Lecture-14

Problem : Find the point(s) closest to the origin on the Hyperbolic Cylinder $x^2 - z^2 - 1 = 0$.

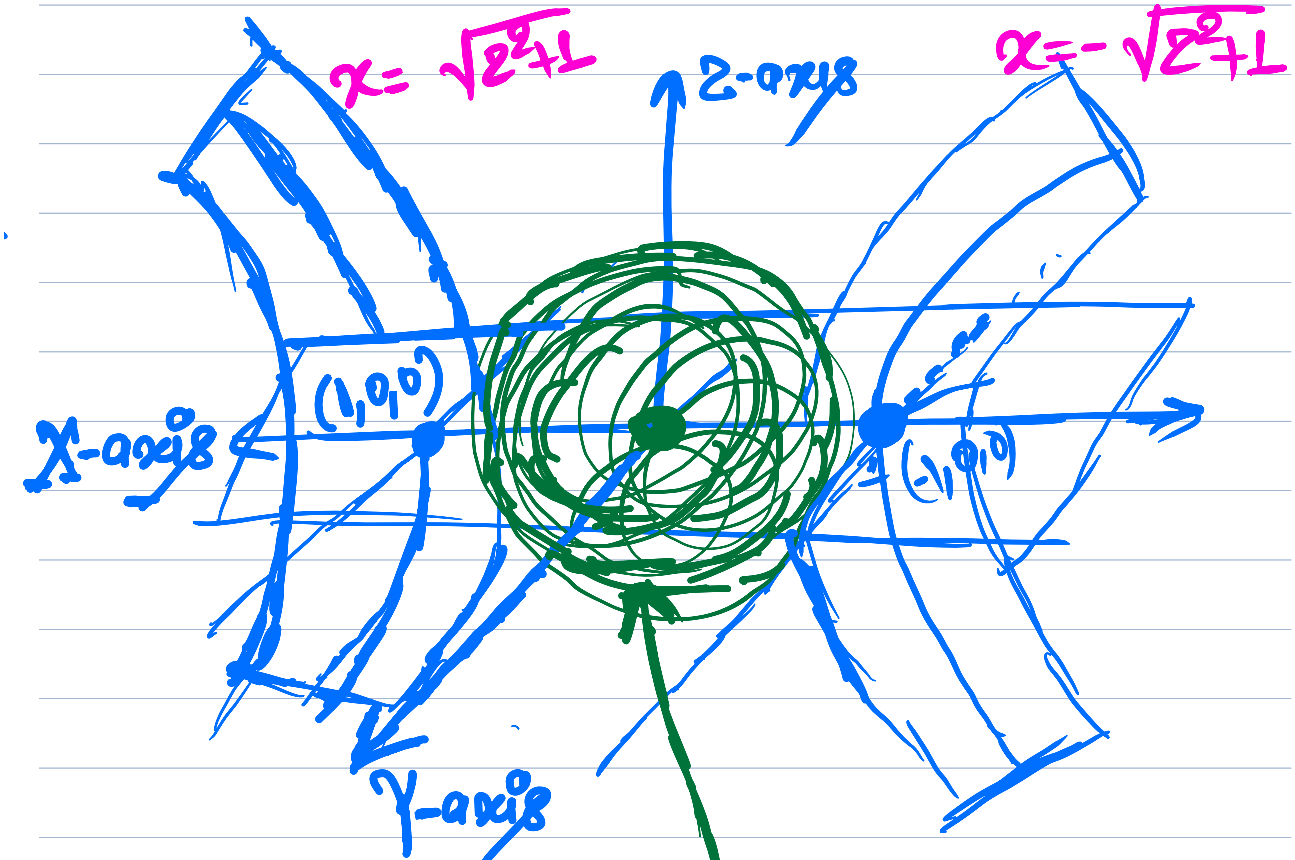
This motivates us to develop a geometric method

Approach III : [The method of Lagrange multipliers]
(Geometric method)

minimize $h(x,y,z) = x^2 + y^2 + z^2$

subject to the constraint $\underbrace{x^2 - z^2 - 1 = 0.}_{\downarrow}$

$(x^2 - z^2 = 1)$



$$f(x,y,z) = x^2+y^2+z^2-a$$

$$g(x,y,z) = x^2-z^2-1$$

[The sphere $f(x,y,z)=0$
touches the surface $g(x,y,z)=0$

What are the properties of points where the sphere $f(x,y,z)=0$ touches the surface $g(x,y,z)=0$?

1. The radius of the sphere = the shortest distance from the origin.

2. Tangent planes to f and g at these points are same.

3. Normal lines are also same.

---

2. and 3. $\Rightarrow$ ∇f and ∇g are scalar multiples of each other.

$\Updownarrow$

$$\boxed{\nabla f = \lambda \nabla g \text{ for some } \lambda \in \mathbb{R}}$$

$$\Rightarrow 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(2x\vec{i} + 0\vec{j} - 2z\vec{k})$$

$$\Rightarrow \left.\begin{array}{l} 2x = \lambda 2x \\ 2y = 0 \\ 2z = -\lambda 2z \end{array}\right] \Rightarrow \boxed{y=0} \text{ --- } (A)$$

$$\Rightarrow \left.\begin{array}{l} x(1-\lambda) = 0; \\ z(1+\lambda) = 0. \end{array}\right]$$

So, $x(1-\lambda)=0 \Rightarrow$ either $x=0$ or $\lambda=1$.

Clearly $x \neq 0$ (why?).

So, $\lambda = 1$.

Thus, we have $\boxed{y=0}$ --- (A)

and $\boxed{\lambda=1}$ --- (B)

Using the final equation $z(1+\lambda)=0$,

we get $z(1+1)=0 \Rightarrow z=0$

$\therefore \boxed{z=0}$ --- (C)

From (A) and (C), and using $x^2 - z^2 = 1$,

we get $x^2 = 1 \Rightarrow \boxed{x = \pm 1}$ — (D)

Hence, the required points are:

$(1,0,0)$ and $(-1,0,0)$. ∎

The method of Lagrange multipliers

Suppose that $f(x,y,z)$ is a differentiable function whose variables are subject to the constraint $g(x,y,z) = 0$ (where g is differentiable and $\nabla g \neq 0$).

To find the constrained local extrema of f, we need to simultaneously solve the equations

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) & \text{--- (1)} \\ g(x,y,z) = 0 & \text{--- (2)} \end{cases}$$

for the variables x, y, z and λ.

