Lecture - 05

Continuity of a function at a point

Let $S \subseteq \mathbb{R}$ and $f: S \to \mathbb{R}$ be a function.
We say that f is continuous at $a \in \mathbb{R}$ if

(i) $a \in \mathrm{dom}(f) = S$;

(ii) $\lim_{x \to a} f(x)$ exists;

(iii) $\lim_{x \to a} f(x) = f(a)$.

$\Updownarrow$ (Equivalently,)

Let $S \subseteq \mathbb{R}$, $a \in S$ and $f: S \to \mathbb{R}$ be a function.
We say that f is continuous at $a \in S$ if
$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that for all
$x \in \mathrm{dom}(f) = S$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Analogously,

Let $S \subseteq \mathbb{R}^2$, $a = (a_1, a_2) \in S$ and $f : S \to \mathbb{R}$ be a function. We say that **f is continuous at $a \in S$** if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $(x_1, x_2) \in dom(f) = S$

$$\underbrace{\|(x_1, x_2) - (a_1, a_2)\|}_{= \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}} < \delta \Rightarrow |f(x_1, x_2) - f(a_1, a_2)| < \varepsilon$$

Note: f is called "continuous" if it is continuous at each point of its domain.

Example 1: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is f continuous at $(0,0)$?

Soln: (See Example 6 from the previous lecture)

Example 2: Let $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$.

→ Prove/Verify that

(a) $(0,0) \notin \text{dom}(f)$

(b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists

[Hint: See Example 8 of the previous lecture

→ define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Example 3: Let $g(x,y) = \dfrac{3x^2y}{x^2+y^2}$.

Define $g(0,0)$ in such a way that extends g to be continuous at the origin.

Example 4: Let $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$.

Define $f(0,0)$ in such a way that extends f to be continuous at the origin.

Homework: (use polar coordinates to find the limit of f at the point $(0,0)$)

Example 5: Let
$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is continuous at every point except $(0,0)$.

Solution: Hint: Try two cases:

Case I: When $(x_0,y_0) \neq (0,0)$. Now, use $\varepsilon-\delta$ argument to show that f is continuous at (x_0,y_0).

Case II: Show that f is not continuous at $(0,0)$.

Question (8): At what points in the plane/space are the following functions continuous?

(1) $g(x,y) = \sin\left(\frac{1}{xy}\right)$

(2) $h(x,y) = \dfrac{x^2+y^2}{x^2-3x+2}$

(3) $g(x,y) = \dfrac{x+y}{2+\cos x}$

(4) $f(x,y,z) = \dfrac{1}{x^2+z^2-1}$

(5) $h(x,y,z) = \dfrac{1}{|y|+|z|}$

Partial derivatives

Let us first recall the following definition in the case of functions of one variable.

defn: Let $D \subseteq \mathbb{R}$, let $x_0 \in D$ be an interior point and let $f : D \to \mathbb{R}$ be a function.
We say that <u>f is differentiable at x_0</u>, or, <u>f has a derivative at x_0</u>, provided that the following limit exists,

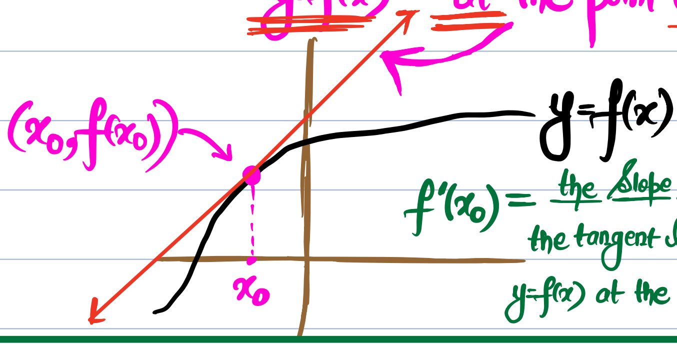
$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad \left(= \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \right).$$

When this limit exists, we denote it by

$$f'(x_0) \text{ or } \left.\frac{df}{dx}\right|_{x=x_0}$$

Question: <u>What is the geometrical interpretation of $f'(x_0)$?</u>

<u>the tangent line to the curve $y=f(x)$ at the point $(x_0, f(x_0))$</u>



$f'(x_0) = \dfrac{\text{the slope of}}{\text{the tangent line to the curve}}$
$y=f(x)$ at the point $(x_0, f(x_0))$

On the language you may encounter

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then the following statements essentially mean the same thing.

- f is differentiable at a point $x_0 \in D$
- f has a derivative at a point $x_0 \in D$

Question

What is the analogous notion of the "derivative" in case of $f: \mathbb{R}^2 \to \mathbb{R}$, or $f: \mathbb{R}^n \to \mathbb{R}$, or $f: \mathbb{R}^n \to \mathbb{R}^m$? [$\longrightarrow$ Later!]

First, let us talk about partial derivatives.

defn: Let $D \subseteq \mathbb{R}^2$, let $(x_0, y_0) \in D$ be an interior point of D and let $f: D \to \mathbb{R}$ be a real-valued function of two variables.

We say that **f has a partial derivative w.r.t x at the point (x_0, y_0)** provided that the following limit exists,

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

When this limit exists, we denote it by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \text{ or } f_x(x_0, y_0) \text{ or } \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}$$

and call it "**the partial derivative of f w.r.t. x at the point (x_0, y_0).**"

Example: $f(x,y) = xy + 3x^2 + y$

Compute $\frac{\partial f}{\partial x}$ at $(1,2)$.

$$\frac{\partial f}{\partial x} = y + 6x$$

$$\frac{\partial f}{\partial x}(1,2) = 2+6 = 8.$$

Example: Calculate the partial derivative $\frac{\partial f}{\partial x}$ of the function

$$f(x,y) = x^3 - 3x^2y^3 + y^2.$$

Soln: $$\frac{\partial f}{\partial x} = 3x^2 - 6xy^3.$$

defn: Let $D \subseteq \mathbb{R}^2$, let $(x_0, y_0) \in D$ be an interior point of D and let $f : D \to \mathbb{R}$ be a real-valued function of two variables.

