12: $\tan u$

13: $\cot u$

14: $\sec u$

15: $\csc u$

16: $\dfrac{1}{\sqrt{a^2-u^2}}$

17: $\dfrac{1}{a^2+u^2}$

18: $\dfrac{1}{u\sqrt{u^2-a^2}}$

19: $\dfrac{1}{a^2-u^2}$

20: $\dfrac{1}{u^2-a^2}$

21: $\sqrt{a^2+u^2}$

63: $\sin^2 u$

64: $\cos^2 u$

65: $\tan^2 u$

66: $\cot^2 u$

71: $\sec^3 u$

72: $\csc^3 u$

82: $u \sin u$

83: $u \cos u$

92: $u \arctan u$

96: ue^{au}

97: ue^{au}

100: $\ln u$

102: $\dfrac{1}{u \ln u}$

1. $\displaystyle\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

2. $\displaystyle\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\displaystyle\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C$

4. $\displaystyle\int e^u\,\mathrm{d}u = e^u + C$

5. $\displaystyle\int a^u\,\mathrm{d}u = \frac{a^u}{\ln a} + C$

6. $\displaystyle\int \sin u\,\mathrm{d}u = -\cos u + C$

7. $\displaystyle\int \cos u\,\mathrm{d}u = \sin u + C$

8. $\displaystyle\int \sec^2 u\,\mathrm{d}u = \tan u + C$

9. $\displaystyle\int \csc^2 u\,\mathrm{d}u = -\cot u + C$

10. $\displaystyle\int \sec u \tan u\,\mathrm{d}u = \sec u + C$

11. $\displaystyle\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$

103. $\displaystyle\int \sinh u\,\mathrm{d}u = \cosh u + C$

104. $\displaystyle\int \cosh u\,\mathrm{d}u = \sinh u + C$

These integrals can be used without explanation

$$I_{102} = \int \frac{1}{u \ln u} \, \mathrm{d}u$$

Let $\quad v = \ln u \quad \longrightarrow \quad \mathrm{d}v = \frac{1}{u} \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln v + C$$

Reverse the substitution

$$= \ln |\ln u|$$

102

$$\int \frac{1}{u \ln u} \, \mathrm{d}u = \ln |\ln u| + C$$

$$I_{12} = \int \tan u \, \mathrm{d}u$$

$$= \int \frac{\sin u}{\cos u} \, \mathrm{d}u$$

Let $\quad v = \cos u \quad \longrightarrow \quad \mathrm{d}v = -\sin u \, \mathrm{d}u \quad \longrightarrow \quad -\mathrm{d}v = \sin u \, \mathrm{d}u$

$$= -\int \frac{1}{v} \, \mathrm{d}v$$

$$= -\ln |v| + C$$

Reverse the substitution

$$= -\ln |\cos u| + C$$

$$\int \tan u \, \mathrm{d}u = \ln |\cos u| + C$$

12

$$I_{13} = \int \cot u \, \mathrm{d}u$$

$$= \int \frac{\cos u}{\sin u} \, \mathrm{d}u$$

Let $v = \sin u \longrightarrow \mathrm{d}v = \cos u \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

13 $\int \cot u \, \mathrm{d}u = \ln|\sin u| + C$

$$I_{100} = \int \ln u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} 1 = (u)' \\ (\ln u)' = \frac{1}{u} \end{cases}$

$$= (u)(\ln u) - \int (u)\left(\frac{1}{u}\right) \mathrm{d}u$$

$$= u \ln u - \int 1 \, \mathrm{d}u$$

$$= u \ln u - u + C$$

$\int \ln u \; \mathrm{d}u \;=\; u \ln u - u + C$ 100

$$I_{96} = \int u e^{au} \,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ e^{au} = \left(\dfrac{e^{au}}{a}\right)' \end{cases}$

$$= (u)\left(\frac{e^{au}}{a}\right) - \int (1)\left(\frac{e^{au}}{a}\right) \mathrm{d}u$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a} \int e^{au} \,\mathrm{d}u$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a^2} e^{au} + C$$

