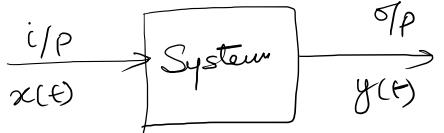


30/8/2024

Systems & Their properties



$$x[n] \longrightarrow y[n]$$

① System : with memory or without memory

$$S_1: y(t) = x(t)$$

$$S_2: y(t) = x^2(t)$$

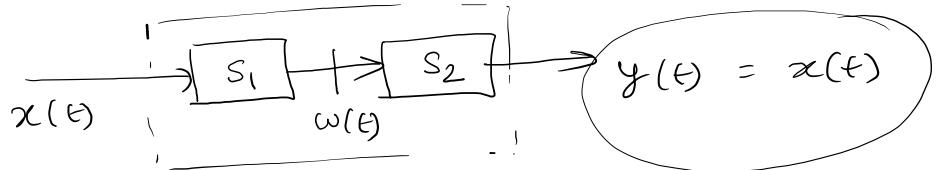
$$S_3: y(t) = x(t-2)$$

$$S_4: y(t) = \frac{1}{2} \{ x(t) + x(t-1) \}$$

$$S_5: y(t) = x(t+2)$$

(2)

Invertible / Non-invertible Systems



$$\text{if } S_2 = \text{inv}(S_1)$$

(a)

$$S_1: w(t) = 2x(t)$$

$$S_2: y(t) = \frac{1}{2}w(t) = \frac{1}{2} \times 2x(t) = x(t)$$

S_1 is invertible & the inverse system is S_2 :

(b)

$$S_1: w[n] = \sum_{k=-\infty}^n x[k]$$

$$S_2: y[n] = w[n] - w[n-1]$$

$$\begin{aligned}
 &= \sum_{k=-\infty}^{n-1} x[k] + x[n] - \sum_{k=-\infty}^{n-1} x[k] \\
 &= x[n]
 \end{aligned}$$

(c)



$$S_1: w[n] = x[n] - x[n-1]$$

$$S_2: y[n] = \sum_{k=-\infty}^n w[k]$$

Invertible

(d)

$$S_1: w(t) = x^2(t)$$

$$S_2: y(t) = \sqrt{w(t)}$$

S_1 is not
invertible

If distinct ips lead to distinct outputs, the system
is invertible, else not.

(3)

Causal Systems —

$$S_1: y(t) = x(t-t_0) \quad t_0 \text{ is the ret. no.}$$

$$S_2: y[n] = x[n+3]$$

S_1 = causal system

S_2 = non-causal system

Causality: Unless

OR unless

there is a cause, there should be no effect.

there is a non-zero i/p, there should not be a non-zero o/p.

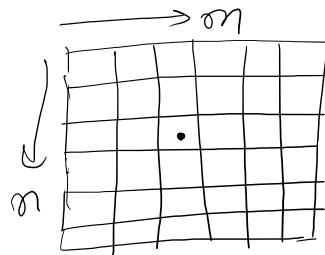
OR a causal system depends on current & past i/p.

Accumulator:

$$S_3: y[n] = \sum_{k=-\infty}^n x[k] \text{ causal.}$$

$$S_4: y[n, m]$$

$$= \frac{1}{5} \left\{ x[n, m] + x[n-1, m] + x[n+1, m] + x[n, m-1] + x[n, m+1] \right\}$$



Non-causal System

(F)

Stability —

Stable System

: A Bounded i/p should lead to a bounded o/p.

BIBO Stability

Bounded i/p $\equiv |x(t)| < M_x < \infty$

\hookrightarrow finite value
for t

is on amplitude

Or $|x[n]| < M_x < \infty$

\hookrightarrow finite value for all n.

this bounded i/p is fed to a system.
 If the system generates an o/p, that is also bounded
 Then, we call it a stable system -

$$x(t) = u(t) ; \quad x[n] = u[n]$$

Examples
of bounded
i/p's.

$$x(t) = t u(t) \quad \text{or} \quad x[n] = t u[n].$$

unbounded
signals

if $|y(n)| \leq M_y < \infty$ $\forall n$

→ finite no.

for bounded $x(n)$, then the system is stable.

S₁: $y[n] = \frac{1}{3} \{ x[n] + x[n-1] + x[n+1] \}$

$$|y[n]| = \left| \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n+1] \right|$$
$$\leq \frac{1}{3} |x[n]| + \frac{1}{3} |x[n-1]| + \frac{1}{3} |x[n+1]|$$

if $|x[n]| \leq M_x \quad \forall n$
 $|y[n]| = \text{bounded}$ for all n .

$$S_2 : y[n] = \frac{1}{3} \left\{ x[n] - x[n-1] - x[n-2] \right\}$$

$$|y[n]| = \frac{1}{3} |x[n] - x[n-1] - x[n-2]|$$

Since $|x[n]| < M_x < \infty$ for n
 Sum is difference of 3 finite nos. will
 again be a finite no.

So $|y[n]| < M_y < \infty \forall n$
 Stable System.

$$(S_3 : y[n] = n x[n])$$

$$\left| \frac{y[n]}{x[n]} \right| < M_x < \infty \quad \text{if } n \text{ bounded if p}$$

$$|y[n]| = |n \underline{x[n]}| \xrightarrow[n \rightarrow \infty]{} \infty$$

Unstable System

$$S_4 : y(t) = x(2t) \quad \text{Stable System}$$

(5)

Time - Invariant Systems

$$x(t) \xrightarrow{S} y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

$$x(t+t_0) \longrightarrow y(t+t_0)$$

$$\underline{x_1(t)} \xrightarrow{\quad} \underline{\underline{y_1(t)}}$$

$$\underline{x_2(t)} \longrightarrow \underline{\underline{y_2(t)}}$$

If

if $\underline{\underline{x_2(t)}} = \underline{\underline{x_1(t-t_0)}}$

$$y_2(t) = \underline{\underline{y_1(t-t_0)}}$$

$$y_2(t) =$$

If yes, it is a time
invariant system
else, not.

$$S: y(t) = x(2t)$$

$$x_1(t)$$



$$y_1(t)$$

$$= x_1(2t)$$

$$x_2(t)$$



$$y_2(t)$$

$$= x_2(2t)$$

$$y_1(t - t_0) = x_1(2t - 2t_0)$$

@

I op

$$x_2(t) = x_1(t - t_0) \Rightarrow y_2(t) = x_1(2t - t_0) \Rightarrow \text{II op}$$

