$$\int$$

The Book of Integrals

https://github.com/heckman/book-of-integrals

1: $\int u \, dv = uv - \int v \, du$

2: $\int u^n \, du = \dfrac{u^{n+1}}{n+1} + C, \; n \neq -1$

3: $\int \dfrac{1}{u} \, du = \ln|u| + C$

4: $\int e^u \, du = e^u + C$

5: $\int a^u \, du = \dfrac{a^u}{\ln a} + C$

6: $\int \sin u \, du = -\cos u + C$

7: $\int \cos u \, du = \sin u + C$

8: $\int \sec^2 u \, du = \tan u + C$

9: $\int \csc^2 u \, du = -\cot u + C$

10: $\int \sec u \tan u \, du = \sec u + C$

11: $\int \csc u \cot u \, du = -\csc u + C$

103: $\int \sinh u \, du = \cosh u + C$

104: $\int \cosh u \, du = \sinh u + C$

12: $\tan u$

13: $\cot u$

14: $\sec u$

15: $\csc u$

16: $\dfrac{1}{\sqrt{a^2-u^2}}$

17: $\dfrac{1}{a^2+u^2}$

18: $\dfrac{1}{u\sqrt{u^2-a^2}}$

19: $\dfrac{1}{a^2-u^2}$

20: $\dfrac{1}{u^2-a^2}$

21: $\sqrt{a^2+u^2}$

63: $\sin^2 u$

64: $\cos^2 u$

65: $\tan^2 u$

66: $\cot^2 u$

67: $\sin^3 u$

68: $\cos^3 u$

69: $\tan^3 u$

70: $\cot^3 u$

71: $\sec^3 u$

72: $\csc^3 u$

73: $\sin^n u$

74: $\cos^n u$

75: $\tan^n u$

76: $\cot^n u$

77: $\sec^n u$

78: $\csc^n u$

82: $u \sin u$

83: $u \cos u$

92: $u \arctan u$

96: ue^{au}

97: $u^n e^{au}$

100: $\ln u$

102: $\dfrac{1}{u \ln u}$

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$$I_{12} = \int \tan u \, \mathrm{d}u$$

$$= \int \frac{\sin u}{\cos u} \, \mathrm{d}u$$

Let $\quad v = \cos u \quad \longrightarrow \quad \mathrm{d}v = -\sin u \, \mathrm{d}u \quad \longrightarrow \quad -\mathrm{d}v = \sin u \, \mathrm{d}u$

$$= -\int \frac{1}{v} \, \mathrm{d}v$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\cos u| + C$$

$$\int \tan u \, \mathrm{d}u = \ln|\cos u| + C$$

$$I_{13} = \int \cot u \, \mathrm{d}u$$

$$= \int \frac{\cos u}{\sin u} \, \mathrm{d}u$$

Let $\quad v = \sin u \quad \longrightarrow \quad \mathrm{d}v = \cos u \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

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$$\int \cot u \, \mathrm{d}u = \ln|\sin u| + C$$

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$$I_{14} = \int \sec u \, \mathrm{d}u$$

$$= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \, \mathrm{d}u$$

$$= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \, \mathrm{d}u$$

Let $v = \sec u + \tan u \longrightarrow \mathrm{d}v = \sec^2 u + \sec u \tan u \, \mathrm{d}u$

$$= \int \frac{\mathrm{d}v}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sec u + \tan u| + C$$

$\int \sec u \, \mathrm{d}u = \ln|\sec u + \tan u| + C$

$$I_{15} = \int \csc u \, \mathrm{d}u$$

$$= \int \csc u \cdot \frac{\csc u - \cot u}{\csc u - \cot u} \, \mathrm{d}u$$

$$= \int \frac{\csc^2 u - \csc u \cdot \cot u}{\csc u - \cot u} \, \mathrm{d}u$$

Let $\quad v = \csc u - \cot u \quad \longrightarrow \quad \mathrm{d}v = \csc^2 u - \csc u \cot u \, \mathrm{d}u$

$$= \int \frac{\mathrm{d}v}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\csc u - \cot u| + C$$