$$= \left(\frac{1}{a} u - \frac{1}{a^2}\right) e^{au} + C$$

Simplify

$$= \frac{1}{a^2}(au - 1)\, e^{au} + C$$

$\int u e^{au} \,\mathrm{d}u = \frac{1}{a^2}(au-1)e^{au} + C$ 96

$$I_{14} = \int \sec u \,\mathrm{d}u$$

$$= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \,\mathrm{d}u$$

$$= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \,\mathrm{d}u$$

Let $v = \sec u + \tan u \longrightarrow \mathrm{d}v = \sec^2 u + \sec u \tan u \,\mathrm{d}u$

$$= \int \frac{\mathrm{d}v}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sec u + \tan u| + C$$

$\int \sec u \,\mathrm{d}u = \ln|\sec u + \tan u| + C$ 14

$$I_{15} = \int \csc u \, \mathrm{d}u$$

$$= \int \csc u \cdot \frac{\csc u - \cot u}{\csc u - \cot u} \, \mathrm{d}u$$

$$= \int \frac{\csc^2 u - \csc u \cdot \cot u}{\csc u - \cot u} \, \mathrm{d}u$$

Let $v = \csc u - \cot u \longrightarrow \mathrm{d}v = \csc^2 u - \csc u \cot u \, \mathrm{d}u$

$$= \int \frac{\mathrm{d}v}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\csc u - \cot u| + C$$

15 $\int \csc u \, \mathrm{d}u = \ln|\csc u - \cot u| + C$

$$I_{92} = \int u \arctan u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} u = \left(\frac{1}{2}u^2\right)' \\ (\arctan u)' = \frac{1}{1+u^2} \end{cases}$

$$= \left(\tfrac{1}{2}u^2\right)(\arctan u) - \int \left(\tfrac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right) \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int \frac{u^2+1-1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int 1 - \frac{1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2}(u - \arctan u) + C$$

$$= \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$$

$\int u \arctan u \, \mathrm{d}u = \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$ 92

$$I_{83} = \int u \cos u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' &= 1 \\ \cos u &= (\sin u)' \end{cases}$

$$= (u)(\sin u) - \int (1)(\sin u)\, \mathrm{d}u$$

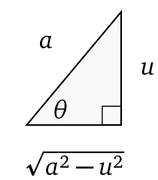
$$= u \sin u - (-\cos u)$$

$$= \cos u + u \sin u$$

83 $\int u \cos u \; \mathrm{d}u \;=\; \cos u + u \sin u + C$

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}}\, \mathrm{d}u$$

Let $\sin\theta = \frac{u}{a}$, $\cos\theta = \frac{\sqrt{a^2-u^2}}{a} \longrightarrow \tan\theta = \frac{u}{\sqrt{a^2-u^2}}$



and so $u = a\sin\theta \longrightarrow \mathrm{d}u = a\cos\theta\,\mathrm{d}\theta$

and $\sqrt{a^2-u^2} = a\cos\theta$

$$= \int \frac{a\cos\theta}{a\cos\theta}\,\mathrm{d}\theta$$

$$= \int 1\,\mathrm{d}\theta$$

$$= \theta + C$$

Reverse the substitution of u, where $\sin\theta = \frac{u}{a} \longrightarrow \theta = \arcsin\frac{u}{a}$

$$= \arcsin\frac{u}{a} + C$$

$\int \frac{1}{\sqrt{a^2-u^2}} \; \mathrm{d}u \;=\; \arcsin\frac{u}{a} + C$ 16

$$I_{17} = \int \frac{1}{a^2+u^2}\,\mathrm{d}u$$

Let $v = \frac{u}{a} \longrightarrow av = u \longrightarrow a\,\mathrm{d}v = \mathrm{d}u$