@ ≠ b

$$y_2(t) \neq y_1(t - t_0)$$

So, "S" is a
time-varying system

$$S_2 : y[n] = n x[n]$$

$$x_1[n] \longrightarrow y_1[n] = n x_1[n]$$

$$y_1[n-n_0] = (n-n_0) x_1[n-n_0] \quad @$$

$$x_2[n] \longrightarrow y_2[n] = n x_2[n]$$

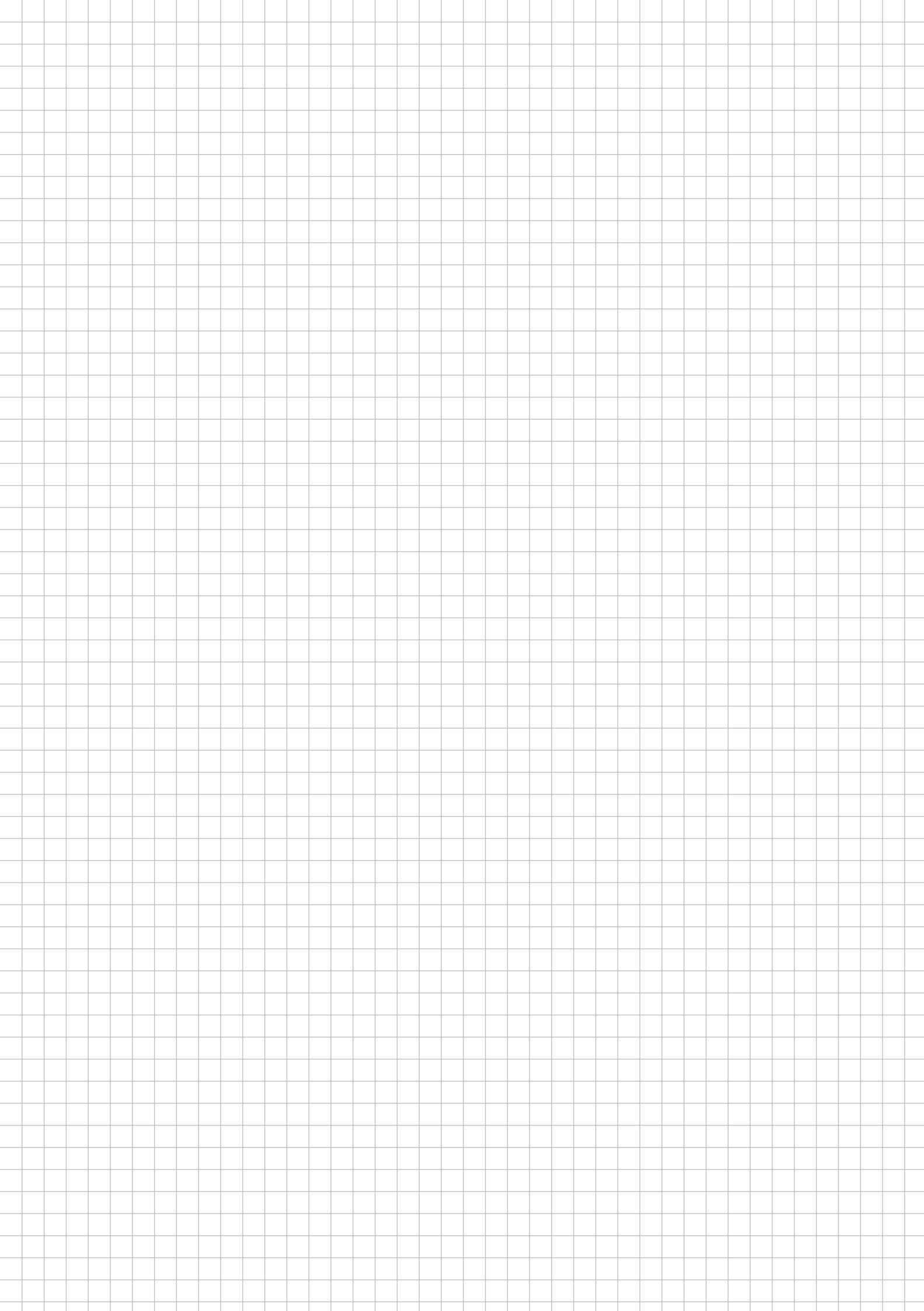
$$\text{let } x_2[n] = x_1[n-n_0]$$

delayed
↑
if

$$y_2[n] = n x_1[n-n_0] \quad b$$

$$\text{But } y_2[n] \neq y_1[n-n_0]$$

Time-varying



(5)

directly —

A system is linear if the properties of additivity & homogeneity holds true.

$$x_1(t) \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow y_2(t)$$

$$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t)$$

Additivity :
homogeneity :

$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$$

$$ax_1(t) \longrightarrow ay_1(t)$$

System
is Linear

$\forall a, b \in \mathbb{R}$

$$S: \quad y(t) = 2x(t) + 3 = 2(i/p) + 3$$

$$x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$$

$$x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$$

$$\boxed{a x_1(t) + b x_2(t)} \xrightarrow{i/p} 2\left(\frac{i}{p}\right) + 3$$

$$= 2(a x_1(t) + b x_2(t)) + 3$$

$$= 2ax_1(t) + 2bx_2(t) + 3$$

$$ay_1(t) + by_2(t) = a(2x_1(t) + 3) + b(2x_2(t) + 3)$$
$$= 2ax_1(t) + 3a + 2bx_2(t) + 3b$$

Not linear

(b)

(a)

A system that is both linear & time-invariant
is called an LTI System

LTI \equiv Linear & Time Invariant

4/9/2024

$$x[n] = \cos\left(\frac{\pi}{8}n^2\right)$$

Periodic if $x[n] = x[n+N]$ $\forall n$ for some positive N .
Smallest +ve N that satisfies this eqn, is called the fundamental time period.

$$x[n+N] = \cos\left(\frac{\pi}{8}(n+N)^2\right) = \cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{2\pi}{8}nN\right)$$

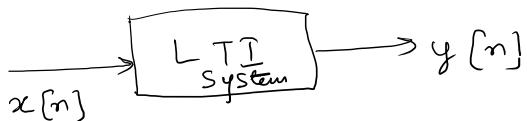
$$\cos(2\pi k + \theta) = \cos\theta \quad \forall k \in \mathbb{Z}$$

Can it be a multiple of 2π

$$\begin{aligned} \text{If } N=4 & \quad \frac{\pi}{8}N^2 \\ \text{If } N=8 & \quad \frac{\pi}{8} \times 8 \times 8^2 = 2\pi \cdot 2 \end{aligned}$$

$$\begin{aligned} \frac{2\pi}{8} \not\approx n &= \pi n \\ \frac{2\pi}{8} \not\approx n &= 2\pi n \end{aligned}$$

Discrete-time Convolution -



$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

$$x[n] = x[-1] \delta[n+1] = \delta[n+1] \Rightarrow$$

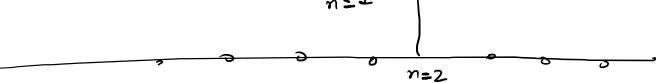
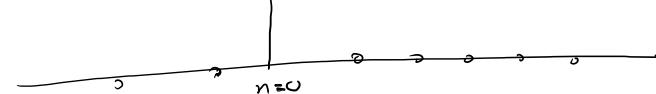
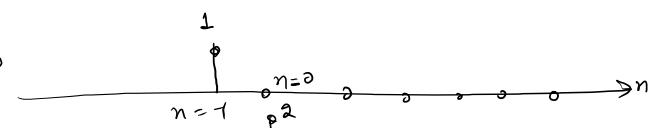
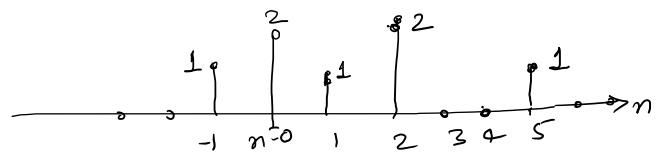
$$x[0] \delta[n] = 2 \delta[n]$$

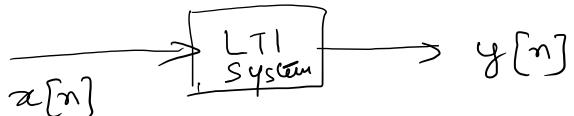
$$x[1] \delta[n-1] = \delta[n-1]$$

$$x[2] \delta[n-2] = 2 \delta[n-2]$$

$$x[5] \delta[n-5] = \delta[n-5]$$

$$x[n]$$





True - meaning
 $h[n, m]$
 where
 the i/p
 was located
 at i/p
 variable

$$x[n] \rightarrow y[n] = h[n]$$

= Impulse response of the
LTI System

$$\delta[n-1] \rightarrow h[n-1]$$

$$x[1] \delta[n-1] \rightarrow x[1] h[n-1]$$

$$x[-1] \delta[n+1] \rightarrow x[-1] h[n+1]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k] h[n-k] = y[n]$$

D.T. convolution of

$$\underline{x[n] * h[n] = y[n]}$$

"Convolution" \rightarrow linear, commutative

$$x[n] * h[n] = h[n] * x[n]$$

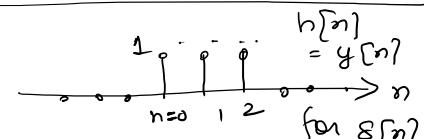
$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Given an LTI
System
whose response
impulse is

$$h[n] = \begin{cases} 1 & n=0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$



$$h[n] = y[n]$$

for $y[n]$

as i/p.

