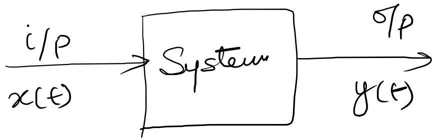
30/8/2024

Systems & Their properties



$x[n] \longrightarrow y[n]$

① System : with memory or without memory

$S_1 : y(t) = x(t)$

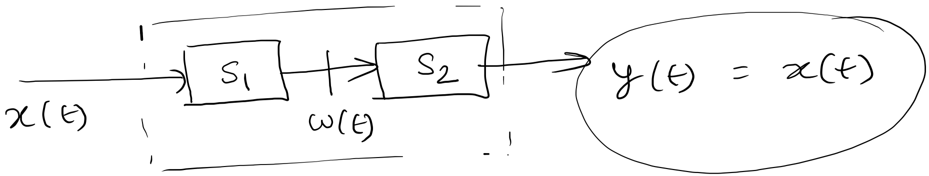
$S_2 : y(t) = x^2(t)$

$S_3 : y(t) = x(t-2)$

$S_4 : y(t) = \frac{1}{2}\{x(t) + x(t-1)\}$

$S_5 : y(t) = x(t+2)$

② Invertible / Non-invertible Systems



if $S_2 = inv(S_1)$

ⓐ $S_1 : \quad w(t) = 2x(t)$

$S_2 : \quad y(t) = \frac{1}{2} w(t) = \frac{1}{2} \times 2x(t) = x(t)$

S_1 is invertible & the inverse system is S_2.

ⓑ $S_1 : \quad w[n] = \sum_{k=-\infty}^{n} x[k]$

$S_2 : \quad y[n] = w[n] - w[n-1]$

$= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$

$= \cancel{\sum_{k=-\infty}^{n-1} x[k]} + x[n] - \cancel{\sum_{k=-\infty}^{n-1} x[k]}$

$= x[n]$

(c) $x[n] \rightarrow \boxed{S_1} \xrightarrow{w[n]} \boxed{S_2} \rightarrow y[n]$

$$S_1 : \quad w[n] = x[n] - x[n-1]$$

$$S_2 : \quad y[n] = \sum_{k=-\infty}^{n} w[k]$$

Invertible

(d)

$$S_1 : \quad w(t) = x^2(t)$$

$$S_2 : \quad y(t) = \sqrt{w(t)}$$

S_1 is not invertible

If distinct i/ps lead to distinct o/ps, the system is invertible, else not.

③ Causal Systems —

$S_1 : \quad y(t) = x(t - t_0)$ $\quad t_0$ is +ve real no.

$S_2 : \quad y[n] = x[n+3]$

$S_1 \equiv$ causal system

$S_2 \equiv$ non-causal system

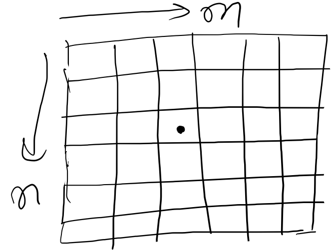
Causality: Unless there is a cause, there should be no effect.

Or unless there is a non-zero i/p, there should not be a non-zero o/p.

Or a causal system depends on current & past i/p.

Accumulator : $S_3 : \quad y[n] = \sum_{k=-\infty}^{n} x[k]$ causal.

S_4: $y[n,m] = \frac{1}{5}\{ x[n,m] + x[n-1,m] + x[n+1,m] + x[n,m-1] + x[n,m+1] \}$



Non-causal System

Stability —

BIBO Stability

A Stable System : A Bounded i/p should lead to a bounded o/p.

is on amplitude

Bounded i/p $\equiv$ $|x(t)| < M_x < \infty$ $\hookrightarrow$ finite value $\forall\ t$

Or $|x[n]| < M_x < \infty$ $\hookrightarrow$ finite value for all n.

this bounded i/p is fed to a system.

If the system generates an o/p, that is also bounded

Then, we call it a stable system —

Examples of bounded i/ps.

$x(t) = u(t)$; $x[n] = u[n]$

unbounded signals

$x(t) = tu(t)$ or $x[n] = tu[n]$

if $|y(t)| \leq M_y < \infty \quad \forall t$

$\hookrightarrow$ finite no.

for bounded i/p, then the system is stable.

S_1: $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n+1]\}$

$$|y[n]| = \left|\frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n+1]\right|$$

$$\leq \frac{1}{3}|x[n]| + \frac{1}{3}|x[n-1]| + \frac{1}{3}|x[n+1]| \quad \forall n$$

if $|x[n]| < M_x \quad \forall n$

$|y[n]| =$ ~~bounded~~ for all n.

$S_2 : \quad y[n] = \frac{1}{3}\left\{ x[n] - x[n-1] - x[n-2] \right\}$

$$|y[n]| = \frac{1}{3}\left| x[n] - x[n-1] - x[n-2] \right|$$

Since $|x[n]| < M_x < \infty \; \forall n$

Sum & difference of 3 finite nos. will again be a finite no.

So $|y[n]| < M_y < \infty \quad \forall n$

Stable System.

S_3 : $y[n] = n\,x[n]$

$|x[n]| < M_x < \infty \quad \forall n$

bounded i/p

$|y[n]| = |n\,x[n]| \longrightarrow \infty$

as $n \longrightarrow \infty$

Unstable system

S_4 : $y(t) = x(2t)$ Stable system

(5) Time - Invariant Systems

$$x(t) \xrightarrow{S} y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

$$x(t+t_0) \longrightarrow y(t+t_0)$$

$$x_1(t) \longrightarrow \boxed{} \longrightarrow y_1(t)$$

If $\quad x_2(t) \longrightarrow y_2(t)$

if $\quad x_2(t) = x_1(t-t_0) \longrightarrow$ Is $y_2(t) = y_1(t-t_0)$?

$y_2(t) =$

If yes, it is a time-invariant system

else, not.

$S: \; y(t) = x(2t)$

$x_1(t) \longrightarrow y_1(t) \boxed{= x_1(2t)}$

$x_2(t) \longrightarrow y_2(t) \boxed{= x_2(2t)}$

$$\boxed{y_1(t-t_0) = x_1(2t-2t_0)} \quad \text{(a)} \quad \text{I o/p}$$

$x_2(t) = x_1(t-t_0) \Rightarrow y_2(t) = x_1(2t-t_0) \Rightarrow$ (b) II o/p

(a) $\neq$ (b)

$y_2(t) \neq y_1(t-t_0)$

So, "S" is a time-varying system

$S_2 : y[n] = n\,x[n]$

$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$

$\downarrow$ delayed o/p

$y_1[n-n_0] = (n-n_0)\,x_1[n-n_0]$ — ⓐ

$x_2[n] \longrightarrow y_2[n] = n\,x_2[n]$

Let $x_2[n] = x_1[n-n_0]$

$\uparrow$ delayed i/p

$y_2[n] = n\,x_1[n-n_0]$ — ⓑ

But $y_2[n] \neq y_1[n-n_0]$

Time-varying

(5) Linearity

A system is linear if the properties of additivity & homogeneity holds true.

$$x_1(t) \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow y_2(t)$$

$$\boxed{a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t) \longrightarrow \text{System is Linear}}$$

$\forall\, a, b \in \mathbb{R}$

Additivity : $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$

homogeneity : $a x_1(t) \longrightarrow a y_1(t)$

$$S: \quad y(t) = 2x(t) + 3 = 2(i/p) + 3$$

$$x_1(t) \longrightarrow y_1(t) = 2x_1(t) + 3$$

$$x_2(t) \longrightarrow y_2(t) = 2x_2(t) + 3$$

$$\boxed{\overset{i/p}{a x_1(t) + b x_2(t)}} \longrightarrow 2\left(i/p \right) + 3$$

$$= 2\left(a x_1(t) + b x_2(t) \right) + 3 \quad \text{———} \quad ⓐ$$

$$= 2a x_1(t) + 2b x_2(t) + 3$$

$$a y_1(t) + b y_2(t) = a\left(2x_1(t) + 3\right) + b\left(2x_2(t) + 3\right)$$

$$= 2a x_1(t) + 3a + 2b x_2(t) + 3b \quad ⓑ$$

Not linear

A system that is both linear & time-invariant is called an LTI system

LTI $\equiv$ Linear & Time Invariant

4/9/2024

$$x[n] = \cos\left(\frac{\pi}{8} n^2\right)$$

Periodic if $x[n] = x[n+N]$ $\forall n$ for some positive N. is called the Smallest +ve N that satisfies this eqn, N is called the fundamental time period.

$$x[n+N] = \cos\left(\frac{\pi}{8}(n+N)^2\right) = \cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{2\pi}{8}nN\right)$$

$$\cos(2\pi k + \theta) = \cos\theta \quad \forall k \in \mathbb{Z}$$

Can it be a multiple of 2π

$\frac{\pi}{8} N^2 \longrightarrow$

$\frac{\pi}{\cancel{8}_{\cancel{2}}} \times \cancel{4} \times \cancel{4}^2 = 2\pi$