15 $\quad \int \csc u \, \mathrm{d}u = \ln|\csc u - \cot u| + C$

$$I_{102} = \int \frac{1}{u \ln u} \, \mathrm{d}u$$

Let $\quad v = \ln u \quad \longrightarrow \quad \mathrm{d}v = \frac{1}{u} \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln v + C$$

Reverse the substitution

$$= \ln|\ln u|$$

$\int \frac{1}{u \ln u} \, \mathrm{d}u \; = \; \ln|\ln u| \; + \; C$ $\quad$ 102

$$I_{100} = \int \ln u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} 1 = (u)' \\ (\ln u)' = \dfrac{1}{u} \end{cases}$

$$= (u)(\ln u) - \int (u)\left(\frac{1}{u}\right) \mathrm{d}u$$

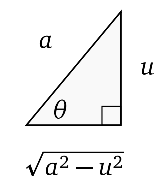
$$= u \ln u - \int 1 \, \mathrm{d}u$$

$$= u \ln u - u + C$$

100 $\int \ln u \; \mathrm{d}u \;=\; u \ln u - u + C$

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}} \mathrm{d}u$$

Let $\sin\theta = \dfrac{u}{a}$, $\cos\theta = \dfrac{\sqrt{a^2-u^2}}{a} \longrightarrow \tan\theta = \dfrac{u}{\sqrt{a^2-u^2}}$



and so $u = a\sin\theta \longrightarrow \mathrm{d}u = a\cos\theta \, \mathrm{d}\theta$

and $\sqrt{a^2-u^2} = a\cos\theta$

$$= \int \frac{a\cos\theta}{a\cos\theta} \mathrm{d}\theta$$

$$= \int 1 \, \mathrm{d}\theta$$

$$= \theta + C$$

Reverse the substitution of u, where $\sin\theta = \dfrac{u}{a} \longrightarrow \theta = \arcsin\dfrac{u}{a}$

$$= \arcsin\frac{u}{a} + C$$

If $a = 1$ this solution can be used with explanation.

$\int \frac{1}{\sqrt{a^2-u^2}} \; \mathrm{d}u \;=\; \arcsin\frac{u}{a} + C$ 16

$$I_{17} = \int \frac{1}{a^2+u^2}\,\mathrm{d}u$$

Let $v = \frac{u}{a} \longrightarrow av = u \longrightarrow a\,\mathrm{d}v = \mathrm{d}u$

$$= \int \frac{a}{a^2+(av)^2}\,\mathrm{d}v$$

$$= \int \frac{a}{a^2(1+v^2)}\,\mathrm{d}v$$

$$= \frac{1}{a}\int \frac{1}{1+v^2}\,\mathrm{d}v$$

$$= \frac{1}{a}\arctan v$$

Reverse the substitution of u

$$= \frac{1}{a}\arctan\frac{u}{a} + C$$

If $a = 1$ this solution can be used with explanation.

17 $\int \frac{1}{a^2+u^2}\,\mathrm{d}u = \frac{1}{a}\arctan\frac{u}{a} + C$

$$I_{97} = \int u^n e^{au}\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u^n)' = n \cdot u^{n-1} \\ e^{au} = \left(\frac{e^{au}}{a}\right)' \end{cases}$

$$= (u^n)\left(\frac{e^{au}}{a}\right) - \int \left(n \cdot u^{n-1}\right)\left(\frac{e^{au}}{a}\right)\mathrm{d}u$$

$$= \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,\mathrm{d}u$$

$\int u^n e^{au}\,\mathrm{d}u = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,\mathrm{d}u$ 97

$$I_{96} = \int u e^{au}\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ e^{au} = \left(\dfrac{e^{au}}{a}\right)' \end{cases}$

$$= (u)\left(\frac{e^{au}}{a}\right) - \int (1)\left(\frac{e^{au}}{a}\right)\mathrm{d}u$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a}\int e^{au}\,\mathrm{d}u$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a^2} e^{au} + C$$