$$= \int \frac{a}{a^2+(av)^2}\,\mathrm{d}v$$

$$= \int \frac{a}{a^2(1+v^2)}\,\mathrm{d}v$$

$$= \frac{1}{a}\int \frac{1}{1+v^2}\,\mathrm{d}v$$

$$= \frac{1}{a}\arctan v$$

Reverse the substitution of u

$$= \frac{1}{a}\arctan\frac{u}{a} + C$$

17 $\int \frac{1}{a^2+u^2}\,\mathrm{d}u = \frac{1}{a}\arctan\frac{u}{a} + C$

$$I_{82} = \int u\sin u\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ \sin u = (-\cos u)' \end{cases}$

$$= (u)(-\cos u) - \int (1)(-\cos u)\,\mathrm{d}u$$

$$= \int \cos u\,\mathrm{d}u - u\cos u$$

$$= \sin u - u\cos u$$

$\int u\sin u\ \mathrm{d}u = \sin u - u\cos u + C$ 82

$$I_{74} = \int \sin^n u \, \mathrm{d}u$$

Integrate by parts $\left\{ \begin{array}{rcl} \left(\cos^{n-1} u\right)' & = & -(n-1)\cos^{n-2} u \sin u \\ \cos u & = & (\sin u)' \end{array} \right.$

$$= \left(\cos^{n-1} u\right)(\sin u) - \int \left(-(n-1)\cos^{n-2} u \sin u\right)(\sin u)\,\mathrm{d}u$$

$$= \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u \sin^2 u \, \mathrm{d}u$$

$$= \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u (1 - \cos^2 u)\, \mathrm{d}u$$

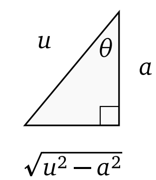
$$= \cos^{n-1} u \, \sin u + (n-1) \left(\int \cos^{n-2} u \, \mathrm{d}u - \int \cos^n u \, \mathrm{d}u \right)$$

$$1 + (n-1) \int \cos^n u \, \mathrm{d}u = \cos^{n-1} u \, \sin u + (n-1) \int \cos^{n-2} u \, \mathrm{d}u$$

$$\int \cos^n u \, \mathrm{d}u = \frac{1}{n} \cos^{n-1} u \, \sin u + \frac{(n-1)}{n} \int \cos^{n-2} u \, \mathrm{d}u$$

$$I_{18} = \int \frac{1}{u\sqrt{u^2 - a^2}} \, \mathrm{d}u$$

Let $\sin\theta = \dfrac{\sqrt{u^2 - a^2}}{u}$, $\cos\theta = \dfrac{a}{u} \longrightarrow \tan\theta = \dfrac{\sqrt{u^2 - a^2}}{a}$



and so $u = a\sec\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2 - a^2} = a\tan\theta$

$$I_{18} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta \cdot a\tan\theta}\,\mathrm{d}\theta$$

$$= \int \frac{1}{a}\,\mathrm{d}\theta$$

$$= \frac{1}{a}\theta + C$$

Reverse the substitution, where $u = a\sec\theta \longrightarrow \theta = \operatorname{arcsec}\dfrac{u}{a}$

$$= \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$$

74 $\int \cos^n u \, \mathrm{d}u = \frac{1}{n} \cos^{n-1} u \, \sin u + \frac{n-1}{n} \int \cos^{n-2} u \, \mathrm{d}u$

$\int \frac{1}{u\sqrt{u^2-a^2}} \, \mathrm{d}u = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$ 18

$$I_{19} = \int \frac{1}{a^2-u^2}\,\mathrm{d}u$$

$$= \int \frac{A}{a+u} + \frac{B}{a-u}\,\mathrm{d}u$$

Where $A(a-u)+B(a+u)=1 \longrightarrow A+B=\frac{1}{a}$ and $B-A=0$
$\longrightarrow B=A \longrightarrow 2A=\frac{1}{a} \longrightarrow A=\frac{1}{2a} \longrightarrow B=\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u}\,\mathrm{d}u$$