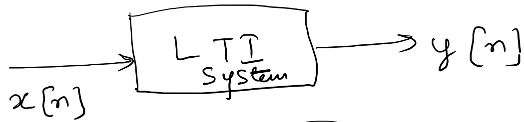
If $N = 4$

If $N = 8$ $\quad 2\pi \cdot 2$

$\frac{\cancel{2}\pi}{\cancel{8}} \cancel{4}\, n = \pi n$

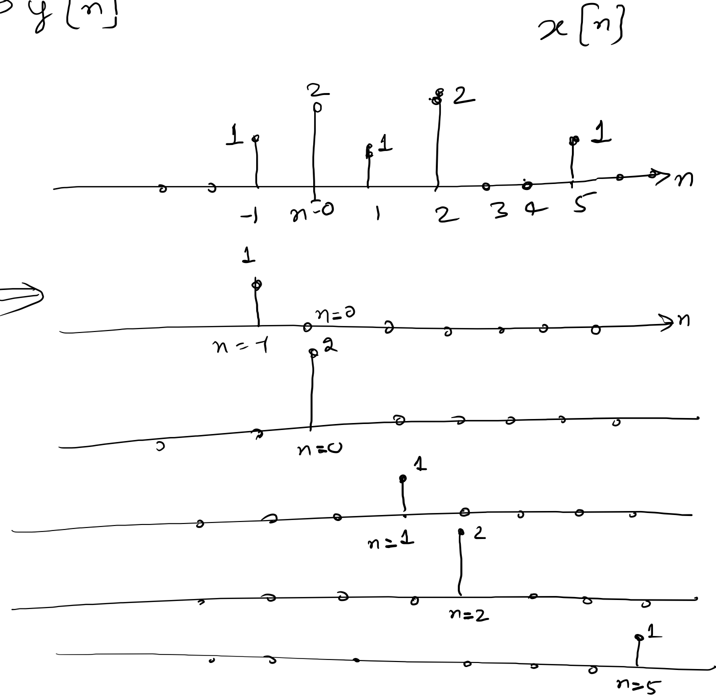
$\frac{2\pi}{\cancel{8}} \cdot \cancel{8}\, n = 2\pi n$

Discrete-time Convolution –



$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

$$
\begin{aligned}
x[n] = \; & x[-1]\,\delta[n+1] = \delta[n+1] \\
+ \; & x[0]\,\delta[n] = 2\,\delta[n] \\
+ \; & x[1]\,\delta[n-1] = \delta[n-1] \\
+ \; & x[2]\,\delta[n-2] = 2\,\delta[n-2] \\
+ \; & x[5]\,\delta[n-5] = \delta[n-5]
\end{aligned}
$$



Time-varying

$h[n, m]$

i/p variable

where the impulse was located at i/p

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$\delta[n] \longrightarrow y[n] = h[n]$$

$\equiv$ Impulse response of the LTI System

$$\delta[n-1] \longrightarrow h[n-1]$$

$$x[1]\,\delta[n-1] \longrightarrow x[1]\,h[n-1]$$

$$x[-1]\,\delta[n+1] \longrightarrow x[-1]\,h[n+1]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}_{\text{D.T. convolution of}} = y[n]$$

D.T. convolution of

$$x[n] * h[n] = y[n]$$

"Convolution" $\longrightarrow$ linear, commutative

$$x[n] * h[n] = h[n] * x[n]$$

$$\left(x_1[n] + x_2[n]\right) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

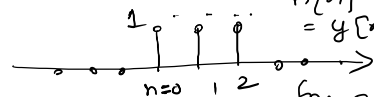
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Ex

Given an LTI System whose impulse response is

$$h[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$



$h[n] = y[n]$ for $\delta[n]$ as i/p.

Given $x[n] = \frac{3}{5}\delta[n] + \frac{1}{2}\delta[n+1]$

Compute $y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1] = \frac{3}{5}\left[\delta[n] + \delta[n-1] + \delta[n-2]\right]$

$+ \frac{1}{2}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right]$

Ex-2

LTI System $h[n] = u[n]$

Given $x[n] = \alpha^n u[n]$ where $0 < \alpha < 1$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$h[n] = 0 \quad n < 0$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=0}^{\infty} \underbrace{h[k]}\, x[n-k] = \sum_{k=0}^{\infty} x[n-k]$$

$$= \sum_{k=0}^{\infty} \alpha^{n-k} u[n-k]$$

$u[n-k] = 0$

$n - k < 0$

$n < k$

or $k > n$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

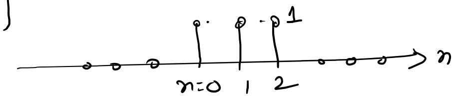
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$x[n]$ $y[n]$

$h[n]$

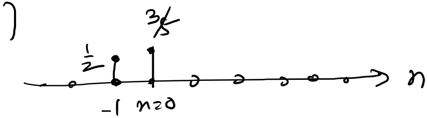
$$h[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$h[n]$



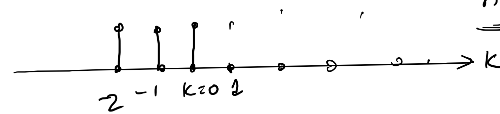
$$x[n] = \frac{3}{5}\,\delta[n] + \frac{1}{2}\,\delta[n+1]$$

$x[n]$



$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$

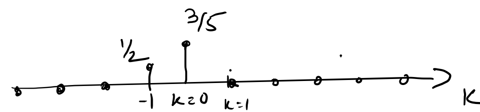
$h[-k]$



$$= 0 + \frac{1}{2}\times 1 + \frac{3}{5}\times 1 + 0 + 0 \ldots$$

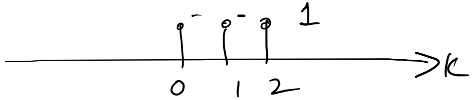
$$= \frac{1}{2} + \frac{3}{5}$$

$x[k]$

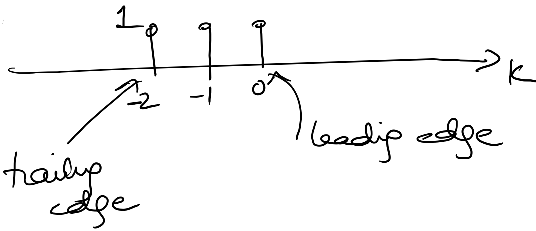


$$h[1-k] = h[-k+1]$$

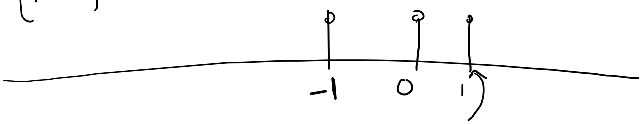
$h[k]$



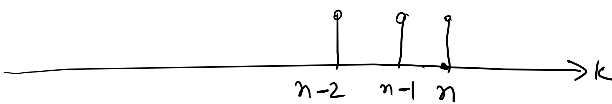
$h[-k]$



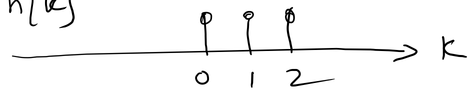
$h[1-k]$



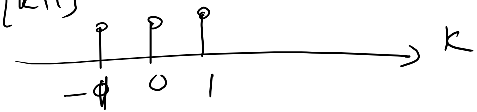
$h[n-k]$ & $n > 0$



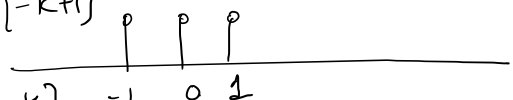
$h[k]$



$h[k+1]$

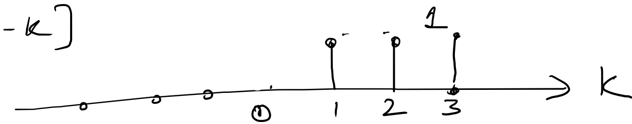


$h[-k+1]$



$= h[1-k]$

$h[3-k]$



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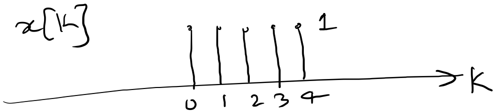
Discrete time convolution

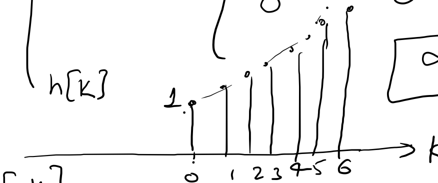
Example:

$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\alpha > 1}$$





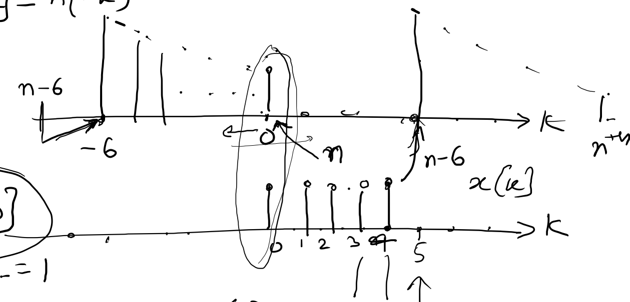
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, g[k]$$

$$= + \ldots x[-1]\,g[-1] + x[0]\,g[0] + x[1]\,g[1] + \ldots = 1$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k]$$

$$g[k] = h[-k]$$

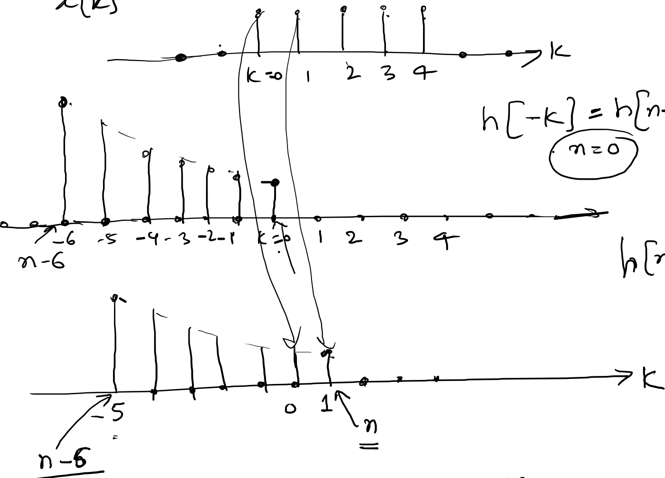


n − 6

$n < 0$

$n = -1, -2, \ldots$

$x[k]$



$h[-k] = h[n-k]$ $n=0$

$h[n-k]\big|_{n=1} = h[1-k]$

$k=1$

$h[0]$

$h[n-k] \neq 0$ $\qquad n-6 \longrightarrow n$

$-1-6=-7$ to -1 ← range of argument for which $h[n-k]$ is non-zero.

$n = -1$

-5 to 1

$n = 1$

-8 to -2

$n = -2$

$$y[n] = \sum_k x[k]\, h[n-k]$$

overlap of the regions where

I $\quad y[n] = 0$, $n < 0$ $\qquad n = -1$

or say $n = -1$

$h[n-k] \equiv -7$ to -1 on k-axis

$x[k] \equiv 0$ to 4 on k-axis

$n = 11$

5 to 11

$h[n-k] \neq 0$

0 to 4

$x[k] \neq 0$

$n > 10$ & $n < 0$

$y[n] = 0$

② $0 \le n < 4$

$$y[n] = \sum_{k=0}^{n} x[k]\,h[n-k]$$

$h[n-k]$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

$$= \alpha^n \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^k$$

$$= \alpha^n \frac{\left(1-(\alpha^{-1})^{n+1}\right)}{1-(\alpha^{-1})}$$

$k=0$	$n=0$	-6 to 0
$k=0,1$	$n=1$	-5 to 1
$k=0,1,2$	$n=2$	-4 to 2
$k=0,1,2,3$	$n=3$	-3 to 3
$k=0,\dots 4$	$n=4$	-2 to 4

$x[k]$

0 to 4

③ $4 \le n \le 6$

$$y[n] = \sum_{k=0}^{4} x[k]\,h[n-k]$$

④ $6 < n \le 10$

$$y[n] = \sum_{k=n-6}^{4} x[k]\,h[n-k]$$

C. T.

$$x(t) \xrightarrow{} \boxed{\text{LTI}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

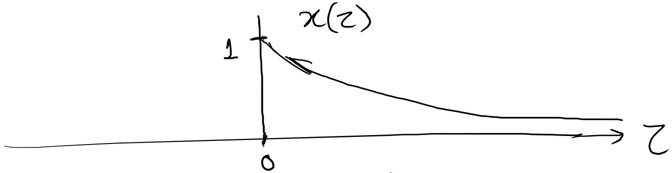
$$\boxed{y(t) = x(t) * h(t)}$$

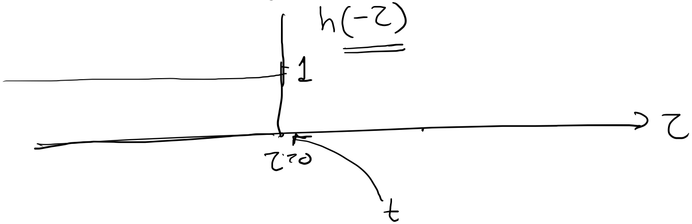
" Convolution " operator

Ex

$x(t) = e^{-at} u(t)$ $\quad a > 0$

$h(t) = u(t)$





$h(t-\tau)$

when $t=0$ $\quad h(-\tau)$

① $t < 0$

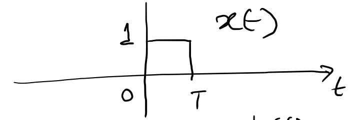
$y(t) = 0$

② $t > 0$

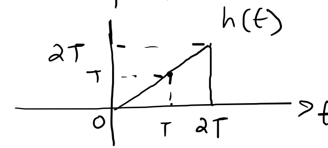
$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

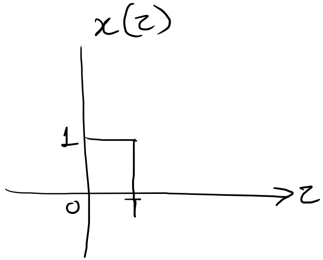
Ex

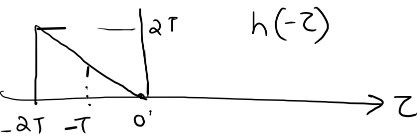
$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$



$$h(t) = \begin{cases} t & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}$$







① $y(t) = 0$ $\quad t < 0$ & $\underline{t > 3T}$

② $0 \le t \le T$

$$y(t) = \int_0^t x(\tau)\,h(t-\tau)\,d\tau$$

③ $T \le t \le 2T$

$$y(t) = \int_0^T x(\tau)\,h(t-\tau)\,d\tau$$

④ $2T \le t \le 3T$

$$y(t) = \int_{t-2T}^{T} x(\tau)\,h(t-\tau)\,d\tau$$

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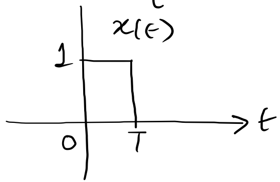
Convolution

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

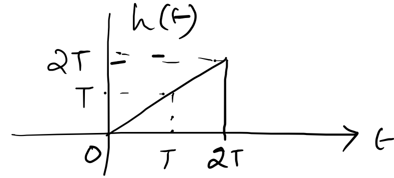
$0 < t < T$

$0 \le t < T$

$0 < t \le T$



$$h(t) = \begin{cases} t & 0 \le t \le 2T \\ 0 & \text{otherwise} \end{cases}$$

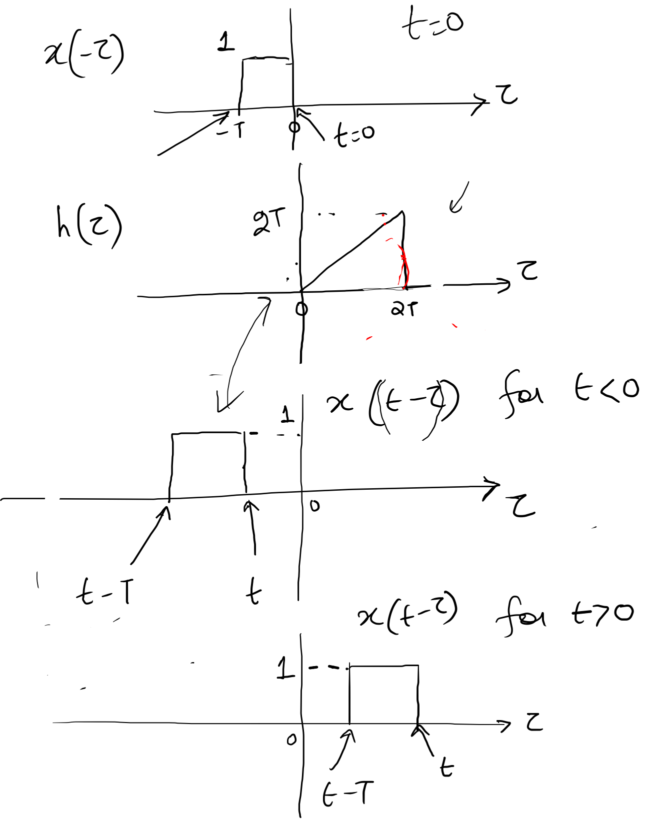


Find $y(t)$?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= x(t) * h(t)$$

$$= h(t) * x(t)$$



$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(0-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(-\tau)\, d\tau$$