Given $x[n] = \frac{3}{5} \delta[n] + \frac{1}{2} \delta[n+1]$

Compute $y[n] = \frac{3}{5} h[n] + \frac{1}{2} h[n+1] = \frac{3}{5} [\delta[n] + \delta[n-1] + \delta[n-2]] + \frac{1}{2} [\delta[n+1] + \delta[n] + \delta[n-1]]$

Ex-2

$$\text{LTI system} \quad h[n] = u[n]$$

$$\text{Given } x[n] = d^n u[n] \quad \text{where } 0 < d < 1$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$\begin{array}{l} h[n] = 0 \quad n < 0 \\ \downarrow \end{array}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{h[k]}_{=} x[n-k]$$

$$= \sum_{k=0}^{\infty} h[k] x[n-k] = \sum_{k=0}^{\infty} x[n-k]$$

(a)

$$y[n] = \sum_{k=0}^n d^{n-k}.$$

$$= \sum_{k=0}^{\infty} d^{n-k} u[n-k]$$

$$u[n-k] = 0$$

$$n-k < 0$$

$$-n < k$$

or $\boxed{k > n}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

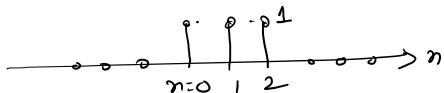
$$h[n] = \begin{cases} 1 & n=0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = \frac{3}{5} 8[n] + \frac{1}{2} 8[n+1]$$

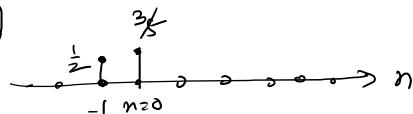
$$y[0] = \sum_{k=-\infty}^{\infty} x[k] h[-k]$$

$$\begin{aligned} h[1-k] &= h[-k+1] \\ &= 0 + \frac{1}{2} \times 1 + \frac{3}{5} \times 1 + 0 + 0 \dots \\ &= \frac{1}{2} + \frac{3}{5} \end{aligned}$$

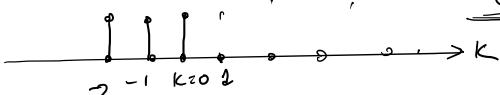
$$h[n]$$



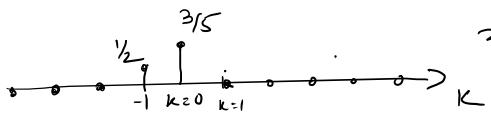
$$x[n]$$

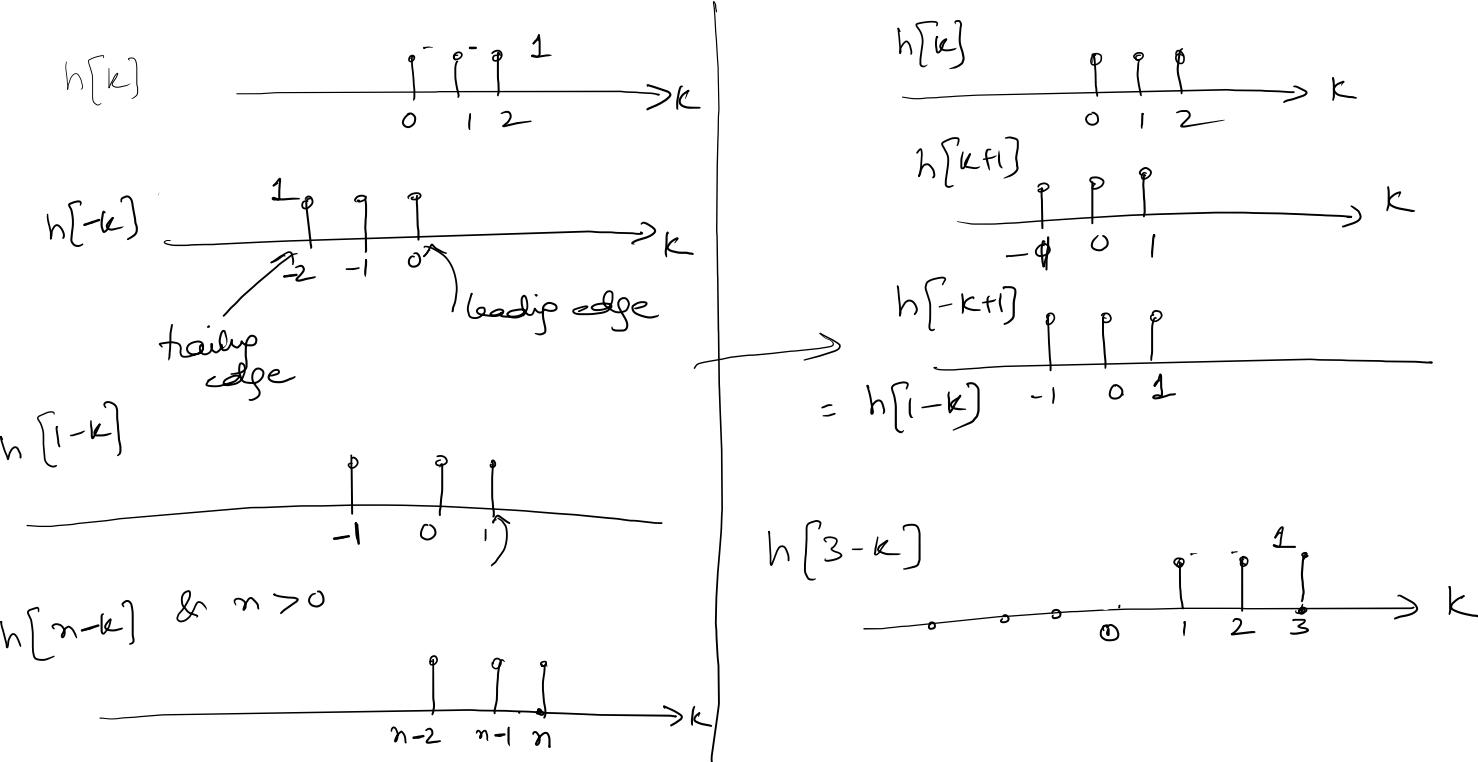


$$h[-k]$$



$$x[k]$$





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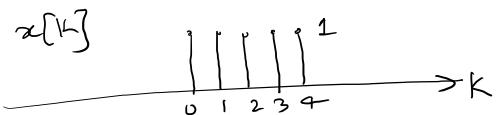
Example:

Discrete-time convolution

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n & 0 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