Explicitly,

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\[2ex] \dfrac{\partial f}{\partial z} = \lambda \dfrac{\partial g}{\partial z} \\[2ex] g(x,y,z) = 0 \end{cases}$$

Question: What happens when f and g are functions of two variables?

Answer:

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y)=0\end{array}\right] \text{ or } \left[\begin{array}{l}\dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x,y)=0\end{array}\right.$$

Problem 3: Find the extreme values of the function

$$f(x,y) = xy$$

that are attained on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

Solution Optimize $f(x,y) = xy$

Subject to $g(x,y) = \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0.$

$$[\nabla f = \lambda \nabla g] \text{ and } \left[\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0\right]$$

$$\Rightarrow y\vec{i} + x\vec{j} = \lambda\left(\frac{x}{4}\vec{i} + y\vec{j}\right)$$

$$\Rightarrow \begin{cases} y = \dfrac{\lambda x}{4} \\ x = \lambda y \end{cases} \Rightarrow y = \dfrac{\lambda^2 y}{4} \Rightarrow y\left(1 - \dfrac{\lambda^2}{4}\right) = 0$$

$\Rightarrow$ Either $y = 0$ or $\lambda = \pm 2$

<u>Case I</u>: $y = 0 \Rightarrow x = 0$. point $\equiv (0,0)$ ✗ Not possible

<u>Case II</u>: $y \neq 0$. That means $\lambda = \pm 2$

$\Rightarrow x = \pm 2y$ and $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$

$$\Rightarrow \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$$

$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$

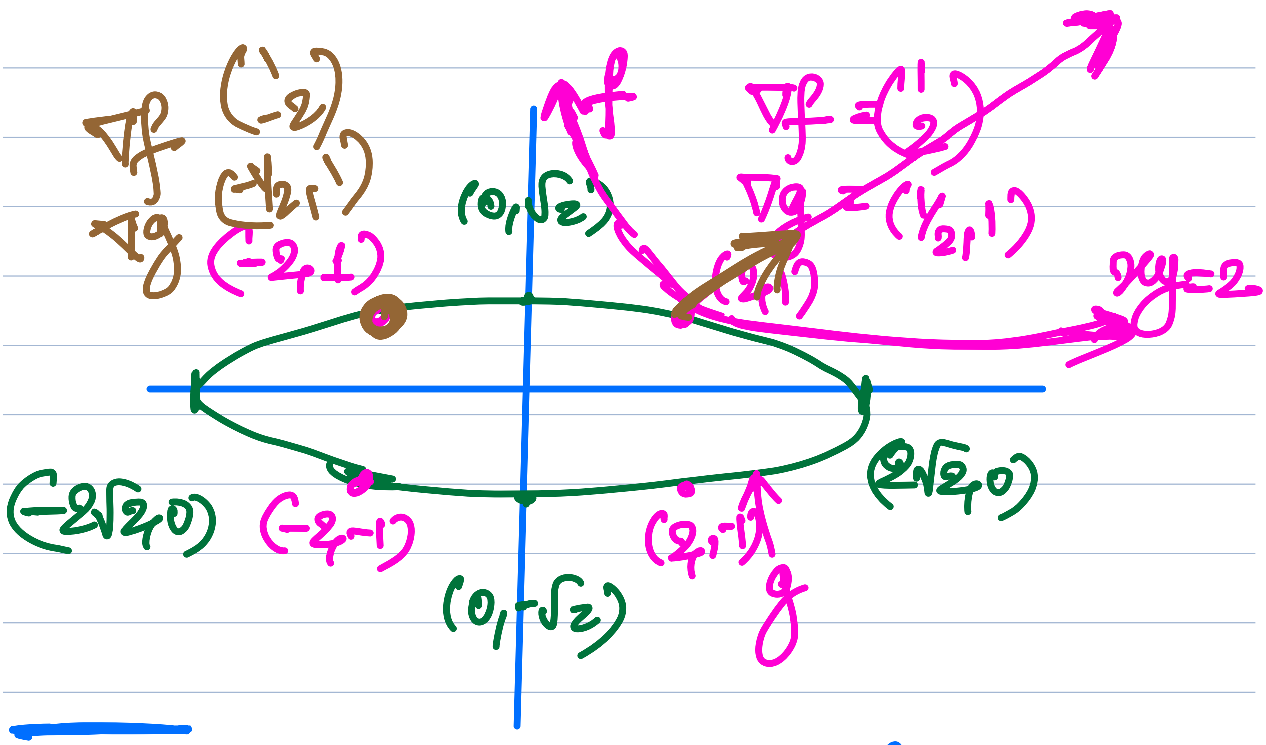
points: $(\pm 2, 1)$ and $(\mp 2, -1)$

or $(2,1)\ (-2,1)\ (-2,-1), (2,-1)$. ■

<u>Now</u>,

$f(2,1) = 2 \qquad f(-2,-1) = 2$

$f(-2,1) = -2 \qquad f(2,-1) = -2$



Problem 4: Find the extreme values of

$f(x,y) = xy$ subject to

$g(x,y) = x^2 + y^2 - 10 = 0$

HW

Problem 5: Find the max and min values of

$f(x,y) = 3x + 4y$ on the

circle $x^2 + y^2 = 1$.

Solution $f(x,y) = 3x+4y$

$g(x,y) = x^2+y^2-1 = 0$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3 = 2\lambda x \\ 4 = 2\lambda y \end{cases}$$

$$\Rightarrow \begin{cases} x = 3/2\lambda \\ y = 4/2\lambda \end{cases}$$

$g(x,y) = 0$

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 1$$

$$\Rightarrow \frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} = 1$$

$$\Rightarrow \frac{25}{4\lambda^2} = 1$$

$$\Rightarrow 4\lambda^2 = 25$$

$$\Rightarrow \lambda = \pm 5/2 .$$

$\left(\frac{3}{5}, 4/5\right)$ and $\left(-3/5, -4/5\right)$ ∎

Lagrange Multipliers with 2 constraints

$f(x,y,z)$ — diff'ble fn. whose variables are subject to constraints

$g_1(x,y,z) = 0$
and $g_2(x,y,z) = 0$

g_1 and g_2 differentiable and ∇g_1 is not parallel to ∇g_2.

Method:

$$\begin{cases} \nabla f = \lambda \nabla g_1 + u \nabla g_2 & \text{— (1)} \\ g_1 = 0 & \text{— (2)} \\ g_2 = 0 & \text{— (3)} \end{cases}$$

x, y, z, λ & u

Problem 6: The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$.

