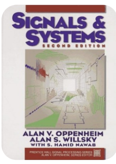
Exercise 22

Chapter 2, Page 141



Signals and Systems

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Step 1

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The convolution of $x(t)$ and $h(t)$ is given by,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \qquad (1)$$

Given,

$$x(t) = e^{-\alpha t}u(t)$$
$$h(t) = e^{-\beta t}u(t)$$

(i) When $\alpha \neq \beta$:

The signals $x(t)$ and $h(t)$ ae as shown in Figure.2.22.1(a) and Figure.2.22.1(b).

Figure.2.22.1(c) and Figure.2.22.1(d) represents $x(\tau)$ and $h(t-\tau)$ respectively. Figure.2.22.1(e) gives the convolution $y(t)$ for $t < 0$, but it can be observed from the figure that there is no overlapping of signals $x(\tau)$ and $h(t-\tau)$ for $t < 0$. Therefore,

$$y(t) = 0, \quad \text{if } t < 0$$

From Figure.2.22.1(f), if $t \geq 0$, there will be overlapping of signals and the limits of overlapping or integration are from $\tau = 0$ to $\tau = t$. Therefore, for $t \geq 0$,

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t (1)e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau$$
$$y(t) = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau$$
$$y(t) = e^{-\beta t}\left[-\frac{e^{-(\alpha-\beta)\tau}}{\alpha-\beta}\right]_0^t = e^{-\beta t}\left[\frac{1-e^{-(\alpha-\beta)t}}{\alpha-\beta}\right]$$
$$y(t) = e^{-\beta t}\frac{1-e^{-(\alpha-\beta)t}}{\alpha-\beta}, \quad \text{if } t \geq 0$$
$$y(t) = e^{-\beta t}\frac{\left(1-e^{-(\alpha-\beta)t}\right)}{\alpha-\beta}u(t)$$

Step 2

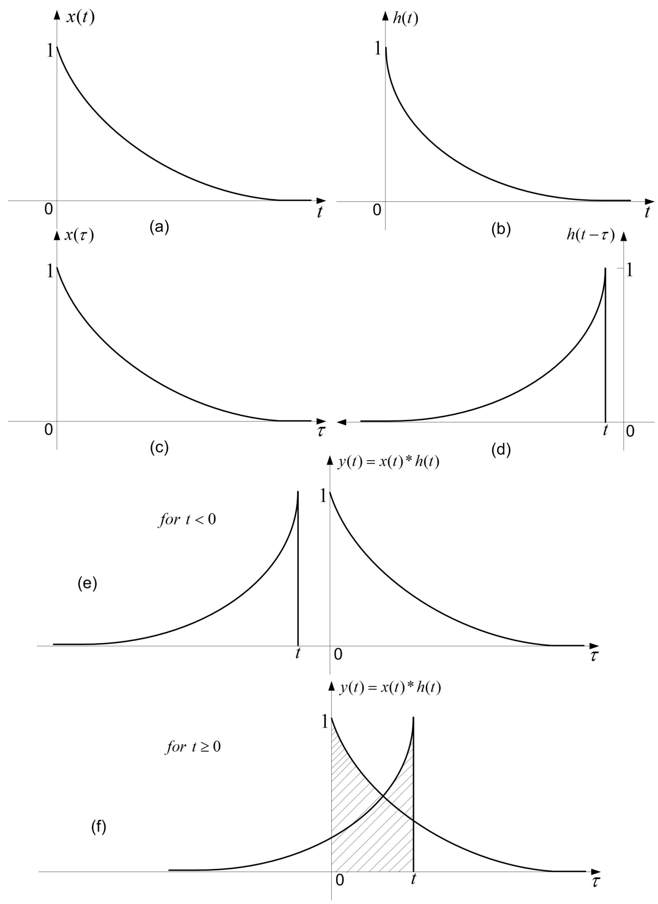


Figure.2.22.1

Step 3

(ii) When $\alpha = \beta$: Figure.2.22.1(e) gives the convolution $y(t)$ for $t < 0$, but it can be observed from the figure that there is no overlapping of signals $x(\tau)$ and $h(t-\tau)$ for $t < 0$.

Therefore,

$$y(t) = 0, \quad \text{if } t < 0$$

From Figure.2.22.1(f), if $t \geq 0$, there will be overlapping of signals and the limits of overlapping or integration are from $\tau = 0$ to $\tau = t$. Therefore, for $t \geq 0$,