$\alpha > 1$

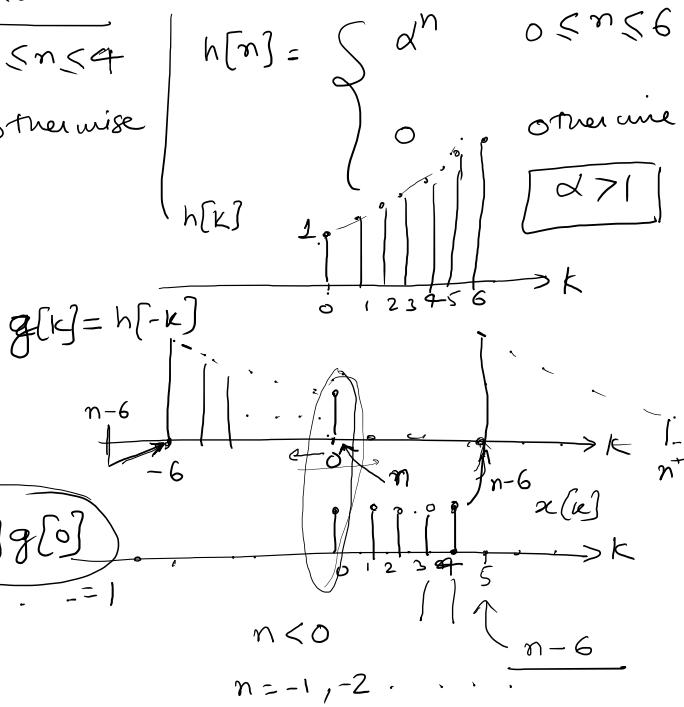


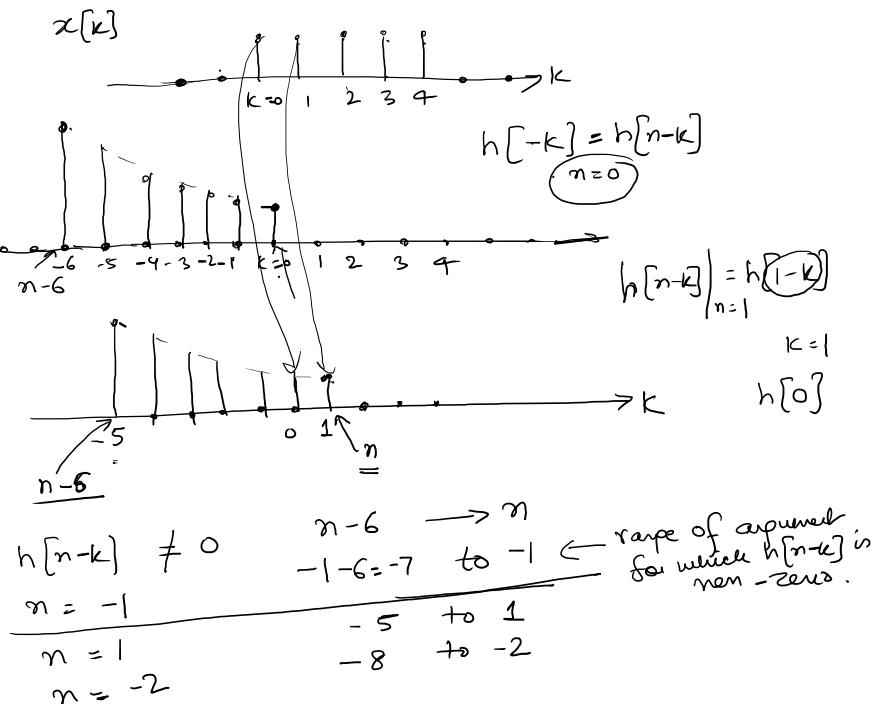
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k] g[k]$$

$$= + \dots x[-1] g[-1] + \boxed{x[0] g[0]} + x[1] g[1] + \dots = 1$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1-k]$$





$$y[n] = \sum_k x[k] h[n-k]$$

overlap or the regions where

(I)

$$\begin{cases} y[n] = 0 & n < 0 \\ \end{cases}$$

or say $n = -1 \rightarrow -1$ on k -axis

$$h[n-k] = \begin{cases} 1 & n = -1 \\ 0 & \text{to } 4 \text{ on } k\text{-axis} \end{cases}$$

$$x[k] = \begin{cases} 1 & n = -1 \\ 0 & \text{to } 4 \text{ on } k\text{-axis} \end{cases}$$

$$\begin{array}{ll} n = 11 & 5 \text{ to } 11 \\ h[n-k] \neq 0 & 0 \text{ to } 4 \\ x[k] \neq 0 & n > 0 \text{ & } n < 0 \\ \end{array}$$

$$y[n] = 0$$

$$\textcircled{2} \quad \underline{0 \leq n \leq 4}$$

$$y[n] = \sum_{k=0}^n x[k] h[n-k]$$

$$\begin{aligned} y[n] &= \sum_{k=0}^n x[k] h^{n-k} \\ &= x^n \sum_{k=0}^n \left(\frac{1}{2}\right)^k \\ &= x^n \frac{\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)}{1 - \left(\frac{1}{2}\right)} \end{aligned}$$

$$k=0$$

$$k=0, 1,$$

$$\vdots$$

$$k=0, 1, 2, \dots$$

$$k=0, 1, 2, 3$$

$$k=0, \dots, 4$$

$$n=0$$

$$\underline{n=1}$$

$$\underline{\underline{n=2}}$$

$$\underline{\underline{\underline{n=3}}}$$

$$\underline{\underline{\underline{\underline{n=4}}}}$$

$$\underline{-6 \text{ to } 0}$$

$$\underline{-5 \text{ to } 1}$$

$$\underline{\underline{-4 \text{ to } 2}}$$

$$\underline{\underline{\underline{-3 \text{ to } 3}}}$$

$$\underline{\underline{\underline{\underline{-2 \text{ to } 4}}}}$$

$$\textcircled{3} \quad 4 \leq n \leq 6$$

$$y[n] = \sum_{k=0}^4 x[k] h[n-k]$$

$$\begin{array}{c} x[k] \\ \textcircled{0 \text{ to } 4} \end{array}$$

$$\textcircled{4} \quad 6 < n \leq 10$$

$$y[n] = \sum_{k=n-6}^4 x[k] h[n-k]$$

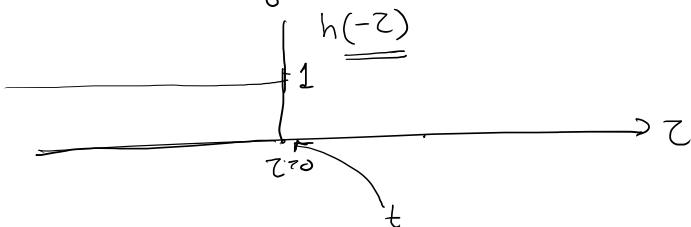
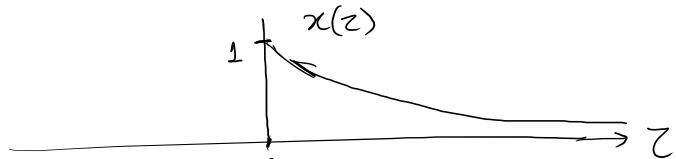
C. T.

$$x(t) \xrightarrow{\text{LTI}} y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Ex
Ex

$$x(t) = e^{-at} u(t) \quad a > 0$$

$$h(t) = 10^{-t}$$



$$\boxed{y(t) = x(t) * h(t)}$$

"Convolution operator"

$$h(t-\tau)$$

t when $t=0$

$$h(-\tau)$$

①

$t < 0$

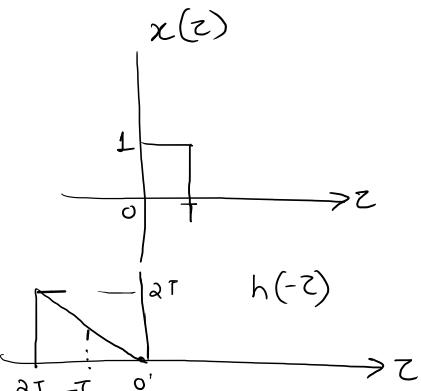
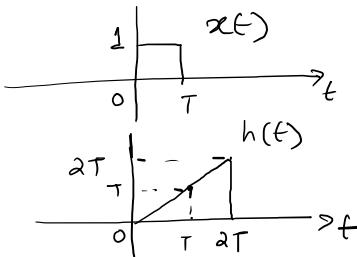
$$y(t) = 0$$

②

$$t > 0$$
$$y(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

Ex

$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$



$$h(t) = \begin{cases} t & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}$$

$$\textcircled{1} \quad y(t) = 0 \quad t < 0$$

$$\textcircled{2} \quad t \geq 3T$$

$$\textcircled{2} \quad 0 \leq t \leq T \quad y(t) = \int_{-\infty}^t x(z)h(t-z)dz$$

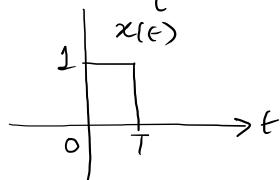
$$\textcircled{3} \quad T \leq t \leq 2T \quad y(t) = \int_0^T x(z)h(t-z)dz$$

$$\textcircled{4} \quad 2T \leq t \leq 3T \quad y(t) = \int_{t-2T}^T x(z)h(t-z)dz$$

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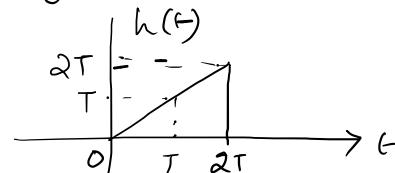
Convolution

$$x(t) = \begin{cases} 1 \\ 0 \end{cases}$$

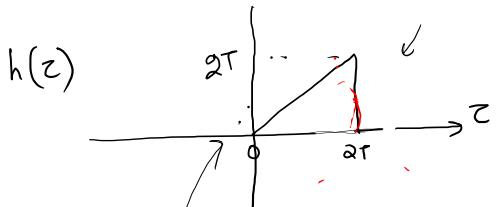
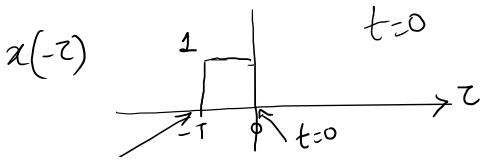


$$\begin{aligned} 0 < t < T \\ 0 \leq t \leq T \\ \text{otherwise} \\ 0 \leq t < T \\ 0 < t \leq T \end{aligned}$$