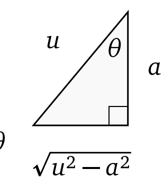
$$= \left(\frac{1}{a}u - \frac{1}{a^2}\right) e^{au} + C$$

Simplify

$$= \frac{1}{a^2}(au - 1)\, e^{au} + C$$

$$I_{18} = \int \frac{1}{u\sqrt{u^2 - a^2}}\,\mathrm{d}u$$

Let $\sin\theta = \dfrac{\sqrt{u^2-a^2}}{u}$, $\cos\theta = \dfrac{a}{u} \longrightarrow \tan\theta = \dfrac{\sqrt{u^2-a^2}}{a}$



and so $u = a\sec\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2-a^2} = a\tan\theta$

$$I_{18} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\cdot a\tan\theta}\,\mathrm{d}\theta$$

$$= \int \frac{1}{a}\,\mathrm{d}\theta$$

$$= \frac{1}{a}\theta + C$$

Reverse the substitution, where $u = a\sec\theta \longrightarrow \theta = \operatorname{arcsec}\dfrac{u}{a}$

$$= \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

$\int u e^{au}\,\mathrm{d}u = \frac{1}{a^2}(au-1)e^{au} + C$

$\int \frac{1}{u\sqrt{u^2-a^2}}\,\mathrm{d}u = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$

$$I_{19} = \int \frac{1}{a^2 - u^2}\,\mathrm{d}u$$

$$= \int \frac{A}{a+u} + \frac{B}{a-u}\,\mathrm{d}u$$

Where $A(a-u) + B(a+u) = 1 \longrightarrow A + B = \frac{1}{a}$ and $B - A = 0$
$\longrightarrow B = A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = \frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u}\,\mathrm{d}u$$

$$= \frac{1}{2a} \cdot \int \frac{1}{a+u} + \frac{1}{a-u}\,\mathrm{d}u$$

Rearranging so u preceeds a for aesthetic reasons

$$= \frac{1}{2a} \cdot \int \frac{1}{u+a} - \frac{1}{u-a}\,\mathrm{d}u$$

$$= \frac{1}{2a}\left(\ln|u+a| - \ln|u-a|\right) + C$$

$$= \frac{1}{2a} \cdot \ln\left|\frac{u+a}{u-a}\right| + C$$

$$I_{92} = \int u \arctan u \,\mathrm{d}u$$

Integrate by parts $\begin{cases} u = \left(\frac{1}{2}u^2\right)' \\ (\arctan u)' = \frac{1}{1+u^2} \end{cases}$

$$= \left(\tfrac{1}{2}u^2\right)(\arctan u) - \int \left(\tfrac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right)\mathrm{d}u$$

$$= \frac{u^2}{2}\arctan u - \frac{1}{2}\int \frac{u^2+1-1}{1+u^2}\,\mathrm{d}u$$

$$= \frac{u^2}{2}\arctan u - \frac{1}{2}\int 1 - \frac{1}{1+u^2}\,\mathrm{d}u$$

$$= \frac{u^2}{2}\arctan u - \frac{1}{2}(u - \arctan u) + C$$

$$= \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C$$

$\int \frac{1}{a^2-u^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

$\int u \arctan u\ \mathrm{d}u = \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C$

$$I_{83} = \int u \cos u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ \cos u = (\sin u)' \end{cases}$

$$= (u)(\sin u) - \int (1)(\sin u)\, \mathrm{d}u$$

$$= u \sin u - (-\cos u)$$

$$= \cos u + u \sin u$$

83 $\int u \cos u \, \mathrm{d}u = \cos u + u \sin u + C$

$$I_{20} = \int \frac{1}{u^2 - a^2} \mathrm{d}u$$

$$= \int \frac{A}{u-a} + \frac{B}{u+a} \mathrm{d}u$$

Where $A(u+a) + B(u-a) = 1 \longrightarrow A + B = 0$ and $A - B = \frac{1}{a}$
$\longrightarrow B = -A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = -\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a} \mathrm{d}u$$

