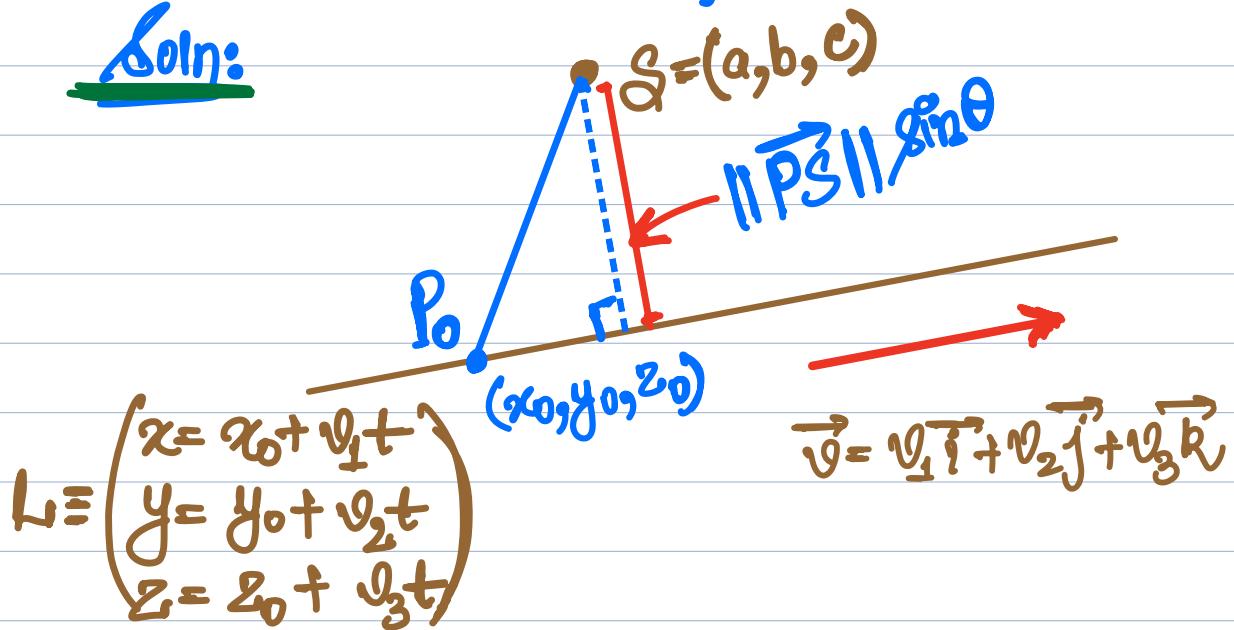
① a line L passing through a point

$P_0 = (x_0, y_0, z_0)$ and parallel to
the vector $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

and ② a point $S = (a, b, c)$ that is not
on the line L .

Calculate the distance from S to L .

Soln:



distance from S to $L = \|\vec{PS}\| \sin \theta$,

where θ is the angle between \vec{PS} and the line L .

However,

Since the line L is parallel to the vector \vec{v} ,

the angle between \vec{PS} and L
is same as
the angle between \vec{PS} and \vec{v} .

$$\therefore \text{distance from } S \text{ to } L = \|\vec{PS}\| \sin \theta$$

$$= \frac{\|\vec{PS}\| \|\vec{v}\| \sin \theta}{\|\vec{v}\|}$$

$$= \frac{\|\vec{PS} \times \vec{v}\|}{\|\vec{v}\|}$$

≡

8. Planes in \mathbb{R}^3

A plane can be uniquely determined in several ways

(a) passing through three noncollinear points

(b) passing through a line and a point not on
the line.

* (c) passing through a point and perpendicular
to a specified direction.

Given

① a point $P_0(x_0, y_0, z_0)$ and

② a vector $\vec{N} = \vec{ai} + \vec{bj} + \vec{ck}$

Let $P = (x, y, z)$ be an arbitrary point in \mathbb{R}^3 .

Then, P lies on the plane passing through P_0
and perpendicular to the vector \vec{N}



$$\vec{N} \cdot \vec{P_0 P} = 0$$

Which is equivalent to

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

or,

$$ax + by + cz = ax_0 + by_0 + cz_0$$

"d"

so that

$$ax + by + cz = d$$

A linear equation in x, y and z

of this form always
represents a plane with
normal vector $\vec{N} = \vec{a} + \vec{b} + \vec{c}$.

*Geometric equation
of the plane*

*Cartesian equation of the
plane through the point
 $P_0 = (x_0, y_0, z_0)$ with normal
vector $\vec{N} = \vec{a} + \vec{b} + \vec{c}$*

Qn: find an equation for the plane that passes through the three points

$$P_0 = (3, 2, -1), P_1 = (1, -1, 3) \text{ and}$$

$$P_2 = (3, -2, 4).$$

Soln: $\vec{N} = \vec{P_0P_1} \times \vec{P_0P_2}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -3 & 4 \\ 0 & -4 & 5 \end{vmatrix}$$
$$= \vec{i} + 10\vec{j} + 8\vec{k}.$$

Use P_0 as the point. Solve w.r.t. (H.W)

Qn: Find the point at which the line

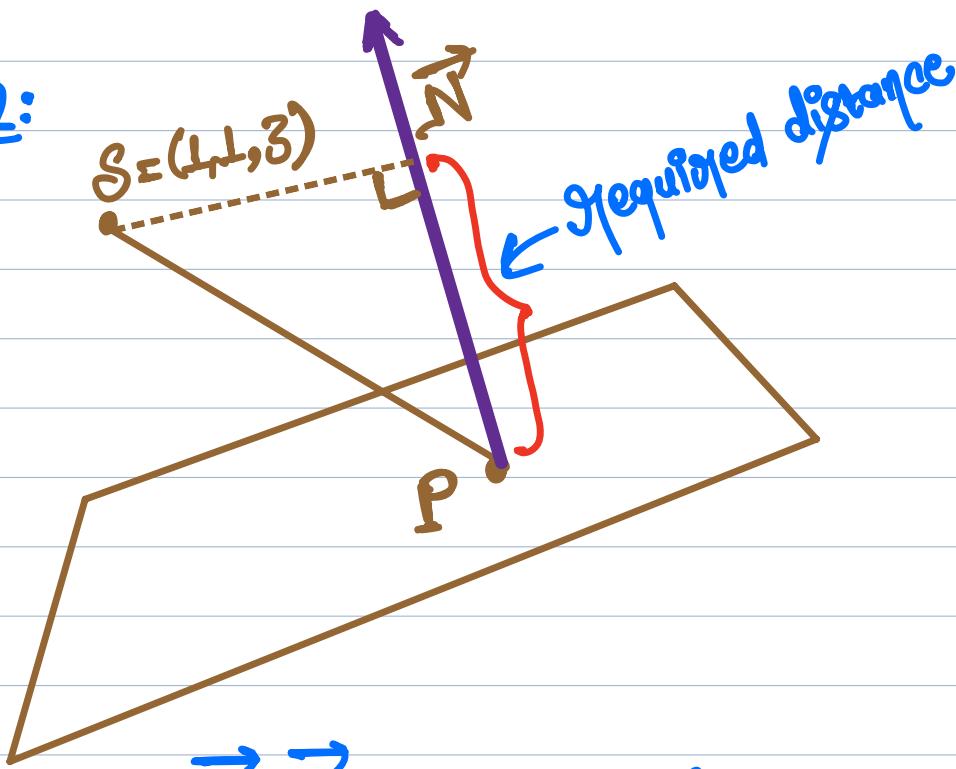
$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2} \text{ intersects the plane}$$

$$x + 2y + 2z = 22.$$

Soln: H.W. $(4, 1, 8)$ ans.

Qn: Find the distance from $S=(1,1,3)$ to the plane $3x+2y+6z=6$.

Soln:



$$\frac{\vec{PS} \cdot \vec{N}}{\|\vec{N}\|}$$

the length of the
projection of \vec{PS} on \vec{N}

How to find P?

$$(3x+2y+6z=6)$$

put $x=0$ and $z=0$. Then, $y=3$

$\therefore (0,3,0)$ is on the plane.

■

Qn: find the cosine of the angle between two planes

$$x + 4y - 4z = 9$$

$$x + 2y + 2z = -3$$

Also, find parametric equations for the line of intersection of these planes.

Soln: (HW)

Practice problems:

Book: Thomas' Calculus (11th Edition)

Chapter 12

Section 12.5

Exercises

Question 63 to Question 72

(End of the digression)

Let's consider the following question.

Problem: Find the tangent line to the ellipse

$$\frac{x^2}{4} + y^2 = 2 \text{ at the point } (-2, 1).$$

Approach I: ~~$\frac{\partial \vec{x}}{4} + \vec{y} \cdot \frac{dy}{dx} = 0$~~

$$\Rightarrow \frac{dy}{dx} = \left(-\frac{x}{4}\right) \left(\frac{1}{y}\right)$$

$$= \frac{-x}{4y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(-2,1)} = \frac{2}{4} = \frac{1}{2}$$

Eqn of the tangent line:

$$(y - y_1) = m(x - x_1)$$

$$\text{or, } y - 1 = \frac{1}{2}(x + 2)$$

$$\text{that is, } \boxed{2y - x - 4 = 0.}$$

Digression: What is the equation of the normal line at the same point?

$$\text{Slope} = \frac{-1}{\text{Slope of the tangent line}} = -2.$$

∴ Eqn of the normal line:

$$(y-1) = -2(x+2)$$
$$\Rightarrow \boxed{y + 2x + 3 = 0}$$

Approach II:

We develop an alternative way to understand the ellipse $\frac{x^2}{4} + y^2 = 1$.