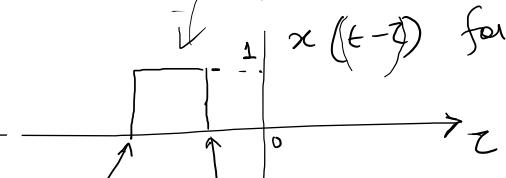
$$h(t) = \begin{cases} t & 0 \leq t \leq 2T \\ 0 & \text{otherwise} \end{cases}$$

Find $y(t) ?$

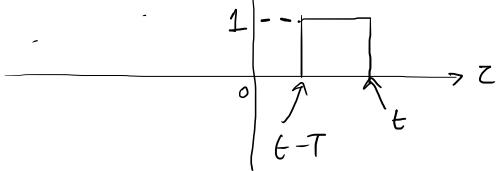
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(z) h(t-z) dz \\ &= x(t) * h(t) \\ &= h(t) * x(t) \end{aligned}$$



$x(t-\tau)$ for $t < 0$



$x(t-\tau)$ for $t > 0$



$$y(0) = \int_{-\infty}^{\infty} h(z) x(2-z) dz$$

$$= \int_{-\infty}^{\infty} h(z) x(2-z) dz$$

① Case - 1 t < 0

$$y(t) = \int_{-\infty}^{\infty} h(z) x(t-z) dz$$

$$= 0$$

② Case - 2 t > 3T

$$y(t) = 0$$

$\text{det } t = \frac{5\pi}{2}$

Case -3 : $0 < t < T$

$$y(t) = \int_0^t h(z) dz = \int_0^t z dz = \frac{z^2}{2} \Big|_0^t = \frac{t^2}{2}$$

Case -4 : $T \leq t < 2T$

$$y(t) = \int_{t-T}^t h(z) dz = \frac{z^2}{2} \Big|_{t-T}^t = \frac{t^2}{2} - \frac{(t-T)^2}{2}$$
$$= -\frac{T^2 + 2tT}{2} = -\frac{T^2}{2} + tT$$

Case -5 $2T \leq t \leq 3T$

$$y(t) = \int_{t-T}^{2T} h(z) dz = \frac{z^2}{2} \Big|_{t-T}^{2T}$$

Ex

$$x(t) = e^{2t} u(-t)$$

$$h(t) = u(t-3)$$

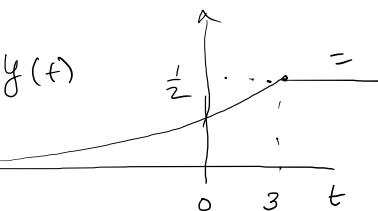
Case -1

$$t \leq 3$$

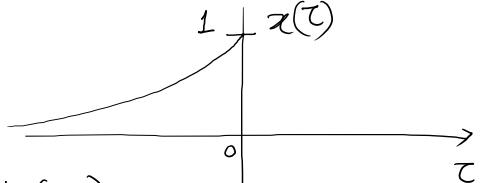
$$y(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz$$

$$y(t) = \int_{-\infty}^{t-3} x(z) dz$$

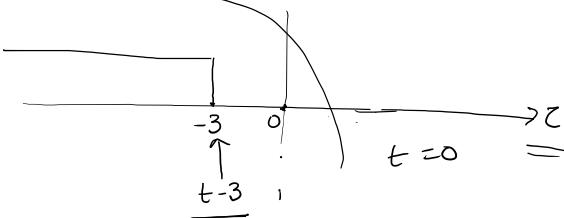
$$= \int_{-\infty}^{t-3} e^{2z} dz = \frac{e^{2z}}{2} \Big|_{-\infty}^{t-3} = \frac{e^{2(t-3)}}{2}$$



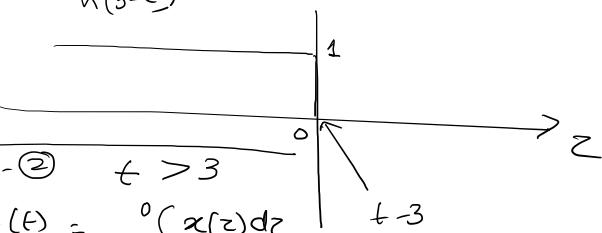
$$x(z)$$



$$h(-z)$$



$$h(3-z)$$



Case -② $t > 3$

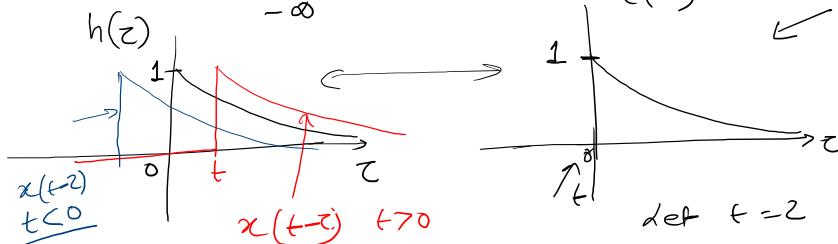
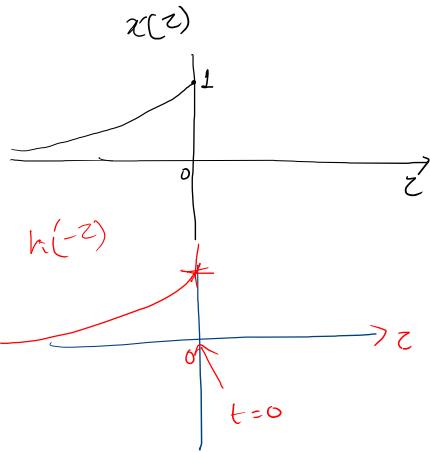
$$y(t) = \int_{-\infty}^0 x(z) dz = \frac{1}{2}$$

Ex

$$x(t) = e^{at} u(-t) \quad a > 0$$

$$h(t) = e^{-at} u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz = \int_{-\infty}^{\infty} \underline{h(z) x(t-z) dz}$$



def $t = -2$

Case-1 $t > 0$

Case-2 $t < 0$

Case-III

$$y(t) = \int_t^{\infty} e^{-az} e^{a(t-z)} dz$$

Case-2

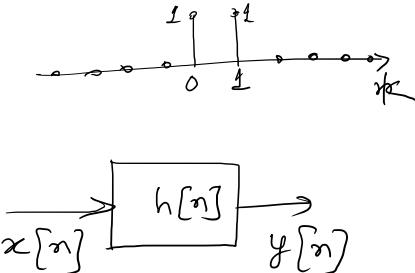
$$y(t) = \int_0^{\infty} e^{-az} e^{a(t-z)} dz$$

there is an
LTI
System

System Properties

$$h[n] = \begin{cases} 1 & n=0,1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{S: } y[n] = x[n] + x[n-1]$$

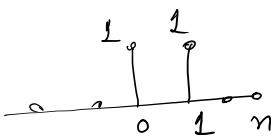


$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

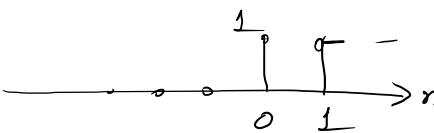
$$h[n] = \delta[n] + \delta[n-1] \Rightarrow h[k] = \delta[k] + \delta[k-1]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1]) x[n-k] \\ &= \underbrace{\sum_{k=-\infty}^{\infty} \delta[k] x[n-k]}_{= x[n]} + \underbrace{\sum_{k=-\infty}^{\infty} \delta[k-1] x[n-k]}_{= x[n-1]} \end{aligned}$$

$$S_2 : \quad y[n] = \left(x[n] + x[n-1] \right)^2$$



$$S_3 : \quad y[n] = \max \{ x[n], x[n-1] \}$$



13/9/2024

System Properties [holds true for both C.T. & D.T. Systems]