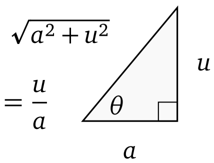
$$= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a} \mathrm{d}u$$

$$= \frac{1}{2a} \left(\ln|u-a| - \ln|u+a| + C \right)$$

$$= \frac{1}{2a} \cdot \ln\left|\frac{u-a}{u+a}\right| + C$$

$\int \frac{1}{u^2-a^2} \mathrm{d}u = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C$ 20

$$I_{21} = \int \sqrt{a^2+u^2}\,\mathrm{d}u$$



Let $\sin\theta = \frac{u}{\sqrt{a^2+u^2}}$, $\cos\theta = \frac{a}{\sqrt{a^2+u^2}}$ $\longrightarrow$ $\tan\theta = \frac{u}{a}$

and so $u = a\tan\theta$ $\longrightarrow$ $\mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2+a^2} = a\sec\theta$

$$= a\sec\theta\, a\sec^2\theta\,\mathrm{d}\theta$$

$$= a^2 \int \sec^3\theta\,\mathrm{d}\theta$$

Apply integal identity 71 :

$$\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

$$= a^2\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C\right)$$

$$= \frac{a^2}{2}\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \frac{a^2}{2}\ln\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|\frac{u+\sqrt{u^2+a^2}}{a}\right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| - \frac{a^2}{2}\ln|a| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C$$

21 $\int\sqrt{a^2+u^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

$$I_{82} = \int u\sin u\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ \sin u = (-\cos u)' \end{cases}$

$$= (u)(-\cos u) - \int (1)(-\cos u)\,\mathrm{d}u$$

$$= \int \cos u\,\mathrm{d}u - u\cos u$$

$$= \sin u - u\cos u$$

$\int u\sin u\ \mathrm{d}u = \sin u - u\cos u + C$ 82

$$I_{78} = \int \csc^n u \,\mathrm{d}u$$

Integrate by parts $\left\{ \begin{array}{rcl} \csc^2 u & = & (-\cot u)' \\ \left(\csc^{n-2} u\right)' & = & -(n-2)\csc^{n-3} u \csc u \cot u \end{array} \right.$

$$= (-\cot u)\left(\csc^{n-2} u\right) - \int (-\cot u)\left(-(n-2)\csc^{n-3} u \csc u \cot u\right) \mathrm{d}u$$

$$= -\cot u \, \csc^{n-2} u - (n-2) \int \csc^{n-2} \cot^2 u \,\mathrm{d}u$$

$$= -\cot u \, \csc^{n-2} u - (n-2) \int \csc^{n-2}(\csc^2 u - 1) \,\mathrm{d}u$$

$$= -\cot u \, \csc^{n-2} u - (n-2)\left(\int \csc^n u \,\mathrm{d}u - \int \csc^{n-2} \mathrm{d}u\right)$$

$$(1 + (n-2)) \int \csc^n u \,\mathrm{d}u = -\cot u \, \csc^{n-2} u + (n-2) \int \csc^{n-2} \mathrm{d}u$$

$$\int \csc^n u \,\mathrm{d}u = \frac{-1}{n-1} \cot u \, \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} \mathrm{d}u$$

$$I_{63} = \int \sin^2 u \,\mathrm{d}u$$

$$= \int \frac{1 - \cos 2u}{2} \mathrm{d}u$$

$$= \frac{1}{2} \cdot \int 1 - \cos 2u \,\mathrm{d}u$$

$$= \frac{1}{2} \cdot \left(u - \frac{1}{2} \sin 2u\right) + C$$

$$= \tfrac{1}{2} u - \tfrac{1}{4} \sin 2u + C$$

78 $\int \csc^n u \ \mathrm{d}u = \frac{-1}{n-1} \cot u \, \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u \ \mathrm{d}u$