① Consider the function $f(x, y) := \frac{x^2}{4} + y^2$.

② Verify that the point $(-2, 1) \in \text{dom}(f)$.

In fact $(-2, 1) \in \text{int}(\text{dom}(f))$.

- ① Also, verify that $f(-2, 1) = 2$.
- ② Recall the definition of a level curve.

If $c \in \text{ran}(f)$, then $f(x, y) = c$ is
a level curve

- ③ In this case,

$$\begin{cases} z = f(x, y) = \frac{x^2}{2} + y^2; \\ (x_0, y_0) = (-2, 1); \text{ and} \\ f(x, y) = f(x_0, y_0) \text{ is a level curve.} \end{cases}$$

The question now boils down to finding
the tangent to the level curve $f(x, y) = f(x_0, y_0)$
at the point (x_0, y_0) .

To solve this, we need the following result.

Thm : At every point (x_0, y_0) in the domain of a differentiable function $Z = f(x, y)$,

the gradient of f at (x_0, y_0)

[i.e., $(\nabla f)_{(x_0, y_0)}$]

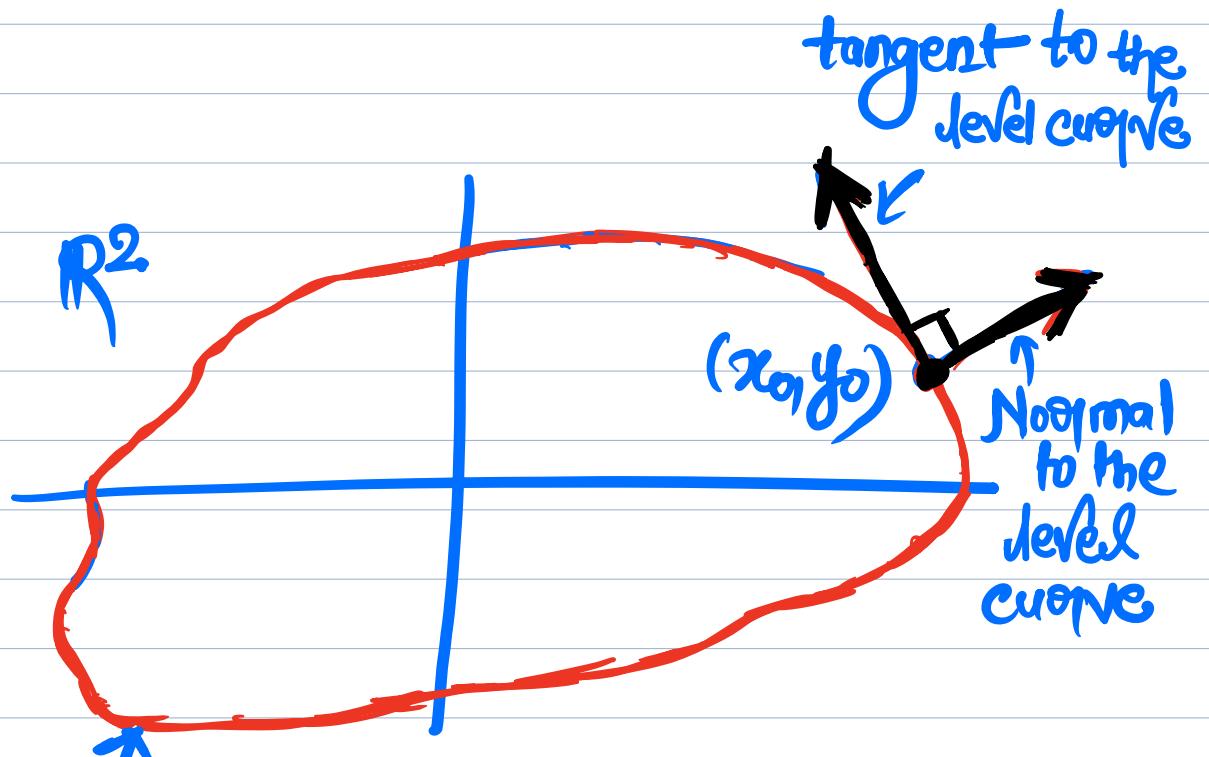
is normal to

the level curve through (x_0, y_0) .

[i.e., $f(x, y) = f(x_0, y_0)$]

proof: (Later)

Pictorially,



Level curve $f(x, y) = f(x_0, y_0),$

i.e. $\frac{x^2}{4} + y^2 = 2$

Now, let's get back to the problem.

Solution:

Step I: Compute $(\nabla f)_{(x_0, y_0)}$

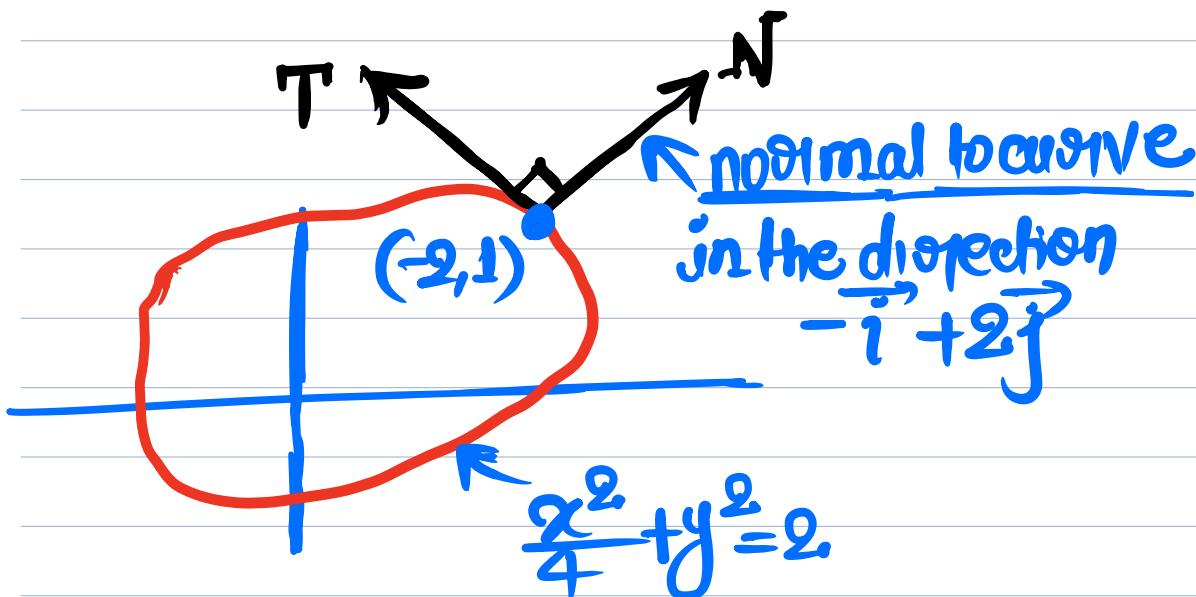
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$\nabla f|_{(-2,1)} = (-1, 2)$$

$$= -\vec{i} + 2\vec{j}$$

Step II: Realise that $-\vec{i} + 2\vec{j}$
is normal to the level curve

$$\frac{x^2}{4} + y^2 = 2 \text{ at the point } (-2, 1)$$



Step III: Need to find the eqn of the tangent line at $(-2, 1)$.

Question: Find the eqn of a line (in \mathbb{R}^2) passing through (x_0, y_0) and orthogonal to $A\vec{i} + B\vec{j}$

Ans: $A(x - x_0) + B(y - y_0) = 0$



Equation of the tangent line

$$y - 1 = -1(x+2) + 2(y-1) = 0$$

$$-x + 2y - 4 = 0$$

$$2y - x = 4.$$



~~Differentiation~~

~~(Additional problem)~~

Suppose we wish to find the equation of the normal line to the curve

$$\frac{x^2}{4} + y^2 = 2 \text{ at } (-2, 1).$$

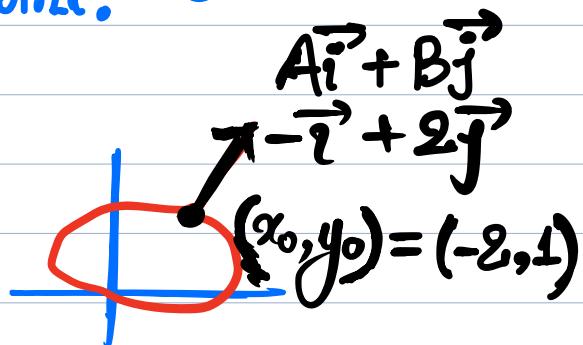
With the new tool at our disposal, how shall we attempt this problem?

Ans: This is same as finding the equation of a line that

- (i) passes through the point (x_0, y_0) &
- (ii) is parallel to the gradient vector at this point.

