Lecture 02

Warm-up problem:

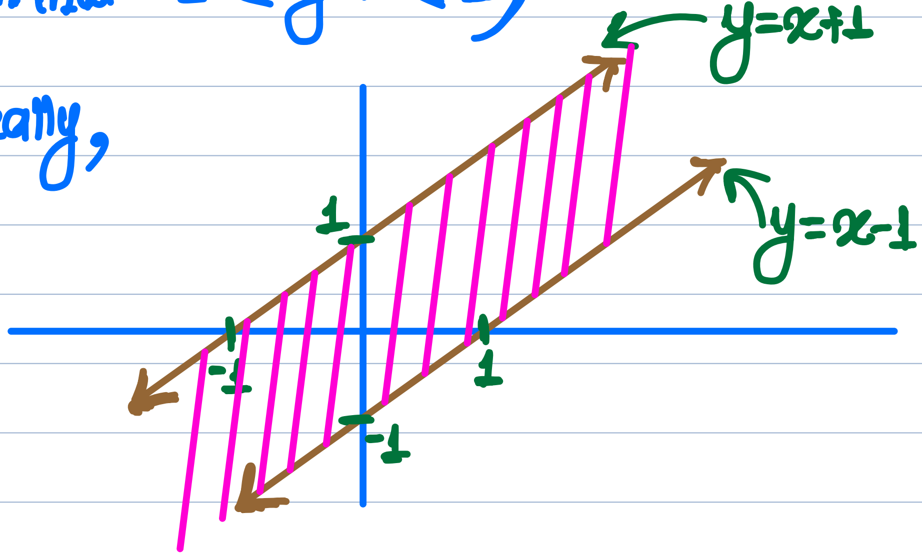
Find the domain and the range of the following functions.

(i) $z = f(x,y) = \sin^{-1}(y-x)$.

Solution: $\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 : -1 \leq y-x \leq 1\}$

(This is the set of all ordered pairs in $\mathbb{R}^2$ such that $-1 \leq y-x \leq 1$)

graphically,



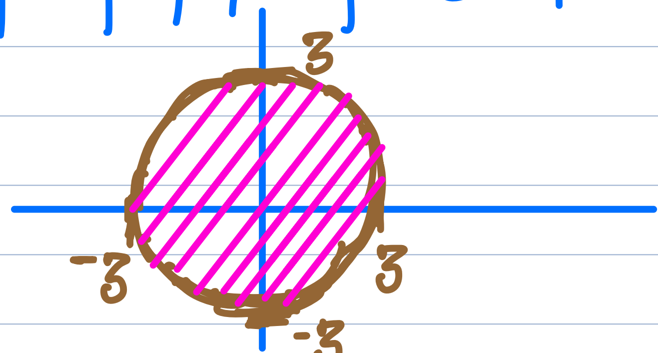
$\text{ran}(f) = [-\pi/2, \pi/2]$.

(ii) $z = g(x,y) = \sqrt{9-x^2-y^2}$

Solution: $\text{dom}(g) = \{(x,y) \in \mathbb{R}^2 : 9-x^2-y^2 \geq 0\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 9\}$

Graphically, the domain is the shaded region including the circumference of the circle.



$\text{ran}(f) = [0,3] \subseteq \mathbb{R}$.

(iii) $\omega = xy\ln z$

Soln: clearly ω is a function of three variables
$\therefore$ domain must be contained in $\mathbb{R}^3$.

Let $w = h(x,y,z) = xy\ln z$. Then,

$\text{dom}(h) = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$

[Graphically, it is the half-space where the z-coordinate is strictly positive (and thus the x-y plane is excluded).]

$\operatorname{Ran}(f) = \mathbb{R}$. (How?)

[Let $\alpha \in \mathbb{R}$ be arbitrary. Then one can choose $x=\alpha$, $y=1$ and $z=e$ so that $xy\ln(z) = \alpha$.]

Exercises

(iv) $\omega = \sqrt{y-x^2}$;

(v) $f(x,y) = \sqrt{y-x}$;

(vi) $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$;

(vii) $g(x,y) = y/x^2$;

(viii) $h(x,y) = \dfrac{1}{x-y}$.

Elements of point-set Topology on $\mathbb{R}^n$

Let $n \in \mathbb{N}$ be fixed but arbitrarily chosen.

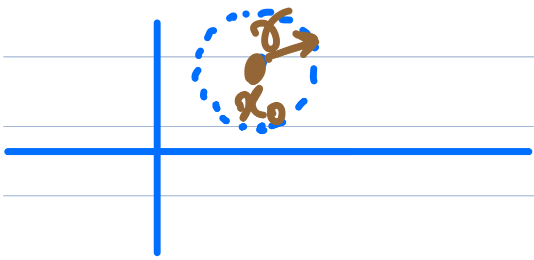
1. Open ball in $\mathbb{R}^n$ of radius $r>0$ centered at $x_0 \in \mathbb{R}^n$

→ If $n=2$, then we are in $\mathbb{R}^2$:

Suppose $x_0 \in \mathbb{R}^2$ is given and $r>0$ is also given. Then the open ball in $\mathbb{R}^2$ of radius r centered at x_0 is

the open disc of radius r centered at x_0

$$= B_r(x_0) = \{x \in \mathbb{R}^2 : \|x - x_0\| < r\} \subseteq \mathbb{R}^2$$



Notation

→ If $n=1$, the space is $\mathbb{R}$.