$\int \sin^2 u \ \mathrm{d}u = \frac{1}{2} u - \frac{1}{4} \sin 2u + C$ 63

$$
\begin{aligned}
I_{64} &= \int \cos^2 u \, du \\
&= \int \frac{1+\cos 2u}{2} \, du \\
&= \frac{1}{2} \cdot \int 1 + \cos 2u \, du \\
&= \frac{1}{2} \cdot \left(u + \frac{1}{2} \sin 2u \right) + C \\
&= \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C
\end{aligned}
$$

$$I_{77} = \int \sec^n u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} \sec^2 u = (\tan u)' \\ \left(\sec^{n-2} u\right)' = (n-2)\sec^{n-3} u \sec u \tan u \end{cases}$$

$$
\begin{aligned}
&= (\tan u)\left(\sec^{n-2} u\right) - \int (\tan u)\left((n-2)\sec^{n-3} u \sec u \tan u\right) du \\
&= \tan u \ \sec^{n-2} u - (n-2) \int \sec^{n-2} \tan^2 u \, du \\
&= \tan u \ \sec^{n-2} u - (n-2) \int \sec^{n-2}(\sec^2 u - 1) \, du \\
&= \tan u \ \sec^{n-2} u - (n-2) \left(\int \sec^n u \, du - \int \sec^{n-2} du \right)
\end{aligned}
$$

$$(1 + (n-2)) \int \sec^n u \, du = \tan u \ \sec^{n-2} u + (n-2) \int \sec^{n-2} du$$

$$\int \sec^n u \, du = \frac{1}{n-1} \tan u \ \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} du$$

64 $\int \cos^2 u \ du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$

$\int \sec^n u \ du = \frac{1}{n-1} \tan u \ \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \ du$ 77

$$I_{76} = \int \cot^n u \, \mathrm{d}u$$

$$= \int \cot^{n-2} u \cot^2 u \, \mathrm{d}u$$

$$= \int \cot^{n-2} u (\csc^2 u - 1) \, \mathrm{d}u$$

$$= \int \cot^{n-2} u \ \csc^2 u \, \mathrm{d}u - \int \cot^{n-2} u \, \mathrm{d}u$$

Let $\quad v = \cot u \quad \longrightarrow \quad \mathrm{d}v = -\csc^2 u \, \mathrm{d}u$

$$= - \int v^{n-2} u \, \mathrm{d}v - \int \cot^{n-2} u$$

$$= \frac{-1}{n-1} v^{n-1} - \int \cot^{n-2} u$$

$$= \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u$$

76 $\quad \int \cot^n u \ \mathrm{d}u \ = \ \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u \ \mathrm{d}u$

$$I_{65} = \int \tan^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \sec^2 u - 1$

$$= \int \sec^2 u - 1 \, \mathrm{d}u$$

$$= \tan u - u + C$$

$\int \tan^2 u \ \mathrm{d}u \ = \ \tan u \ - \ u \ + \ C$ $\quad$ 65

$$I_{66} = \int \cot^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \csc^2 u - 1$

$$= \int \csc^2 u - 1 \, \mathrm{d}u$$

$$= -\cot u - u + C$$

66 $\int \cot^2 u \ \mathrm{d}u \ = \ -\cot u \ - \ u \ + \ C$

$$I_{75} = \int \tan^n u \, \mathrm{d}u$$

$$= \int \tan^{n-2} u \tan^2 u \, \mathrm{d}u$$

$$= \int \tan^{n-2} u (\sec^2 u - 1) \, \mathrm{d}u$$

$$= \int \tan^{n-2} u \ \sec^2 u \, \mathrm{d}u - \int \tan^{n-2} u \, \mathrm{d}u$$