That is,

$$\begin{cases} x(t) = x_0 + At \\ y(t) = y_0 + Bt \end{cases}$$



Thus, the equation of the normal line (in the parameterized form) is

$$\begin{cases} x(t) = -2 - t \\ y(t) = 1 + 2t \end{cases}$$

(Verify: $\frac{x+2}{-1} = \frac{y-1}{2}$,

$$\Rightarrow 2x + 4 = 1 - y$$

$$\Rightarrow 2x + y + 3 = 0.$$

End of
digression

The power of this approach (Approach II)
is witnessed in higher dimension.

Let us deal with functions of three variables.

① Let $w = f(x, y, z)$ be a function

② Let $(x_0, y_0, z_0) \in \text{dom}(f)$

③ Let f be differentiable at (x_0, y_0, z_0) .

④ Recall that

If $c \in \text{ran}(f)$, then $f(x, y, z) = c$

is referred to as a level surface.

⑤ Thus $f(x, y, z) = f(x_0, y_0, z_0)$

is a level surface through (x_0, y_0, z_0)

① We need to determine the tangent plane
at the point (x_0, y_0, z_0) on the level
surface $f(x, y, z) = f(x_0, y_0, z_0)$

* The tangent plane at the point (x_0, y_0, z_0)
on the level surface $f(x, y, z) = f(x_0, y_0, z_0)$
is the plane that

- (i) passes through (x_0, y_0, z_0) and
- (ii) is perpendicular to the
gradient vector at this point,
i.e., $\nabla f|_{(x_0, y_0, z_0)}$.

Consequently, the tangent plane is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

⑥ We also need to find the normal line of the surface $f(x, y, z) = f(x_0, y_0, z_0)$ at the point (x_0, y_0, z_0) .

* The normal line of the surface $f(x, y, z) = f(x_0, y_0, z_0)$ at the point (x_0, y_0, z_0) is the line that

- (i) passes through (x_0, y_0, z_0) and
- (ii) is parallel to the gradient vector at this point, i.e., $\nabla f|_{(x_0, y_0, z_0)}$

Therefore, the normal line is given by

$$\left\{ \begin{array}{l} x(t) = x_0 + f_x(x_0, y_0, z_0)t \\ y(t) = y_0 + f_y(x_0, y_0, z_0)t \\ z(t) = z_0 + f_z(x_0, y_0, z_0)t \end{array} \right.$$



Qn: given a function $f: D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$, and a point $(x_0, y_0, z_0) \in \text{int}(D)$, compare the following.

(1) The equation of the tangent plane at the point (x_0, y_0, z_0) on the surface $f(x, y, z) = f(x_0, y_0, z_0)$.

(2) The linearization of the function f (i.e., $f(x, y, z)$) at the point (x_0, y_0, z_0) .

[HW]

Qn: Exercise 59 (Section 14.6) [HW]

The linearization of $Z = f(x, y)$ at the point (x_0, y_0) is the tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ on the surface $Z = f(x, y)$

Qn: Find the tangent plane and normal line
of the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at
the point $P_0(1,2,4)$. (HW)

Practice problems: (Section 14.6)

Exercises 59, 60, 61, 62, 63

Lecture 11

Warm-up problems

(1) What does the following equation
 $ax+by+c=0$

represent in the geometry of

- (i) three dimensions;
- (ii) two dimensions.

(2) Find the plane tangent to the surface

$$z = x \cos y - y e^x \text{ at } (0, 0, 0).$$

Soln: define $f(x, y, z) := x \cos y - y e^x - z$.

Clearly, $(0, 0, 0) \in \text{Int}(\text{dom}(f))$.

Then $f(x, y, z) = f(0, 0, 0)$ is a level surface.

(In fact, it is the surface in question,
since $f(0, 0, 0) = 0$.)

$$\nabla f|_{(0,0,0)} = (1, -1, -1).$$

tangent plane = plane passing through $(0,0,0)$
and \perp to $\vec{i} - \vec{j} - \vec{k}$.

$$= x - y - z = 0.$$

[Generalization of the previous question]

(3) Find the plane tangent to a surface

$$z = f(x, y) \text{ at } (x_0, y_0, z_0).$$

Solution: Define $F(x, y, z) := f(x, y) - z$.

Observe that

① $F(x, y, z) = F(x_0, y_0, z_0)$ is the
surface in question (and is a level
surface)

④ $\nabla F|_{(x_0, y_0, z_0)}$

$$= \left(F_x|_{(x_0, y_0, z_0)}, F_y|_{(x_0, y_0, z_0)}, F_z|_{(x_0, y_0, z_0)} \right)$$

$$= \left(f_x|_{(x_0, y_0)}, f_y|_{(x_0, y_0)}, -1 \right)$$

⑤ Eqn of the tangent plane =

The plane passing through (x_0, y_0, z_0)

and \perp to $\nabla F|_{(x_0, y_0, z_0)}$ ↴

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - 1(z - z_0) = 0$$



Prerequisites for lectures this week:

Surface (Section 12.6 of Thomas' Calculus)

Cylinders

① $\rightarrow y = x^2$ (The parabolic cylinder)

\rightarrow Any curve $f(x, y) = c$ in the xy -plane defines a cylinder in \mathbb{R}^3 parallel to the z -axis whose equation is also $f(x, y) = c$.

\rightarrow Sketch the following cylinders

② $\frac{x^2}{9} + \frac{y^2}{4} = 1$

③ $z = x^2$

Quadratic Surfaces

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iy + Jz + K = 0$$

2. The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

3. The hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

4. The hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

5. The elliptic cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

6. The elliptic paraboloid

$$z = ax^2 + by^2$$

7. The hyperbolic paraboloid

$$z = by^2 - ax^2$$

Extrema of real-valued functions

of one variable

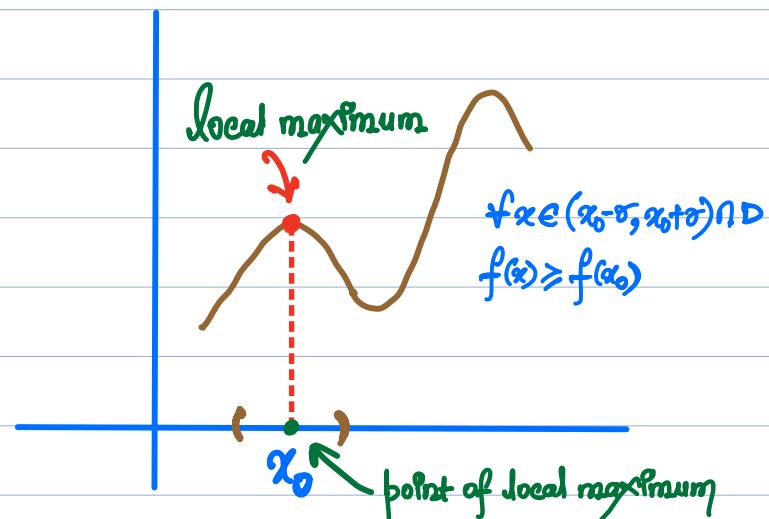
$$y = f(x)$$

of two/several variables

$$z = f(x, y)$$

Recapitulation (functions of one variable)

① defn: given a function $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and a point $x_0 \in \text{Int}(D)$, we say that f has a local maximum at x_0 if there exists $\sigma > 0$ such that $\forall x \in (x_0 - \sigma, x_0 + \sigma) \cap D$, we have $f(x) \leq f(x_0)$.



○ Defn: If $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$,
then f is said to have a local minimum at x_0 .

○ Theorem: [First derivative test for local extrema values]

Given
○ a function $y = f(x)$ and
○ a point $x_0 \in \text{int}(\text{dom}(f))$.

If
i) f has a local extremum at x_0 and
ii) $f'(x_0)$ exists,
then $f'(x_0) = 0$.

○ The converse of the above result is not always true!

Consider $f(x) = x^3$ at $x=0$.

Verify that $f'(0)=0$ but f does not have a local maximum or a local minimum at $x=0$

① Question: At which points f may assume (if it assumes) a local extremum?

Ans:

- ① interior points where $f' = 0$
- ② interior points where f' does not exist
- ③ end points of the domain of f

④ defn: An interior point, say x_0 , of the domain of a function f is called a critical point of f if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

⑤ Remark: The only points in the domain of f where f can possibly assume a local extremum are:

- ① critical points of f ;
- ② end points of f .

Also called
"boundary points"

End of
Recapitulation

Extrema of real-valued functions

of two/several variables

① defn: given a function $f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$,
and a point $(x_0, y_0) \in \text{int}(D)$,

We say that f has a local maximum at (x_0, y_0)

if there exists $\sigma > 0$ such that

$\forall (x, y) \in B_\sigma((x_0, y_0)) \cap D$, we have

$$f(x, y) \leq f(x_0, y_0).$$

that is, there exists an open ball $B_\sigma((x_0, y_0))$
of radius $\sigma > 0$ centered at (x_0, y_0)
such that $f(x, y) \leq f(x_0, y_0)$
whenever $(x, y) \in B_\sigma((x_0, y_0)) \cap D$.

① Exercise: define the following:

① "f has a local minimum at (x_0, y_0) "

② "f has a local extremum at (x_0, y_0) "

② Note: If $f(x, y) = \text{constant}$, then every point
in the domain of f is a point of local maximum.

③ Theorem: [First derivative test for
local extremum values]

Given

④ a function $z = f(x, y)$ and

⑤ a point $(x_0, y_0) \in \text{int}(\text{dom}(f))$.