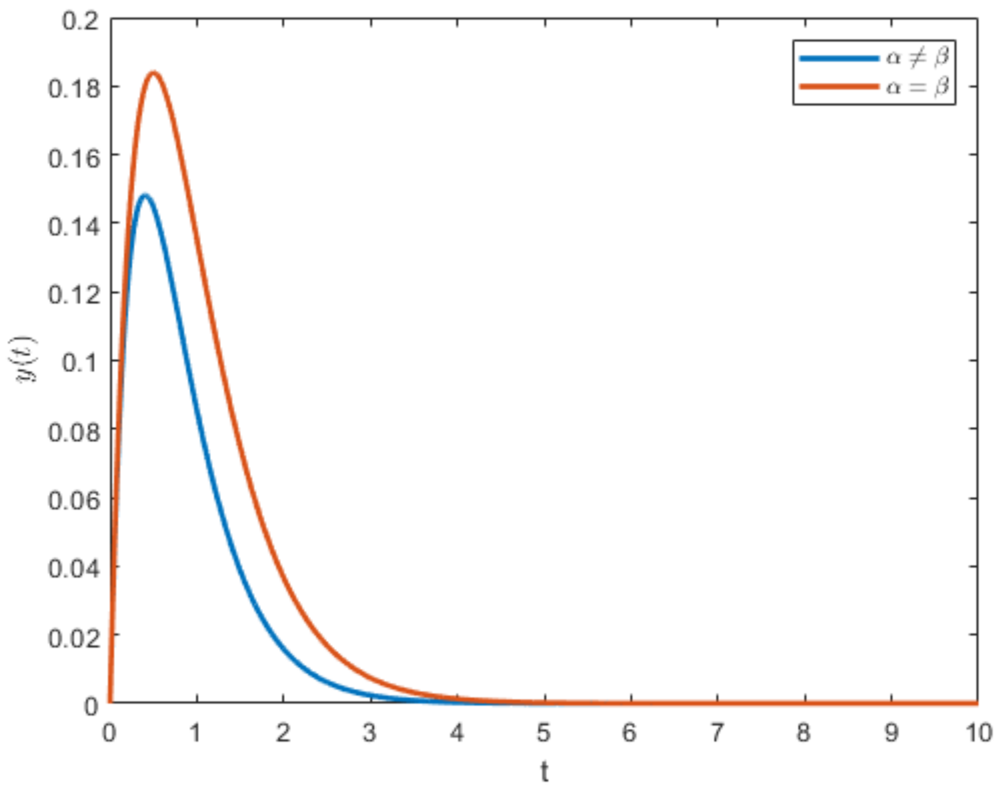
$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t (1)e^{-\alpha\tau}e^{-\alpha(t-\tau)}d\tau = e^{-\alpha t}\int_0^t e^{-(\alpha-\alpha)\tau}d\tau$$
$$y(t) = e^{-\alpha t}\int_0^t e^{-(0)\tau}d\tau = e^{-\alpha t}\int_0^t (1)\tau d\tau$$
$$y(t) = e^{-\alpha t}\left[\tau\right]_0^t = e^{-\beta t}\left[t-0\right]$$
$$y(t) = te^{-\alpha t}, \quad \text{if } t \geq 0$$
$$y(t) = te^{-\alpha t}u(t)$$

Therefore,

$$y(t) = \begin{cases} e^{-\beta t}\left(\frac{1-e^{-(\alpha-\beta)t}}{\alpha-\beta}\right)u(t), & \alpha \neq \beta \\ te^{-\alpha t}u(t), & \alpha = \beta \end{cases}$$

Step 4

The plot of the output signal $y(t)$ of part (a) is given below,



Step 5

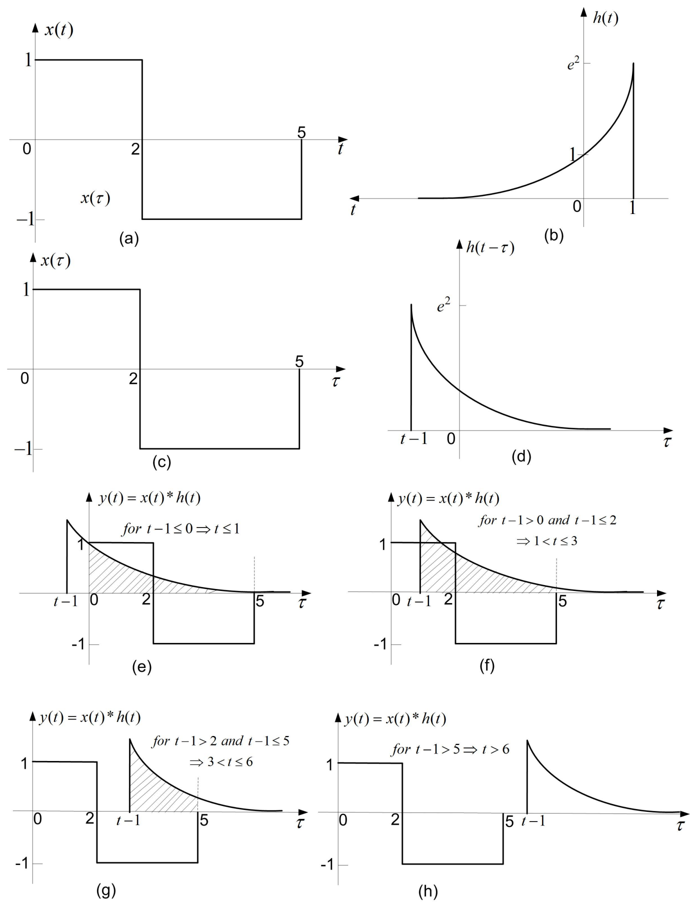


Figure.2.22.2

\textbf{b) Given,

$$x(t) = u(t) - 2u(t-2) + u(t-5)$$
$$h(t) = e^{2t}u(1-t)$$

The signals $x(t)$ and $h(t)$ ae as shown in Figure.2.22.2(a) and Figure.2.22.2(b).