Let $\quad v = \tan u \quad \longrightarrow \quad \mathrm{d}v = \sec^2 u \, \mathrm{d}u$

$$= \int v^{n-2} u \, \mathrm{d}v - \int \tan^{n-2} u$$

$$= \frac{1}{n-1} v^{n-1} - \int \tan^{n-2} u$$

$$= \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u$$

$\int \tan^n u \ \mathrm{d}u \ = \ \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \ \mathrm{d}u$ 75

$$I_{74} = \int \cos^n u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} \left(\cos^{n-1} u\right)' = -(n-1)\cos^{n-2} u \sin u \\ \cos u = (\sin u)' \end{cases}$

$$= \left(\cos^{n-1} u\right)(\sin u) - \int \left(-(n-1)\cos^{n-2} u \sin u\right)(\sin u)\,\mathrm{d}u$$

$$= \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u \sin^2 u \, \mathrm{d}u$$

$$= \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u (1 - \cos^2 u)\,\mathrm{d}u$$

$$= \cos^{n-1} u \, \sin u + (n-1) \left(\int \cos^{n-2} u \, \mathrm{d}u - \int \cos^n u \, \mathrm{d}u \right)$$

$$1 + (n-1) \int \cos^n u \, \mathrm{d}u = \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u \, \mathrm{d}u$$

$$\int \cos^n u \, \mathrm{d}u = \frac{1}{n} \cos^{n-1} u \, \sin u + \frac{(n-1)}{n} \int \cos^{n-2} u \, \mathrm{d}u$$

74 $\int \cos^n u \ \mathrm{d}u \ = \ \frac{1}{n} \cos^{n-1} u \ \sin u + \frac{n-1}{n} \int \cos^{n-2} u \ \mathrm{d}u$

$$I_{67} = \int \sin^3 u \, \mathrm{d}u$$

$$= \int (1 - \cos^2 u) \sin u \, \mathrm{d}u$$

$$= \int \sin u \, \mathrm{d}u - \int \cos^2 u \ \sin u \, \mathrm{d}u$$

Let $v = \cos u \longrightarrow \mathrm{d}v = -\sin u \, \mathrm{d}u$

$$= \int v^2 \, \mathrm{d}v - \cos u$$

$$= \frac{1}{3} v^3 - \cos u + C$$

Reverse the substitution

$$= \frac{1}{3} \cos^3 u - \cos u + C$$

(which we could leave right there)

$$= \left(\frac{1}{3} \cos^2 u - 1 \right) \cos u + C$$

$$= \left(\frac{1}{3} (1 - \sin^2 u) - 1 \right) \cos u + C$$

$$= -\frac{1}{3} \left(2 + \sin^2 u\right) \cos u + C$$

$\int \sin^3 u \ \mathrm{d}u \ = \ -\frac{1}{3} (2 + \sin^2 u) \cos u \ + \ C$ 67

$$I_{68} = \int \cos^3 u \, du$$

$$= \int (1 - \sin^2 u) \cos u \, du$$

$$= \int \cos u \, du - \int \sin^2 u \cos u \, du$$

Let $v = \sin u \longrightarrow dv = \cos u \, du$

$$= \sin u - \int v^2 \, dv$$

$$= \sin u - \frac{1}{3} v^3 + C$$

Reverse the substitution

$$= \sin u - \frac{1}{3} \sin^3 u + C$$

(which we could leave right there)

$$= \left(1 - \frac{1}{3} \sin^2 u\right) \sin u + C$$

$$= \left(1 - \frac{1}{3}(1 - \cos^2 u)\right) \sin u + C$$

$$= \frac{1}{3}\left(2 + \cos^2 u\right) \sin u + C$$

68 $\int \cos^3 u \; du \; = \; \frac{1}{3}(2 + \cos^2 u) \sin u \; + \; C$

$$I_{73} = \int \sin^n u \, du$$

Integrate by parts $\begin{cases} \left(\sin^{n-1} u\right)' = (n-1)\sin^{n-2} u \cos u \\ \sin u = (-\cos u)' \end{cases}$

$$= \left(\sin^{n-1} u\right)(-\cos u) - \int \left((n-1)\sin^{n-2} u \cos u\right)(-\cos u) \, du$$

$$= -\sin^{n-1} u \cos u + (n-1) \int \sin^{n-2} u \cos^2 u \, du$$

$$= -\sin^{n-1} u \cos u + (n-1) \int \sin^{n-2} u (1 - \sin^2 u) \, du$$

$$= -\sin^{n-1} u \cos u + (n-1) \left(\int \sin^{n-2} u \, du - \int \sin^n u \, du \right)$$

$$1 + (n-1) \int \sin^n u \, du = -\sin^{n-1} u \cos u + (n-1) \int \sin^{n-2} u \, du$$

$$\int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{(n-1)}{n} \int \sin^{n-2} u \, du$$

