Lecture 04

Recall:

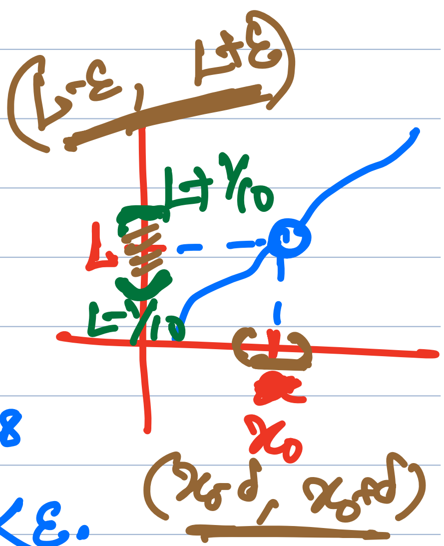
Let $f: \mathbb{R} \to \mathbb{R}$, be a function.

$$\lim_{x \to a} f(x) = L$$

(read as, "the limit of f at a")

[We can get $f(x)$ as close to the real number L as desired, by choosing x sufficiently close to (but not equal to) a]

$$\Updownarrow$$



[Given any $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $x \in \mathrm{dom}(f)$ that satisfies $\underbrace{0 < |x-a| < \delta}_{[a-\delta < x < a+\delta \text{ and } x \neq x_0]}$, we have $\underbrace{|f(x) - L| < \varepsilon}_{[L-\varepsilon < f(x) < L+\varepsilon]}$.

definition: Let $S \subseteq \mathbb{R}$, and $f: S \to \mathbb{R}$ be a function. If $a \in \mathbb{R}$ be a limit point of S, ??
then a point $L \in \mathbb{R}$, is the limit of f at a if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $x \in \mathrm{dom}(f) = S$ and $0 < |x - a| < \delta$,

and we write
$$\lim_{x \to a} f(x) = L.$$

Note: **(1)** $a \in \mathbb{R}$, but
a need not necessarily belong to $\mathrm{dom}(f)$.

(2) Even if $a \in \mathrm{dom}(f)$, it may be the case that
$$f(a) \neq \lim_{x \to a} f(x) = L$$

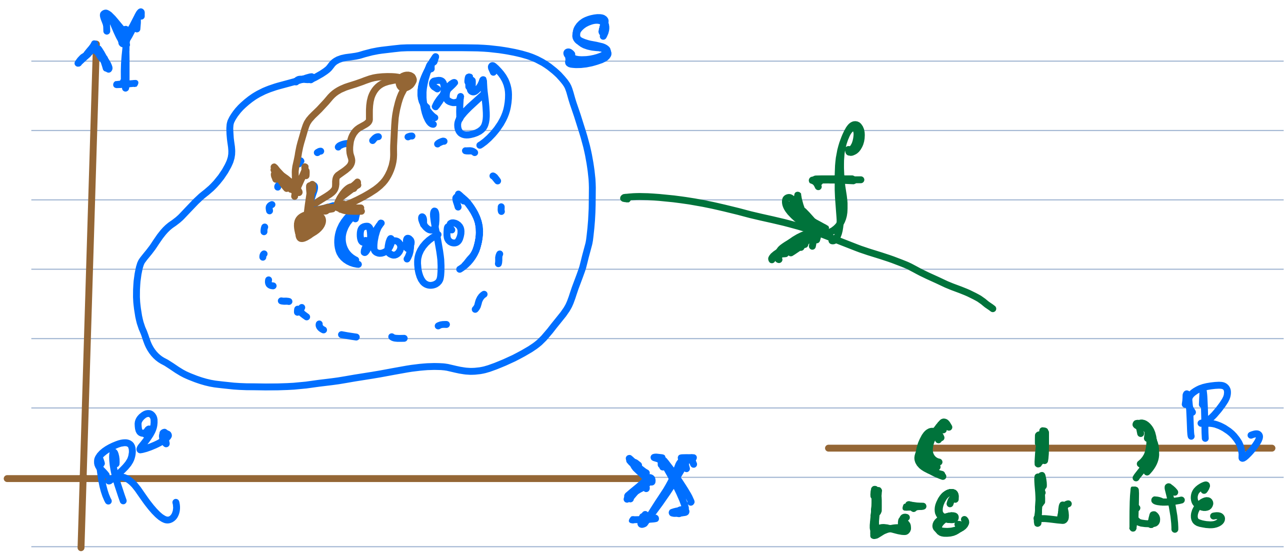
(3) If $a \in \mathrm{dom}(f)$ and if $f(a) = \lim_{x \to a} f(x)$,
then f is continuous at the point a.