Figure.2.22.2(c) and Figure.2.22.2(d) represents $x(\tau)$ and $h(t-\tau)$ respectively. Figure.2.22.2(e) gives the convolution $y(t)$ for $t-1 \le 0 \implies t \le 1$, there will be overlapping of signals and the limits of overlapping or integration are from $\tau = 0$ to $\tau = 5$. Therefore,

$$y(t) = \int_0^5 x(\tau)h(t-\tau)d\tau = \int_0^2 (1)e^{2(t-\tau)}d\tau + \int_2^5 (-1)e^{2(t-\tau)}d\tau$$
$$y(t) = e^{2t}\int_0^2 e^{-2\tau}d\tau - e^{2t}\int_2^5 e^{-2\tau}d\tau$$

$$y(t) = e^{2t}\left[-\frac{e^{-2\tau}}{2}\right]_0^2 - e^{2t}\left[-\frac{e^{-2\tau}}{2}\right]_2^5$$
$$y(t) = e^{2t}\left[-\frac{e^{-2(2)}}{2} + \frac{e^{-2(0)}}{2} + \frac{e^{-2(5)}}{2} - \frac{e^{-2(2)}}{2}\right]$$
$$y(t) = \left[\frac{e^{-10} - 2e^{-4} + 1}{2}\right]e^{2t}, \qquad \text{for } t \leq 1$$

From Figure.2.22.2(f), if ($t - 1 > 0 \implies t > 1$ and $t - 1 \leq 2 \implies t \leq 3$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau = t - 1$ to $\tau = 5$. Therefore, for $1 < t \leq 3$,

$$y(t) = \int_{t-1}^{5} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{2} (1)e^{2(t-\tau)}d\tau + \int_{2}^{5} (-1)e^{2(t-\tau)}d\tau$$
$$y(t) = e^{2t}\int_{t-1}^{2} e^{-2\tau}d\tau - e^{2t}\int_{2}^{5} e^{-2\tau}d\tau$$
$$y(t) = e^{2t}\left[-\frac{e^{-2\tau}}{2}\right]_{t-1}^{2} - e^{2t}\left[-\frac{e^{-2\tau}}{2}\right]_2^5$$
$$y(t) = e^{2t}\left[-\frac{e^{-2(2)}}{2} + \frac{e^{-2(t-1)}}{2} + \frac{e^{-2(5)}}{2} - \frac{e^{-2(2)}}{2}\right]$$
$$y(t) = \left[\frac{e^{-10} - 2e^{-4} + e^{-2t}e^{2}}{2}\right]e^{2t}$$
$$y(t) = \frac{e^{2(t-5)} - 2e^{2(t-2)} + e^{2}}{2}, \qquad \text{for } 1 < t \leq 3$$

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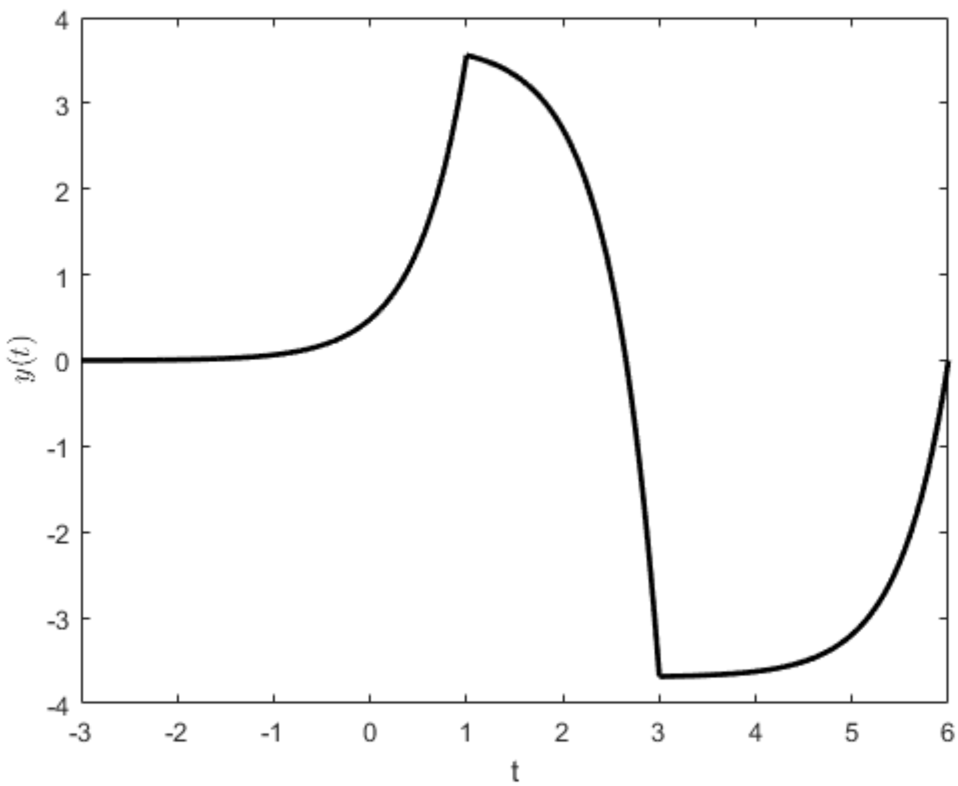
From Figure.2.22.2(g), if ($t - 1 > 2 \implies t > 3$ and $t - 1 \leq 5 \implies t \leq 6$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau = t - 1$ to $\tau = 5$. Therefore, for $3 < t \leq 6$,

$$y(t) = \int_{t-1}^{5} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{5} (-1)e^{2(t-\tau)}d\tau$$
$$y(t) = -e^{2t}\int_{t-1}^{5} e^{-2\tau}d\tau$$
$$y(t) = -e^{2t}\left[-\frac{e^{-2\tau}}{2}\right]_{t-1}^{5}$$