① Case - 1 $\quad t < 0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= 0$$

② Case - 2 $\quad t > 3T$

$$y(t) = 0$$

Let $t = \frac{5T}{2}$

Case-3 : $0 < t < T$

$$y(t) = \int_0^t h(\tau)\,d\tau = \int_0^t \tau\,d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Case-4 : $T \leq t < 2T$

$$y(t) = \int_{t-T}^{t} h(\tau)\,d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{t} = \frac{t^2}{2} - \frac{(t-T)^2}{2}$$

$$= \frac{-T^2 + 2tT}{2} = -\frac{T^2}{2} + tT$$

Case-5 $2T \leq t \leq 3T$

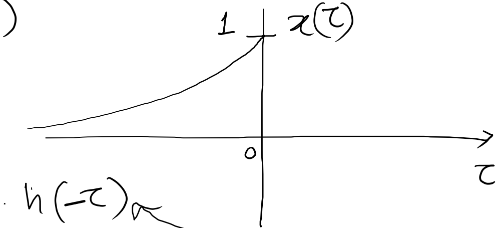
$$y(t) = \int_{t-T}^{2T} h(\tau)\,d\tau = \frac{\tau^2}{2}\Big|_{t-T}^{2T}$$

Ex

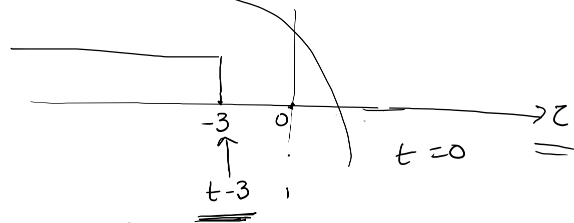
$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$

$x(\tau)$



$h(-\tau)$



Case-1 $\quad t \leq 3$

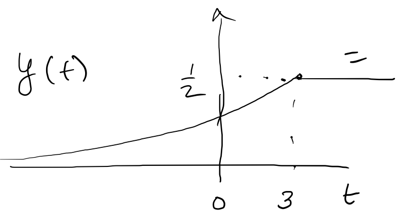
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{t-3} x(\tau)\, d\tau$$

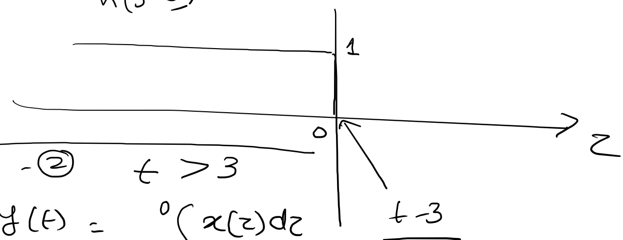
$$= \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \left.\frac{e^{2\tau}}{2}\right|_{-\infty}^{t-3}$$

$$= \frac{e^{2t-6}}{2}$$

$y(t)$



$h(3-\tau)$



Case-2 $\quad t > 3$

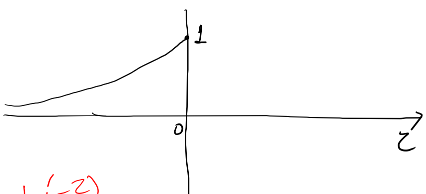
$$y(t) = \int_{-\infty}^{0} x(\tau)\, d\tau = \frac{1}{2}$$

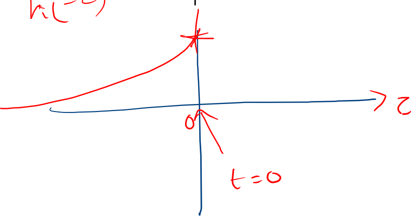
Ex

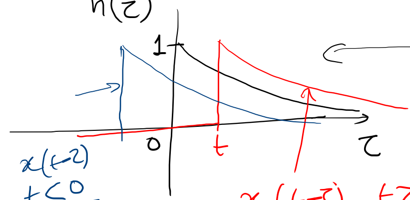
$$x(t) = e^{at}\,u(-t) \qquad a>0$$

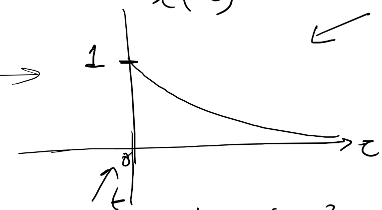
$$h(t) = e^{-at}\,u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

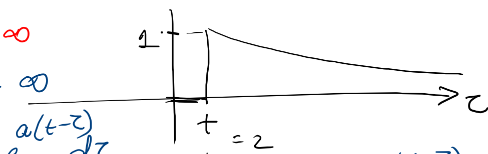








def $t=2$



Case-1 $t \geq 0$ $\qquad t$ — ∞

Case-2 $t<0$ $\qquad 0$ — ∞

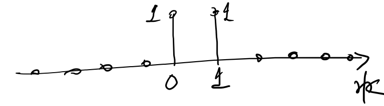
Case-1 $\quad y(t) = \int_{t}^{\infty} e^{-a\tau}\,e^{a(t-\tau)}\,d\tau$

Case-2 $\quad y(t) = \int_{0}^{\infty} e^{-a\tau}\,e^{a(t-\tau)}\,d\tau$

System Properties

there is an LTI System

$$h[n] = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$



S_1: $y[n] = x[n] + x[n-1]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

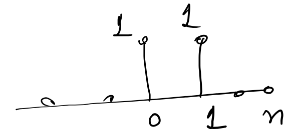
$$h[n] = \delta[n] + \delta[n-1] \implies h[k] = \delta[k] + \delta[k-1]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\delta[k] + \delta[k-1]\right) x[n-k]$$

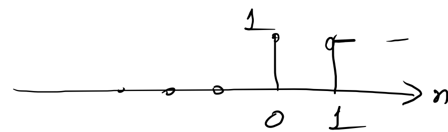
$$= \sum_{k=-\infty}^{\infty} \delta[k]\, x[n-k] + \sum_{k=-\infty}^{\infty} \delta[k-1]\, x[n-k]$$

$$= x[n] + x[n-1]$$

$$S_2 : \quad y[n] = \left(x[n] + x[n-1] \right)^2$$



$$S_3 : \quad y[n] = \max \left\{ x[n],\ x[n-1] \right\}$$



System Properties [holds true for both C.T. & D.T. Systems]

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(2) (i) Commutativity —

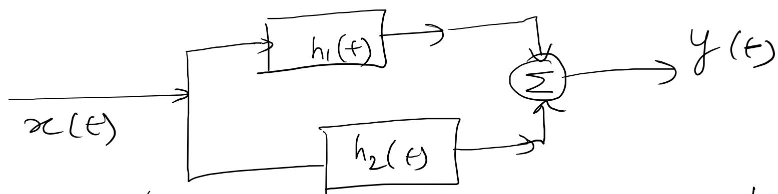
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

Prove that

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

(ii) Convolution distributes over addition

$$y(t) = x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$



$$y(t) = h(t) * \left(x_1(t) + x_2(t)\right) = h(t) * x_1(t) + h(t) * x_2(t)$$

Ex

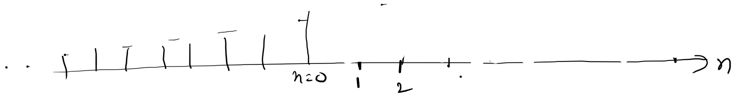
$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + 2^n u[-n] = x_1[n] + x_2[n]$$

$$y[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$x_1[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$+$

$$x_2[n] = 2^n u[-n]$$



(iii) Associativity

$$x(t) * [h_1(t) * h_2(t)]$$

$$= [x(t) * h_1(t)] * h_2(t)$$

order of these systems can also be changed for LTI systems

Invertibility —

$$x(t) \to \boxed{h_1(t)} \xrightarrow{w(t)} \boxed{h_2(t)} \to y(t)$$

If $h_1(t) * h_2(t) = \delta(t)$

$$y(t) = x(t) * \left(h_1(t) * h_2(t)\right)$$
$$= x(t) * \delta(t)$$
$$= x(t)$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

Operational definition of $\delta(t)$

$\delta(t) = 0 \quad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Non-LTI

$h(t, \tau)$

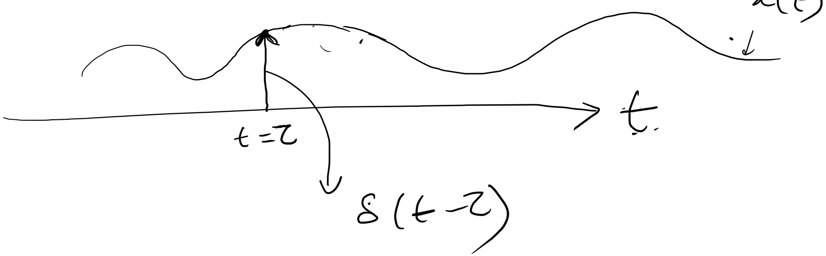
o/p at t

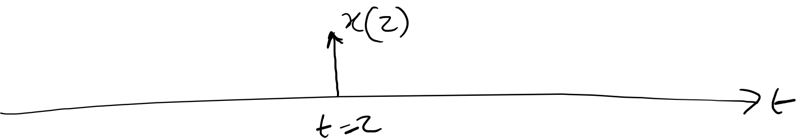
Unit impulse i/p applied at τ

D.T.