Find the points on the ellipse that lie closest to and farthest from the origin.

Soln: find the extreme values of

$$f(x,y,z) = x^2+y^2+z^2$$

Subject to constraints

$$g_1(x,y,z) = x^2+y^2-1 = 0 \leftarrow \text{I}$$

$$g_2(x,y,z) = x+y+z-1 = 0 \leftarrow \text{II}$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x = \lambda 2x + \mu \\ 2y = \lambda 2y + \mu \\ 2z = \mu \end{cases}$$

$$\Rightarrow \begin{cases} 2x = \lambda 2x + 2z \\ 2y = \lambda 2y + 2z \end{cases} \Rightarrow \begin{cases} (1-\lambda)x = z \\ (1-\lambda)y = z \end{cases}$$

Either $\boxed{\lambda = 1}$ then $z = 0$

or, $\boxed{\lambda \neq 1} \Rightarrow x = y \left(= \frac{z}{1-\lambda} \right)$

<u>Case I</u>: $\boxed{\lambda = 1}$, $z = 0$

$\Rightarrow x^2 + y^2 = 1$ and $x + y = 1$

$\Rightarrow x^2 + (1-x)^2 = 1$

$\Rightarrow 2x^2 - 2x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow \underbrace{x=0}_{\Downarrow\ y=1}$ & $\underbrace{x=1}_{\Downarrow\ y=0}$.

$$\boxed{(0,1,0) \ \& \ (1,0,0)}$$

closest to the origin

<u>Case II</u> $\boxed{\lambda \neq 1} \Rightarrow x=y$

$2x^2=1$

$\Downarrow$

$x = \pm \frac{1}{\sqrt{2}}$

&

$2x+z=1$

$\Rightarrow z = 1-2x$

$\Rightarrow z = 1-2\left(\pm\frac{1}{\sqrt{2}}\right)$

$= 1 \mp \sqrt{2}$

$\underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right)}$ and $\underbrace{\left(-1/\sqrt{2}, -1/\sqrt{2}, 1+\sqrt{2}\right)}$

X (HW)   farthest (HW)

Partial derivatives with constrained Variables (Section 14.9 of Thomas' Calc.)

Qn: find $\frac{\partial w}{\partial x}$ if $w = x^2+y^2+z^2$

and $z = x^2+y^2$

Soln: $\frac{\partial w}{\partial x}$

dependent

independent

	Depend	Ind.
①	w, z	x, y
②	w, y	x, z

$$\omega = x^2 + y^2 + z^2$$

$$= x^2 + y^2 + (x^2 + y^2)^2$$

$$= x^2 + y^2 + x^4 + y^4 + 2x^2y^2$$

$$\boxed{\frac{\partial \omega}{\partial x} = 2x + 4x^3 + 4xy^2}$$

$$\omega = x^2 + \underbrace{y^2}_{z - x^2} + z^2$$

$$= z + z^2$$

$$\boxed{\frac{\partial \omega}{\partial x} = 0}$$

find $\frac{\partial w}{\partial x}$ at $(x,y,z)=(2,-1,1)$

if $w = x^2+y^2+z^2$

$z^3 - xy + yz + y^3 = 1$

and x & y are ind. variable

Soln: $w = x^2+y^2+z^2$

$$\Rightarrow \frac{\partial w}{\partial x} = 2x + 2z \cdot \frac{\partial z}{\partial x} \longrightarrow \text{(A)}$$

$z^3 - xy + yz + y^3 = 1$

$$\Rightarrow 3z^2 \cdot \frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} + 0 = 0 \longrightarrow \text{(B)}$$

$$\frac{\partial z}{\partial x}(3z^2+y) = y$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{y}{3z^2+y}$$

$$\therefore \frac{\partial w}{\partial x} = 2x + 2z \cdot \left(\frac{y}{3z^2+y}\right)\Bigg|_{(2,-1,1)}$$

$$= 3 \quad \underline{\underline{\text{ans.}}}$$

A word on notation:

$\left(\frac{\partial w}{\partial x}\right)_y \leftarrow$ "x" and "y" ind. variables; rest every variable ~~is~~ dependent.

$\left(\frac{\partial w}{\partial x}\right)_{y,t} \leftarrow$ "x", "y", and "t" are independent and rest dependent.

Lecture 15

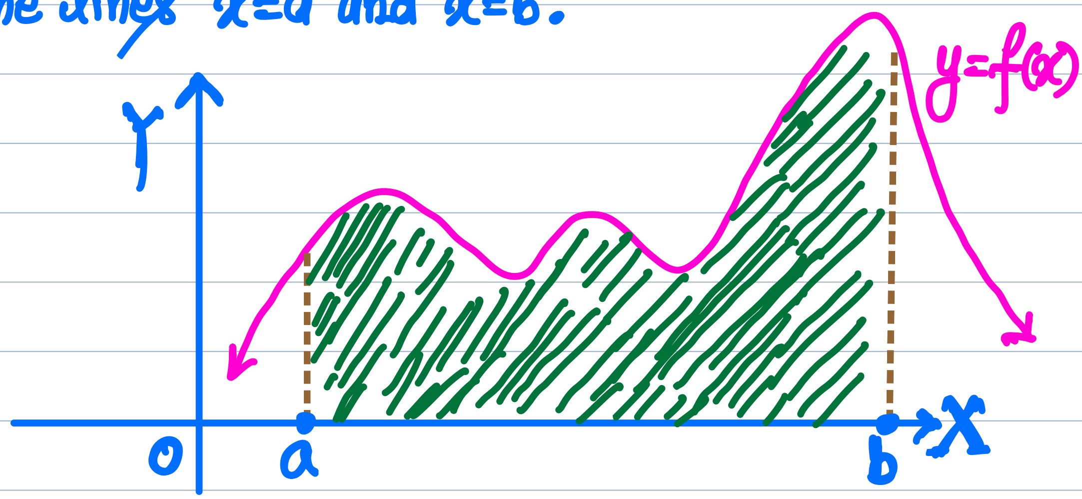
Multiple Integrals

Recapitulation:

Given a continuous function $f:[a,b]\to\mathbb{R}$, recall that

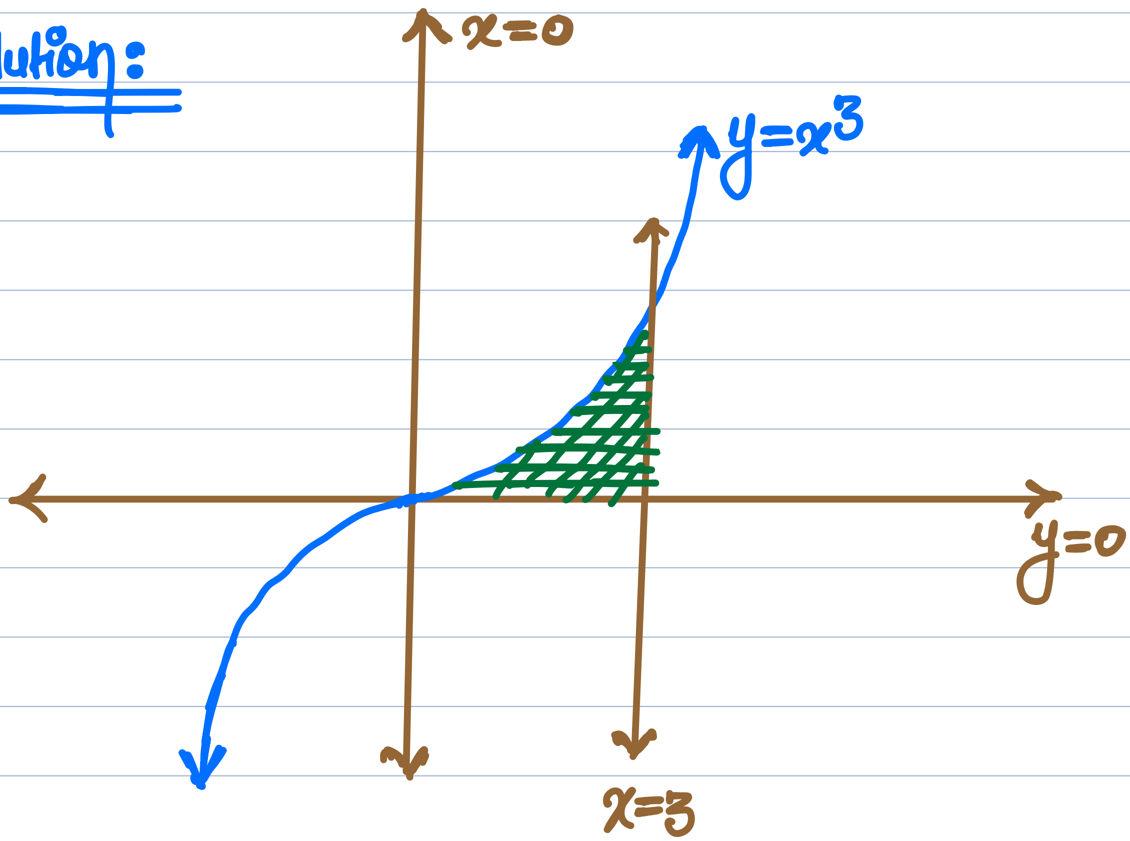
$$\int_a^b f(x)\,dx$$

represents the (signed) area of the region bounded by the curve $y=f(x)$, the x-axis and the lines $x=a$ and $x=b$.



Example: Find the area bounded by the curve $y=x^3$, the x-axis and the line $x=3$.

Solution:



$$\int_{x=0}^{x=3} f(x)\,dx = \int_0^3 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{(3)^4}{4} - 0$$

$= 81/4$ square units. ■

Remark: ⊙ The function $f:[a,b]\to\mathbb{R}$, is called the integrand.

⊙ $[a,b] \equiv$ the interval of integration.

⊙ $\int_a^b f(x)\,dx \equiv$ the (Riemann) integral of the function f over $[a,b]$

Observation:

⊙ the interval of integration is assumed to be closed and bounded.

⊙ the integrand is assumed to be continuous on the interval of integration

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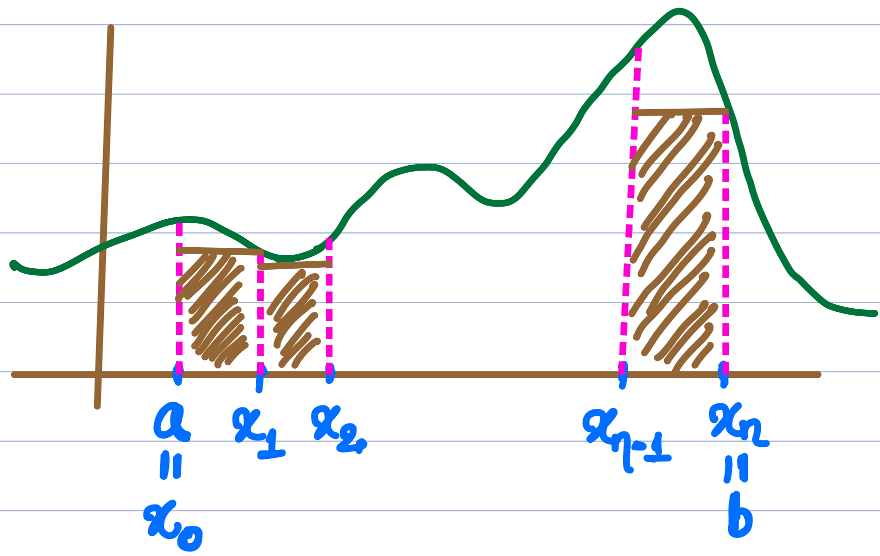
Steps for evaluating Riemann integral

(1) Consider a partition P of $[a,b]$, that is,

$$P=\{x_0, x_1, \dots, x_n\}, \text{ where}$$

$$a=x_0<x_1<x_2<\dots<x_{n-1}<x_n=b$$

(2)



$$\Delta x_i = \text{length of the } i\text{-th interval}$$
$$= (x_i - x_{i-1})$$

- The norm or width of the partition P is given by $\|P\| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}$

(3) Choose a point x_k in the k^{th} subinterval $[x_{k-1}, x_k]$. In the above graph, x_k is the point of minimum of the k^{th} subinterval

(4) Compute $f(x_k)$ for each k.

(5) Compute $\sum_{k=1}^{n} f(x_k)\Delta x_k$.

This is the crude approximation of the area under the curve.
It, of course depends on the partition P that we started with.