Step 7

The plot of the output signal $y(t)$ of part (b) is given below,



Step 8

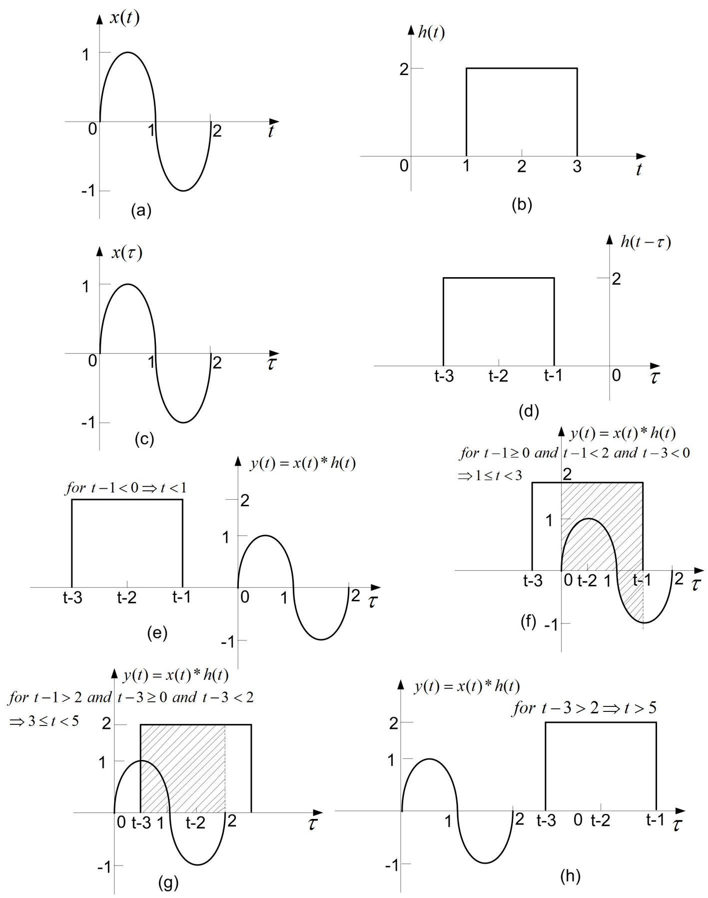


Figure.2.22.3

The signals $x(t)$ and $h(t)$ ae as shown in Figure.2.22.3(a) and Figure.2.22.3(b).

Figure.2.22.3(c) and Figure.2.22.3(d) represents $x(\tau)$ and $h(t-\tau)$ respectively.
Figure.2.22.3(e) gives the convolution $y(t)$ for $t-1<0 \implies t<1$, but it can be observed from the figure that there is no overlapping of signals $x(\tau)$ and $h(t-\tau)$ for $t<1$.
Therefore,

$$y(t) = \quad 0, \quad \text{if } t < 1$$

From Figure.2.22.3(f), if ($t-1 \geq 0 \implies t \geq 1$ and $t-1<2 \implies t<3$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau = t-1$ to $\tau = 5$. Therefore, for $1 \leq t < 3$,

$$y(t) = \int_0^{t-1} x(\tau)h(t-\tau)d\tau = \int_0^{t-1} (2)\sin(\pi\tau)d\tau$$

$$y(t) = 2\int_0^{t-1} \sin(\pi\tau)d\tau = 2\left[-\frac{\cos(\pi\tau)}{\pi}\right]_0^{t-1}$$
$$y(t) = 2\left[\frac{\cos(0) - \cos(\pi(t-1))}{\pi}\right]$$
$$y(t) = \frac{2}{\pi}\left[1 - \cos(\pi(t-1))\right], \qquad \text{for } 1 \le t < 3$$

From Figure.2.22.3(g), if ($t - 1 \geq 2 \implies t \geq 3$ and $t - 3 \leq 3 \implies t \leq 5$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau = t - 1$ to $\tau = 5$. Therefore, for $3 \leq t \leq 5$,

$$y(t) = \int_{t-3}^{2} x(\tau)h(t-\tau)d\tau = \int_{t-3}^{2} (2)\sin(\pi\tau)d\tau$$
$$y(t) = 2\int_{t-3}^{2} \sin(\pi\tau)d\tau = 2\left[-\frac{\cos(\pi\tau)}{\pi}\right]_{t-3}^{2}$$
$$y(t) = 2\left[\frac{\cos(\pi(t-3)) - \cos(2\pi)}{\pi}\right]$$
$$y(t) = \frac{2}{\pi}\left[\cos(\pi(t-3)) - 1\right], \qquad \text{for } 3 \le t \le 5$$

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Figure.2.22.3(h) gives the convolution $y(t)$ for $t - 3 > 2 \implies t > 5$, but it can be observed from the figure that there is no overlapping of signals $x(\tau)$ and $h(t-\tau)$ for $t > 5$.