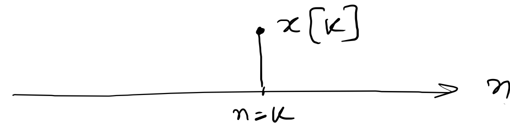
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$
$$= x[n] * \delta[n]$$

$$x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$$





$$\underline{x[k]\,\delta[n-k]}$$



Causality –

Claim: If a system is causal, $h(t) = 0$ for $t < 0$ (C.T)
or $h[n] = 0$ for $n < 0$ (D.T)

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k].$$

For causal

C.T.

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

D.T

$$\boxed{y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]} \Leftarrow \text{ if } h[n] = 0 \text{ for } n < 0$$

$$\boxed{y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]} \Leftarrow$$

Memory or memoryless systems

$S_1: \; y(t) = k\,x(t) \implies h(t) = k\,\delta(t)$

$S_2: \; y[n] = k\,x[n] \implies h[n] = k\,\delta[n]$

Stability —

BIBO Stability condition

For every bounded i/p signal, if we get bounded o/p signal system is stable.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

Let i/n signal is bounded &n $|x(t)| < M_x \;\; \forall\, t$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)\,x(t-\tau)|\,d\tau$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau$$

$$|y(t)| \leq M_x \int_{-\infty}^{\infty} |h(\tau)| d\tau \leq M_x \cdot \text{some finite no.} < \infty \quad \forall t$$

$h(t) = u(t)$
LTI, causal, with memory, unstable

If the system is absolutely integrable —

C.T. System

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

absolute integrability

D. T. System $\quad |y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

absolute summability
↓
system is stable

$h[n] = u[n]$
LTI, unstable, causal, with memory

Constant Coefficient difference equation for Causal LTI Systems

$$\sum_{k=0}^{N} b_k y[n-k] = \sum_{k=0}^{M} a_k x[n-k]$$

① N = order of the system

② The system is at rest — auxilliary conditions

If $x[n] = 0$ $\quad n < n_0$

$y[n] = 0$ $\quad n < n_0$

$y[n-1] = 0$ $\quad n < n_0$ $\quad$ if $N > 0$

$\vdots$

$y[n-N] = 0$ $\quad n < n_0$

(MA system)

moving average system

Case I : $N = 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \cdots + \frac{a_M}{b_0} x[n-M]$$

System is called Finite Impulse response FIR System

Case-2 Let $N = 1$, $M = 0$

a_0, b_0, b_1 are constants

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$\alpha_0 = a_0/b_0$

$\alpha_1 = b_1/b_0$

$$\boxed{y[n] = \alpha_0 x[n] - \alpha_1 y[n-1]}$$

Is it FIR?

If $x[n] = \delta[n]$

$y[n] = h[n]$

$h[0] = \alpha_0$

$h[1] = -\alpha_1 \alpha_0$

$h[2] = +\alpha_1^2 \alpha_0$

$h[3] = \vdots$

IIR system
Infinite Impulse response system

Autoregressive System
(AR) system

Case 3

$N > 0 \,,\; M > 0$

ARMA Systems

18|9|2024 $\boxed{\text{C.T.}}$ Constant coefficient differential equ. representation of Causal LTI system

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)$$

Condition of initial rest is satisfied

if $x(t) = 0$ for $t < t_0$

$\frac{d^k}{dt^k} y(t) = 0$ for $t < t_0$

$\forall \; k = 0, 1, \ldots N$

$N \equiv$ order of the system

Let $N = 2$ & $M = 3$

$$b_2 \frac{d^2}{dt^2} y(t) + b_1 \frac{d}{dt} y(t) + b_0 y(t) = a_0 x(t) + a_1 \frac{d}{dt} x(t) + a_2 \frac{d^2}{dt^2} x(t) + a_3 \frac{d^3}{dt^3} x(t)$$

Block Diagram representation

D.T.

$a_0 x[n] \equiv$ Scalar multiplier

$x[n-1] \equiv$ Delay element

$S_1: y[n] = a_0 x[n] + a_1 x[n-1]$

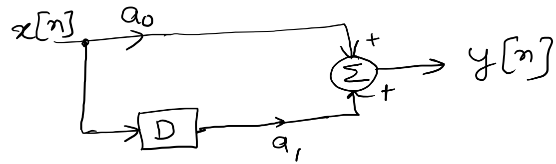
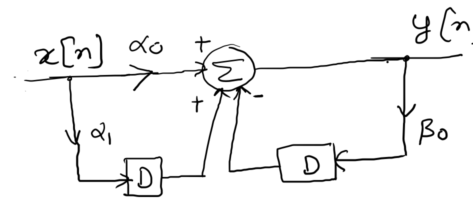


Fig.: S_1

$S_2 : b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$

$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$

$= \alpha_0 x[n] + \alpha_1 x[n-1] - \beta_0 y[n-1]$



where $\alpha_0 = \frac{a_0}{b_0}$

$\vdots$

constants

Fourier Series

C.T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \forall t$$

for Periodic Signal $x(t)$

Synthesis equation $\Rightarrow$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)}$$

vector

$\rightarrow$ basis fns.

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$$

$$\boxed{\omega_0 = \frac{2\pi}{T}}$$

$T \equiv$ fundamental time period of $x(t)$

$$\phi_1(t) = e^{j\omega_0 t} \qquad \phi_2(t) = e^{2j\omega_0 t}$$

$$\phi_{-1}(t) = e^{-j\omega_0 t} \qquad \phi_{-2}(t) = e^{-j2\omega_0 t}$$

. . . .

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$= \sum_{k=-\infty}^{\infty} a_k\,\delta[n-k]$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\phi_0(t) = e^{j0} = \cos 0 + j \sin 0$$

$$= 1$$

$$\boxed{\langle \phi_k(t), \phi_l(t) \rangle_T \triangleq \frac{1}{T}\int_T \phi_k(t)\,\phi_l^*(t)\,dt}$$ = Defn inner product

$$= \frac{1}{T}\int_T e^{jk\omega_0 t}\, e^{-jl\omega_0 t}\,dt$$

$= 1$ if $m = 0$

$$= \frac{1}{T}\int_T e^{j(k-l)\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T e^{jm\omega_0 t}\,dt = \left.\frac{e^{jm\omega_0 t}}{jm\omega_0 T}\right|_0^T = \frac{e^{jm2\pi} - e^{j0}}{jm\omega_0 T} = 0$$

$= 0$ if $m \neq 0$ if $k \neq l$

$= 1$ if $k = l$

Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{\phi_k(t)\}_{k\in\mathbb{Z},\ T}$ = span a vector space

form an orthonormal basis (O.N.B)

Analysis : $\boxed{a_k = \langle x(t), \phi_k(t) \rangle_T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_\ell(t)\rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_\ell(t) \right\rangle_T = \langle \cdots + a_{-10}\phi_{-10}(t) + \cdots + a_0\phi_0(t) + \cdots a_9\phi_9(t) + \cdots, \phi_\ell(t)\rangle_T$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \langle \phi_k(t), \phi_\ell(t)\rangle_T \right)$$

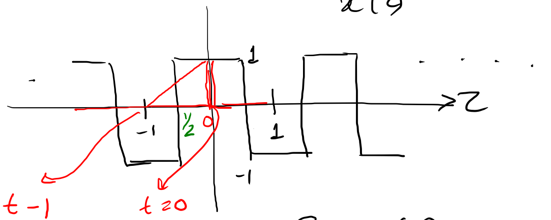
$$= \cdots \langle a_{-10}\phi_{-10}(t), \phi_\ell(t)\rangle_T + \cdots + \langle a_0\phi_0(t), \phi_\ell(t)\rangle_T + \cdots + \langle a_9\phi_9(t), \phi_\ell(t)\rangle_T + \cdots$$