If ⑥ f has a local extremum at (x_0, y_0) and

⑦ both the partial derivatives of f exist
at the point (x_0, y_0) , that is,
 $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist,

then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

④ **Remark:** If $f(x,y)$ has an extreme value at a point in the domain of f , then that point must be

- ① an interior point where $f_x = f_y = 0$
- ② an interior point where at least one of the two partial derivatives f_x and f_y does not exist
- ③ a boundary point of the domain of f .

④ **definition:** An interior point, say (x_0, y_0) , of the domain of a function f is called a critical point of f if

(i) either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or

(ii) at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ does not exist.

① Thus, the only points in the domain of f where f can possibly assume a local extremum are:

- ① critical points of f ;
- ② boundary points of f .

Example: find the local extreme values of the function $f(x,y) = x^2 + y^2$.

Solution: Clearly, $\text{① } f: \mathbb{R}^2 \rightarrow \mathbb{R}$
② $\text{dom}(f)$ is \mathbb{R}^2 — the whole 2-dimensional Euclidean space

and hence there are no boundary points!

So, we shall only focus on the critical points of f .

However, since $x^2 + y^2$ is a polynomial, both the partial derivatives of f exist at every point in the domain of f . Consequently, we shall only

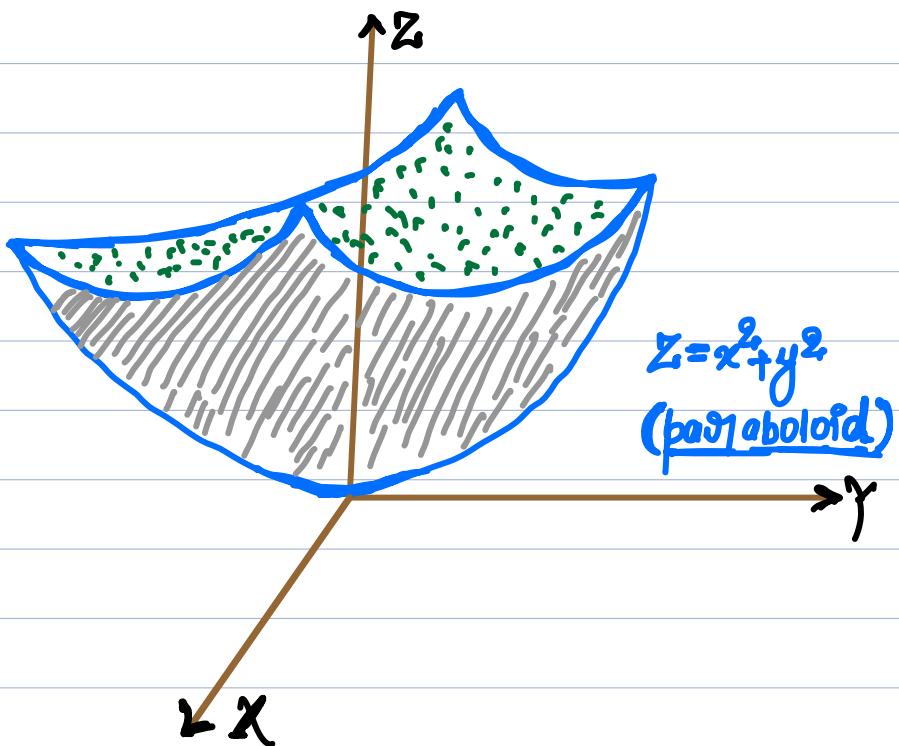
Search for the interior points (x_0, y_0) in the domain of f where $f_x = f_y = 0$.

Now, $f_x = 2x$ and $f_y = 2y$.

So, $f_x = 0 \Rightarrow x = 0$ and $f_y = 0 \Rightarrow y = 0$.

The only critical point of f is $(0,0)$.

Now convince yourself that $(0,0)$ is a point of local minimum. (Hw)



© It is worth mentioning that there exists a function $f(x,y)$ with a point (x_0, y_0) in the interior of its domain such that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

but f has neither a local maximum nor a local minimum at the point (x_0, y_0) .

— Such points are called
"saddle points" (Next lecture)

Example: Find the local extreme values (if any) of the function $f(x,y) = x^2 + y^2$.

[H.W.]

Lecture 18

Example 1: find the local extreme values (if any) of the function $f(x,y) = y^2 - x^2$.

Solution: $\rightarrow \textcircled{1} \text{ dom}(f) = \mathbb{R}^2$. (No boundary points)

$$\rightarrow \textcircled{2} f_x = -2x = 0 \Rightarrow x=0.$$

$$f_y = 2y = 0 \Rightarrow y=0.$$

$\rightarrow \textcircled{3}$ $(0,0)$ is a critical point of f .

$\rightarrow \textcircled{4}$ local extrema can occur only at $x=0, y=0$.

$$\rightarrow \textcircled{5} f(0,0)=0.$$

However,

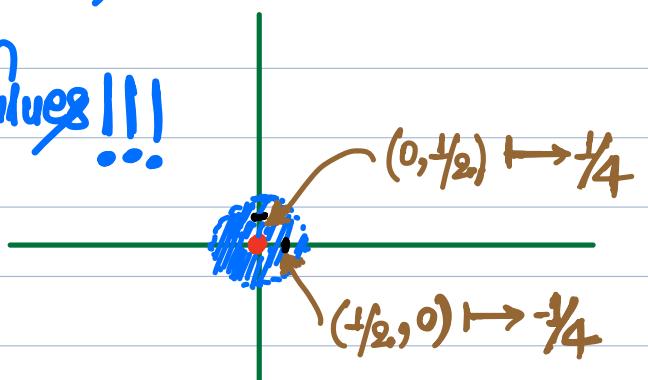
$$\boxed{f(x,0) = -x^2 < 0 \text{ (along the positive } x\text{-axis)}}$$

$$\boxed{f(0,y) = y^2 > 0 \text{ (along the positive } y\text{-axis)}}$$

\therefore every open disc in the xy -plane centered at $(0,0)$ has points where $f > 0$ and points where $f < 0$. — $(0,0)$ is a saddle point.

\therefore No extreme value at $(0,0)$. ■

\therefore No local extreme values!!!



Definition: Let $f(x,y)$ be a differentiable function. A critical point $(x_0, y_0) \in \text{Int}(\text{dom}(f))$ is called a "saddle point" if $\forall \delta > 0$, $\exists (x_1, y_1)$ and $(x_2, y_2) \in B_\delta((x_0, y_0)) \cap \text{dom}(f)$ such that

$$f(x_1, y_1) > f(x_0, y_0) \text{ and}$$

$$f(x_2, y_2) < f(x_0, y_0)$$

○ Theorem: [Second derivative test for local extremum values]

Let $f(x, y)$ be a real-valued function of two variables and let (x_0, y_0) be a critical point of f .

If the second-order partial derivatives of f (i.e., f_{xx}, f_{yy}, f_{xy} and f_{yx}) exist and

are continuous in some open ball centered at the critical point (x_0, y_0) , then

(1) f has a local maximum at (x_0, y_0) if

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 > 0 \text{ and } f_{xx}(x_0, y_0) < 0$$

(2.) f has a local minimum at (x_0, y_0) if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 > 0 \text{ and } f_{xx}(x_0, y_0) > 0$$

(3.) f has a saddle point at (x_0, y_0) if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 < 0$$

(4.) The test is inconclusive at (x_0, y_0) if

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 = 0$$

(In this case, any of the behaviour described
in (1) to (3) may occur.)

definition: The expression

$$f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

Can be expressed as

$$\begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

and is called

the discriminant of f at (x_0, y_0)

or

the Hessian of f at (x_0, y_0) .

Example 2: Find the local extreme values

of $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$.

Ans. Critical point $\equiv (15, -8)$,
point of local minima | Value = -63

Example 3: Find the local extreme values

$$\text{of } f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Example 4: Find the local extreme values

$$\text{of } f(x,y) = xy.$$

Example 5: Find the local extreme values

$$\text{of } f(x,y) = x^3 - y^3 - 2xy + 6$$

Answer:

Critical points $\begin{cases} (0,0) \\ (-\frac{2}{3}, \frac{2}{3}) \end{cases}$

saddle point

local max

$$\underline{\text{Value}} = \frac{170}{27}$$

Example 6: $f(x,y) = y \sin x$

Critical points $\equiv \{(n\pi, 0) : n \in \mathbb{Z}\}$

(Saddle points)

