$$I_{12} = \int \tan u \, \mathrm{d}u$$

$$= \int \frac{\sin u}{\cos u} \, \mathrm{d}u$$

Let $\quad v = \cos u \quad \longrightarrow \quad \mathrm{d}v = -\sin u \, \mathrm{d}u \quad \longrightarrow \quad -\mathrm{d}v = \sin u \, \mathrm{d}u$

$$= -\int \frac{1}{v} \, \mathrm{d}v$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\cos u| + C$$

$$\int \tan u \, \mathrm{d}u = \ln|\cos u| + C$$

$$I_{13} = \int \cot u \, \mathrm{d}u$$

$$= \int \frac{\cos u}{\sin u} \, \mathrm{d}u$$

Let $\quad v = \sin u \quad \longrightarrow \quad \mathrm{d}v = \cos u \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

$$\int \cot u \, \mathrm{d}u = \ln|\sin u| + C$$

$$I_{14} = \int \sec u \, \mathrm{d}u$$

$$= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \, \mathrm{d}u$$

$$= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \, \mathrm{d}u$$

Let $v = \sec u + \tan u \longrightarrow \mathrm{d}v = \sec^2 u + \sec u \tan u \, \mathrm{d}u$

$$= \int \frac{1}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sec u + \tan u| + C$$

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14 $\quad \int \sec u \, \mathrm{d}u = \ln|\sec u + \tan u| + C$

1. $\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$

2. $\int u^n \, \mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{1}{u} \, \mathrm{d}u = \ln|u| + C$

4. $\int e^u \, \mathrm{d}u = e^u + C$

5. $\int a^u \, \mathrm{d}u = \frac{a^u}{\ln a} + C$

6. $\int \sin u \, \mathrm{d}u = -\cos u + C$

7. $\int \cos u \, \mathrm{d}u = \sin u + C$

8. $\int \sec^2 u \, \mathrm{d}u = \tan u + C$

9. $\int \csc^2 u \, \mathrm{d}u = -\cot u + C$

10. $\int \sec u \tan u \, \mathrm{d}u = \sec u + C$

11. $\int \csc u \cot u \, \mathrm{d}u = -\csc u + C$

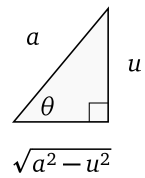
103. $\int \sinh u \, \mathrm{d}u = \cosh u + C$

104. $\int \cosh u \, \mathrm{d}u = \sinh u + C$

These integrals can be used without explanation

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}} \, \mathrm{d}u$$

Let $\sin\theta = \frac{u}{a}$, $\cos\theta = \frac{\sqrt{a^2-u^2}}{a} \longrightarrow \tan\theta = \frac{u}{\sqrt{a^2-u^2}}$



and so $u = a\sin\theta \longrightarrow \mathrm{d}u = a\cos\theta \, \mathrm{d}\theta$

and $\sqrt{a^2-u^2} = a\cos\theta$

$$= \int \frac{a\cos\theta}{a\cos\theta} \, \mathrm{d}\theta$$

$$= \int 1 \, \mathrm{d}\theta$$

$$= \theta + C$$

Reverse the substitution of u, where $\sin\theta = \frac{u}{a} \longrightarrow \theta = \arcsin\frac{u}{a}$

$$= \arcsin\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}} \, \mathrm{d}u = \arcsin\frac{u}{a} + C$$

$$I_{17} = \int \frac{1}{a^2 + u^2} \, \mathrm{d}u$$

Let $v = \frac{u}{a} \longrightarrow av = u \longrightarrow a\,\mathrm{d}v = \mathrm{d}u$

$$= \int \frac{a}{a^2 + (av)^2} \, \mathrm{d}v$$

$$= \int \frac{a}{a^2(1+v^2)} \, \mathrm{d}v$$

$$= \frac{1}{a} \int \frac{1}{1+v^2} \, \mathrm{d}v$$

$$= \frac{1}{a} \arctan v$$

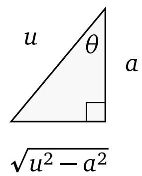
Reverse the substitution of u

$$= \frac{1}{a} \arctan\frac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2} \, \mathrm{d}u = \frac{1}{a} \arctan\frac{u}{a} + C$$

$$I_{18} = \int \frac{1}{u\sqrt{u^2 - a^2}}\,\mathrm{d}u$$

Let $\sin\theta = \dfrac{\sqrt{u^2-a^2}}{u}$, $\cos\theta = \dfrac{a}{u} \longrightarrow \tan\theta = \dfrac{\sqrt{u^2-a^2}}{a}$