$\int \sin^n u \; du \; = \; -\frac{1}{n} \sin^{n-1} u \; \cos u \; + \; \frac{n-1}{n} \int \sin^{n-2} u \; du$ 73

$$I_{72} = \int \csc^3 u \,\mathrm{d}u$$

Integrate by parts $\begin{cases} (\csc u)' &= -\csc u \cot u \\ \csc^2 u &= (-\cot u)' \end{cases}$

$$= (\csc u)(-\cot u) - \int (-\csc u \cot u)(-\cot u)\,\mathrm{d}u$$

$$= -\csc u \cot u - \int \csc u \cot^2 \mathrm{d}u$$

$$= -\csc u \cot u - \int \csc u (\csc^2 u - 1)\,\mathrm{d}u$$

$$= -\csc u \cot u - \int \csc^3 u \,\mathrm{d}u + \int \csc u \,\mathrm{d}u$$

$$2\int \csc^3 u \,\mathrm{d}u = -\csc u \cot u + \int \csc u \,\mathrm{d}u$$

$$\int \csc^3 u \,\mathrm{d}u = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\int \csc u \,\mathrm{d}u$$

$$= \frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$$

(72) $\int \csc^3 u \ \mathrm{d}u = -\frac{1}{2}\csc u \ \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

$$I_{69} = \int \tan^3 u \,\mathrm{d}u$$

$$= \int (\sec^2 u - 1)\tan u \,\mathrm{d}u$$

$$= \int (\sec^2 u \ \tan u)\,\mathrm{d}u - \int \tan u \,\mathrm{d}u$$

Let $\quad v = \tan u \quad \longrightarrow \quad \mathrm{d}v = \sec^2 u \,\mathrm{d}u$

$$= \int v \,\mathrm{d}v - \ln|\sec u|$$

$$= \frac{1}{2}v^2 - \ln|\sec u| + C$$

Reverse the substitution

$$= \frac{1}{2}\tan^2 u - \ln|\sec u| + C$$

(which we could leave right there)

$$= \frac{1}{2}\tan^2 u + \ln|\cos u| + C$$

$\int \tan^3 u \ \mathrm{d}u = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$ (69)

$$I_{70} = \int \cot^3 u \, du$$

$$= \int (\csc^2 u - 1) \cot u \, du$$

$$= \int (\csc^2 u \cot u) \, du - \int \cot u \, du$$

Let $v = \cot u \longrightarrow dv = -\csc^2 u \, du$

$$= -\int v \, dv - \ln|\sin u|$$

$$= -\frac{1}{2} v^2 - \ln|\sin u| + C$$

Reverse the substitution

$$= -\frac{1}{2} \cot^2 u - \ln|\sin u| + C$$

(which we could leave right there)

$$= -\frac{1}{2} \cot^2 u - \ln|\sin u| + C$$

70 $\int \cot^3 u \, du = -\frac{1}{2} \cot^2 u - \ln|\sin u| + C$

$$I_{71} = \int \sec^3 u \, du$$

Integrate by parts $\begin{cases} (\sec u)' = \sec u \tan u \\ \sec^2 u = (\tan u)' \end{cases}$

$$= (\sec u)(\tan u) - \int (\sec u \tan u)(\tan u) \, du$$

$$= \sec u \tan u - \int \sec u \tan^2 du$$

$$= \sec u \tan u - \int \sec u (\sec^2 u - 1) \, du$$

$$= \sec u \tan u - \int \sec^3 u \, du + \int \sec u \, du$$

$$2 \int \sec^3 u \, du = \sec u \tan u + \int \sec u \, du$$

$$\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u \, du$$

$$= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$$

$\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$ 71