Example 7: $f(x,y) = x^4 + y^4 + 4xy$

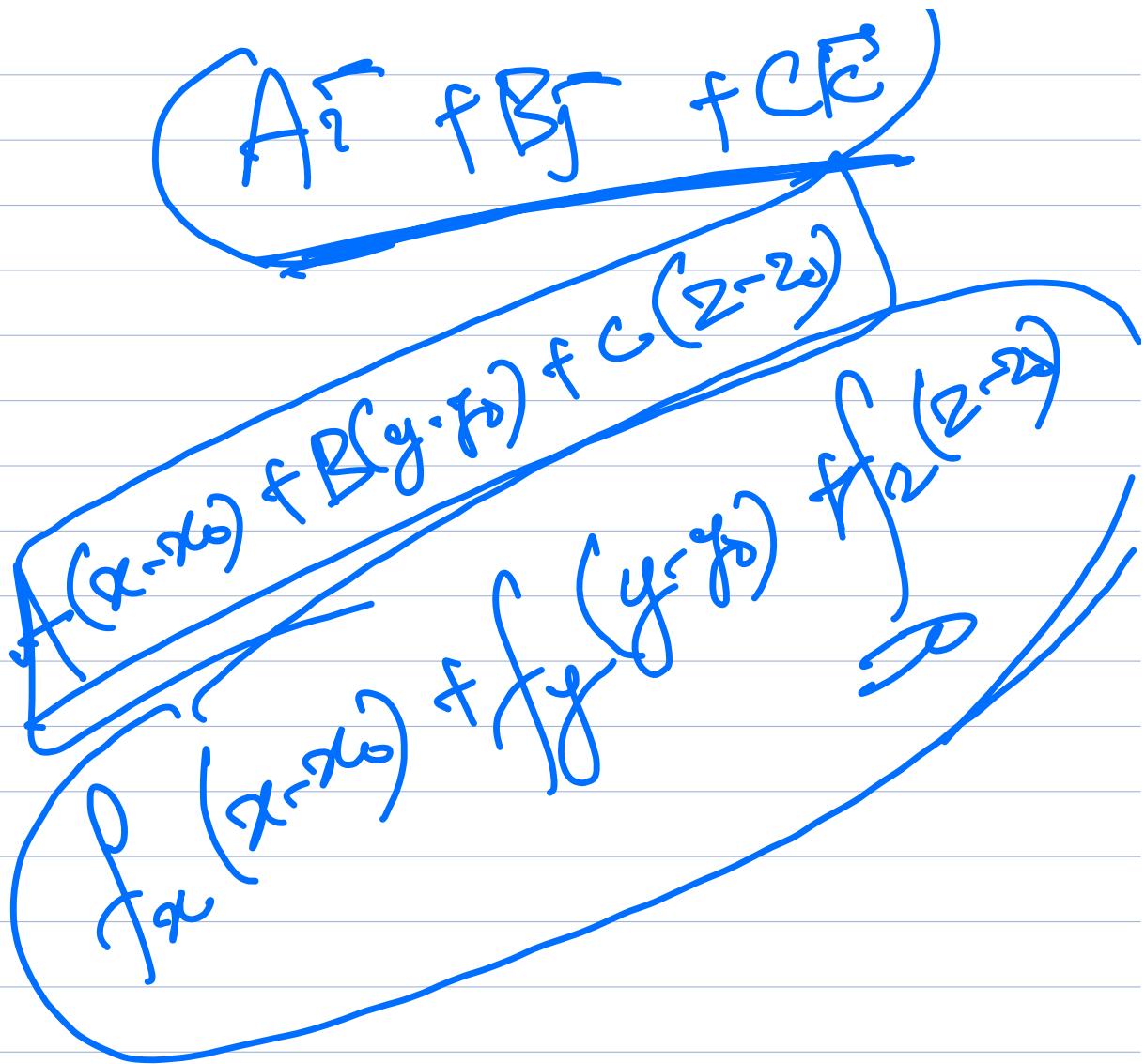
Critical points $\rightarrow (0,0), (1,-1)$ and $(-1,1)$

Saddle point

local min
 $(=-2)$

local min
 $(=-2)$





* $f(x,y) = y^2 - x^2$

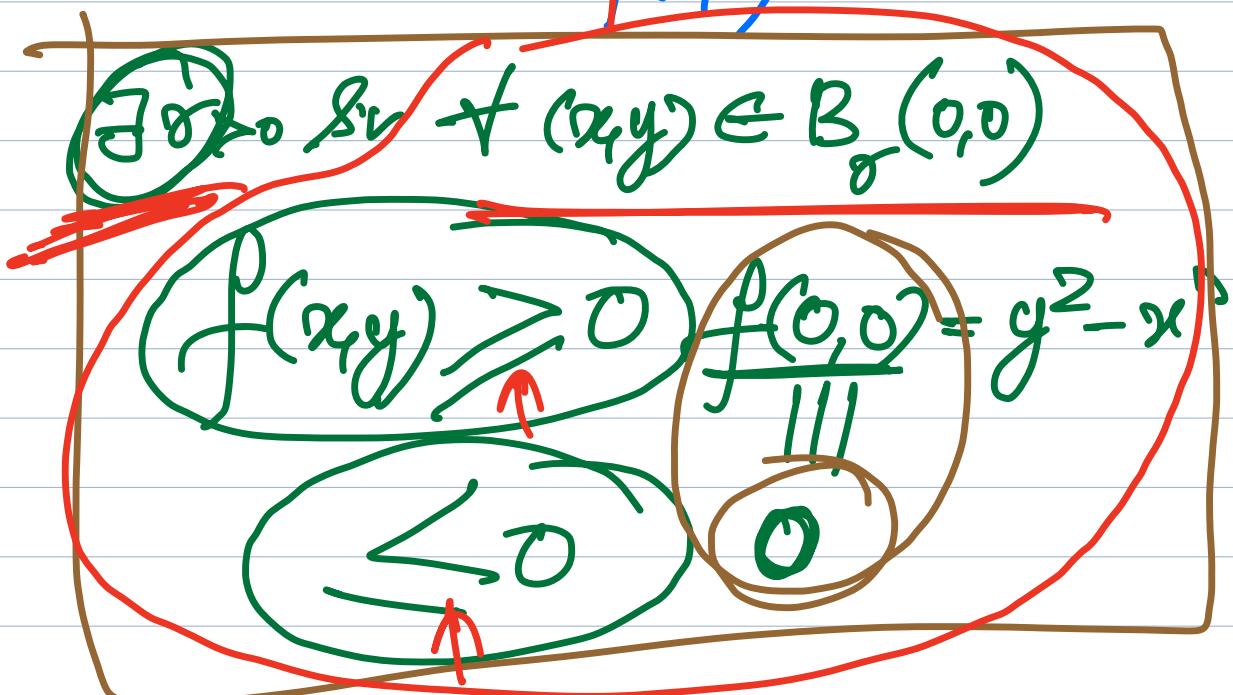
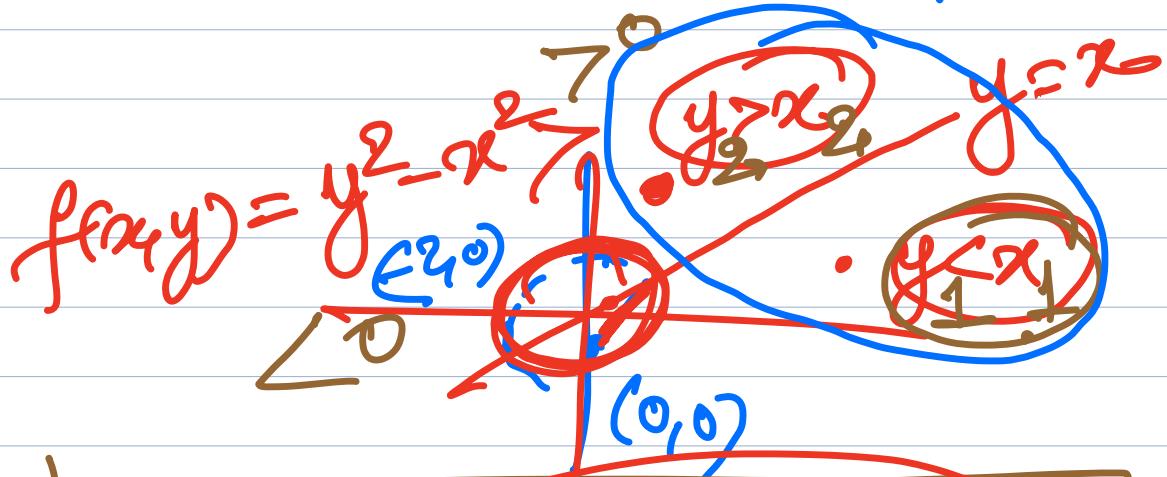
$$\text{dom}(f) = \mathbb{R}^2$$