(2) (i) Commutativity -

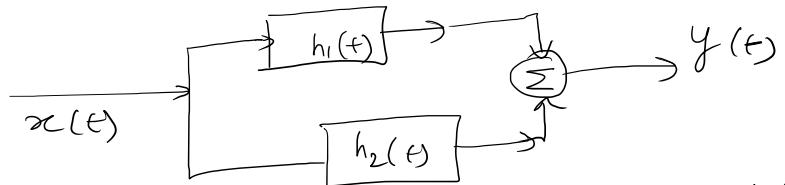
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$\int_{-\infty}^{\infty} x(z) h(t-z) dz = \int_{-\infty}^{\infty} h(z) x(t-z) dz$$

Prove that

(ii) Convolution distributes over addition

$$y(t) = x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



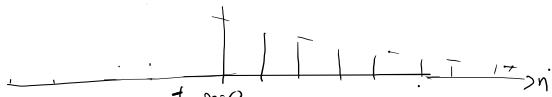
$$y(t) = h(t) * (x_1(t) + x_2(t)) = h(t) * x_1(t) + h(t) * x_2(t)$$

Ex

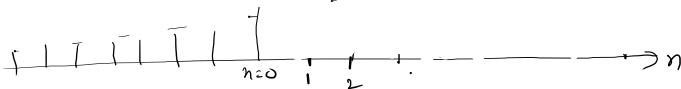
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n] = x_1[n] + x_2[n]$$

$$y[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$



$$x_2[n] = 2^n u[-n]$$



(iii) Associativity

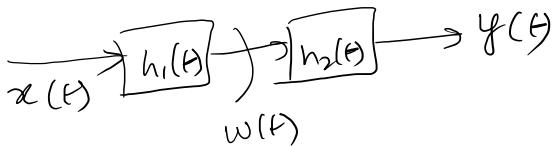
$$x(t) * \left[h_1(t) * h_2(t) \right]$$

$$= \left[x(t) * h_1(t) \right] * h_2(t)$$



order of these systems can
also be changed for LTI systems

Invertibility —



If $h_1(t) * h_2(t) = \delta(t)$

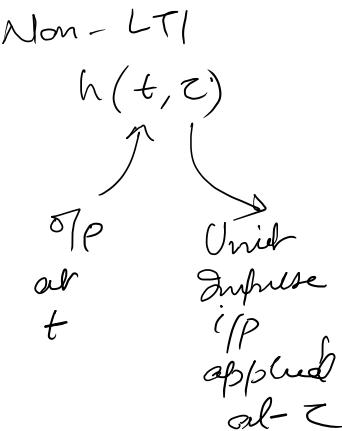
$$\begin{aligned} y(t) &= x(t) * (h_1(t) * h_2(t)) \\ &= \textcircled{x(t) * \delta(t)} \\ &= x(t) \end{aligned}$$

$$x(t) = \int_{-\infty}^{\infty} x(z) \delta(t-z) dz$$

Operational definition
of $\delta(t)$

$$\delta(t) = 0 \quad t \neq 0$$

$$\int \delta(t) dt = 1$$



D.T.

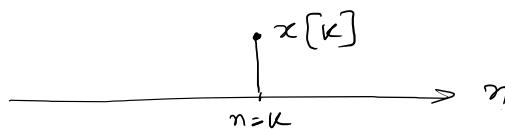
$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

$$= x[n] * \delta[n]$$

$$x(t) \delta(t-z) = x(z) \delta(t-z)$$



$$\underline{x[k] \delta[n-k]}$$



Causality -

Claim: If a system is causal, $h(t) = 0$ for $t < 0$ (C.T)
 or $h[n] = 0$ for $n < 0$ (D.T)

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k]. \end{aligned}$$

For causal

$$\begin{aligned} y(t) &= \int_{-\infty}^t x(z) h(t-z) dz. \\ &= \int_0^t h(z) x(t-z) dz \end{aligned}$$

C.T. \Rightarrow D.T.

$y[n] = \sum_{k=0}^{\infty} h[k] x[n-k]$

$y[n] = \sum_{k=-\infty}^n x[k] h[n-k]$

if $h[n] = 0$ for $n < 0$

Memory or meaningless Systems

$$S_1: y(t) = kx(t)$$

$$\Rightarrow h(t) = k\delta(t)$$

$$S_2: y[n] = kx[n]$$

$$\Rightarrow h[n] = k\delta[n]$$

Stability -

BIBO

Stability condition

For every bounded i/p signal, if we get bounded o/p signal
then system is stable.

$$y(t) = \int_{-\infty}^{\infty} x(z)h(t-z)dz = \int_{-\infty}^{\infty} h(z)x(t-z)dz$$

Let i/n signal is bounded & $|x(t)|_{\infty} < M_x \quad \forall t$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(z)x(t-z)dz \right| \leq \int_{-\infty}^{\infty} |h(z)x(t-z)| dz \\ \leq \int_{-\infty}^{\infty} |h(z)| |x(t-z)| dz$$

$$|y(t)| \leq M_x \int_{-\infty}^{\infty} |h(z)| dz \leq M_x \cdot \text{some finite no.} < \infty$$

$h(t) = u(t)$
 LTI / causal, with memory
 unstable

If the system is absolutely integrable -

C.R.
System

$$\int_{-\infty}^{\infty} |h(z)| dz < \infty$$

absolute Integrability

$$\text{D.T. System} \quad |y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

absolute summability
 ↓ system is stable

$h[n] = u[n]$
 LTI / unstable, causal, with memory

Constant Coefficient difference equation for
Causal LTI Systems

$$\sum_{k=0}^N b_k y[n-k] = \sum_{k=0}^M a_k x[n-k]$$

① $N \geq$ order of the system

② the system is at most -
 auxiliary conditions

(MA System)

Moving average system

Case I: $N = 0$

System is called
 Finite Impulse response
 FIR System

$$\left\{ \begin{array}{l} \text{if } x[n] = 0 \quad n < n_0 \\ \text{if } y[n] = 0 \quad n < n_0 \\ \text{if } y[n-1] = 0 \quad n < n_0 \\ \vdots \\ \text{if } y[n-N] = 0 \quad n < n_0 \end{array} \right.$$

if
 $N > 0$

$$b_0 y[n] = a_0 x[n] + a_1 x[n-1] + \dots + a_M x[n-M]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \dots + \frac{a_M}{b_0} x[n-M]$$

Case -2

Let $N=1$, $M=0$

$$b_1 y[n-1] + b_0 y[n] = a_0 x[n]$$

$$y[n] = \frac{a_0}{b_0} x[n] - \frac{b_1}{b_0} y[n-1]$$

$$\boxed{y[n] = d_0 x[n] - d_1 y[n-1]}$$

Is it FIR?

If $x[n] = \delta[n]$
 $y[n] = h[n]$

IIR System
Infinite Impulse
Response System

$$\begin{aligned}h[0] &= d_0 \\h[1] &= -d_1 d_0 \\h[2] &= +d_1^2 d_0 \\h[3] &= :\end{aligned}$$

Autoregressive System
(AR) system

a_0, b_0, b_1
are
constants

$$d_0 = a_0/b_0$$

$$d_1 = b_1/b_0$$

Case 3

$N > 0, M > 0$

ARMA Systems

18/9/2024

C.T.

Constant coefficient differential eqn. representation of
Causal LTI system

$$\sum_{k=0}^N b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M a_k \frac{d^k}{dt^k} x(t)$$

N = order of the system

$$\frac{dx}{dt} \quad N=2 \quad \& \quad M=3$$

$$b_2 \frac{d^2}{dt^2} y(t) + b_1 \frac{dy}{dt} + b_0 y(t) = a_0 x(t) + a_1 \frac{dx}{dt}$$

$$+ a_2 \frac{d^2}{dt^2} x(t) + a_3 \frac{d^3}{dt^3} x(t)$$

Condition of initial
rest is satisfied

if $x(t) = 0$ for $t < 0$

$\frac{d^k}{dt^k} y(t) = 0$ for $t < 0$
 $\forall k=0, 1, \dots, N$

Block Diagram representation

D.T.

