

$$I_{92} = \int u \arctan u \, du$$

$$\begin{aligned} \text{Integrate by parts} \quad & \begin{cases} u = \left(\frac{1}{2}u^2\right)' \\ (\arctan u)' = \frac{1}{1+u^2} \end{cases} \\ & = \left(\frac{1}{2}u^2\right)(\arctan u) - \int \left(\frac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right) du \\ & = \frac{u^2}{2} \arctan u - \frac{1}{2} \int \frac{u^2 + 1 - 1}{1 + u^2} du \\ & = \frac{u^2}{2} \arctan u - \frac{1}{2} \int 1 - \frac{1}{1 + u^2} du \\ & = \frac{u^2}{2} \arctan u - \frac{1}{2} (u - \arctan u) + C \\ & = \frac{u^2 + 1}{2} \arctan u - \frac{u}{2} + C \end{aligned}$$

$$(92) \quad \int u \arctan u \, du = \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$$

1.  $\int u \, dv = uv - \int v \, du$
2.  $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
3.  $\int \frac{1}{u} \, du = \ln|u| + C$
4.  $\int e^u \, du = e^u + C$
5.  $\int a^u \, du = \frac{a^u}{\ln a} + C$
6.  $\int \sin u \, du = -\cos u + C$
7.  $\int \cos u \, du = \sin u + C$
8.  $\int \sec^2 u \, du = \tan u + C$
9.  $\int \csc^2 u \, du = -\cot u + C$
10.  $\int \sec u \tan u \, du = \sec u + C$
11.  $\int \csc u \cot u \, du = -\csc u + C$
103.  $\int \sinh u \, du = \cosh u + C$
104.  $\int \cosh u \, du = \sinh u + C$

These integrals can be used without explanation

$$I_{12} = \int \tan u \, du$$

$$= \int \frac{\sin u}{\cos u} \, du$$

$$\text{Let } v = \cos u \longrightarrow dv = -\sin u \, du \longrightarrow -dv = \sin u \, du$$

$$= - \int \frac{1}{v} \, dv$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\cos u| + C$$

$$I_{83} = \int u \cos u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} (u)' = 1 \\ \cos u = (\sin u)' \end{cases}$$

$$= (u)(\sin u) - \int (1)(\sin u) \, du$$

$$= u \sin u - (-\cos u)$$

$$= \cos u + u \sin u$$

12

$$\int \tan u \, du = \ln|\cos u| + C$$

83

$$\int u \cos u \, du = \cos u + u \sin u + C$$

$$I_{82} = \int u \sin u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} (u)' = 1 \\ \sin u = (-\cos u)' \end{cases}$$

$$= (u)(-\cos u) - \int (1)(-\cos u) \, du$$

$$= \int \cos u \, du - u \cos u$$

$$= \sin u - u \cos u$$

$$\int u \sin u \, du = \sin u - u \cos u + C \quad (82)$$

$$I_{13} = \int \cot u \, du$$

$$= \int \frac{\cos u}{\sin u} \, du$$

$$\text{Let } v = \sin u \longrightarrow dv = \cos u \, du$$

$$= \int \frac{1}{v} \, dv$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C \quad (13)$$

$$\begin{aligned}
 I_{14} &= \int \sec u \, du \\
 &= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \, du \\
 &= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \, du
 \end{aligned}$$

Let  $v = \sec u + \tan u \longrightarrow dv = \sec^2 u + \sec u \tan u \, du$

$$\begin{aligned}
 &= \int \frac{dv}{v} \\
 &= \ln|v| + C
 \end{aligned}$$

Reverse the substitution

$$= \ln|\sec u + \tan u| + C$$

14

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$I_{72} = \int \csc^3 u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} (\csc u)' = -\csc u \cot u \\ \csc^2 u = (-\cot u)' \end{cases}$$