Therefore,

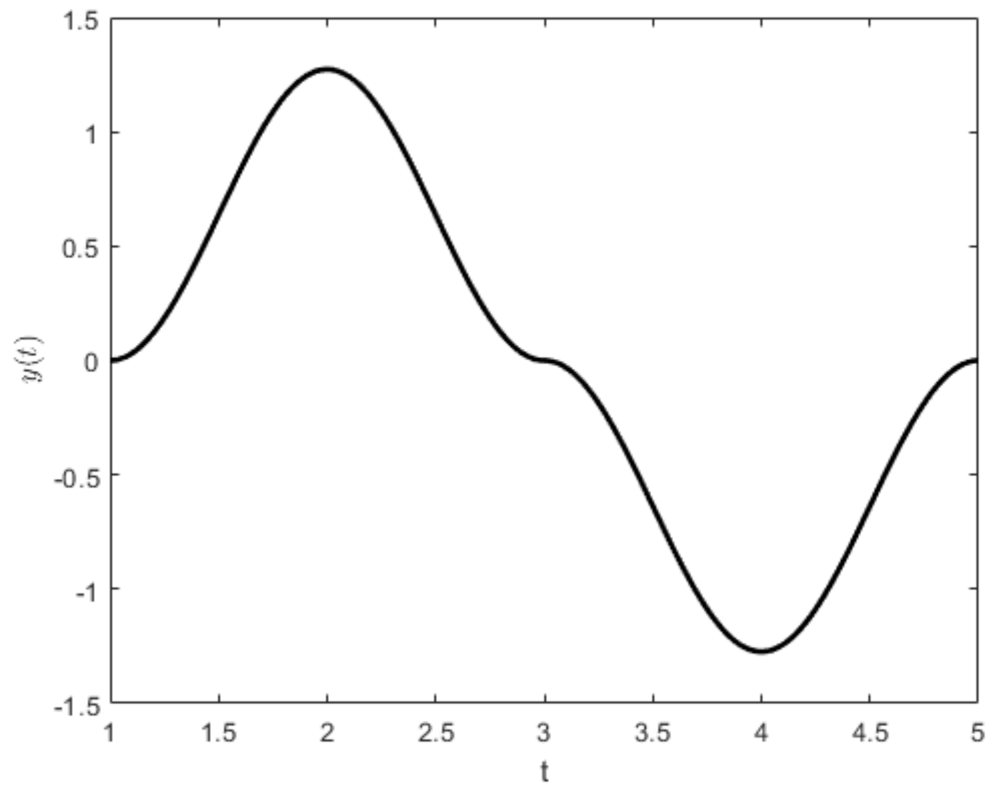
$$y(t) = \quad 0, \quad \text{if } t > 5$$

Therefore,

$$y(t) = \begin{cases} 0, & t < 1 \\ \frac{2}{\pi}\left[1 - \cos(\pi(t-1))\right], & 1 \le t < 3 \\ \frac{2}{\pi}\left[\cos(\pi(t-3)) - 1\right], & 3 \le t \le 5 \\ 0, & t > 5 \end{cases}$$

Step 10

The plot of the output signal $y(t)$ of part (c) is given below,



Step 11

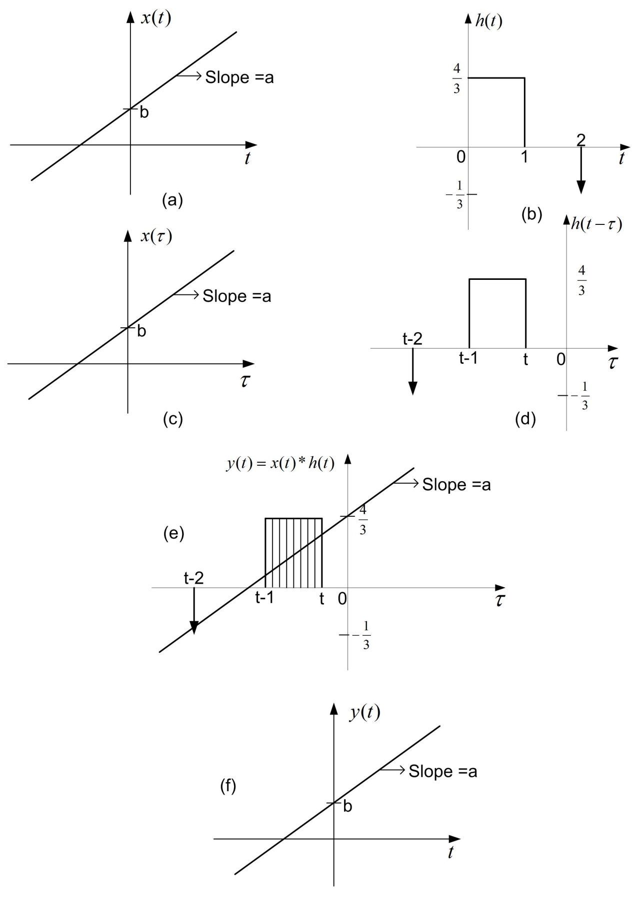


Figure.2.22.4

d) The signals $x(t)$ and $h(t)$ are shown in Figure.2.22.4(a) and Figure.2.22.4(b) respectively. From the Figures, the signals can be represented mathematically as,

$$\begin{aligned} x(t) &= at+b \\ h(t) &= \frac{4}{3}\{u(t)-u(t-1)\}-\frac{1}{3}\delta(t-2) \end{aligned}$$