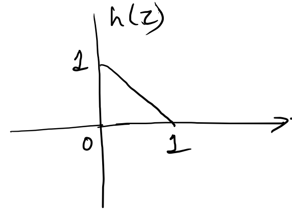
$$= \sum_{k=-\infty}^{\infty} a_k \delta[k-\ell]$$

$$\langle x(t), \phi_\ell(t)\rangle_T = a_\ell$$

=

Ex

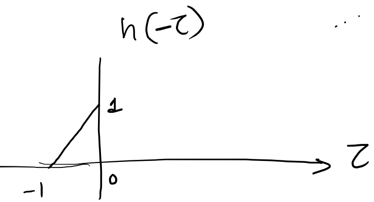




$h(-\tau) = 1+\tau$

$h(-\tau+t) = 1+\tau-t$

Convolve $x(t)$ & $h(t)$



$$y(0) = \int_{-1}^{-1/2} x(\tau)h(t-\tau)d\tau + \int_{-1/2}^{0} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-1}^{-1/2} (-1)(1+\tau)d\tau + \int_{-1/2}^{0} (1)(1+\tau)d\tau$$

$$= -\left(\tau + \frac{\tau^2}{2}\right)\Big|_{-1}^{1/2} + \left(\tau+\frac{\tau^2}{2}\right)\Big|_{-1/2}^{0}$$

<u>Case 1</u>: $0 \le t \le 1/2$

$$y(t) = -\int_{t-1}^{-1/2}(1+\tau-t)d\tau + \int_{-1/2}^{t}(1+\tau-t)d\tau$$

C.T. Fourier Series

20/9/2024

periodic signals

Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$$

$\omega_0 \rightarrow$ rad/s

$f_0 = \frac{1}{T}$ $\equiv$ per sec or Hz

$e^{jk\omega_0 t} = \phi_k(t)$ = pure imaginary exponential (in power)

$T \equiv$ fundamental time period

$$x(t) = 3 + 2e^{j\omega_0 t} + e^{j2\omega_0 t}$$

$$= 3 + 2(\cos\omega_0 t + j\sin\omega_0 t) + \cos 2\omega_0 t + j\sin 2\omega_0 t$$

$$= \sum_{k=0}^{2} a_k e^{jk\omega_0 t}$$

$a_0 = 3$

$a_1 = 2$

$a_2 = 1$

all other a_k's for $k \neq 0,1,2$ are zero

Analysis equation

$$a_k = \langle x(t), \phi_k(t) \rangle_T = \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt$$

$$a_k = \frac{1}{T}\int_T x(t)\,\phi_k^*(t)\,dt = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

$$\{\phi_k(t)\}_{k\in\mathbb{Z}} \equiv \text{orthogonal basis} \equiv \int_T \phi_m(t)\,\phi_k^*(t)\,dt = T\,\delta[m-k]$$

$$\text{R.H.S} = \frac{1}{T}\int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t)\right)\phi_k^*(t)\,dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \left(\int_T \phi_m(t)\,\phi_k^*(t)\right)dt\right]$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\left[a_m \underbrace{\int_T e^{j(m-k)\omega_0 t}\,dt}\right]$$

$$= \frac{1}{\cancel{T}}\sum_{m=-\infty}^{\infty} a_m \cdot \cancel{T}\,\delta[m-k] = a_k = \text{L.H.S.}$$

$\phi_1(t)$, $\phi_{-1}(t)$

$\downarrow$

I harmonic

$\phi_2(t)$, $\phi_{-2}(t)$

$\downarrow$

2nd harmonics

&

so on

Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Analysis : $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

e^{st} ≡ act as eigenfunctions of an LTI system

where $H(S)$ ≡ corresponding eigen value

$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

Let us assume $\boxed{x(t) = e^{st}}$ $\boxed{\text{where } s = j\omega_0}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{st} \cdot e^{-s\tau}\, d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$$

$$\boxed{y(t) = e^{st} H(s)}$$

$e^{st} \rightarrow \boxed{\text{LTI}} \rightarrow e^{st} \cdot H(S)$

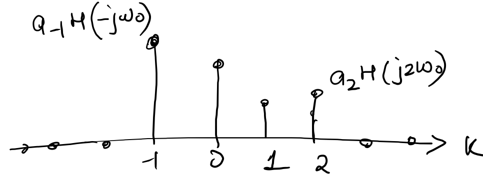
$A\underline{v} = \lambda \underline{v}$

Given an Line Spectrum

$x(t)$

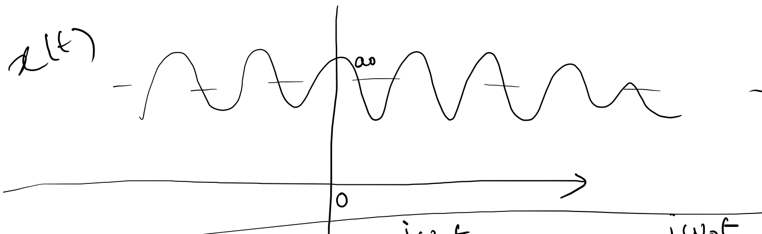
frequency domain representation

$\xrightarrow[\text{System}]{\text{LTI}}$



Line Spectrum of $y(t)$

$x(t)$



if $H(-j\omega_0) = 0$

$a_{-1}H(-j\omega_0) = 0$

& this freq. will disappear in o/p

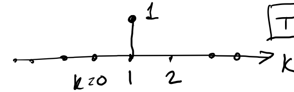
Given $\boxed{x(t) = a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}}$

Ex

$S: \quad y(t) = x(t-2)$ let $x(t) = e^{j2t}$

$\omega_0 = 2\ \text{rad/s} = \frac{2\pi}{T}$

$\boxed{T = \pi}$



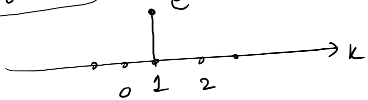
$$y(t) = e^{j2(t-2)}$$

$$= e^{j2t} \cdot e^{-j4}$$

$$= e^{j2t} H(j2)$$

$\boxed{H(j2) = e^{-j4}}$ $\boxed{s = j2}$

line spectrum of $y(t)$



Ex

What is $h(t)$? $h(t) = \delta(t-2)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = \int_{-\infty}^{\infty} \delta(\tau-2) e^{-j2\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau-2) \cdot e^{-j2.2} d\tau$$

$$= e^{-j4} \boxed{\int_{-\infty}^{\infty} \delta(\tau-2) d\tau}$$

$$= e^{-j4}$$

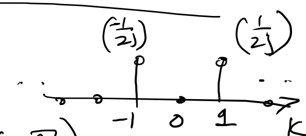
$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Ex

$$x(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \left(\frac{1}{2j}\right)e^{j\omega_0 t} + \left(\frac{-1}{2j}\right)e^{-j\omega_0 t}$$

Find and plot the line spectrum of $x(t)$



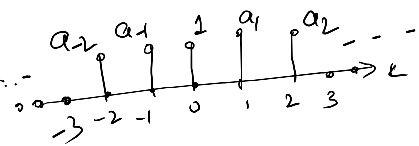
Ex

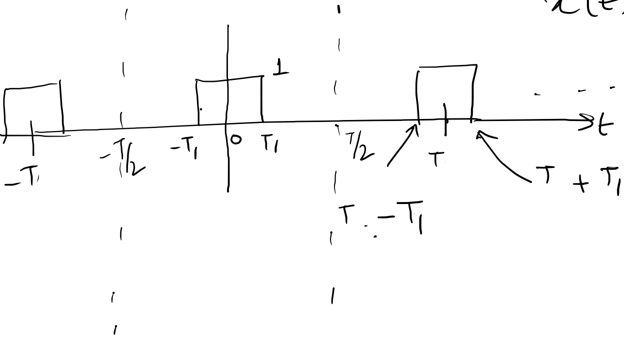
$$x(t) = 1 + \sin\omega_0 t + \cos\left(2\omega_0 t + \pi/4\right)$$

$$= 1 + \frac{e^{j\omega_0 t}}{2j} + \left(\frac{-1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$= 1 + \left(\frac{1}{2j}\right)\phi_1(t) + \left(\frac{-1}{2j}\right)\phi_{-1}(t) + \left(\frac{1}{2}e^{j\pi/4}\right)\phi_2(t) + \left(\frac{1}{2}e^{-j\pi/4}\right)\phi_{-2}(t)$$

$$= a_0 + a_1\phi_1(t) + a_{-1}\phi_{-1}(t) + a_2\phi_2(t) + a_{-2}\phi_{-2}(t)$$





$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

periodic

1. if $T \longrightarrow \infty$, signal will become aperiodic
2. there should be a dc bias term, so $a_0 \neq 0$
3.

$\omega_0 = \dfrac{2\pi}{T}$

↓

fundamental freq.

Solve it, plot the line spectrum.

25/9

C.T. F.S

1 Linearity

$x(t) \underset{}{\xrightleftharpoons{F.S}} a_k$

$y(t) \underset{}{\xrightleftharpoons{F.S.}} b_k$

$x(t)$ & $y(t)$ are periodic with fundamental time period $= T$.