and so $u = a\sec\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2-a^2} = a\tan\theta$

$$I_{18} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\cdot a\tan\theta}\,\mathrm{d}\theta$$

$$= \int \frac{1}{a}\,\mathrm{d}\theta$$

$$= \frac{1}{a}\theta + C$$

Reverse the substitution, where $u = a\sec\theta \longrightarrow \theta = \operatorname{arcsec}\dfrac{u}{a}$

$$= \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,\mathrm{d}u = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

$$I_{15} = \int \csc u\,\mathrm{d}u$$

$$= \int \csc u \cdot \frac{\csc u + \cot u}{\csc u + \cot u}\,\mathrm{d}u$$

$$= \int \frac{\csc^2 u + \csc u\cdot\cot u}{\csc u + \cot u}\,\mathrm{d}u$$

Let $v = \csc u + \cot u \longrightarrow \mathrm{d}v = -\csc^2 u + \csc u\cot u\,\mathrm{d}u$

$\longrightarrow -\mathrm{d}v = \csc^2 u + \csc u\cot u\,\mathrm{d}u$

$$= \int \frac{-\mathrm{d}v}{v}$$

$$= -\int \frac{1}{v}\,\mathrm{d}v$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\csc u + \cot u| + C$$

$$\int \csc u\,\mathrm{d}u = -\ln|\csc u + \cot u| + C$$

$$I_{20} = \int \frac{1}{u^2 - a^2}\,\mathrm{d}u$$

$$= \int \frac{A}{u-a} + \frac{B}{u+a}\,\mathrm{d}u$$

Where $A(u+a) + B(u-a) = 1 \longrightarrow A + B = 0$ and $A - B = \frac{1}{a}$
$\longrightarrow B = -A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = -\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a}\,\mathrm{d}u$$

$$= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a}\,\mathrm{d}u$$

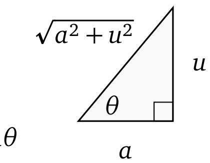
$$= \frac{1}{2a}\left(\ln|u-a| - \ln|u+a| + C\right)$$

$$= \frac{1}{2a} \cdot \ln\left|\frac{u-a}{u+a}\right| + C$$

20 $$\int \frac{1}{u^2 - a^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$I_{21} = \int \sqrt{a^2 + u^2}\,\mathrm{d}u$$

Let $\sin\theta = \frac{u}{\sqrt{a^2+u^2}}$, $\cos\theta = \frac{a}{\sqrt{a^2+u^2}} \longrightarrow \tan\theta = \frac{u}{a}$



and so $u = a\tan\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$
and $\sqrt{u^2+a^2} = a\sec\theta$

$$= a\sec\theta\, a\sec^2\theta\,\mathrm{d}\theta$$

$$= a^2 \int \sec^3\theta\,\mathrm{d}\theta$$

Apply integal identity 71 : $\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

$$= a^2\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C\right)$$

$$= \frac{a^2}{2}\frac{\sqrt{u^2+a^2}}{a} \cdot \frac{u}{a} + \frac{a^2}{2}\ln\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|\frac{u + \sqrt{u^2+a^2}}{a}\right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2+a^2}\right| + \frac{a^2}{2}\ln|a| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2+a^2}\right| + C$$

$$\int \sqrt{a^2+u^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$ 21

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$$I_{19} = \int \frac{1}{a^2 - u^2}\,\mathrm{d}u$$

$$= \int \frac{A}{a+u} + \frac{B}{a-u}\,\mathrm{d}u$$

Where $A(a-u)+B(a+u)=1 \longrightarrow A+B=\frac{1}{a}$ and $B-A=0$
$\longrightarrow B=A \longrightarrow 2A=\frac{1}{a} \longrightarrow A=\frac{1}{2a} \longrightarrow B=\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u}\,\mathrm{d}u$$

$$= \frac{1}{2a} \cdot \int \frac{1}{a+u} + \frac{1}{a-u}\,\mathrm{d}u$$

Rearranging so u preceeds a for aesthetic reasons

$$= \frac{1}{2a} \cdot \int \frac{1}{u+a} - \frac{1}{u-a}\,\mathrm{d}u$$

$$= \frac{1}{2a}\left(\ln|u+a| - \ln|u-a|\right) + C$$

$$= \frac{1}{2a} \cdot \ln\left|\frac{u+a}{u-a}\right| + C$$

$$\int \frac{1}{a^2 - u^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

$$I_{64} = \int \cos^2 u \, \mathrm{d}u$$

$$= \int \frac{1+\cos 2u}{2} \, \mathrm{d}u$$

$$= \frac{1}{2} \cdot \int 1 + \cos 2u \, \mathrm{d}u$$

$$= \frac{1}{2} \cdot \left(u + \frac{1}{2} \sin 2u \right) + C$$

$$= \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C$$

$$\int \cos^2 u \, \mathrm{d}u = \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C$$