$B_r(x_0) =$ open interval (x_0-r, x_0+r)

$= \{x \in \mathbb{R} : |x-x_0| < r\}$

$\therefore B_2(5) = \{x \in \mathbb{R} : |x-5| < 2\} = (3,7)$

→ **In $\mathbb{R}^n$**: The open ball (in $\mathbb{R}^n$) of radius r centered at $x_0 \in \mathbb{R}^n$ is given by

$$B_r(x_0) = \{x \in \mathbb{R}^n : \|x-x_0\| < r\}$$

Question: Find $B_2(4)$;

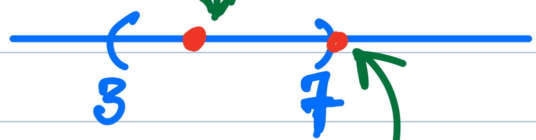
$B_3((2,3))$;

$B_1((1,0,0))$.

2. Interior point of a given subset of $\mathbb{R}^n$

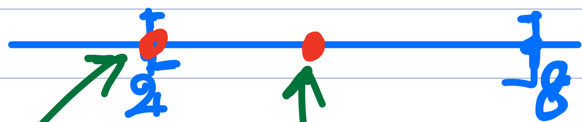
⊙ Consider the open interval $(3,7) \subseteq \mathbb{R}$,

an interior point of $(3,7)$



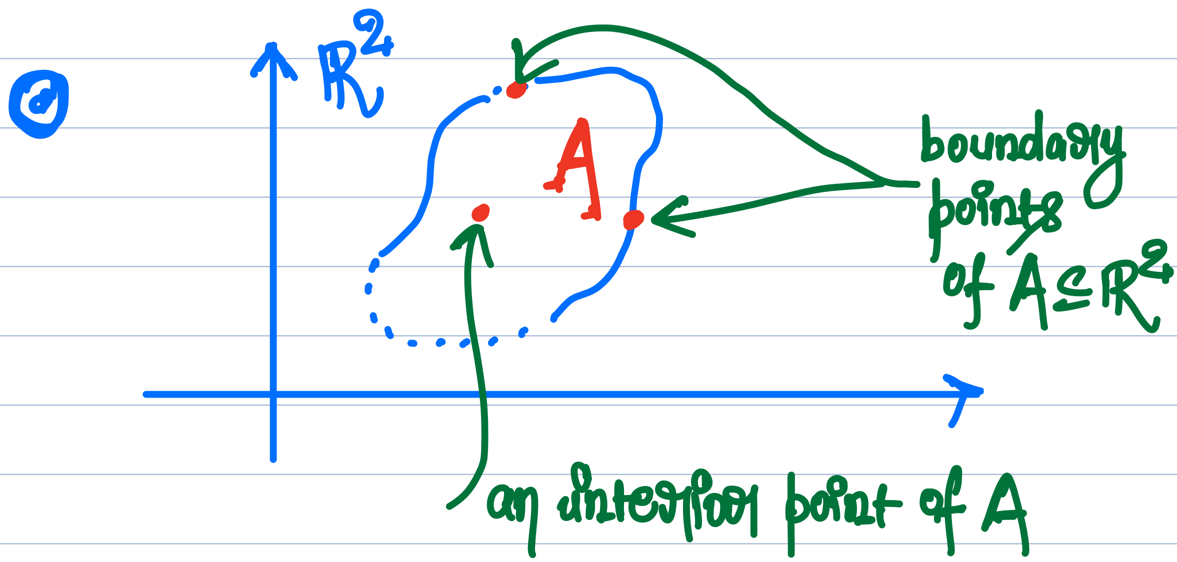
Not an interior point of $(3,7)$
It is, in fact, a boundary point of $(3,7)$

⊙ Consider $[2,8]$



a boundary point of $[2,8]$

an interior point of $[2,8]$



definition Let $A \subseteq \mathbb{R}^n$. A point $x \in A$ is said to be **an interior point of A** if $\exists\, r > 0$ such that $B_r(x) \subseteq A$

[symbol for "there exist(s)"]

[In definition "if" is essentially "if and only if"]

Alternatively, A point $x \in A$ is called an interior point of A if it is the centre of some open ball contained in A.

3. Boundary point of a given subset of $\mathbb{R}^n$

definition: Let $A \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a boundary point of A if $\forall r > 0$, $B_r(x) \cap A \neq \phi$ and $B_r(x) \cap A^c \neq \phi$

[Symbol for "For every"]

[Again, this means "if and only if"]

Alternatively, a point $x \in \mathbb{R}^n$ is called a boundary point of A if every open ball centered at x contains points of A as well as points of A^c

④ Interior of a set

Let $A \subseteq \mathbb{R}^n$. The interior of A (denoted by $\operatorname{int}(A)$) is the set of all interior points of A.