$$(x,y) \in \mathbb{R}^2$$

$$f_x = -2x = 0$$

$$f_y = \cancel{2y} = 0$$

$(0,0) - \text{CP}$



$\forall \varepsilon > 0 \exists (x_1, y_1) \text{ & } (x_2, y_2)$

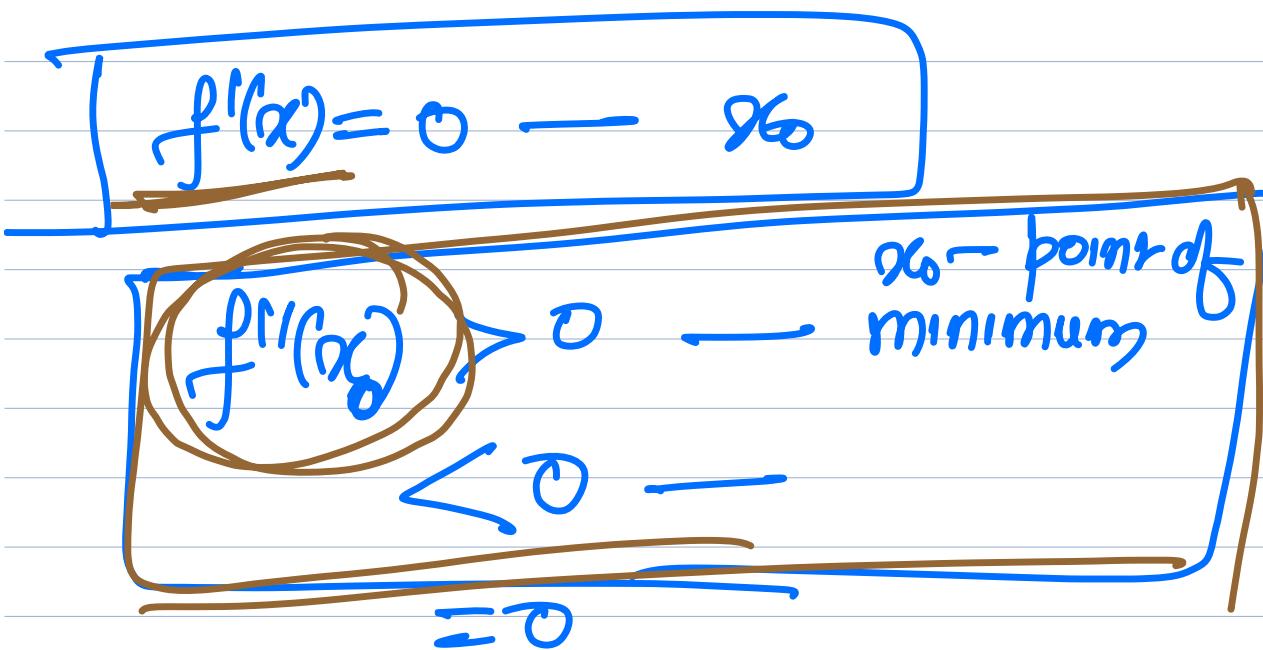
$\in B_\rho(0,0)$ s.t.

$f(x_1, y_1) > f(0,0);$

$f(x_2, y_2) < f(0,0).$

$(0,0) \approx (x_0, y_0)$

[Second derivative test]



→ $\textcircled{1} Z = f(x, y)$ — be a function

→ $\textcircled{2} (x_0, y_0)$ is a critical point.

→ $\textcircled{3} f_x$ and f_y exist at (x_0, y_0)

→ $\textcircled{4} f_x$ and f_y continuous at x_0

↗

throughout some open disc

↙

centered at (x_0, y_0)



$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

Then

① If f has a
local max. at

(x_0, y_0) w/

$f_{xx} < 0$ and
 $f_{xx} f_{yy} - f_{xy}^2 > 0$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y^2}$$

② Local min

$f_{xx} > 0$ and

$$f_{xx}f_{yy} - f_{xy}^2 > 0$$

$$f_{xy} = \begin{cases} f_x \\ f_y \end{cases}$$

of
 $\frac{\partial}{\partial y} \frac{\partial}{\partial x}$

③

f has a "saddle point" at (x_0, y_0)

$$f_{xx}f_{yy} - f_{xy}^2 = 0$$

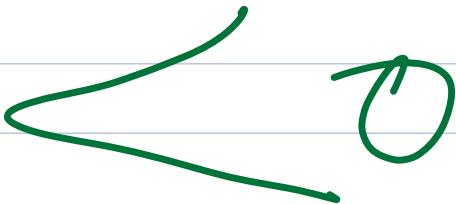
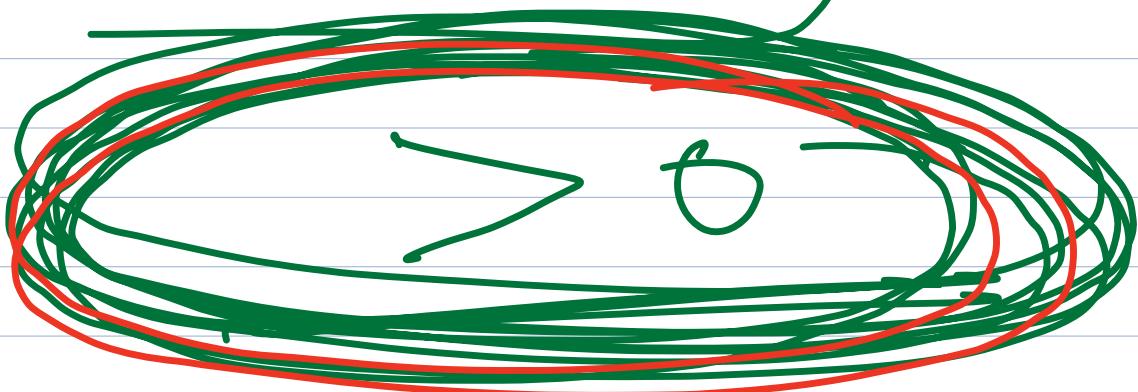
④

$$f_{xx}f_{yy} - f_{xy}^2 = 0$$

→ The test is unconclusive.

$$\begin{array}{|c|c|c|c|} \hline
 & f_{xx} & f_{xy} \\ \hline
 \text{der} & f_{xy} & f_{yy} \\ \hline
 & (x, y) & \\ \hline
 \end{array}$$

$$= f_{xx}f_{yy} - f_{xy}^2$$



$$f(x,y) = x^2 + y^2$$

(0,0)

$$f_x = 2x$$

$$f_y = 2y$$

$$f_{xx} = 2$$

$$f_{yy} = 2$$

$$(gx)_y = 2$$

$$(fa)_y =$$

$$f_{xy}$$

$$(fa)_y =$$

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = \boxed{4} \Rightarrow$$

$$f(x,y) = y^2 - x^2$$

$$f_{xx} = \cancel{y^2} - 2$$

$$f_{yy} = 2$$

$$f_{xy} = 0$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\leq -4 < 0$

\leq

\geq

$f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

C.P. = $(15, -8)$

local min.

GB

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

$$f(x,y) = xy$$

$$f(x,y) \underset{x^2 - y^2}{=} - 2xy + 6$$

$$f(x,y) = y \sin x$$

\mathbb{R}^2

C.P. = ?

$$f_x = y \cos x = 0$$

$$f_y = \sin x = 0$$

$$x = n\pi, n \in \mathbb{Z}$$

$$(2m+1)\frac{\pi}{2}, m \in \mathbb{Z}$$

C.P.:

$$\{(n\pi, 0) : n \in \mathbb{Z}\}$$

$$f(x,y) = x^4 + y^4 + 4xy$$

$$(0,0)$$

$$(1,-1)$$

$$(-1,1)$$

$f_{\text{p.}}$

$f_{\text{loc. min.}}$

$f_{\text{loc. min.}}$



$$\textcircled{1} f(x,y) = \underline{2 + 2x + 2y - x^2 - y^2}$$

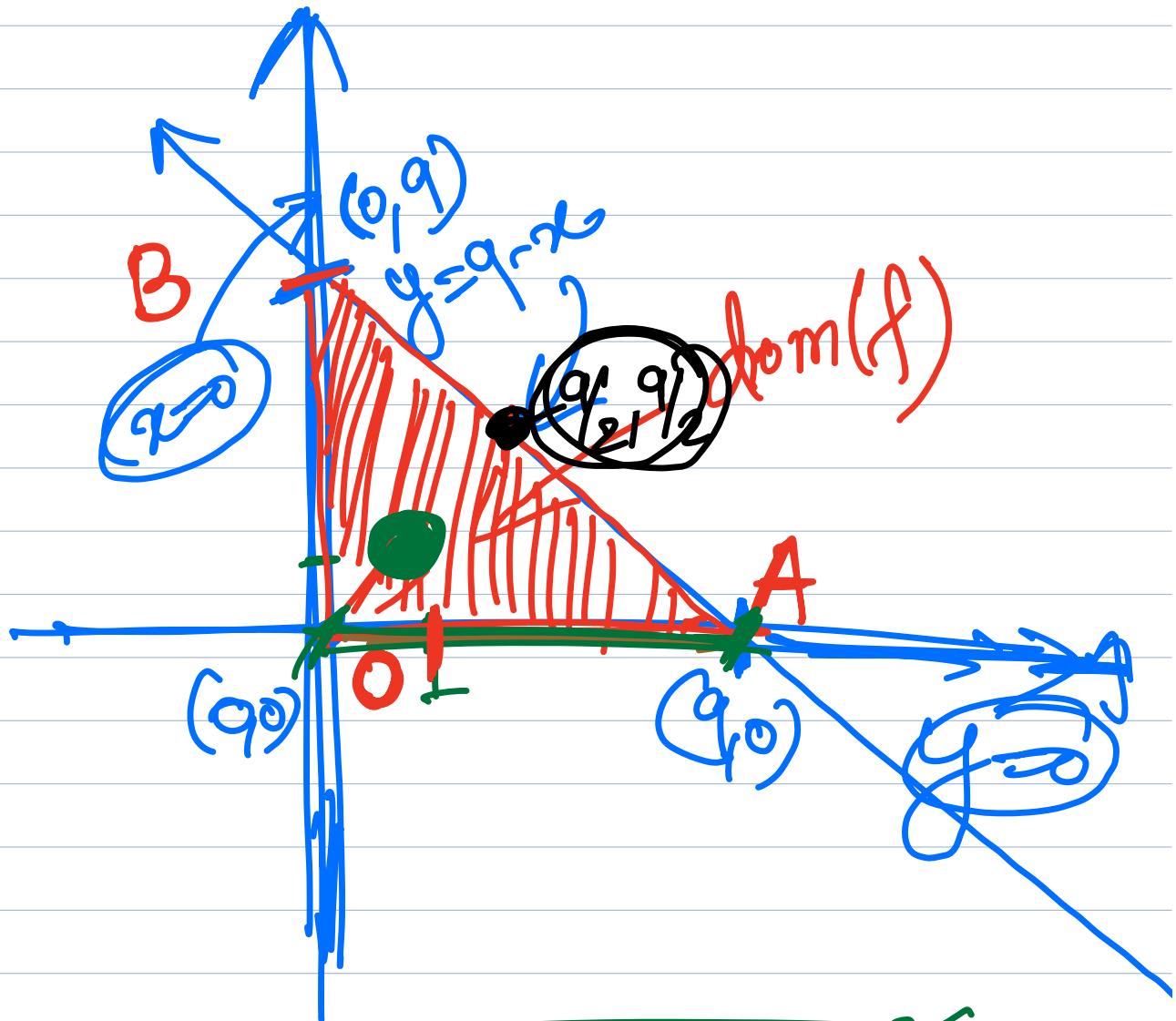
Find the absolute max. and min.