$$I_{65} = \int \tan^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \sec^2 u - 1$

$$= \int \sec^2 u - 1 \, \mathrm{d}u$$

$$= \tan u - u + C$$

$$\int \tan^2 u \, \mathrm{d}u = \tan u - u + C$$

$$I_{71} = \int \sec^3 u \,\mathrm{d}u$$

Integrate by parts $\begin{cases} (\sec u)' &= \sec u \tan u \\ \sec^2 u &= (\tan u)' \end{cases}$

$$= (\sec u)(\tan u) - \int (\sec u \tan u)(\tan u)\,\mathrm{d}u$$

$$= \sec u \tan u - \int \sec u \tan^2 \mathrm{d}u$$

$$= \sec u \tan u - \int \sec u(\sec^2 u - 1)\,\mathrm{d}u$$

$$= \sec u \tan u - \int \sec^3 u \,\mathrm{d}u + \int \sec u \,\mathrm{d}u$$

$$2\int \sec^3 u \,\mathrm{d}u = \sec u \tan u + \int \sec u \,\mathrm{d}u$$

$$\int \sec^3 u \,\mathrm{d}u = \frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u \,\mathrm{d}u$$

$$= \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

(71) $\displaystyle\int \sec^3 u \,\mathrm{d}u = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

$$I_{63} = \int \sin^2 u \,\mathrm{d}u$$

$$= \int \frac{1-\cos 2u}{2}\,\mathrm{d}u$$

$$= \frac{1}{2}\cdot \int 1 - \cos 2u \,\mathrm{d}u$$

$$= \frac{1}{2}\cdot\left(u - \frac{1}{2}\sin 2u\right) + C$$

$$= \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$$

$\displaystyle\int \sin^2 u \,\mathrm{d}u = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$ (63)

$$I_{96} = \int ue^{au}\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' &= 1 \\ e^{au} &= \left(\dfrac{e^{au}}{a}\right)' \end{cases}$

$$= (u)\left(\frac{e^{au}}{a}\right) - \int (1)\left(\frac{e^{au}}{a}\right)\mathrm{d}u$$

$$= \frac{1}{a}ue^{au} - \frac{1}{a}\int e^{au}\,\mathrm{d}u$$

$$= \frac{1}{a}ue^{au} - \frac{1}{a^2}e^{au} + C$$

$$= \left(\frac{1}{a}u - \frac{1}{a^2}\right)e^{au} + C$$

Simplify

$$= \frac{1}{a^2}(au-1)e^{au} + C$$

$$\int ue^{au}\,\mathrm{d}u = \frac{1}{a^2}(au-1)e^{au} + C$$

$$I_{100} = \int \ln u\,\mathrm{d}u$$

Integrate by parts $\begin{cases} 1 &= (u)' \\ (\ln u)' &= \dfrac{1}{u} \end{cases}$

$$= (u)(\ln u) - \int (u)\left(\frac{1}{u}\right)\mathrm{d}u$$

$$= u\ln u - \int 1\,\mathrm{d}u$$

$$= u\ln u - u + C$$

$$\int \ln u\,\mathrm{d}u = u\ln u - u + C$$

$$I_{102} = \int \frac{1}{u \ln u} \, \mathrm{d}u$$

Let $\quad v = \ln u \quad \longrightarrow \quad \mathrm{d}v = \frac{1}{u} \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln v + C$$

Reverse the substitution

$$= \ln|\ln u|$$

$$\int \frac{1}{u \ln u} \, \mathrm{d}u = \ln|\ln u| + C$$

$$I_{92} = \int u \arctan u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} u &= \left(\frac{1}{2}u^2\right)' \\ (\arctan u)' &= \frac{1}{1+u^2} \end{cases}$

$$= \left(\tfrac{1}{2}u^2\right)(\arctan u) - \int \left(\tfrac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right) \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int \frac{u^2+1-1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int 1 - \frac{1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} (u - \arctan u) + C$$

$$= \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$$

$$\int u \arctan u \, \mathrm{d}u = \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$$