Lecture 07

A word on notation:

- We use $\vec{x}$ to denote an element of $\mathbb{R}^n$ where $n \geq 2$;

  The "arrow" is used to indicate that the element is an n-tuple!

- $\vec{f}$ is used when the range of the function is $\mathbb{R}^n$ with $n \geq 2$;

  when $n=1$, we simply use f.

- $x, \vec{x}, f, \vec{f}$

The Jacobian matrix

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{Int}(D)$, let $\vec{f} : D \to \mathbb{R}^m$ be a (vector-valued) function (of n independent variables $x_1, x_2, \ldots, x_n$) and let $f_i : D \to \mathbb{R}$, $1 \leq i \leq m$, denote the component functions of $\vec{f}$.

Now, suppose that $\vec{f}$ is differentiable at $\vec{x_0}$. Let $T_{\vec{x_0}} = \vec{f}'(\vec{x_0})$ be the total derivative of $\vec{f}$ at $\vec{x_0}$. Of course, $T_{\vec{x_0}}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

definition:

The $m \times n$ matrix of the linear transformation $T_{\vec{x_0}}$, with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, is called **the Jacobian matrix of the function $\vec{f}$ at the point $\vec{x_0}$.**

Example 1: $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \sin x .$$

Compute $f'((2,3))$

Ans: $f'((2,3)) = [\cos 2, 0]$

Example 2: Let $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$\vec{f}(x,y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$.

Determine the Jacobian matrix of $\vec{f}$ at an arbitrary point $(x,y) \in \mathbb{R}^2$.

Solution: (HW) $\vec{f}'(x,y) = [\quad]_{3\times 2}$

$$f_1(x,y) = \sin x \cos y$$
$$f_2(x,y) = \sin x \sin y$$
$$f_3(x,y) = \cos x \cos y$$

Thus,

$$\vec{f}'((x,y)) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix} \blacksquare$$

Example 3: (when range is 1-dimensional)

Let $D \subseteq \mathbb{R}^n$, $\vec{x}_0 \in \text{int}(D)$, let $f: D \to \mathbb{R}$ be a function that is differentiable at $\vec{x}_0$. Then $f'(\vec{x}_0)$ is a $1 \times n$ matrix given by

$$\left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \frac{\partial f}{\partial x_2}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0)\right].$$

In this case, the Jacobian matrix of f at $\vec{x}_0$ is a row-matrix, which is also called "the gradient of f at $\vec{x}_0$", and is denoted by $\nabla f\big|_{\vec{x}_0}$ or $\nabla f(\vec{x}_0)$.

Remarks:

1. The symbol ∇ is read as "<u>del</u>".

2. $\nabla f(\vec{x}_0) = \left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0)\right]$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ and is called "<u>the gradient of f at $\vec{x}_0$</u>".

3. $\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$ is called "<u>the gradient of f</u>", or simply, "<u>the del f</u>".

Note that, the gradient of f — ∇f, is **NOT** a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$;

4. Instead, $\nabla f : E \subseteq \mathbb{R}^n \longrightarrow M_{1\times n}(\mathbb{R})$

where $E \subseteq D$ is the set of all points in D where the partial derivatives

$\frac{\partial f}{\partial x_j}$ exists for every $j \in \{1, \ldots, n\}$.

(5) Consider the function $f(x,y) = x^2 y$.

Then,

(i) $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \end{bmatrix}$

(This is a function from $\mathbb{R}^2$ to $M_{1,2}(\mathbb{R})$)

(ii) $\nabla f \Big|_{(1,2)} = \begin{bmatrix} \frac{\partial f}{\partial x}(1,2) & \frac{\partial f}{\partial y}(1,2) \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix}$

(This is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$)

(iii) the gradient of f = $\begin{bmatrix} 2xy & x^2 \end{bmatrix}$;

the gradient of f at $(1,2)$ = $\begin{bmatrix} 4 & 4 \end{bmatrix}$.

Example 4: (When the domain is 1-dimensional)

Let $D \subseteq \mathbb{R}$,

$x_0 \in \text{int}(D)$,

$\vec{f} : D \longrightarrow \mathbb{R}^m$ be a function

that is differentiable at x_0.

Then $\vec{f}'(x_0)$ is the $m \times 1$ matrix given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) \end{bmatrix} = \left[\frac{\partial f_i}{\partial x_1}(x_0) \right].$$

In this case,

(a) We omit the ∂ notation and
(b) We use the ordinary derivative sign

(Why?)

and the Jacobian matrix of $\vec{f}$ at x_0 is a column matrix, which in calculus, is denoted by $\frac{d\vec{f}}{dx}(x_0)$.