So, let us denote it by

$$S(f,P)=\sum_{k=1}^{n} f(x_k)\Delta x_k .$$

(6) Consider the following limit

$$\lim_{\|P\|\to 0} S(f,P)$$

(7) If the above limit exists (which always does in case of continuous functions), then f is said to be Riemann integrable and we set

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(f,P)$$

and call it

the (Riemann) integral of f over $[a,b]$.

Remark: Every continuous function on $[a,b]$ is Riemann integrable.

$\to$ Not a trivial result!

$\to$ I will try to pen down the discussion in class about this in an addendum to this lecture. However, that won't be part of the prescribed syllabus.

Question Let $f:[a,b]\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is f Riemann integrable?

End of recapitulation

Double integral

- the one-dimensional interval $[a,b]$ of integration is replaced by a two-dimensional subset $D \subseteq \mathbb{R}^2$, called the **region of integration**.

- Of course, then, f must be a function of two variables and f must also be real-valued.

- integrand $\equiv$ a scalar field defined and continuous on a region $D \subseteq \mathbb{R}^2$.

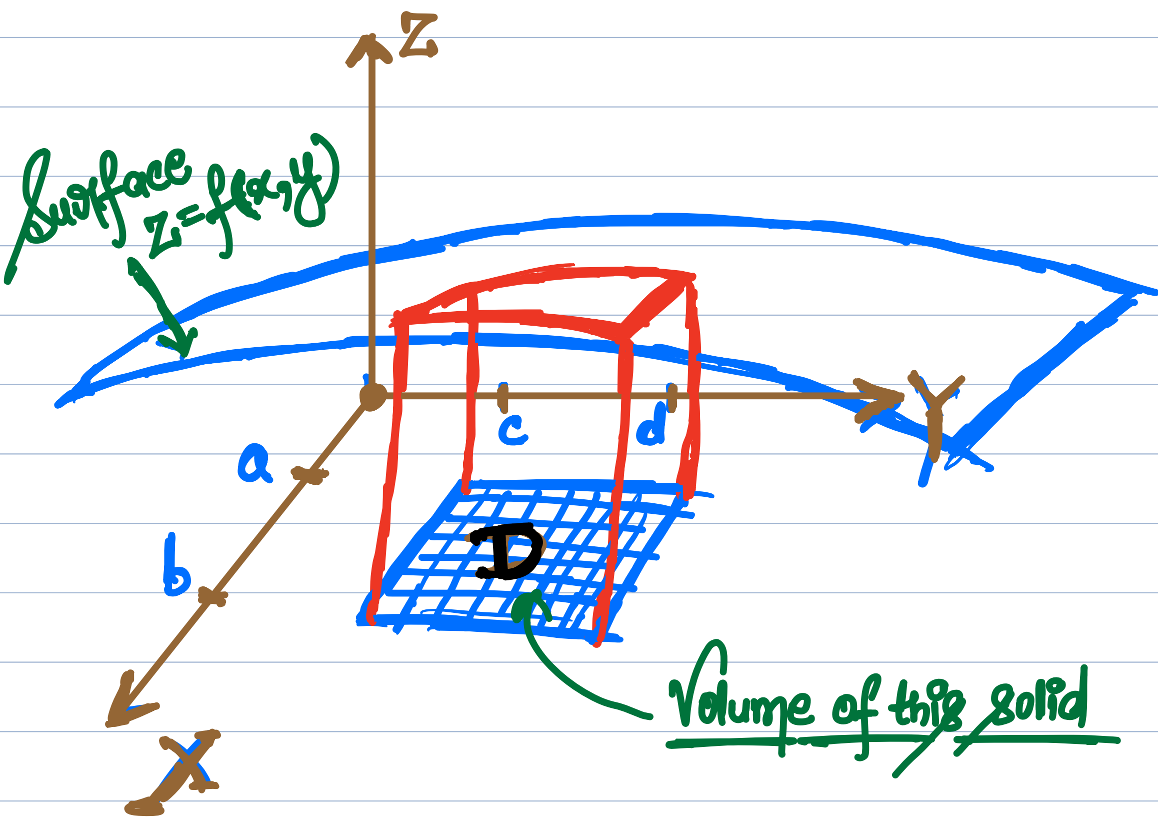
- the integral (or, the double integral)

$$\equiv \iint_D f \quad \text{or} \quad \iint_D f(x,y)\,dx\,dy$$

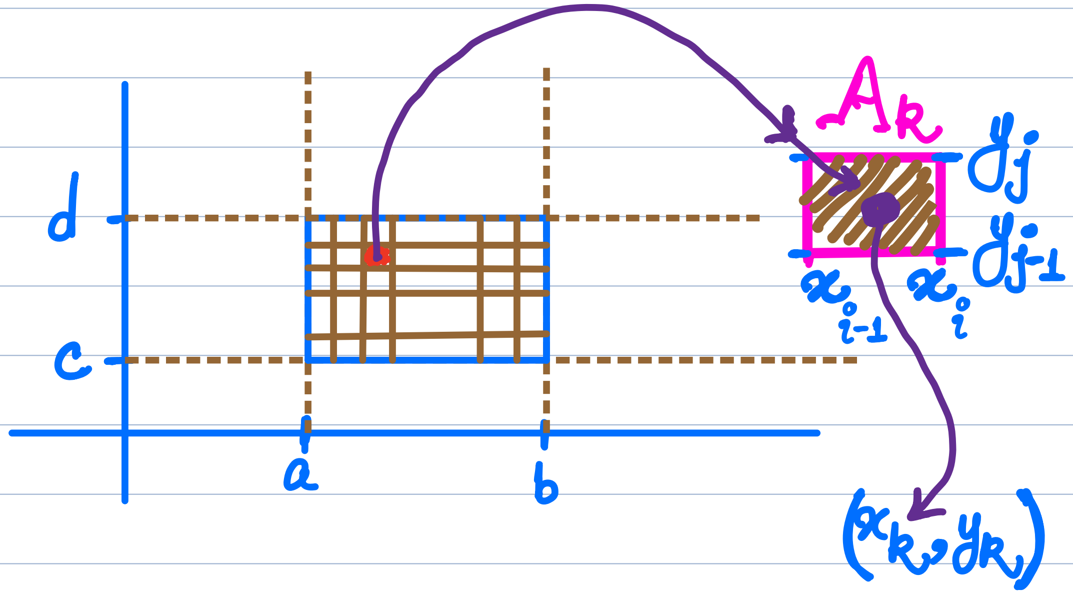
Double Integral over Rectangles

Let $f(x,y)$ be defined on a rectangular region

$D: a \le x \le b, \ c \le y \le d$



Step I: partition the rectangle D into n subrectangles, $A_1, A_2, \ldots, A_n$



(one can consider partitioning the interval $[a,b]$ into m_1 subintervals and the interval $[c,d]$ into m_2 subintervals, so that

$$P_1 = \{x_0, \dots, x_{m_1}\} \text{ and}$$
$$P_2 = \{y_0, \dots, y_{m_2}\}$$

So that the partition $P_1 \times P_2$ yields a partition of the rectangle D into $m_1 m_2$ subrectangles. Set $n = m_1 m_2$.)