$$= \frac{1}{2a}\cdot\int \frac{1}{a+u} + \frac{1}{a-u}\,\mathrm{d}u$$

Rearranging so u preceeds a for aesthetic reasons

$$= \frac{1}{2a}\cdot\int \frac{1}{u+a} - \frac{1}{u-a}\,\mathrm{d}u$$

$$= \frac{1}{2a}\left(\ln|u+a| - \ln|u-a|\right) + C$$

$$= \frac{1}{2a}\cdot\ln\left|\frac{u+a}{u-a}\right| + C$$

$$I_{73} = \int \sin^n u\,\mathrm{d}u$$

Integrate by parts $\begin{cases} \left(\sin^{n-1} u\right)' = (n-1)\sin^{n-2} u\cos u \\ \sin u = (-\cos u)' \end{cases}$

$$= \left(\sin^{n-1} u\right)(-\cos u) - \int \left((n-1)\sin^{n-2} u\cos u\right)(-\cos u)\,\mathrm{d}u$$

$$= -\sin^{n-1} u\,\cos u + (n-1)\int \sin^{n-2} u\cos^2 u\,\mathrm{d}u$$

$$= -\sin^{n-1} u\,\cos u + (n-1)\int \sin^{n-2} u(1-\sin^2 u)\,\mathrm{d}u$$

$$= -\sin^{n-1} u\,\cos u + (n-1)\left(\int \sin^{n-2} u\,\mathrm{d}u - \int \sin^n u\,\mathrm{d}u\right)$$

$$1+(n-1)\int \sin^n u\,\mathrm{d}u = -\sin^{n-1} u\,\cos u + (n-1)\int \sin^{n-2} u\,\mathrm{d}u$$

$$\int \sin^n u\,\mathrm{d}u = -\frac{1}{n}\sin^{n-1} u\,\cos u + \frac{(n-1)}{n}\int \sin^{n-2} u\,\mathrm{d}u$$

19 $\int \frac{1}{a^2-u^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

$\int \sin^n u\,\mathrm{d}u = -\frac{1}{n}\sin^{n-1} u\,\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,\mathrm{d}u$ 73

$$I_{72} = \int \csc^3 u \, \mathrm{d}u$$

$$\text{Integrate by parts} \quad \begin{cases} (\csc u)' &= -\csc u \cot u \\ \csc^2 u &= (-\cot u)' \end{cases}$$

$$\begin{aligned}
&= (\csc u)(-\cot u) - \int (-\csc u \cot u)(-\cot u) \, \mathrm{d}u \\
&= -\csc u \cot u - \int \csc u \cot^2 \mathrm{d}u \\
&= -\csc u \cot u - \int \csc u (\csc^2 u - 1) \, \mathrm{d}u \\
&= -\csc u \cot u - \int \csc^3 u \, \mathrm{d}u + \int \csc u \, \mathrm{d}u
\end{aligned}$$

$$2 \int \csc^3 u \, \mathrm{d}u = -\csc u \cot u + \int \csc u \, \mathrm{d}u$$

$$\int \csc^3 u \, \mathrm{d}u = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \int \csc u \, \mathrm{d}u$$

$$= \frac{1}{2} \csc u \cot u + \frac{1}{2} \ln|\csc u - \cot u| + C$$

$$I_{20} = \int \frac{1}{u^2 - a^2} \, \mathrm{d}u$$

$$= \int \frac{A}{u-a} + \frac{B}{u+a} \, \mathrm{d}u$$

Where $A(u+a) + B(u-a) = 1 \longrightarrow A + B = 0$ and $A - B = \frac{1}{a}$
$\longrightarrow B = -A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = -\frac{1}{2a}$

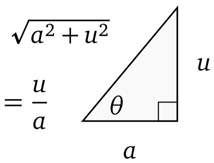
$$\begin{aligned}
&= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a} \, \mathrm{d}u \\
&= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a} \, \mathrm{d}u \\
&= \frac{1}{2a} \left(\ln|u-a| - \ln|u+a| + C \right) \\
&= \frac{1}{2a} \cdot \ln\left| \frac{u-a}{u+a} \right| + C
\end{aligned}$$