values of f

on the triangular plate

in the first quadrant bold.

by the lines $x=0$, $y=0$, $y=9-x$



$$f(x) = 2 + 2x + 2y - x^2 - y^2$$

$$f_x = 2 - 2x = 0 \Rightarrow x=1$$

$$f_y = 2 - 2y = 0 \Rightarrow y=1$$

$(1,1)$ is a ~~crit.~~

$$\begin{aligned} f_{xx} &= -2 \\ f_{yy} &= -2 \end{aligned}$$

$$f_{xy} = 0$$

$$\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 \geq 0$$

$(1,1)$ is a pt. of local maximum

$$f(1,1) = 4$$



Along OA

($y=0$)

$$\underline{-2 + 2x + 2y} \quad \underline{x^2 - y^2}$$

$$g(x) = 2 + 2x - x^2, \quad x \in [0, 9]$$

$$g'(x) = 2 - 2x = 0 \Rightarrow x_0 = 1$$

$$g''(x) = -2 < 0$$

local max

$$g(1) = 2 + 2 - 1 = 3$$

$$g(1)$$

$$(1, 0)$$

$$f(1, 0) = 3$$

$$f(x, y)$$

Along OB

($x=0$)

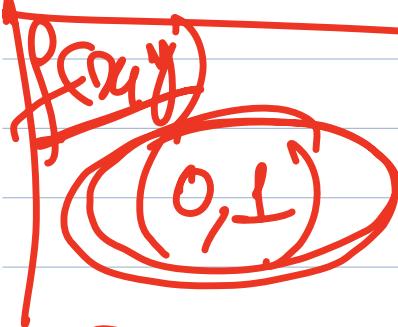
$$2 + 2x + 2y - x^2 \cdot y^2$$

$$h(y) = 2 + 2y - y^2$$

$0 \leq y \leq 9$

$$\rightarrow y_0 = 1$$

$$h(1) = 3$$



$$f(0,1) = 3$$

Local max

Along AB

$$(y=9-x)$$

$$2+2x+2y - x^2 - y^2$$

$$\begin{aligned} P(x) &= 2+2x+2(9-x) \\ &\quad - x^2 - (9-x)^2 \end{aligned}$$

$$\begin{aligned} &= 2+2x+18-2x \\ &\quad - x^2 - [81+x^2] \\ &\quad - 18x \end{aligned}$$

$$20-x^2-81-x^2$$

$$+18x$$

$$\Rightarrow -2x^2 - 61 + 18x$$

$k(x) =$

$$x \in [0, 9]$$

$$x \in (0, 9)$$

$$k'(x) = -4x + 18 = 0$$

$$x = 18/4 = 9/2$$

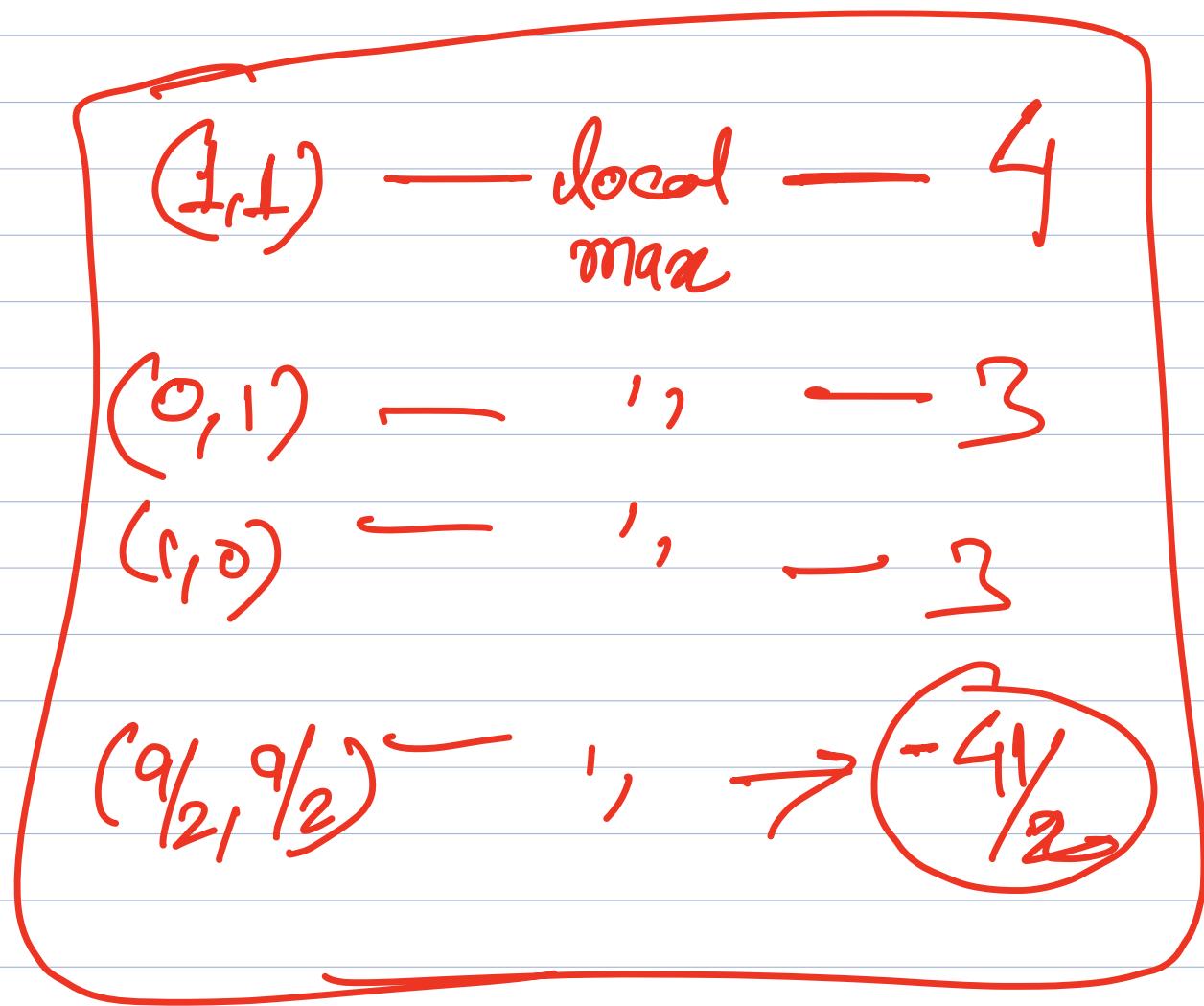
$$k''(x) = -4$$

local max.

$$f(x) \equiv (9/2, 9/2)$$

$$f(9/2, 9/2) = -4/2$$

local max.



Abs max = 4

Abs min = ??

Lecture 15

Recall the problem from the last lecture:

Find the absolute maximum and minimum value of the function

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2.$$

On the triangular plane in the first quadrant bounded by the lines

$$x=0, y=0, \text{ and } x+y=9.$$

→ Natural domain ?

→ Restricted domain ?

→ This problem is an example of
constrained optimization [discussion]

Problem 1: find the point $P(x, y, z)$

closest to the origin on the plane $2x+y-z-5=0$.

Solution:

Minimize $g(x, y, z) := \sqrt{x^2 + y^2 + z^2}$

Subject to the constraint $2x+y-z-5=0$.

Alternatively,

$$\text{minimize } h(x, y, z) = x^2 + y^2 + z^2$$

Subject to the constraint $2x+y-z-5=0$.

Alternatively,

$$\text{minimize } f(x, y) = x^2 + y^2 + (2x+y-5)^2$$

(No constraint!)

To this end,

$$f_x = 0 \Rightarrow x = 5/3$$

$$f_y = 0 \Rightarrow y = 5/6 \quad (\text{Verify!})$$

Use the second derivative test to ensure that $(5/3, 5/6)$ is a point of minima for f .

Thus, $z = 2x + y - 5 = 2(5/3) + (5/6) - 5 = -5/6$.