Step II: Area of the subrectangle $A_k = \Delta A_k$

and

$$\text{diam}(A_k) = \sqrt{l^2 + b^2}, \text{ so that}$$

$$\|P\| = \max\{\text{diam}(A_k) : 1 \le k \le n\}$$

Step III: Choose a point (x_k, y_k) in A_k

Step IV: Compute $f((x_k, y_k))$

Step V: Compute $\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$.

This is the crude approximation of the volume beneath the surface $f(x,y)$. It, of course, depends on the partition P that we started with. So, let us denote it by

$$S(f, P) = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

Step VI: Consider the following limit

$$\lim_{\|P\| \to 0} S(f, P)$$

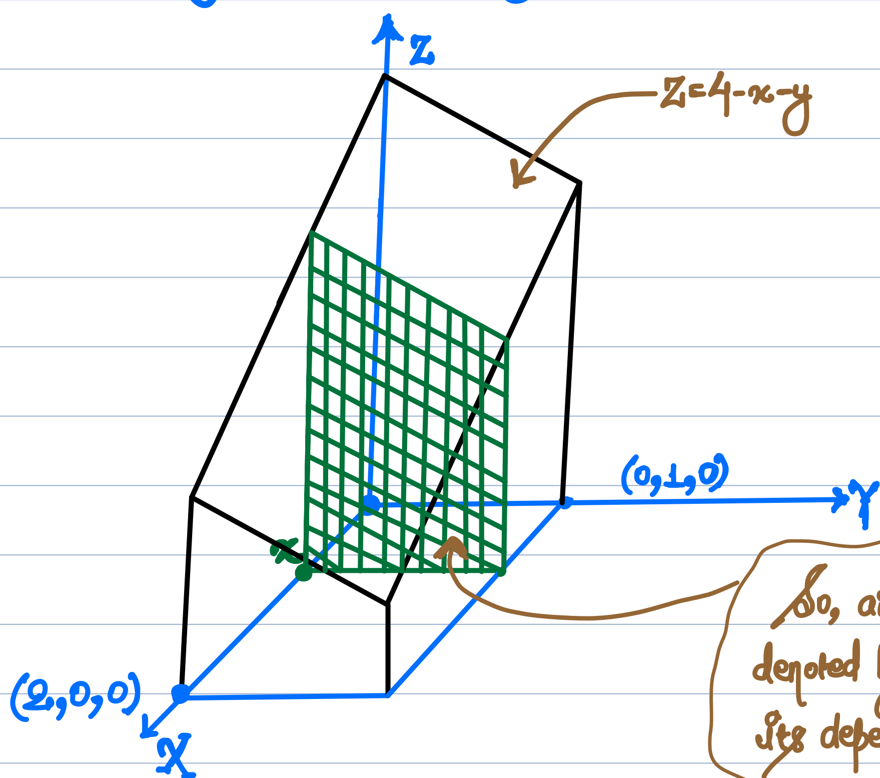
Step VII: If the above limit exists (which always does in case of continuous functions), then f is said to be Riemann integrable and we set

$$\iint_D f(x,y)\,dx\,dy = \lim_{\|P\|\to 0} S(f,P)$$ and

call it **the double integral of f over the region D.**

Remark: Every continuous function $f(x,y)$ on a closed bounded region of $\mathbb{R}^2$ is integrable.

Problem 1: Integrate $f(x,y) = 4-x-y$ over the region $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 1\}$.



Sliced the solid with planes that are perpendicular to the x-axis. So, area of this slice is denoted by $A(x)$ to emphasize its dependence on x.

Thus, Volume of the solid beneath the surface

$$= \int_{x=0}^{2} A(x)\,dx$$

How to compute $A(x)$ for an arbitrary $x \in [0,2]$?

$$A(x) = \int_{y=0}^{1} (4-x-y)\,dy \qquad \text{(why?)}$$

$$= 4y - xy - y^2/2 \Big]_{y=0}^{y=1}$$

$$= 4 - x - 1/2$$

$$= 7/2 - x.$$

$$\therefore \text{Volume} = \int_0^2 7/2 - x \; dx = 7/2\,x - \frac{x^2}{2}\Big]_{x=0}^{x=2}$$

$$= 7 - 2$$

$= 5$ cubic units. ■

$$\therefore \text{Volume} = \int_0^2 \int_0^1 (4-x-y)\,dy\,dx$$

What happens if we would have sliced the solid with planes that are perpendicular to the y-axis instead of the x-axis?

(Approach II:)

$$\text{Volume} = \int_{y=0}^{1} A(y)\,dy, \text{ where}$$

$$A(y) = \int_{x=0}^{2} (4-x-y)\,dx.$$

Consequently, $\text{Volume} = \int_0^1 \int_0^2 (4-x-y)\,dx\,dy$

$= 5$ cubic units. ■

Question Does the order of integration matter?