 $\int \csc^3 u \ \mathrm{d}u = -\frac{1}{2} \csc u \ \cot u + \frac{1}{2} \ln|\csc u - \cot u| + C$

$\int \frac{1}{u^2-a^2} \, \mathrm{d}u = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C$

$$I_{21} = \int \sqrt{a^2+u^2}\,\mathrm{d}u$$

Let $\sin\theta = \frac{u}{\sqrt{a^2+u^2}}$, $\cos\theta = \frac{a}{\sqrt{a^2+u^2}} \longrightarrow \tan\theta = \frac{u}{a}$



and so $u = a\tan\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2+a^2} = a\sec\theta$

$$=a\sec\theta\, a\sec^2\theta\,\mathrm{d}\theta$$

$$=a^2\int \sec^3\theta\,\mathrm{d}\theta$$

Apply integal identity (71) :

$$\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u+\tan u| + C$$

$$=a^2\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| + C\right)$$

$$=\frac{a^2}{2}\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \frac{a^2}{2}\ln\left|\frac{\sqrt{u^2+a^2}}{a}+\frac{u}{a}\right| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|\frac{u+\sqrt{u^2+a^2}}{a}\right| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| - \frac{a^2}{2}\ln|a| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C$$

(21) $\int\sqrt{a^2+u^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

$$I_{71} = \int \sec^3 u\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (\sec u)' &= \sec u\tan u \\ \sec^2 u &= (\tan u)' \end{cases}$

$$=(\sec u)(\tan u) - \int (\sec u\tan u)(\tan u)\,\mathrm{d}u$$

$$=\sec u\tan u - \int \sec u\tan^2\mathrm{d}u$$

$$=\sec u\tan u - \int \sec u(\sec^2 u - 1)\,\mathrm{d}u$$

$$=\sec u\tan u - \int \sec^3 u\,\mathrm{d}u + \int \sec u\,\mathrm{d}u$$

$$2\int \sec^3 u\,\mathrm{d}u = \sec u\tan u + \int \sec u\,\mathrm{d}u$$

$$\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\int \sec u\,\mathrm{d}u$$

$$=\frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u+\tan u| + C$$

$\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u+\tan u| + C$ (71)

$$I_{66} = \int \cot^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \csc^2 u - 1$

$$= \int \csc^2 u - 1 \, \mathrm{d}u$$

$$= -\cot u - u + C$$

66 $\int \cot^2 u \ \mathrm{d}u \ = \ -\cot u - u + C$

$$I_{63} = \int \sin^2 u \, \mathrm{d}u$$

$$= \int \frac{1 - \cos 2u}{2} \mathrm{d}u$$

$$= \frac{1}{2} \cdot \int 1 - \cos 2u \, \mathrm{d}u$$

$$= \frac{1}{2} \cdot \left(u - \frac{1}{2} \sin 2u \right) + C$$

$$= \tfrac{1}{2} u - \tfrac{1}{4} \sin 2u + C$$

$\int \sin^2 u \ \mathrm{d}u \ = \ \tfrac{1}{2} u - \tfrac{1}{4} \sin 2u + C$ 63

$$
\begin{aligned}
I_{64} &= \int \cos^2 u \, \mathrm{d}u \\
&= \int \frac{1+\cos 2u}{2} \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \int 1 + \cos 2u \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \left(u + \frac{1}{2} \sin 2u \right) + C \\
&= \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C
\end{aligned}
$$

$$I_{65} = \int \tan^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \sec^2 u - 1$

$$
\begin{aligned}
&= \int \sec^2 u - 1 \, \mathrm{d}u \\
&= \tan u - u + C
\end{aligned}
$$

$\int \cos^2 u \ \mathrm{d}u \ = \ \tfrac{1}{2} u \ + \ \tfrac{1}{4} \sin 2u \ + \ C$

$\int \tan^2 u \ \mathrm{d}u \ = \ \tan u \ - \ u \ + \ C$