Let us consider, $h_1(t)=\frac{4}{3}\{u(t)-u(t-1)\}$ and $h_2(t)=\frac{1}{3}\delta(t-2)$, then $h(t)=h_1(t)-h_2(t)$. Now consider, the convolution of $x(t)$ and $h_1(t)$. From Figure.2.22.4(e), it is clear that the signals $x(t)$ and $h_1(t)$ overlap and the limits of overlapping are from $\tau=t-1$ to $\tau=t$. Therefore,

$$y_1(t)=\int_{t-1}^{t} x(\tau)h_1(t-\tau)d\tau=\int_{t-1}^{t}\frac{4}{3}(a\tau+b)d\tau$$

$$y_1(t)=\frac{4}{3}\left[a\frac{\tau^2}{2}+b\tau\right]^t \quad =\frac{4}{3}\left[a\frac{t^2}{2}+bt-a\frac{(t-1)^2}{2}-b(t-1)\right]$$

$$y_1(t) = \frac{4}{3}\left[a\frac{t^2}{2} + bt - a\frac{t^2}{2} + a\frac{2t}{2} - a\frac{1}{2} - bt + b\right]$$
$$y_1(t) = \frac{4}{3}\left[at - \frac{a}{2} + b\right]$$

Now consider, the convolution of $x(t)$ and $h_2(t)$. i.e.,

$$y_2(t) = x(t) * h_2(t) = x(t) * \frac{1}{3}\delta(t-2)$$
$$y_2(t) = \frac{1}{3}x(t-2) \qquad (\because x(t) * \delta(t-t_0) = x(t-t_0))$$
$$y_2(t) = \frac{1}{3}(a(t-2) + b)$$

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Therefore, $x(t) * h(t)$ is,

$$y(t) = x(t) * h(t) = x(t) * \{h_1(t) - h_2(t)\}$$
$$y(t) = x(t) * h(t) = [x(t) * h_1(t)] - [x(t) * h_2(t)] \quad (\because \text{Distributive property of Convol}$$
$$y(t) = y_1(t) - y_2(t)$$
$$y(t) = \frac{4}{3}\left[at - \frac{a}{2} + b\right] - \frac{1}{3}(a(t-2) + b)$$
$$y(t) = at\left(\frac{4}{3} - \frac{1}{3}\right) + b\left(\frac{4}{3} - \frac{1}{3}\right) - \frac{2a}{3} + \frac{2a}{3} = at + b$$
$$y(t) = x(t), \text{ for all } t$$

Therefore, $y(t) = x(t)$ for all t $y(t)$ is plotted in Figure.2.22.4(f)

Step 13

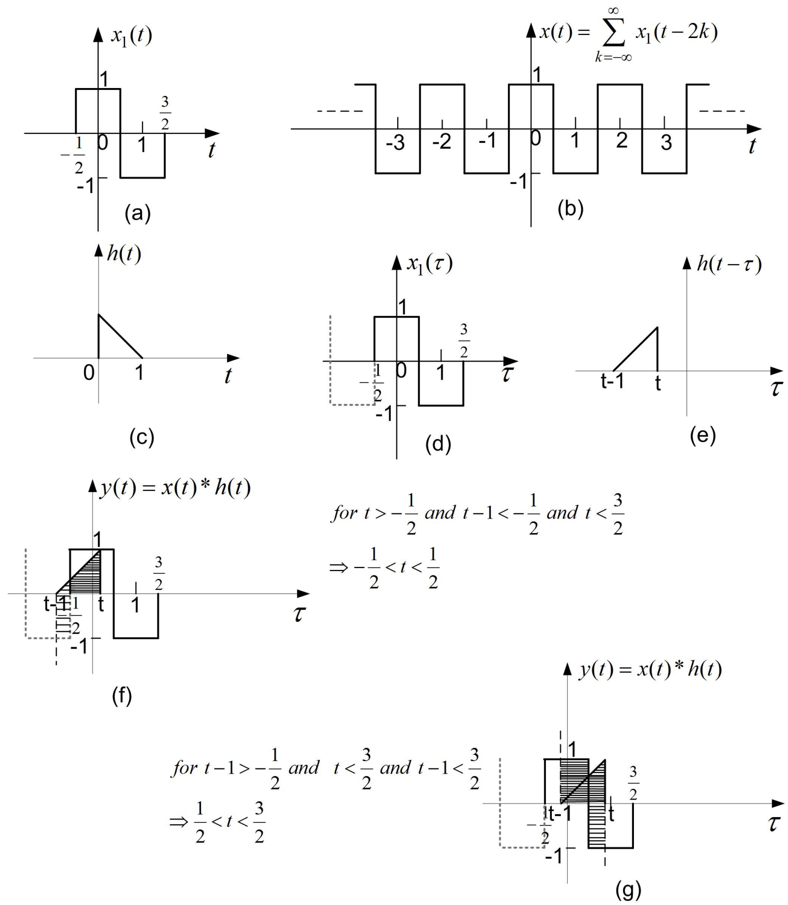

Figure.2.22.5

(e) The signals $x(t)$ and $h(t)$ are as shown in Figure.2.22.5(b) and Figure.2.22.5(c) respectively. It can be clearly seen that the signal $x(t)$ is a periodic signal with period 2 and the fundamental signal to construct this periodic signal is $x_1(t)$ and is shown in Figure.2.22.5(a). Therefore,