$a_0 x[n]$ \equiv Scalar multiplier

$x[n-1]$ \equiv Delay element

$$S_1: y[n] = a_0 x[n] + a_1 x[n-1]$$

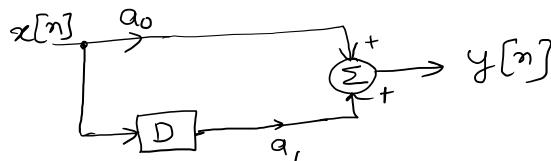


Fig.: S_1

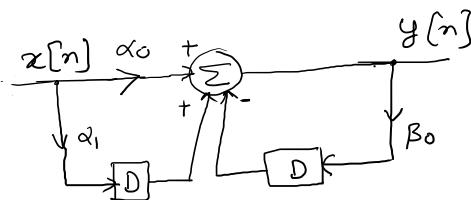
$$S_2: b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$

$$= d_0 x[n] + d_1 x[n-1] - \beta_0 y[n-1]$$

where
 $d_0 = \frac{a_0}{b_0}$

constants



C.T

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

for
Periodic
Signal
 $x(t)$

$$\downarrow x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

Synthesis
equation \Rightarrow

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

vector

$j\omega_0 t$

$$e^{j\omega_0 t} = G \cos \omega_0 t + j \sin \omega_0 t$$

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$\boxed{w_0 = \frac{2\pi}{T}}$$

$T \equiv$ fundamental time
period of $x(t)$

$$\phi_1(t) = e^{j\omega_0 t}$$

$$\phi_2(t) = e^{j2\omega_0 t}$$

$$\phi_{-1}(t) = e^{-j\omega_0 t}$$

$$\phi_{-2}(t) = e^{-j2\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$$

$$x(t) = \int_{-\infty}^{\infty} x(z) \delta(t-z) dz$$

$$\phi_0(t) = e^{j0} = G_0 + j S_0$$

$$= 1$$

$$\langle \phi_k(t), \phi_l(t) \rangle \triangleq \frac{1}{T} \int_0^T \phi_k(t) \phi_l^*(t) dt$$

≡ Defining inner product

$$\begin{aligned}
 &= \frac{1}{T} \int_0^T e^{jk\omega_0 t} e^{-jl\omega_0 t} dt \\
 &= \frac{1}{T} \int_0^T e^{j(k-l)\omega_0 t} dt \\
 &= \frac{1}{T} \int_0^T e^{jm\omega_0 t} dt = \left. \frac{e^{jm\omega_0 t}}{jm\omega_0 T} \right|_0^T = \frac{e^{jm\omega_0 T} - e^{j0}}{jm\omega_0 T} = 0
 \end{aligned}$$

$\Rightarrow 1 \quad \text{if } m=0$

$= 0 \quad \text{if } m \neq 0$

$= 1 \quad \text{if } k=l$

Synthesis : $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$

$\{ \phi_k(t) \}_{k \in \mathbb{Z}}$ = Span a vector space T

Analysis :

$$a_k = \langle x(t), \phi_k(t) \rangle_T$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \right\rangle_T = \langle \dots + a_{-10} \phi_{-10}(t) + \dots + a_0 \phi_0(t) + \dots + a_9 \phi_9(t) + \dots, \phi_l(t) \rangle_T$$

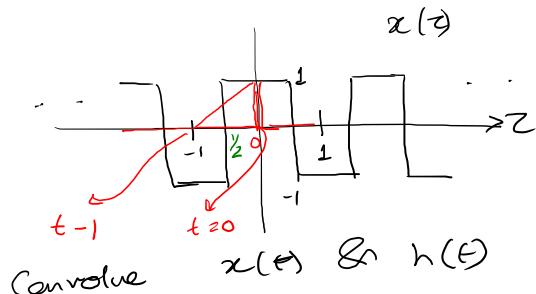
$$= \sum_{k=-\infty}^{\infty} \underbrace{a_k \langle \phi_k(t), \phi_l(t) \rangle_T}$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta[k-l]$$

$$\langle x(t), \phi_l(t) \rangle_T = a_l$$

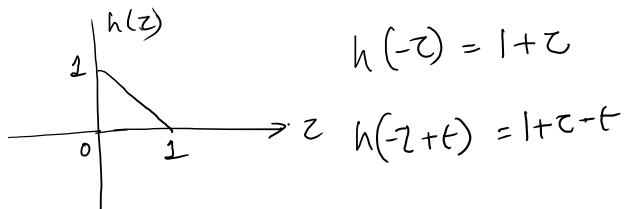
from an orthonormal basis (O.N.B)

~~Ex~~



Convolve

$x(t) \otimes h(t)$

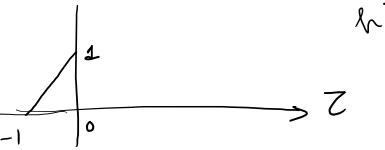


$$h(-t) = 1 + t$$

$$h(-t+1) = 1 + t - 1$$

$$\begin{aligned}
 y(0) &= \int_{-1}^{1/2} x(z)h(t-z)dz + \int_{-1/2}^0 x(z)h(t-z)dz \\
 &= \int_{-1}^{-1/2} (-1)(-1)dz + \int_{-1/2}^0 (1)(1)dz + \int_{-1/2}^0 2p(z+1)(1)dz \\
 &= \left(\frac{z^2}{2} + z \right) \Big|_{-1}^{-1/2} + \left(\frac{z^2}{2} + z \right) \Big|_{-1/2}^0
 \end{aligned}$$

$(2-t)h$



$$\begin{cases}
 \text{Case 1: } 0 \leq t \leq 1/2 \\
 y(t) = \int_{-1}^{-1/2} (1+z)dz + \int_{-1/2}^t (1+z-2)dz
 \end{cases}$$

20/9/2024

Synthesis

periodic signals

C.T. Fourier Series

$$\underline{x(t)} = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

pure imaginary exponential (in power)

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$$

rad/s

$$f_0 = \frac{1}{T}$$

 \equiv
per sec
or Hz

 $T \equiv$ fundamental time period

$e^{jk\omega_0 t}$

$e^{jk\omega_0 t} = \phi_k(t) = \text{pure imaginary exponential (in power)}$

$j\omega_0 t \quad j2\omega_0 t$

$x(t) = 3 + 2e + e$

$= 3 + 2(\underbrace{e^{j\omega_0 t} + e^{j2\omega_0 t}}_{j\omega_0 t \quad j2\omega_0 t}) + e^{j8\omega_0 t}$
 $= \sum_{k=0}^2 a_k e^{jk\omega_0 t}$

$a_0 = 3$	all other
$a_1 = 2$	
$a_2 = 1$	

a_k 's for $k \neq 0, 1, 2$ are zero

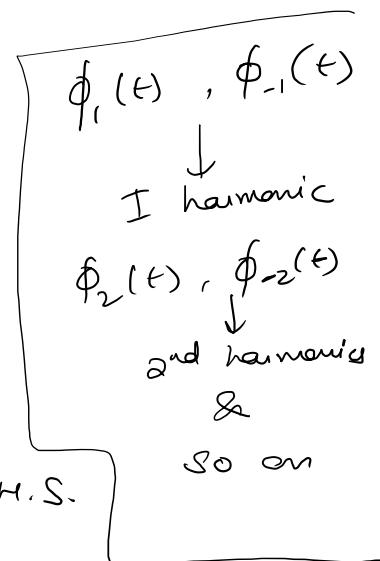
Analysis
equation

$a_k = \langle x(t), \phi_k(t) \rangle_T = \frac{1}{T} \int_T x(t) \phi_k^*(t) dt$

$$a_k = \frac{1}{T} \int_T x(t) \phi_k^*(t) dt = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$\left\{ \phi_k(t) \right\}_{k \in \mathbb{Z}} \equiv \text{orthogonal basis} \equiv \int_T \phi_m(t) \phi_k^*(t) dt = T \delta[m-k]$$

$$\begin{aligned} R.H.S. &= \frac{1}{T} \int \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t) \right) \phi_k^*(t) dt \\ &= \frac{1}{T} \sum_{m=-\infty}^{\infty} a_m \left[\int_T \phi_m(t) \phi_k^*(t) dt \right] \\ &= \frac{1}{T} \sum_{m=-\infty}^{\infty} a_m \left[\int_T e^{j(m-k)\omega_0 t} dt \right] \\ &= \frac{1}{T} \sum_{m=-\infty}^{\infty} a_m T \delta[m-k] = a_k = L.H.S. \end{aligned}$$



$$\text{Synthesis: } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\text{Analysis: } a_k = \frac{1}{T} \int_T \int x(t) e^{-jk\omega_0 t} dt$$

e^{st} act as eigenfunctions of an LTI system where $H(s)$ = corresponding eigenvalue