Fubini's theorem:

If $f(x,y)$ is continuous on a rectangular region
$D: a \leq x \leq b, \; c \leq y \leq d$,
then

$$\iint_D f = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy$$

$$= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx$$

Problem 2: Calculate $\iint_D f$,

where

$f(x,y) = 1-6x^2y$ and

$D: 0 \leq x \leq 2, -1 \leq y \leq 1.$

Homework

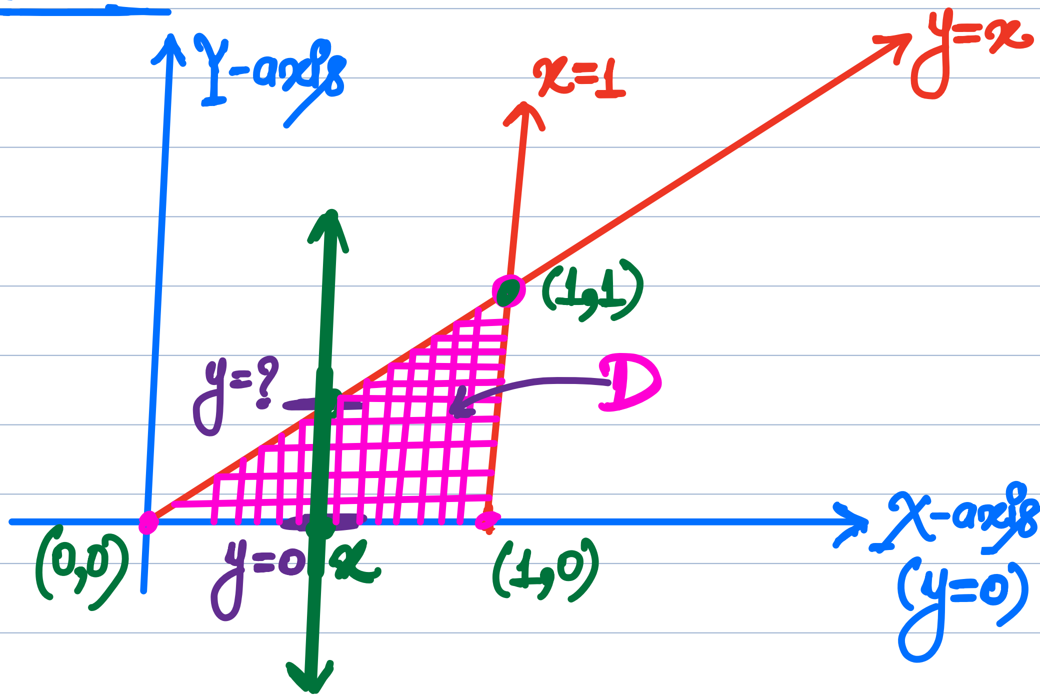
Question: What if the region of integration, $D \subseteq \mathbb{R}^2$ is nonrectangular?

Instead of going through the theory, let's understand this via a problem.

Double integral over bounded non-rectangular regions

Problem 3: Find the integral of $z = f(x,y) = 3 - x - y$ over the triangular region D in the X-Y plane bounded by the x-axis and the lines $y = x$ and $x = 1$.

Solution:



$$\text{Volume} = \int_{x=0}^{x=1} A(x)\,dx \quad \text{, where}$$

$$A(x) = \int_{y=0}^{y=x} (3-x-y)\,dy.$$

So that

$$\text{Volume} = \int_{x=0}^{x=1} \left[3y - xy - y^2/2\right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \left(\left[3x - x^2 - x^2/2\right] - [0]\right) dx$$

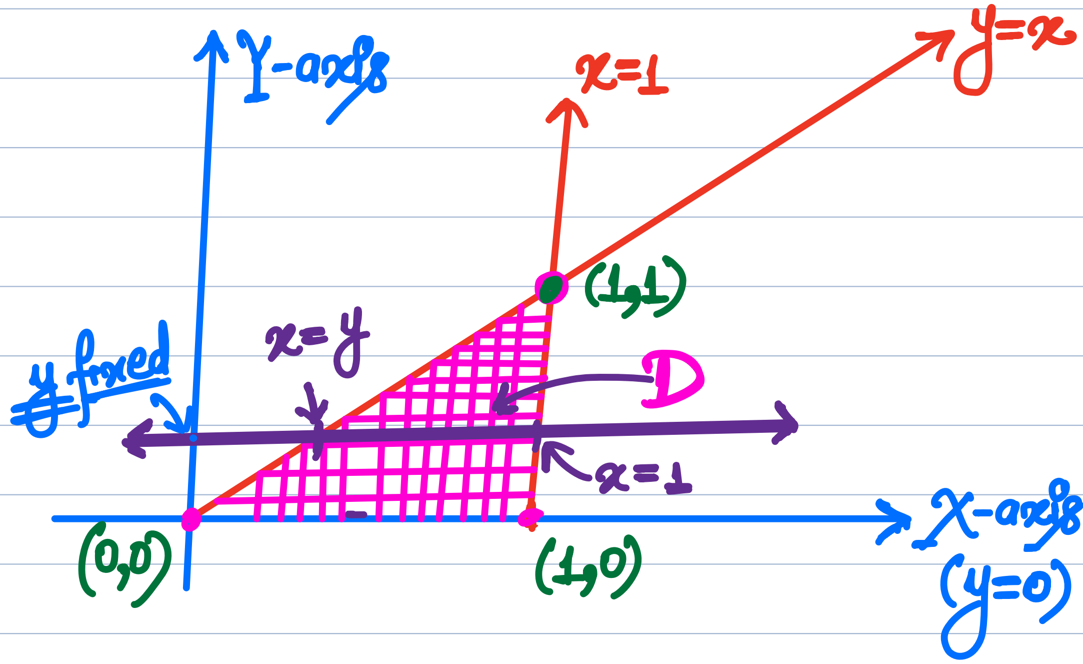
$$= \int_{x=0}^{x=1} \left(3x - \frac{3}{2}x^2\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1}$$

$$= \left(3/2 - 1/2\right) - (0)$$

$= 1$ cubic units.

Approach II: Slicing with planes that are perpendicular to the y-axis.