$$x(t)=\sum_{k=-\infty}^{\infty} x_1(t-2k)$$

$$x(t)=\sum_{k=-\infty}^{\infty} [x_1(t)*\delta(t-2k)] \qquad (\because x(t)*\delta(t-t_0)=x(t-t_0))$$

$$x(t)=x_1(t)*\sum_{k=-\infty}^{\infty} \delta(t-2k)$$

Let us consider, $y_1(t)=x_1(t)*h(t)$ Now the convolution of $x(t)$ and $h(t)$ is given as,

∞

$$y(t) = x(t) * h(t) = x_1(t) * \sum_{k=-\infty} \delta(t-2k) * h(t)$$

$$y(t) = x_1(t) * \left[\sum_{k=-\infty}^{\infty} \delta(t-2k) * h(t)\right] \qquad (\because \text{Associative property of Convolutio}$$

$$y(t) = x_1(t) * \left[h(t) * \sum_{k=-\infty}^{\infty} \delta(t-2k)\right] \qquad (\because \text{Commutative property of Convolu}$$

$$y(t) = [x_1(t) * h(t)] * \sum_{k=-\infty}^{\infty} \delta(t-2k) \qquad (\because \text{Associative property of Convolution})$$

$$y(t) = y_1(t) * \sum_{k=-\infty}^{\infty} \delta(t-2k)$$

$$y(t) = \sum_{k=-\infty}^{\infty} y_1(t-2k)$$

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Therefore, $y(t)$ is also a periodic signal with period 2. Let us now, try to obtain the convolution for one period of $y(t)$.

The signals $x_1(t)$ and $h(t)$ are as shown in Figure.2.22.5(a) and Figure.2.22.5(c).

Figure.2.22.5(d) and Figure.2.22.5(e) represents $x_1(\tau)$ and $h(t-\tau)$ respectively.

Figure.2.22.5(f) gives the convolution of one period of $y(t)$ if ($t > -\frac{1}{2}$ and $t - 1 < -\frac{1}{2} \implies t < \frac{1}{2}$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau = -\frac{1}{2}$ to $\tau = t$ and $\tau = t-1$ to $\tau = -\frac{1}{2}$ ($\because$ Periodic Signal). Therefore, for $-\frac{1}{2} < t < \frac{1}{2}$,

$$y(t) = \int_{t-1}^{t} x(\tau) h(t-\tau) d\tau$$

$$y(t) = \int_{t-1}^{-\frac{1}{2}} (-1)(1-t+\tau) d\tau + \int_{-\frac{1}{2}}^{t} (1)(1-t+\tau) d\tau$$

$$y(t) = (t-1)\int_{t-1}^{-\frac{1}{2}} 1 d\tau - \int_{t-1}^{-\frac{1}{2}} \tau d\tau + (1-t)\int_{-\frac{1}{2}}^{t} 1 d\tau + \int_{-\frac{1}{2}}^{t} \tau d\tau$$

$$y(t) = (t-1)[\tau]_{t-1}^{-\frac{1}{2}} - \left[\tau^2\right]^{-\frac{1}{2}} + (1-t)[\tau]_{-\frac{1}{2}}^{t} + \left[\tau^2\right]^{t}$$

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Figure.2.22.5(g) gives the convolution of one period of $y(t)$ if ($t-1>-\frac{1}{2} \implies t>\frac{1}{2}$ and $t<\frac{3}{2}$), there will be overlapping of signals and the limits of overlapping or integration are from $\tau=t-1$ to $\tau=\frac{1}{2}$ and $\tau=\frac{1}{2}$ to $\tau=t$. Therefore, for $\frac{1}{2}<t<\frac{3}{2}$,

