Lecture 06

Relationship between Continuity and existence of partial derivatives

Example 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

$\longrightarrow$ ⊙ $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = 0$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = 0$

$\longrightarrow$ ⊙ f is **not** continuous at $(0,0)$

[use 2-path test to show that the limit of f at $(0,0)$ does not exist, and hence f is not continuous. Compute the limit along $y = cx^2$.]

Example 2: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

(1) f ~~is~~ not cont. at $(0,0)$

(2) $\dfrac{\partial f}{\partial x}$

$\parallel$

partial derivative of f w.r.t. x

$\parallel$

$\boxed{\text{partial derivative of } f \text{ w.r.t. the first ind. variable}}$

Now, $\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = 0$?? (How)

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = 0.$$

Next, $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{1 - 1}{h}$$

$$= 0.$$

∴ The partial derivatives exist at $(0,0)$

(3) However, f is not continuous at $(0,0)$.
(Verify!!!)

Thus, unlike in one-variable case, where existence of derivatives guarantees continuity, existence of (all) partial derivatives at a point does not even guarantee that the function is continuous at that point!

Need a new notion of derivative!

[Total Derivative]

Differentiability (single-variable case)

Let us recall the following definition from the last lecture.

Definition 1: Let $D \subseteq \mathbb{R}$, $x_0 \in \operatorname{int}(D)$ and $f : D \to \mathbb{R}$ be a (real-valued) function (of one variable). We say that f is differentiable at $x_0 \in \operatorname{int}(D)$

$$\Updownarrow$$

the following limit exists, $\displaystyle\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

When this limit exists, we denote it by $f'(x_0)$, and call it "the derivative of f at x_0".

(End of the definition)

$\Updownarrow$ (convince yourself)

$\exists$ a real number, denoted by $f'(x_0)$,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0).$$

or, equivalently,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-\cancel{hf'(x_0)}^{\,f'(x_0)h}}{h} = 0$$

[Definition 2: (reformulation of the first defn)]

Let $D \subseteq \mathbb{R}$, $x_0 \in \operatorname{int}(D)$ and $f: D \to \mathbb{R}$ be a (real-valued) function (of one variable).

We say that f is differentiable at $x_0 \in \operatorname{int}(D)$

$\Updownarrow$

$\exists$ a real number, denoted by $f'(x_0)$, such that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0$$

Notation:

Let $f'(x_0)$ exists. Define a function E_{x_0} as follows:

(*) $E_{x_0}(h) := f(x_0+h) - f(x_0) - f'(x_0)h.$

(*) $$\boxed{f(x_0+h) - f(x_0) = \underbrace{f'(x_0)h}_{\substack{\text{Approximation of} \\ f(x_0+h)-f(x_0)}} + \underbrace{E_{x_0}(h)}_{\text{Error}}}$$

Observations:

① $f'(x_0)$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

proof: define $T: \mathbb{R} \to \mathbb{R}$ by

$$T(h) = f'(x_0)h. \quad (\text{for every } h \in \mathbb{R})$$

Check: • $T(h_1+h_2) = T(h_1) + T(h_2)$

• $T(\lambda h) = \lambda T(h)$

[Note: In fact, every real number λ can be thought of as a linear mapping of $\mathbb{R}$ into $\mathbb{R}$ (How?)]

② If f is differentiable at x_0, then

$$\frac{E_{x_0}(h)}{h} \to 0 \text{ as } h \to 0$$

i.e., not only the error $E_{x_0}(h)$ tends to 0 as h tends to 0, but it does so rapidly that it still tends to 0 when divided by h !!!

(The error $E_{x_0}(h)$ is <u>of smaller order than h</u> (when h is small))

⊛ The "<u>total derivative</u>" of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ will now be defined in such a way that it preserves these two properties.

[But prior to that, let us re-write the definition of differentiability of a function at a point, in the form which will be generalized to functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

[**<u>Definition 3 (Yet another reformulation)</u>**]

Let $D \subseteq \mathbb{R}$, $x_0 \in \text{int}(D)$ and $f : D \to \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in \text{int}(D)$

$$\Updownarrow$$

there exists a linear transformation $T_{x_0} : \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - T_{x_0}(h)}{h} = 0$$

When f is differentiable at x_0, We set $f'(x_0) = T_{x_0}$ and call it "the derivative of f at x_0"

Now, let us generalize this version of the definition to the case of a real-valued function in several variables.