We say that <u>f has a partial derivative w.r.t y at the point (x_0, y_0)</u> provided that the following limit exists,

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.$$

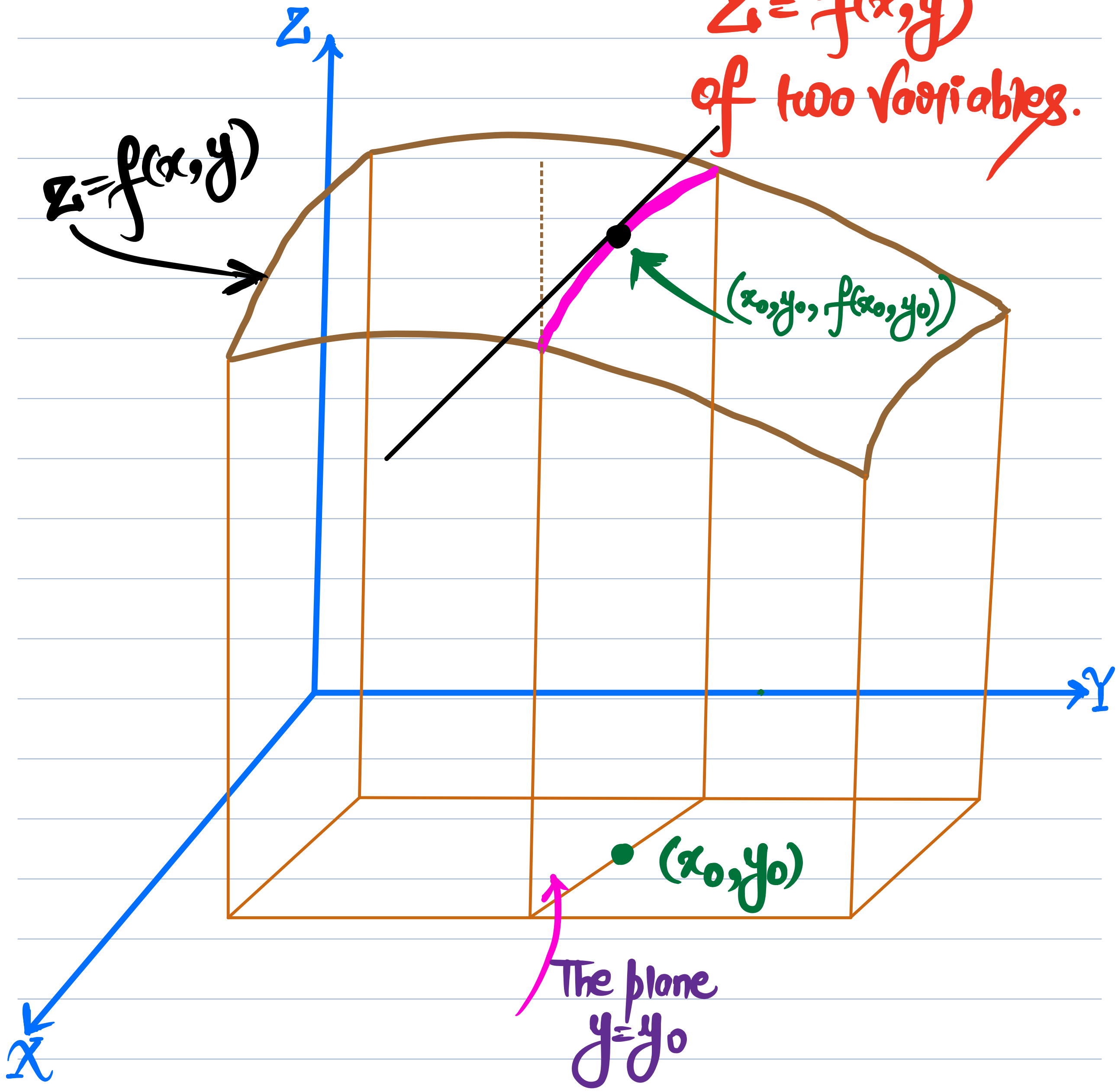
When this limit exists, we denote it by

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \text{ or } f_y(x_0, y_0) \text{ or } \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0}$$

and call it "<u>the partial derivative of f w.r.t. y at the point (x_0, y_0)</u>.

Geometric interpretation of partial derivatives —

for a function
$z = f(x,y)$
of two variables.



- The graph of $z = f(x,y)$ is a surface — the one sketched above.

- (x_0, y_0) is a given point in the interior of the domain of the function (this is a point in the X-Y plane, such that $(x_0, y_0, f(x_0, y_0))$ is a point on the surface.

- We wish to interpret $\frac{\partial f}{\partial x}$ at (x_0, y_0)
- Treat the variable "y" as constant, and the constant must of course be y_0, i.e., $y = y_0$.

- To hold y fixed at the value y_0 means To intersect the surface $z = f(x,y)$ with the plane $y = y_0$.

- The intersection of the surface $z=f(x,y)$ and the plane $y=y_0$ is the curve $z=f(x,y_0)$.

(this curve is on the plane $y=y_0$)

- What does then the value

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) \text{ denote ?}$$

It is the slope of the tangent line to the curve $z=f(x,y_0)$ at the point $x=x_0$.