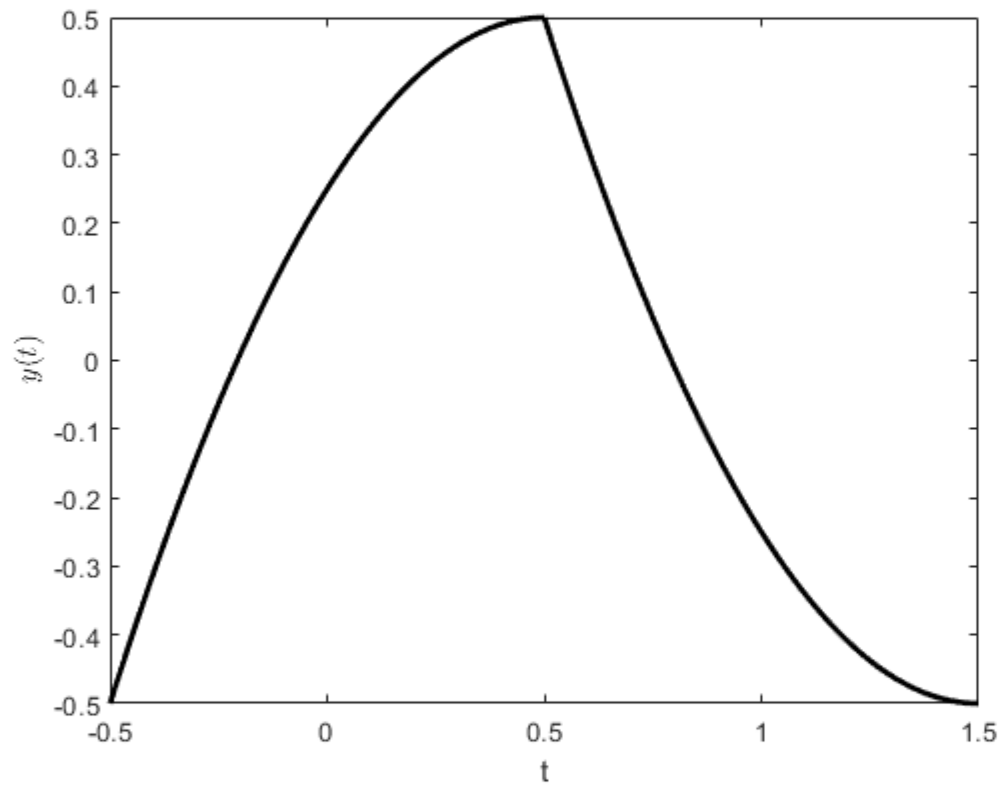
$$y(t)=\int_{t-1}^{t} x(\tau)h(t-\tau)d\tau$$
$$y(t)=\int_{t-1}^{\frac{1}{2}}(1)(1-t+\tau)d\tau+\int_{\frac{1}{2}}^{t}(-1)(1-t+\tau)d\tau$$
$$y(t)=(1-t)\int_{t-1}^{\frac{1}{2}}1d\tau+\int_{t-1}^{\frac{1}{2}}\tau d\tau+(t-1)\int_{\frac{1}{2}}^{t}1d\tau-\int_{\frac{1}{2}}^{t}\tau d\tau$$
$$y(t)=(1-t)\left[\tau\right]_{t-1}^{\frac{1}{2}}+\left[\frac{\tau^2}{2}\right]_{t-1}^{\frac{1}{2}}+(t-1)\left[\tau\right]_{\frac{1}{2}}^{t}-\left[\frac{\tau^2}{2}\right]_{\frac{1}{2}}^{t}$$
$$y(t)=(1-t)\left[\frac{1}{2}-t+1\right]+\left[\frac{(\frac{1}{2})^2}{2}-\frac{(t-1)^2}{2}\right]+(t-1)\left[t-\frac{1}{2}\right]-\left[\frac{t^2}{2}-\frac{(\frac{1}{2})^2}{2}\right]$$
$$y(t)=(1-t)\left[\frac{3}{2}-t\right]+\frac{1}{8}-\frac{t^2}{2}-\frac{1}{2}+\frac{2t}{2}+(t-1)\left[t-\frac{1}{2}\right]-\frac{t^2}{2}+\frac{1}{8}$$
$$y(t)=t^2-\frac{5t}{2}+\frac{3}{2}+\frac{1}{8}-\frac{t^2}{2}-\frac{1}{2}+\frac{2t}{2}+t^2+\frac{1}{2}-\frac{3t}{2}-\frac{t^2}{2}+\frac{1}{8}$$
$$y(t)=t^2-3t+\frac{7}{4}, \qquad \text{for } \frac{1}{2}<t<\frac{3}{2}$$

Therefore, $y(t)$ is periodic signal with period 2 and one period of signal $y(t)$ is,

$$y(t)=\begin{cases}\frac{1}{4}+t-t^2, & -\frac{1}{2}<t<\frac{1}{2}\\ t^2-3t+\frac{7}{4}, & \frac{1}{2}<t<\frac{3}{2}\end{cases}$$

Step 16

The plot of one period of the output signal $y(t)$ of part (e) is given below,



Result

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a)

$$y(t) = \begin{cases} e^{-\beta t}\left(\frac{1-e^{-(\alpha-\beta)t}}{\alpha-\beta}\right)u(t), & \alpha \neq \beta \\ te^{-\alpha t}u(t), & \alpha = \beta \end{cases}$$

b)

$$y(t) = \begin{cases} \left[\frac{e^{-10}-2e^{-4}+1}{2}\right]e^{2t}, & t \leq 1 \\ \frac{e^{2(t-5)}-2e^{2(t-2)}+e^{2}}{2}, & 1 < t \leq 3 \\ \frac{e^{2(t-5)}-e^{2}}{2}, & 3 < t \leq 6 \\ 0, & Otherwise \end{cases}$$

c)

$$y(t) = \begin{cases} 0, & t < 1 \\ \frac{2}{\pi}\left[1-\cos(\pi(t-1))\right], & 1 \leq t < 3 \\ \frac{2}{\pi}\left[\cos(\pi(t-3))-1\right], & 3 \leq t \leq 5 \\ 0, & t > 5 \end{cases}$$

d)

$$y(t) = x(t), \text{ for all } t$$

e) One period of signal $y(t)$ is,

$$y(t) = \begin{cases} \frac{1}{4}+t-t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ t^2-3t+\frac{7}{4}, & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

Exercise 21

Rate this solution

Exercise 23