Let us assume



$$y(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz = \int_{-\infty}^{\infty} h(z) x(t-z) dz$$

$$x(t) = e^{st}$$

$$\text{where } s = j\omega_0$$

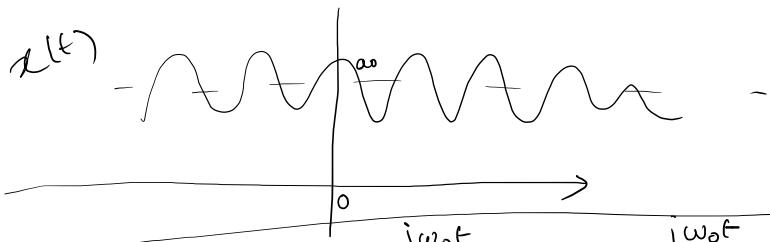
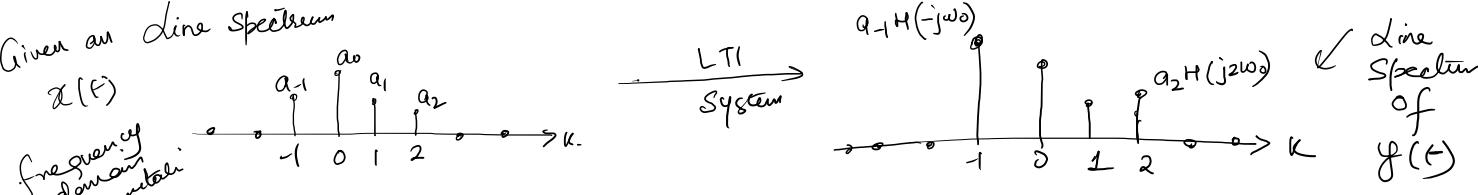
$$y(t) = \int_{-\infty}^{\infty} h(z) e^{s(t-z)} dz = \int_{-\infty}^{\infty} h(z) e^{st} e^{-sz} dz$$

$$= e^{st} \int_{-\infty}^{\infty} h(z) e^{-sz} dz$$

$$y(t) = e^{st} H(s)$$

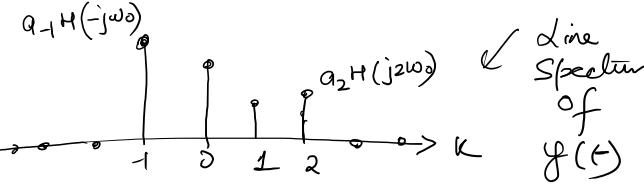


$$\lambda v = \bar{\lambda} v$$



Given

$$x(t) = a_{-1}e^{-jw_0t} + a_0 + a_1e^{jw_0t} + a_2e^{j2w_0t}$$



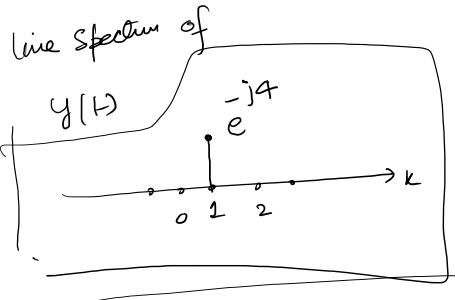
if $H(-jw_0) = 0$
 $a_{-1}H(-jw_0) = 0$
 & this freq. will
 disappear in δ/ρ

Ex

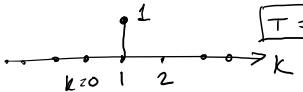
$$s: \quad y(t) = x(t-2)$$

$$\text{det } x(t) = e^{j\omega t}$$

$$\begin{aligned} \omega_0 &= 2\pi f_0 \\ &= \frac{2\pi}{T} \end{aligned}$$



$$\begin{aligned} y(t) &= e^{j\omega(t-2)} \\ &= e^{j\omega t} \cdot e^{-j4} \\ &= e^{j\omega t} H(j\omega) \end{aligned}$$



$$H(j\omega) = e^{-j4}$$

$$S = j\omega$$

Ex

X
what is $h(t)$?

$$h(t) = s(t-2)$$

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(z) e^{-sz} dz = \int_{-\infty}^{\infty} s(z-2) e^{-sz} dz \\ &= \int_{-\infty}^{\infty} s(z-2) \cdot e^{-z} dz \\ &= -j4 \int_{-\infty}^{\infty} s(z-2) dz \\ &= -j4 e^{-j2s} \end{aligned}$$

$$x(t) * s(t) = x(t)$$

$$x(t) * s(t-t_0) \equiv x(t-t_0)$$

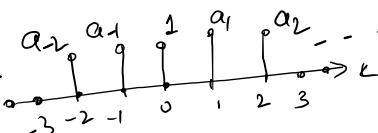
Ex

$$x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \left(\frac{1}{2j}\right) e^{j\omega_0 t} + \left(\frac{-1}{2j}\right) e^{-j\omega_0 t}$$

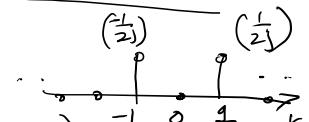
Find and plot the line spectrum of $x(t)$

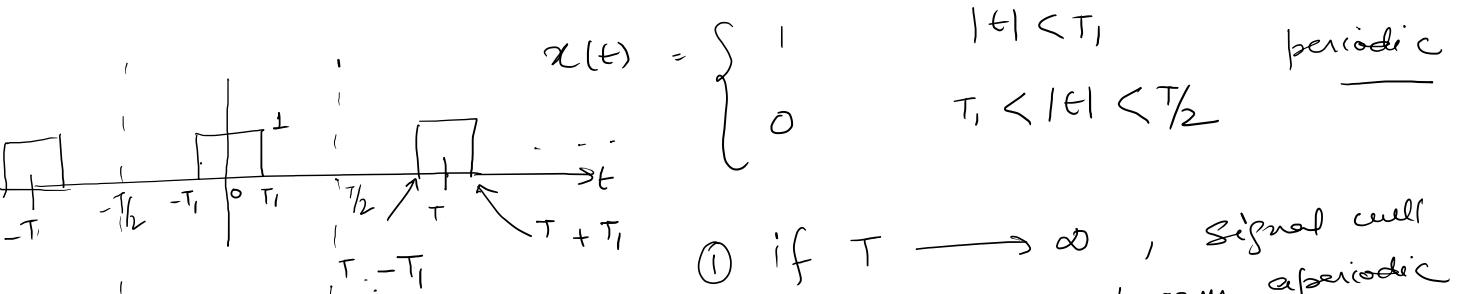
Ex

$$x(t) = 1 + \sin \omega_0 t + \cos(2\omega_0 t + \pi/4)$$



$$\begin{aligned} x(t) &= 1 + \frac{e^{j\omega_0 t}}{2j} + \left(\frac{-1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} \left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)} \right] \\ &= 1 + \left(\frac{1}{2j}\right) \phi_1(t) + \left(\frac{-1}{2j}\right) \phi_{-1}(t) + \left(\frac{1}{2} e^{j\pi/4}\right) \phi_2(t) + \left(\frac{1}{2} e^{-j\pi/4}\right) \phi_{-2}(t) \\ &= a_0 + a_1 \phi_1(t) + a_{-1} \phi_{-1}(t) + a_2 \phi_2(t) + a_{-2} \phi_{-2}(t) \end{aligned}$$





① if $T \rightarrow \infty$, signal will become aperiodic

② there should be a dc bias term, so $a_0 \neq 0$

③

$$\omega_0 = \frac{2\pi}{T}$$

\downarrow
fundamental freq.

Solve it, for the line spectrum.