That is,

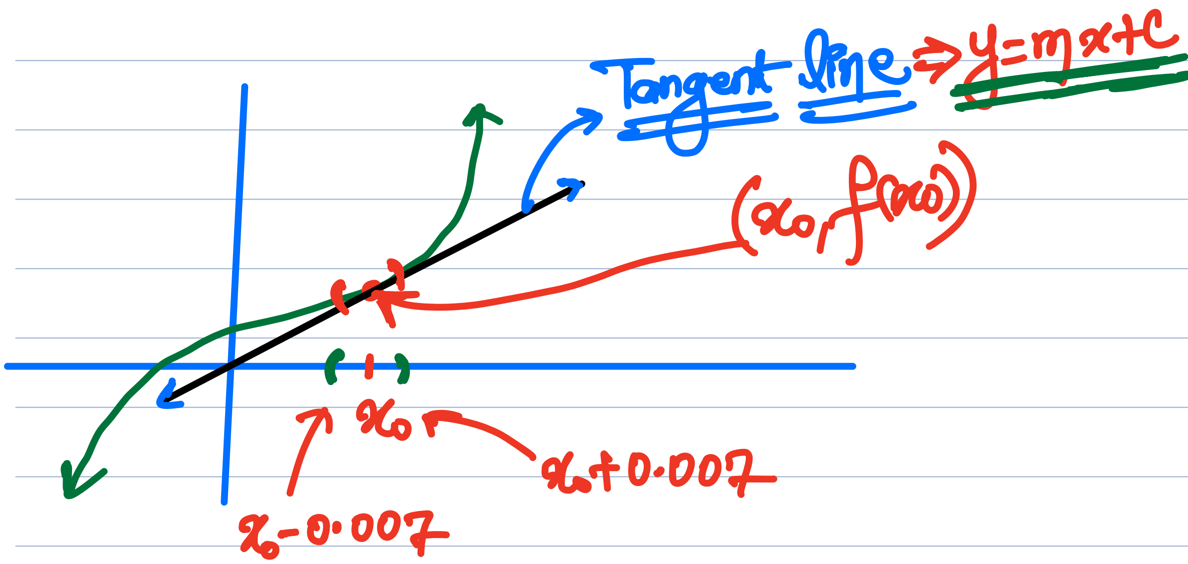
$$\vec{f}\,'(x_0) = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \vdots \\ \frac{df_m}{dx}(x_0) \end{bmatrix} = \frac{d\vec{f}}{dx}(x_0).$$

Standard linear approximation

$$f : \mathbb{R} \to \mathbb{R}$$
$$\Downarrow$$
$$x_0$$

f is differentiable at x_0.

Let $f(x) = 99x^{87} + 6x^{85} + 90x^{23} + 2023.$



$$f(x_0 + 0.003) = ?$$

$$\boxed{L(x) = mx + c}$$

Standard linear approximation of $f(x)$ by $L(x)$

In Calculus, linear function refers to a function whose graph is line.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

$$\boxed{f(x) \approx f(x_0) + f'(x_0)[x - x_0]}$$

$L(x)$

[A polynomial of degree 1]

I. Linearization in multivariable setting

Let $D \subseteq \mathbb{R}^n$ and $\vec{x_0} \in \text{int}(D)$. Let $f: D \to \mathbb{R}$ be a function. Then

f is differentiable at $\vec{x_0}$

$$\Updownarrow$$

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x_0}+\vec{h}) - f(\vec{x_0}) - f'(\vec{x_0})h}{\|h\|} = 0$$

or, equivalently, by replacing $\vec{x_0}+\vec{h}$ by $\vec{x}$,

$$\lim_{\vec{x} \to \vec{x_0}} \frac{f(\vec{x}) - f(\vec{x_0}) - f'(\vec{x_0})(\vec{x}-\vec{x_0})}{\|\vec{x}-\vec{x_0}\|} = 0$$

$$\boxed{\begin{array}{l} f(\vec{x}) \approx f(\vec{x_0}) - f'(\vec{x_0})(\vec{x}-\vec{x_0}) \\ \text{if } \vec{x} \text{ is "very" close to } \vec{x_0} \end{array}}$$

$$\boxed{\therefore\ L(\vec{x}) := f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0)}$$

Linearization of f at $\vec{x}_0$

or

Standard linear approximation of f at $\vec{x}_0$

Remark:

(1) $f'(\vec{x}_0)$ is $1 \times n$ matrix given by

$$\begin{bmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) & \frac{\partial f}{\partial x_2}(\vec{x}_0) & \cdots & \frac{\partial f}{\partial x_n}(\vec{x}_0) \end{bmatrix}$$

(2) $\vec{x} - \vec{x}_0$ is an $n \times 1$ column matrix, given by

$$\begin{bmatrix} x_1 - x_{0,1} \\ x_2 - x_{0,2} \\ \vdots \\ x_n - x_{0,n} \end{bmatrix}$$

(3) Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\longrightarrow f(x,y)$$

Let $\vec{x_0} = (a,b) \in \mathbb{R}^2$ and $\vec{x} = (x,y)$

Now,

$$f(\vec{x_0}) + f'(\vec{x_0})(\vec{x} - \vec{x_0})$$

$$= f((a,b)) + \begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ x-b \end{bmatrix}$$

$$= f((a,b)) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

$$\therefore\ L(x,y) = f((a,b)) + f_x((a,b))(x-a) + f_y((a,b))(y-b)$$

[The usual version found in Calculus text-books]

(4) Let $f : \mathbb{R}^3 \to \mathbb{R}$

$\qquad\qquad \hookrightarrow f(x, y, z)$

Let $\vec{x_0} = (a, b, c) \in \mathbb{R}^3$ and $\vec{x} = (x, y, z)$

Then show that

$$\boxed{\begin{aligned} L(x,y,z) = {} & f((a,b,c)) + f_x(a,b,c)(x-a) \\ & + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) \end{aligned}}$$

Example: Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point $(3, 2)$.

Soln: $$L(x,y) = f((3,2)) + \begin{bmatrix} 2x - y\big|_{(3,2)} & -x + y\big|_{(3,2)} \end{bmatrix} \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$$

To be done later...

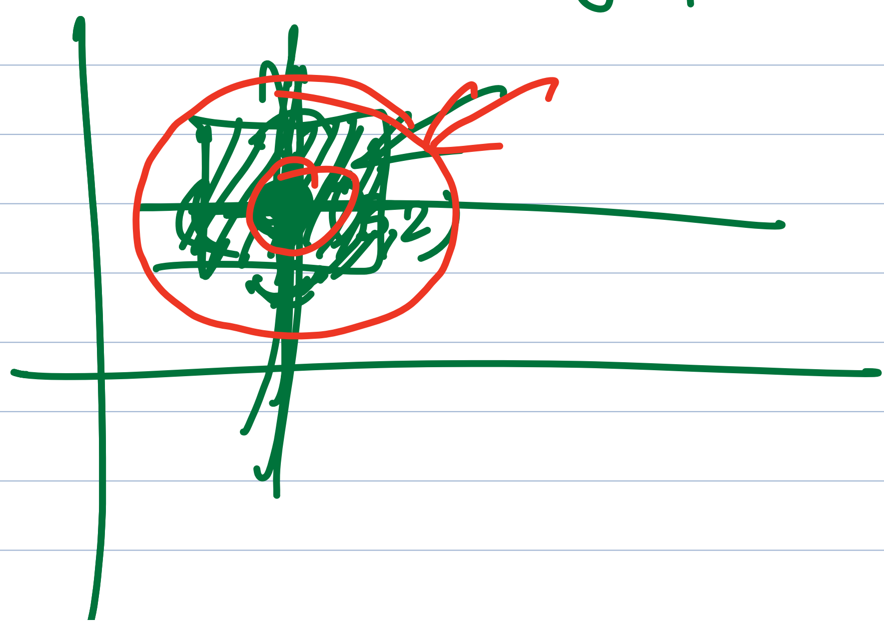
Qn:- Linearize the function

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$$

at the point $(3,2)$.

Find an upper bound for the error incurred in replacing f by L on the rectangle R: $|x-3| \leq 0.1$

$|y-2| \leq 0.1$



Error

$$\left| f(x,y) - L(x,y) \right|$$

Later

(once 2^{nd} derivative test, extreme points, etc. are done!)

Qn: Find the linearization of

$$f(x,y,z) = x^2 - xy + 3\sin z$$

at the point $(x_0, y_0, z_0) = (2,1,0)$.

Find an upper bound for the error incurred in replacing f by L on the rectangle

$$R: |x-2| \le 0.01, \quad |y-1| \le 0.02, \quad |z| \le 0.01.$$