$$\therefore \text{Volume} = \int_{y=0}^{y=1}\int_{x=y}^{x=1} (3-x-y)\,dx\,dy$$

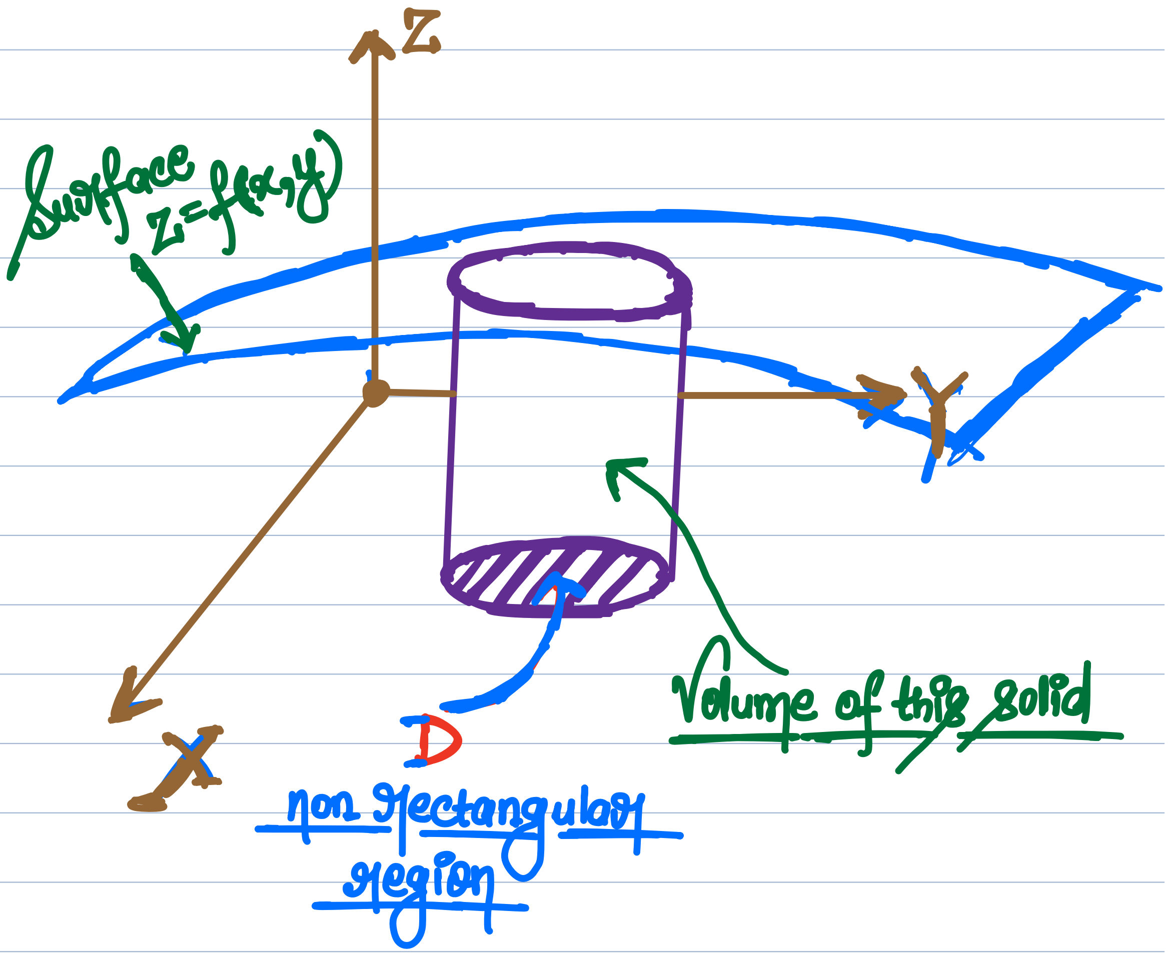
$$= \ldots = 1 \text{ cubic units.}$$

$\blacksquare$

So, the order of integration didn't alter the result.

Can we have an analogous result/theorem for nonrectangular regions of integration?

Double integral over bounded non-rectangular regions



Compute $\iint_D f(x,y)\,dx\,dy$.

Fubini's theorem: (Stronger form)

Let $f(x,y)$ be a continuous function on a region D

① If D is defined by

$$a \leq x \leq b \; ; \; g_1(x) \leq y \leq g_2(x)$$

with g_1 and g_2 being continuous on $[a,b]$, then

$$\iint_D f = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy \right) dx$$

$$= \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy\, dx .$$

② If D is defined by

$$c \leq y \leq d \; ; \; h_1(y) \leq x \leq h_2(y)$$

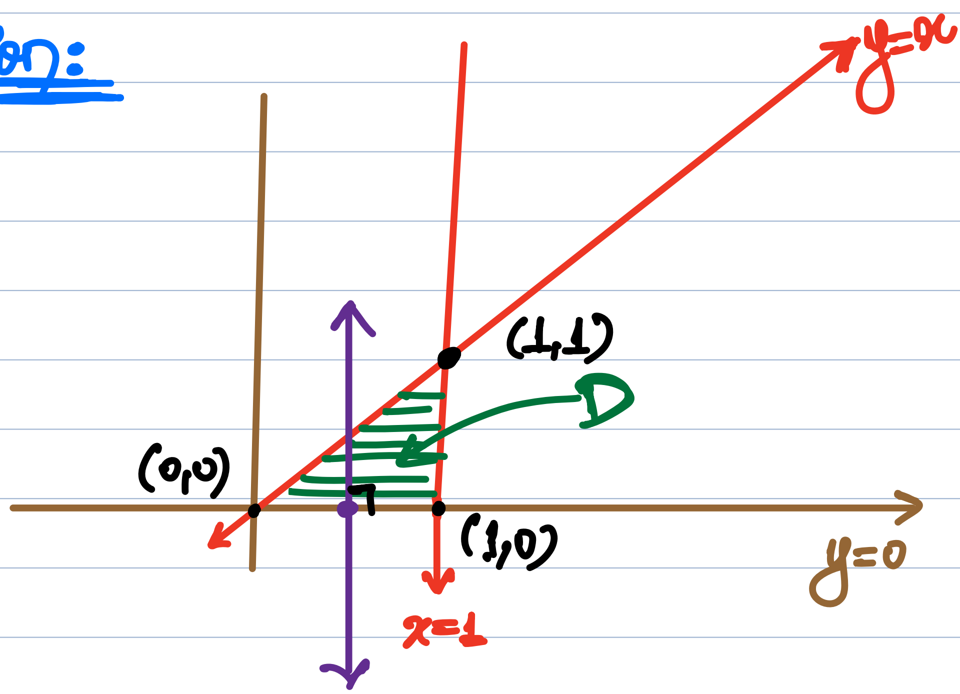
with h_1 and h_2 being continuous on $[c,d]$, then

$$\iint_D f = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$

Problem 4: Calculate $\displaystyle\iint_D \frac{\sin x}{x}\,dx\,dy$,

where D is a subset of the X-Y plane bounded by the x-axis, the line $y=x$, and the line $x=1$.

Solution:



$$A(x) = \int_{y=0}^{y=x} \frac{\sin x}{x}\, dy$$

$$\int_{x=0}^{1}\int_{y=0}^{y=x} \frac{\sin x}{x}\, dy\, dx = \int_{x=0}^{1} \left[\frac{\sin x}{x} \cdot y \right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \sin x \, dx$$

$$= \left[-\cos x \right]_{x=0}^{x=1}$$

$$= 1 - \cos 1 \text{ cubic units.}$$

Question: What happens if you reverse the order of integration?

(HW)