Let's move on to functions of several variables.

definition: Let $S \subseteq \mathbb{R}^n$ and $f: S \to \mathbb{R}$, be a function from S to $\mathbb{R}$. If $a \in \mathbb{R}^n$ is a limit point of S, then a point $L \in \mathbb{R}$, is called the limit of f at a, if $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$x \in S \text{ and } 0 < \|x - a\| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

$\left[\text{If } x = (x_1, x_2, \dots, x_n) \text{ and } a = (a_1, a_2, \dots, a_n), \text{ then } \|x-a\| = \sqrt{\sum_{i=1}^n (x_i - a_i)^2} \right]$ or $\left[x \in B_\delta(a) \text{ and } x \neq a \right]$



Example 1: $f(x,y) = x.$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0.$$

Example 2: $f(x,y) = c$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = c.$$

Example 3: $$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x-y-1}$$

Solution: Note that

$$\text{dom}(f) = \{(x,y)\in\mathbb{R}^2 : x \neq y+1\}.$$

$\therefore$ Clearly $(4,3) \notin \text{dom}(f)$

$$\Rightarrow \lim_{(x,y)\to(4,3)} \frac{(\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})}{(x-y-1)\ (\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{\cancel{(x-y-1)}}{\cancel{(x-y-1)}\ (\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{1}{(\sqrt{x}+\sqrt{y+1})} = 1/4 \text{ ans.}$$

Properties of Limits of fns. of Several Variables

Let f and g be functions from $\mathbb{R}^2$ to $\mathbb{R}$.

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and

$\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$, then

(1) $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M$

(2) $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$

(3) $\lim\limits_{(x,y)\to(x_0,y_0)} [c\,f(x,y)] = cL$

(4) $$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$$

provided that $M \neq 0$.

(5) If $r, s \in \mathbb{Z}$ such that there are no common factors of r and s (except 1) and if $s \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number.

Example 4: $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3}$$

Solution: (use quotient rule)

Example 5: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Solution: (Can't use quotient rule right away!
Why? $\because \sqrt{x}-\sqrt{y}\to 0$ as $(x,y)\to(0,0)$)

Remark: Can you see that the definition of the limit of a function does not help you "find" the limit; it only helps you verify whether or not L is the limit of f at a.

How to show that

(i) L is not the limit of f at a

(ii) the limit of f at a does not exist

For (i) you need to "negate" the implication in the definition of the limit

Hint: How to negate $A \Rightarrow B$?
Negate "$\neg A \vee B$," and you shall get "$A \wedge \neg B$."

"L is not the limit of f at a"

$$\Updownarrow$$

" $\lim_{(x_1,x_2)\to(a_1,a_2)} f(x_1,x_2) \neq L$ "