$z(t) = \boxed{\alpha x(t) + \beta y(t)} \rightleftharpoons \alpha a_k + \beta b_k$

& $z(t) \equiv$ Same fundamental time period T.

$$\boxed{z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

$\omega_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \left(\alpha x(t) + \beta y(t)\right) e^{-jk\omega_0 t} dt = \alpha \cdot \underbrace{\frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt}_{a_k} + \beta \cdot \underbrace{\frac{1}{T}\int_T y(t) e^{-jk\omega_0 t} dt}_{b_k}$$

$$= \alpha a_k + \beta b_k \longrightarrow \text{Proved.}$$

(2) Time shifting

$x(t) \xrightarrow[T]{F.S.} a_k$ $\qquad \omega_0 = \frac{2\pi}{T}$

Find F.S. coeff. of $x(t-t_0)$, where t_0 = constant.

$$z(t) = x(t-t_0) \xrightarrow[T]{} \; ? = C_k = a_k e^{-jk\omega_0 t_0}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} z(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t-t_0) e^{-jk\omega_0 t} dt$$

Let $t - t_0 = \tau$

$dt = d\tau$

$$= \frac{1}{T}\int_{-T/2-t_0}^{T/2-t_0} x(\tau) e^{-jk\omega_0(t_0+\tau)} d\tau$$

$$C_k = e^{-jk\omega_0 t_0} \underbrace{\left(\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau\right)}_{} = a_k e^{-jk\omega_0 t_0}$$

3

Time - reversal

$$x(t) \xrightarrow[T]{F.S.} a_k$$

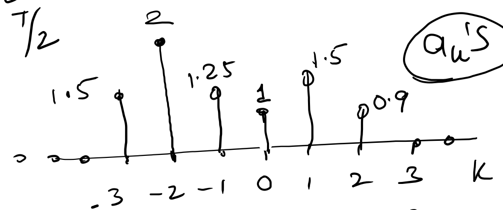
$$\boxed{\begin{aligned} \tau &= -t \\ d\tau &= -dt \end{aligned}}$$

$$z(t) = x(-t) \xrightarrow[T]{F.S.} c_k = a_{-k}$$

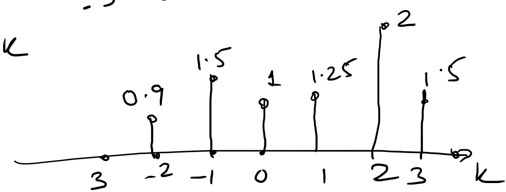
$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(-t)\, e^{-jk\omega_0 t}\, dt = -\frac{1}{T}\int_{T/2}^{-T/2} x(\tau)\, e^{jk\omega_0 \tau}\, d\tau$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{jk\omega_0 \tau}\, d\tau$$

$$c_{-k} = \frac{1}{T}\int_{T} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau = a_k$$



$$\boxed{c_k = a_{-k}}$$



$$c_0 = a_0$$

$$c_1 = a_{-1}$$

4

$x(t) \xrightarrow[T]{F.S.} a_k$ Time period T

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

$z(t) = x^*(t) \longrightarrow a_{-k}^*$

Case-1 Let $x(t)$ is real.

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$z(t) = x^*(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$x(t) = x^*(t)$

$$\boxed{a_k = a_{-k}^*}$$

$$\boxed{a_k^* = a_{-k}}$$

$x(t) = x(-t)$

Case-2 Let $x(t)$ is real & even.

$a_k^* = a_{-k}$ $\boxed{a_k = a_{-k}}$

$\boxed{a_k^* = a_k}$

Conclusion: a_k's are also real & even.

Case 3

$x(t)$ is real & odd

$x(t) = x^*(t) \;\rightarrow\; a_k^* = a_{-k}$

$x(-t) = -x(t) \;\rightarrow\; -a_k = +a_{-k}$

$$a_k^* = -a_k$$

coeff. are purely imaginary.

5 Time scaling

$x(t) \xrightarrow[T]{F.S.} a_k$ $\quad \alpha > 0 \quad \alpha \neq -1$

$z(t) = x(\alpha t) \xrightarrow[T/\alpha,\ \alpha\omega_0]{F.S.} c_k = ?$

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(\alpha\omega_0)t}$$

$$c_k = \frac{1}{T/\alpha}\int_{-T/2\alpha}^{T/2\alpha} x(\alpha t)\, e^{-jk(\alpha\omega_0)t}\, dt$$

def $\alpha t = \tau$

$\alpha\, dt = d\tau$

$dt = \frac{d\tau}{\alpha}$

$$= \frac{1}{T/\cancel{\alpha}}\int_{-T/2}^{T/2} x(\tau)\, e^{-jk\omega_0\tau}\, \frac{d\tau}{\cancel{\alpha}} = a_k$$

(6) Multiplication

$$x(t) \xrightarrow[T]{F.S.} a_k \qquad x(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}$$

$$y(t) \xrightarrow[T]{F.S.} b_k \qquad y(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}$$

$$z(t) = x(t)y(t) \xrightarrow{F.S.} c_k = ?$$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \left(x(t)y(t)\right) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T \sum_l \sum_m a_l b_m e^{j(l+m)\omega_0 t} e^{-jk\omega_0 t} dt$$

$$= \sum_l \sum_m a_l b_m \left(\frac{1}{T}\int_T e^{-j(k-l-m)\omega_0 t} dt \right) = 1$$

$$k-l-m = 0$$

$$l = k-m$$

$$= \boxed{\sum_m a_{k-m} b_m = \sum_l a_l b_{k-l}}$$

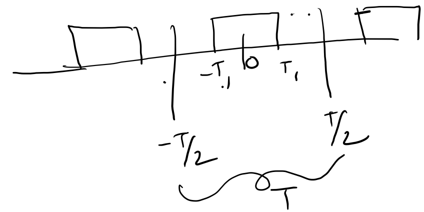
multiplication in the time domain $\Longleftrightarrow$ Convolution in the frequency domain

Fix T_1

① $T = 4T_1$

② $T = 8T_1$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$



$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} dt = \boxed{\frac{2T_1}{T} = a_0} = \frac{1}{2}$$

$\Longleftarrow$ $k \neq 0$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

$$a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1)$$

$k \neq 0$

$$= \frac{1}{k\pi}\sin\left(k\frac{2\pi}{4\cancel{T_1}}\cancel{T_1}\right)$$

$$= \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

7 Parseval's relation $\quad x(t) \xrightarrow[T]{F.S} a_k$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |a_k|^2$$

<u>D.T. F.S</u>

Given a discrete-time periodic signal $x[n]$

$N =$ Fundamental time period

Synthesis $\equiv$ $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

$0 \quad \omega_0, 2\omega_0, \ldots (N-1)\omega_0, N\omega_0$

$0, \frac{2\pi}{N}, \frac{2.2\pi}{N} \ldots (N-1)\cdot\frac{2\pi}{N}, 2\pi = 0$

$= \sum_{k=\langle N\rangle} a_k \phi_k[n]$

$\phi_{k+N}[n] = \phi_k[n] = e^{jk\omega_0 n}$

$\phi_k[n] = e^{jk\omega_0 n}$

$$\langle x[n], \phi_l[n] \rangle_N = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jl\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \left(\sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \right) e^{-jl\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-l)\omega_0 n}$$

$$= \sum_{k=\langle N \rangle} a_k \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} e^{j(k-l)\omega_0 n}}_{=1 \text{ when } k=l}$$

$$= \boxed{a_l = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \phi_l^*[n]}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

Lecture 13

Recall the problem from the last lecture:

Find the absolute maximum and minimum value of the function $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular plane in the first quadrant bdd by the lines $x=0$, $y=0$, and $x+y=9$.

→ Natural domain ?

→ restricted domain ?

→ This problem is an example of constrained optimization [discussion]

Problem 1: Find the point $P(x,y,z)$ closest to the origin on the plane $2x+y-z-5=0$.

Solution:

Minimize $g(x,y,z) := \sqrt{x^2+y^2+z^2}$

Subject to the constraint $2x+y-z-5=0$.

Alternatively,

minimize $h(x,y,z) = x^2+y^2+z^2$

Subject to the constraint $2x+y-z-5=0$.

Alternatively,

minimize $f(x,y) = x^2+y^2+(2x+y-5)^2$

(No Constraint!)

To this end,

$f_x = 0 \Rightarrow x = 5/3$

$f_y = 0 \Rightarrow y = 5/6$ (Verify!)

Use the second derivative test to ensure that $(5/3, 5/6)$ is a point of minima for f.