⑤ Boundary of a set

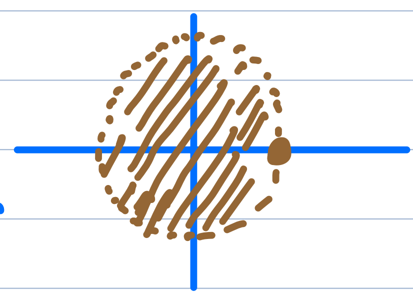
Let $A \subseteq \mathbb{R}^n$. The boundary of A (denoted by ∂A) is the set of all boundary points of A.

Exercise. Let $A \subseteq \mathbb{R}^n$. Prove that $\operatorname{int}(A) \subseteq A$.

Example Let $A = B_1((0,0)) \cup \{(1,0)\} \subseteq \mathbb{R}^2$.

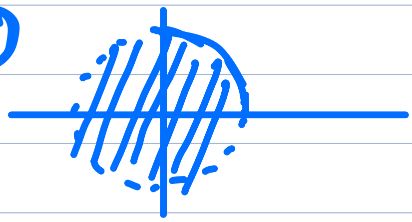
Then, $\operatorname{int}(A) = B_1((0,0))$,

$\partial A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$.



6: Open set in $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. We say that **A is open in $\mathbb{R}^n$** or, **A is an open subset of $\mathbb{R}^n$** if $A = \text{int}(A)$.

- $[2,3)$ is NOT open in $\mathbb{R}$.
- $(2,5)$ is open in $\mathbb{R}$.
- $B_1((0,0)) \cup \{(1,0)\}$ is not open in $\mathbb{R}^2$.
-  Is this open in $\mathbb{R}^2$?

7: Closed set in $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. We say that **A is closed in $\mathbb{R}^n$** or, **A is a closed subset of $\mathbb{R}^n$** if $\partial A \subseteq A$.

- $(2,3]$ is not closed in $\mathbb{R}$.
- $[5,7)$ is not closed in $\mathbb{R}$.
- $[a,b]$, where $a,b \in \mathbb{R}$, is closed in $\mathbb{R}$.
- Let $A = (3,\infty) \subseteq \mathbb{R}$. Compute ∂A.
- Give an example of a subset of $\mathbb{R}$, that is neither open nor closed in $\mathbb{R}$.
- Is $\mathbb{R}^n$ an open subset of $\mathbb{R}^n$?
- Is $\mathbb{R}^n$ a closed subset of $\mathbb{R}^n$?
- Let $A \subseteq \mathbb{R}^n$. Can we assert that $A = \operatorname{int}(A) \cup \partial A$? If yes, prove it. If no, give a counterexample.

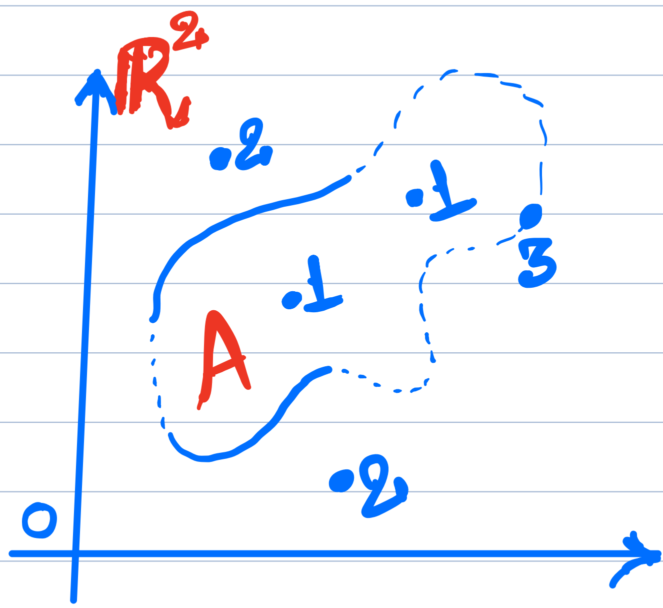
Proposition. Let $A \subseteq \mathbb{R}^n$.

If $x \in \mathbb{R}^n$, then one, and only one, of the following three possibilities holds.

(1) $\exists\, r>0$ s.t. $B_r(x) \subseteq A$. ← interior pt. of A

(2) $\underbrace{\exists\, r>0 \text{ s.t. } B_r(x) \subseteq A^c.}_{x \text{ is an exterior point of } A}$

(3) $\underbrace{\forall\, r>0,\ B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap A^c \neq \emptyset.}_{\text{boundary point of } A}$



Exercise: Find the exterior, interior and the boundary of the following sets.

- $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$
- $\{x \in \mathbb{R}^n : \|x\| = 1\}$
- $\{x \in \mathbb{R}^n : \|x\| < 1\}$
- $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \forall j \in \{1, \dots, n\}\ x_j \in \mathbb{Q}\}$