$$\Updownarrow$$

$\exists \varepsilon > 0$ such that $\forall \delta > 0$, we have

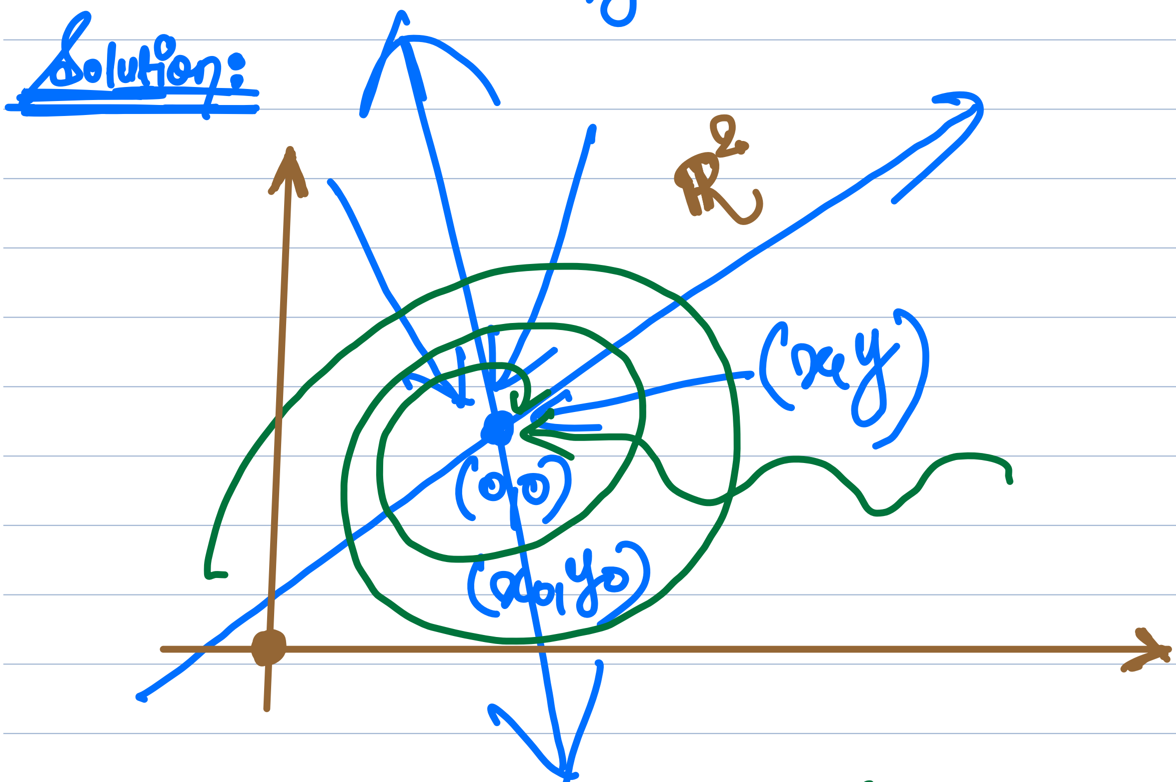
- $x = (x_1, x_2) \in dom(f)$,
- $0 < \|x - a\| < \delta$, and
- $|f(x) - L| \geq \varepsilon$

For (ii), we have, what we call, **two-path test!**

Example 6: Compute

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$ if it exists.

Solution:



→ Let us approach $(0,0)$ along the line $y=2x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{2x(2x)}{x^2 + (2x)^2}$$

$$= \lim_{x \to 0} \frac{4x^2}{5x^2} = 4/5$$

→ Let us approach (0,0) along the line $y = 5x$

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = 5x}} \frac{2xy}{x^2 + y^2}$$

$$= \lim_{x \to 0} \frac{2x(5x)}{x^2 + (5x)^2}$$

$$= \lim_{x \to 0} \frac{10x^2}{26x^2} = 10/26 = 5/13.$$

So, it is evident that the limit must not exist!!!

Reason: Different paths of approach to the origin (0,0) can lead to different results!

In general,

→ Let us approach (0,0) along the line $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{2xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{2x(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{2mx^2}{(1+m^2)x^2}$$

$$= \frac{2m}{1+m^2}$$ which depends on "m", that is, it depends on the path!!!

Example 7: $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = ?$

Hint: Try the path $y = cx^2$.

(By the two-path test, f has no limit as (x,y) approaches $(0,0)$.)

Two-path test (for the nonexistence of limit)

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \to (x_0,y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ does not exist}$$

Example 8: Compute

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} \quad \text{if it exists.}$$

Remark: **(1)** Let $f(x,y) := \dfrac{4xy^2}{x^2+y^2}$.

Then, $\operatorname{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$. f is not defined at $(0,0)$. However, it makes sense to discuss the limit of the function f as (x,y) approaches $(0,0)$.

(Why? Since f is defined at every point around some "neighborhood" of (0,0).)

(2) quotient rule does not apply! (why?)

(3) Can you come up with some simplification? The denominator is not an irrational function.

(4) What if we try 2-path test?

Path 1: (along x-axis)

$$\lim_{(x,0)\to(0,0)} \frac{4x\cdot 0^2}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Path 2: (along the y-axis)

$$\lim_{(0,y)\to(0,0)} \frac{4\cdot 0\cdot y^2}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

Claim: The limit is indeed "0." Prove it!!!