$\therefore (5/3, 5/6, -5/6)$ is the required point. ■

Dilection / Remark

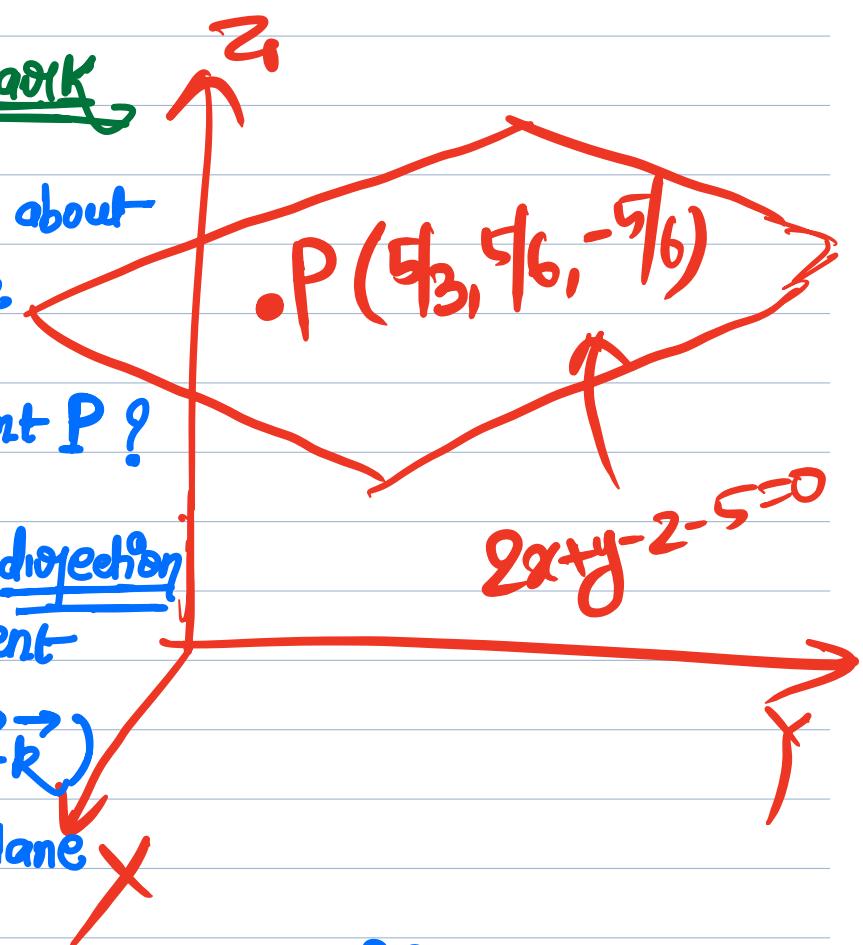
What can be said about the normal line through the above-found-point P?

(a) It is in the direction of the gradient

(i.e., $\vec{i} + \vec{j} - \vec{k}$)

of the given plane

(b) It passes through the origin.



Question: Let Q be another point on the above plane (i.e., $Q \neq P$).

Does the normal line passing through Q also pass through the origin? [HW]

End of the Remark.

[Constrained Optimization]

Problem 2: Find the point(s) closest to the origin on the Hyperbolic Cylinder $x^2 - z^2 - 1 = 0$.

The idea is to re-phrase the problem of constrained optimization into a problem without constraints by using the method of substitution.

Optimize $g(x, y, z) = x^2 + y^2 + z^2$
subject to the constraint $x^2 - z^2 - 1 = 0$.

Approach I: Substitute $x^2 = z^2 + 1$. Then,

We need to minimize $f(y, z) = (z^2 + 1) + y^2 + z^2$.

That is,

$$f(y, z) = 1 + y^2 + 2z^2$$

$$\begin{cases} f_y = 2y = 0 \Rightarrow y=0; \text{ and} \\ f_z = 4z = 0 \Rightarrow z=0 \end{cases}$$

Exercise: Verify, using the second derivative test, that the point $(y, z) = (0, 0)$ is a point of minimum for the function $f(y, z) = 1 + y^2 + z^2$.

Then, we find x by using $x^2 = 1 + z^2$.

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Consequently, the points on the hyperbolic cylinder that are closest to the origin are $(1, 0, 0)$ and $(-1, 0, 0)$.



Now, there is nothing special (at least not in the first glance; or is there?) in choosing to substitute $x^2 = z^2 + 1$. Why can't one choose to substitute $z^2 = x^2 - 1$?

Approach II: Substitute $z^2 = x^2 - 1$.

$$\begin{aligned} \text{Minimize } h(x, y) &= x^2 + y^2 + (x^2 - 1) \\ &= 2x^2 + y^2 - 1. \end{aligned}$$

Then

$$h_x = 4x = 0 \Rightarrow x = 0;$$

$$h_y = 2y = 0 \Rightarrow y = 0.$$

Exercise: Verify that $(0, 0)$ is a point of local minimum for $h(x, y)$.

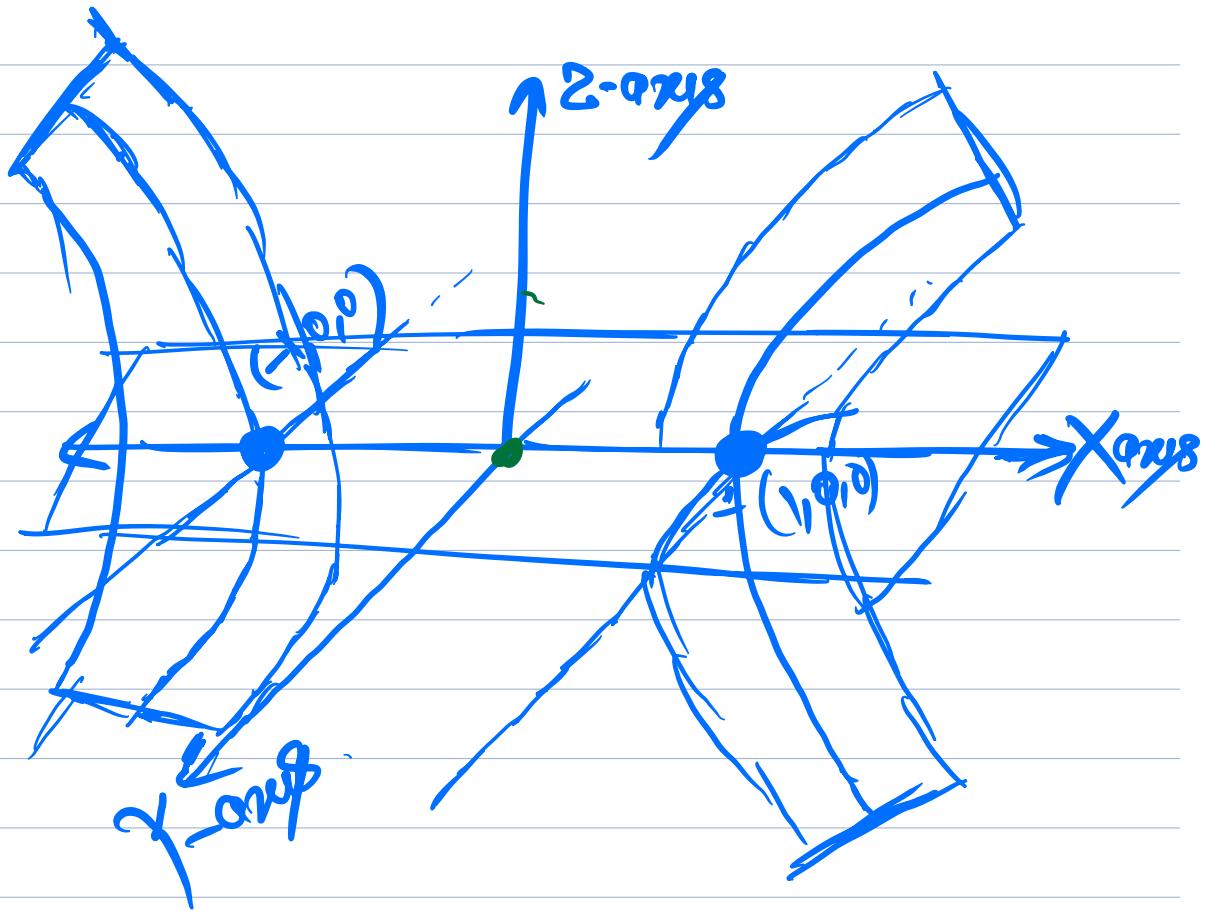
$$\text{But then, } z^2 = x^2 - 1 \Rightarrow z^2 = -1$$

which is an impossible situation!

The point $(0, 0, z)$ doesn't lie in \mathbb{R}^3 , let alone on the hyperbolic cylinder!

What went wrong?

Let's look at the surface:



→ Clearly, we couldn't choose values of x that lie between -1 and 1 , for those won't yield points that are on the cylinder.

→ Approach I worked because we chose "y" and "z" to be the 3rd variable and "x" to be the dependent variable. (Elaborate.)

→ Approach II failed since "x" was chosen to be an independent variable — which created the problem!

Conclusion: You need to be cautious when tackling problems of constrained optimization using the substitution method.

This motivates us to develop a geometric method

Approach III:
(Geometric method)

The method of
Lagrange multipliers

$$\text{minimize } h(x, y, z) = x^2 + y^2 + z^2$$

$$\text{subject to the constraint } x^2 + z^2 - 1 = 0.$$

$$(x^2 - z^2 = 1)$$

... next lecture