Differentiability of a function $f: \mathbb{R}^n \to \mathbb{R}$, at a point $\vec{x}_0 \in \mathbb{R}^n$ (The Total Derivative)

defn: Let $D \subseteq \mathbb{R}^2$, $\vec{x}_0 \in \text{int}(D)$, let $f: D \to \mathbb{R}$. We say that f is differentiable at $\vec{x}_0 \in \text{int}(D) \subseteq \mathbb{R}^2$ (or, f has a total derivative at $\vec{x}_0 \in \mathbb{R}^2$)

$\Updownarrow$

there exists a linear transformation $T_{\vec{x}_0}: \mathbb{R}^2 \to \mathbb{R}$, such that

$$\lim_{\vec{h}=(h_1,h_2)\to(0,0)} \frac{f(\vec{x}_0+\vec{h}) - f(\vec{x}_0) - T_{\vec{x}_0}(\vec{h})}{\|\vec{h}\|} = 0$$

When the **total derivative** exists, we set $f'(\vec{x_0}) = T_{x_0}$ and call it "**the total derivative of f at $\vec{x_0}$**"

■

(End of the definition)

Remarks:

1. $$\lim_{\vec{h} \to \vec{0}} \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|_2} = 0 \qquad \left(E_{x_0} : \mathbb{R}^2 \to \mathbb{R}\right)$$

2. Reformulation of the above defn.

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{int}(D)$, let $f: D \to \mathbb{R}$. Then f is differentiable at $\vec{x_0}$ <u>iff</u>

$$\forall \varepsilon > 0 \left(\exists \delta > 0 \left(0 < \|\vec{h}\| < \delta \Rightarrow \left| \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|} \right| < \varepsilon \right) \right).$$

■

8. Reformulation: (Yet again)

Let $D \subseteq \mathbb{R}^n$, let $\vec{x_0} \in \text{int}(D)$, let $f: D \to \mathbb{R}$. Then f is differentiable at $\vec{x_0}$ iff $\forall \varepsilon > 0 \left(\exists \delta > 0 \left(\right.\right.$

$$\|\vec{h}\| < \delta \Rightarrow \left(|E_{x_0}(\vec{h})| < \|\vec{h}\| \varepsilon \right). \blacksquare$$

Thm 1: If $f: \mathbb{R}^n \to \mathbb{R}$ is diff'ble at $\vec{x_0} \in \mathbb{R}^n$, then $\exists$ a unique linear transformation $T_{x_0}: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\lim_{\vec{h} \to 0} \frac{E_{x_0}(\vec{h})}{\|\vec{h}\|_2} = 0$$

Thm 2: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be diff'ble at $\vec{x_0} \in \mathbb{R}^2$ with the total derivative

$T_{\vec{x_0}} : \mathbb{R}^2 \to \mathbb{R}$, then

① both the partial derivatives of f at $\vec{x_0}$ exist,

and we have

② $$T_{\vec{x_0}} = \left[\frac{\partial f}{\partial x}(\vec{x_0}), \frac{\partial f}{\partial y}(\vec{x_0}) \right].$$

analogous to $f'(\vec{x_0})$

So that,

$$T_{x_0}(\vec{h}) = \left[\frac{\partial f}{\partial x}(\vec{x_0}), \frac{\partial f}{\partial y}(\vec{x_0})\right]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}$$

analogous to $f'(\vec{x_0})h$

End of Theorem 2.

Theorem 3: (Differentiability $\Rightarrow$ continuity)

Let $D \subseteq \mathbb{R}^n$, $\vec{x_0} \in \operatorname{int}(D)$ and $f: D \to \mathbb{R}$ be a function.

If f is differentiable at $\vec{x_0}$, then f is continuous at $\vec{x_0}$.

Proof:- (Beyond the scope of this course, but I would be more than happy to walk you through it if you are really interested!)

Generalization

Recall the definition of a vector-valued function of several variables (or, a vector field).

Let $D \subseteq \mathbb{R}^n$ and let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n variables).
The function $\vec{f}$ (in fact, any such function) has the form

$$\vec{f}(\vec{x}) = \left(f_1(\vec{x}), \dots, f_m(\vec{x})\right)$$

where we call the functions

$f_i : D \to \mathbb{R}$, the "**component functions**".

Differentiability (The Total Derivative)

defn: Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \operatorname{Int}(D)$ and let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n variables).

We say that $\vec{f}$ is differentiable at $\vec{x_0}$ (or, $\vec{f}$ has a total derivative at $\vec{x_0}$) if there exists a linear transformation $T_{\vec{x_0}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{\vec{h} \to \vec{0}} \frac{\left\| \vec{f}(\vec{x_0}+\vec{h}) - \vec{f}(\vec{x_0}) - T_{\vec{x_0}}(\vec{h}) \right\|}{\|\vec{h}\|} = 0.$$

When the total derivative exists, we set
$\vec{f}\,'(\vec{x_0}) = T_{\vec{x_0}}$, and call it "the total derivative of $\vec{f}$ at $\vec{x_0}$."

The Total derivative expressed in terms of partial derivatives

Thm: Let $D \subseteq \mathbb{R}^n$, let $\vec{x}_0 \in \operatorname{Int}(D)$, let $\vec{f}: D \to \mathbb{R}^m$ be a (vector-valued) function (of n independent variables $x_1, x_2, \ldots, x_n$) and let $f_i : D \to \mathbb{R}$, $1 \le i \le m$, denote the component functions of $\vec{f}$.

If $\vec{f}$ is differentiable at $\vec{x}_0$, then for all $i \in \{1, \ldots, m\}$ and for all $j \in \{1, \ldots, n\}$

$$\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \text{ exists and}$$

$$\vec{f}\,'(x_0) = T_{\vec{x}_0} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0) \right].$$

proof: Trivial.

A clever use of matrix multiplication!!!

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