Classroom proof:

$$\left.\begin{array}{l}\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t.}\\ (x,y)\in dom(f)=\mathbb{R}^2\setminus\{(0,0)\}\\ \qquad\text{and}\\ 0<\|(x,y)-(x_0,y_0)\|<\delta\end{array}\right] \Rightarrow |f(x,y)-L|<\varepsilon.$$

<u>Equivalently,</u>

$$\begin{array}{l}\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t.}\\ (x,y)\neq(0,0) \text{ and}\\ \sqrt{x^2+y^2}<\delta\end{array} \Rightarrow \left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon$$

Now, let $\varepsilon>0$ be given.
We are required to find some $\delta>0$, depending only on $\varepsilon>0$, such that

$$\left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon \quad \text{whenever} \quad 0<\sqrt{x^2+y^2}<\delta.$$

To find some $\delta>0$, we need to estimate $\left|\frac{4xy^2}{x^2+y^2}\right|$.
In particular, we want

$$\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon.$$

[Note that, we don't know for which points (x,y) the above inequality is satisfied.

We want this inequality to be satisfied by $(x,y) \neq (0,0)$ whenever $\sqrt{x^2+y^2} < \delta$ for a chosen δ.]

Observe that

$$\left|\frac{4xy^2}{x^2+y^2}\right| = \frac{4|x|y^2}{x^2+y^2}$$

and since $y^2 \leq x^2+y^2$, we get

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x|.$$

So, it suffices to estimate $4|x|$, (that is, if we establish that $4|x| < \varepsilon$, we are through.)

Let's not forget that we want $4|x| < \varepsilon$; we don't already know it. We want $4|x| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$ (the delta that we are looking for!)

So, can we estimate $4|x|$ in terms of $\sqrt{x^2+y^2}$?

Of course, $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$, so that $4|x| \leq 4\sqrt{x^2+y^2}$. So, it is sufficient to make $4\sqrt{x^2+y^2} < \varepsilon$. (why?)

But expecting $4\sqrt{x^2+y^2} < \varepsilon$ is same as expecting $\sqrt{x^2+y^2} < \varepsilon/4$. This suggests us to choose $\delta = \varepsilon/4$.

Now, clearly, for any given $\varepsilon > 0$, we choose $\delta = \varepsilon/4$ to obtain the following implication:

$$0<\sqrt{x^2+y^2}<\delta \Rightarrow \left|\frac{4xy^2}{x^2+y^2}\right| = \frac{4|x|y^2}{x^2+y^2}$$

$$\leq 4|x|$$

$$\leq 4\sqrt{x^2+y^2}$$

$$< 4\delta$$

$$= 4\,\varepsilon/4$$

$$= \varepsilon.$$

Textbook proof:

Let $\varepsilon>0$ be given. We are required to find $\delta>0$, which depends on the given $\varepsilon>0$, such that $0<\|(x,y)-(x_0,y_0)\|<\delta \Rightarrow |f(x,y)-L|<\varepsilon.$

Since $(x_0,y_0)=(0,0)$ and $L=0$, we are required to find $\delta>0$, which depends on $\varepsilon>0$, such that

$$0<\sqrt{x^2+y^2}<\delta \Rightarrow \left|\frac{4xy^2}{x^2+y^2}\right|<\varepsilon.$$

To this end, let us estimate $\left|\frac{4xy^2}{x^2+y^2}\right|$.
Now observe that,

$$\left|\frac{4xy^2}{x^2+y^2}\right|=\frac{4|x|y^2}{x^2+y^2}\leqslant 4|x|\leq 4\sqrt{x^2+y^2}.$$

This suggests that we should choose δ to be $\varepsilon/4$, for then,

$$\sqrt{x^2+y^2}<\delta=\varepsilon/4 \Rightarrow 4\sqrt{x^2+y^2}<\varepsilon$$

and since

$$\left|\frac{4xy^2}{x^2+y^2}\right|\leqslant 4\sqrt{x^2+y^2},$$

it follows that

$$\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon.$$

This completes the proof. $\blacksquare$

Now, please go through the classroom proof once again!