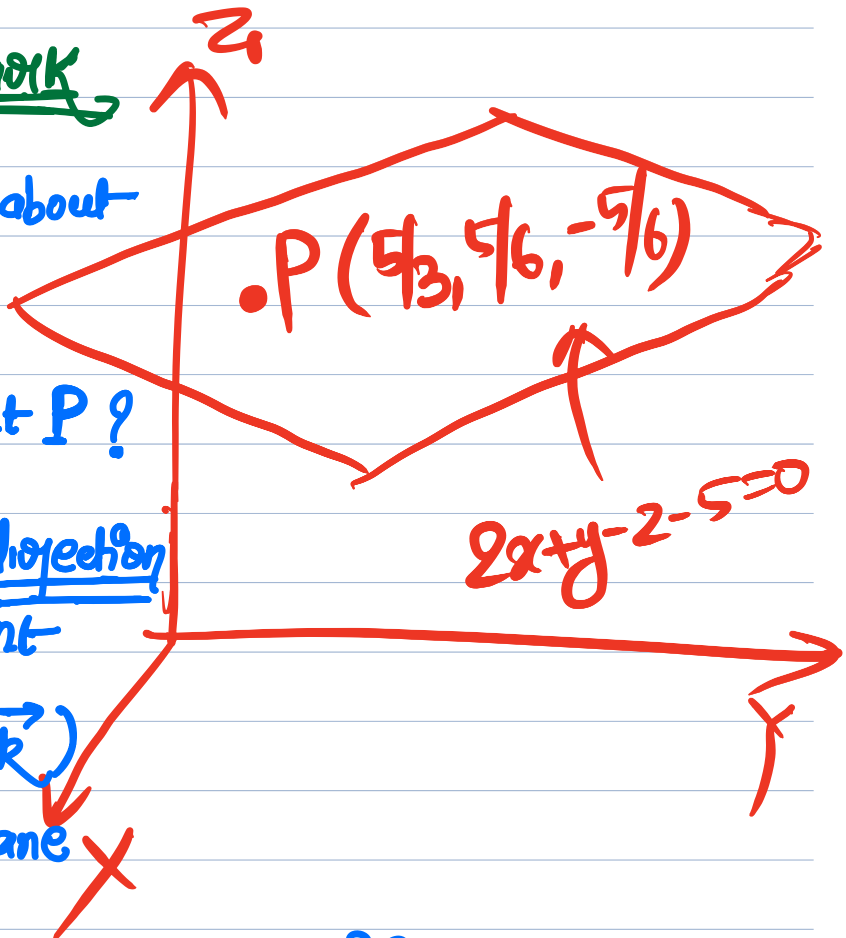
Thus, $z = 2x + y - 5 = 2(5/3) + (5/6) - 5 = -5/6$.

$\therefore$ $(5/3, 5/6, -5/6)$ is the required point. ■

Digression/Remark

What can be said about the normal line through the above-found-point P?

(a) It is in the direction of the gradient (i.e., $2\vec{i} + \vec{j} - \vec{k}$) of the given plane



(b) It passes through the origin.

Question: Let Q be another point on the above plane (i.e., $Q \neq P$).
Does the normal line passing through Q also pass through the origin? [HW]

End of the remark

[Constrained Optimization]

Problem 2: Find the point(s) closest to the origin on the Hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

The idea is to re-phrase the problem of constrained optimization into a problem without constraints by using the method of substitution.

Optimize $g(x,y,z) = x^2 + y^2 + z^2$
subject to the constraint $x^2 - z^2 - 1 = 0$.

Approach II: Substitute $x^2 = z^2 + 1$. Then, we need to minimize $f(y,z) = (z^2+1) + y^2 + z^2$. That is,

$$f(y,z) = 1 + y^2 + 2z^2$$

$$\begin{cases} f_y = 2y = 0 \Rightarrow y = 0; \text{ and} \\ f_z = 4z = 0 \Rightarrow z = 0 \end{cases}$$

Exercise: Verify, using the second derivative test, that the point $(y,z) = (0,0)$ is a point of minimum for the function $f(y,z) = 1 + y^2 + z^2$.

Then, we find x by using $x^2 = 1 + z^2$.

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Consequently, the points on the hyperbolic cylinder that are closest to the origin are $(1,0,0)$ and $(-1,0,0)$. ■

[Now, there is nothing special (at least not in the first glance; or is there?) in choosing to substitute $x^2 = z^2 + 1$. Why can't one choose to substitute $z^2 = x^2 - 1$?

Approach II: Substitute $z^2 = x^2 - 1$.

Minimize $h(x,y) = x^2 + y^2 + (x^2 - 1)$

$$= 2x^2 + y^2 - 1.$$

Then

$$h_x = 4x = 0 \Rightarrow x = 0;$$

$$h_y = 2y = 0 \Rightarrow y = 0.$$

[**Exercise:** Verify that $(0,0)$ is a point of local minimum for $h(x,y)$.

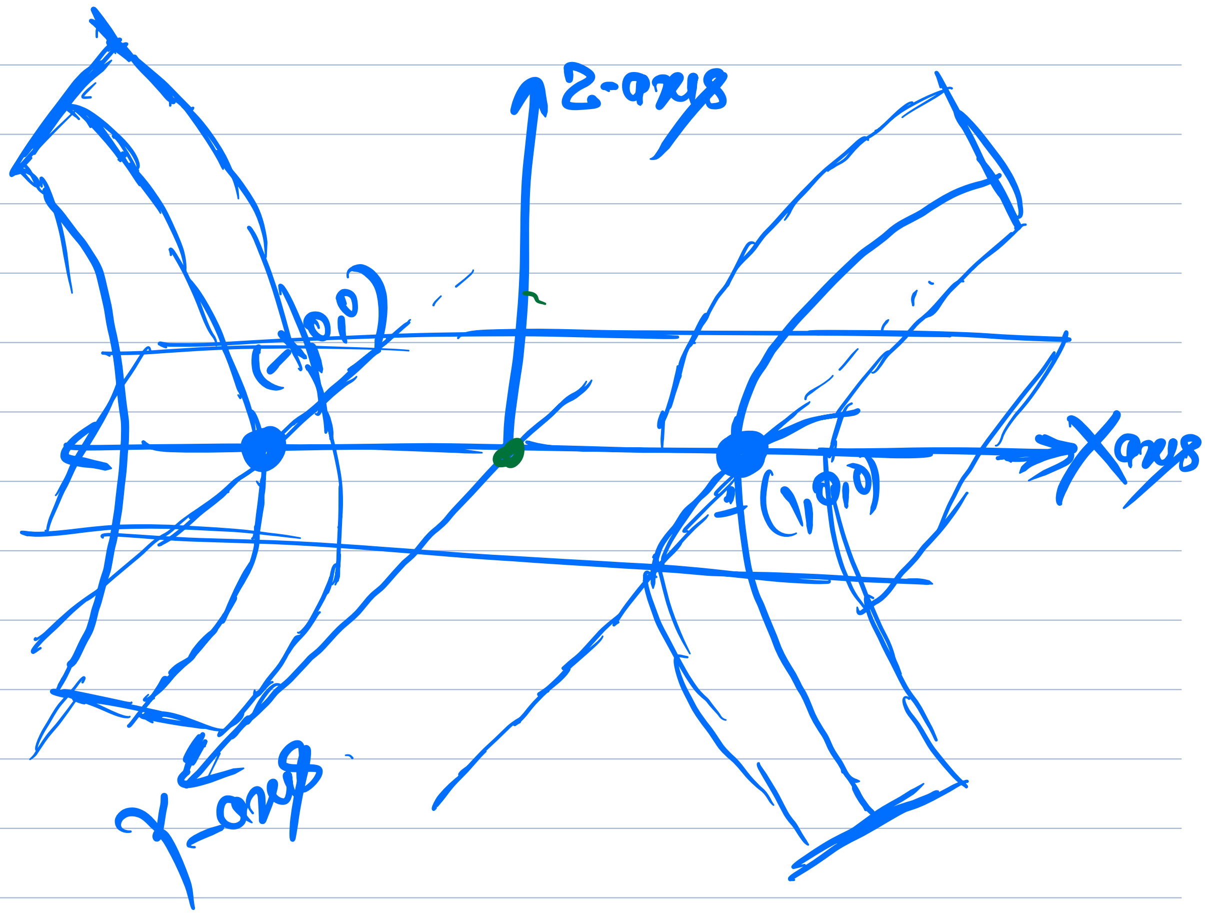
But then, $z^2 = x^2 - 1 \Rightarrow z^2 = -1$

which is an impossible situation!

The point $(0,0,z)$ doesn't lie in $\mathbb{R}^3$, let alone on the hyperbolic cylinder!

What went wrong?

Let's look at the Surface:



→ Clearly, we couldn't choose values of x that lie between -1 and 1, for those won't yield points that are on the cylinder.

→ Approach II worked because we chose "y" and "z" to be the ind. variable and "x" to be the dependent variable. (Elaborate.)

→ Approach II failed since "x" was chosen to be an independent variable — which created the problem!

Conclusion: You need to be cautious when tackling problems of constrained optimization using the substitution method.

This motivates us to develop a geometric method

Approach III: [**The method of Lagrange multipliers**]
(Geometric method)

minimize $h(x,y,z) = x^2+y^2+z^2$

subject to the constraint $\underbrace{x^2-z^2-1=0}_{(x^2-z^2=1)}$.

... next lecture