$$\begin{aligned}
 &= (\csc u)(-\cot u) - \int (-\csc u \cot u)(-\cot u) \, du \\
 &= -\csc u \cot u - \int \csc u \cot^2 u \, du \\
 &= -\csc u \cot u - \int \csc u (\csc^2 u - 1) \, du \\
 &= -\csc u \cot u - \int \csc^3 u \, du + \int \csc u \, du \\
 &2 \int \csc^3 u \, du = -\csc u \cot u + \int \csc u \, du \\
 &\int \csc^3 u \, du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \int \csc u \, du \\
 &= \frac{1}{2} \csc u \cot u + \frac{1}{2} \ln|\csc u - \cot u| + C
 \end{aligned}$$

$$\int \csc^3 u \, du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln|\csc u - \cot u| + C \quad 72$$

$$I_{71} = \int \sec^3 u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} (\sec u)' = \sec u \tan u \\ \sec^2 u = (\tan u)' \end{cases}$$

$$= (\sec u)(\tan u) - \int (\sec u \tan u)(\tan u) \, du$$

$$= \sec u \tan u - \int \sec u \tan^2 u \, du$$

$$= \sec u \tan u - \int \sec u (\sec^2 u - 1) \, du$$

$$= \sec u \tan u - \int \sec^3 u \, du + \int \sec u \, du$$

$$2 \int \sec^3 u \, du = \sec u \tan u + \int \sec u \, du$$

$$\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u \, du$$

$$= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$(71) \quad \int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$I_{15} = \int \csc u \, du$$

$$= \int \csc u \cdot \frac{\csc u - \cot u}{\csc u - \cot u} \, du$$

$$= \int \frac{\csc^2 u - \csc u \cdot \cot u}{\csc u - \cot u} \, du$$

$$\text{Let } v = \csc u - \cot u \longrightarrow dv = \csc^2 u - \csc u \cot u \, du$$

$$= \int \frac{dv}{v}$$

$$= \ln |v| + C$$

Reverse the substitution

$$= \ln |\csc u - \cot u| + C$$

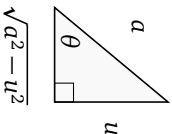
$$\int \csc u \, du = \ln |\csc u - \cot u| + C$$

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}} du$$

$$\text{Let } \sin \theta = \frac{u}{a}, \quad \cos \theta = \frac{\sqrt{a^2 - u^2}}{a} \quad \longrightarrow \quad \tan \theta = \frac{u}{\sqrt{a^2 - u^2}}$$

$$\text{and so } u = a \sin \theta \quad \longrightarrow \quad du = a \cos \theta \, d\theta$$

$$\text{and } \sqrt{a^2 - u^2} = a \cos \theta$$



$$= \int \frac{a \cos \theta}{a \cos \theta} d\theta$$

$$= \int 1 d\theta$$

$$= \theta + C$$

Reverse the substitution of  $u$ , where  $\sin \theta = \frac{u}{a} \longrightarrow \theta = \arcsin \frac{u}{a}$

$$= \arcsin \frac{u}{a} + C$$

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$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin \frac{u}{a} + C$$

$$I_{66} = \int \cot^2 u \, du$$

Rewrite using trigonometric identity  $\tan^2 u = \csc^2 u - 1$

$$= \int \csc^2 u - 1 \, du$$

$$= -\cot u - u + C$$

66

$$\int \cot^2 u \, du = -\cot u - u + C$$

$$I_{65} = \int \tan^2 u \, du$$

Rewrite using trigonometric identity  $\tan^2 u = \sec^2 u - 1$

$$\begin{aligned} &= \int \sec^2 u - 1 \, du \\ &= \tan u - u + C \end{aligned}$$

(65)

$$\int \tan^2 u \, du = \tan u - u + C$$

$$I_{17} = \int \frac{1}{a^2 + u^2} du$$

Let  $v = \frac{u}{a} \longrightarrow av = u \longrightarrow a \, dv = du$

$$\begin{aligned} &= \int \frac{a}{a^2 + (av)^2} dv \\ &= \int \frac{a}{a^2(1 + v^2)} dv \\ &= \frac{1}{a} \int \frac{1}{1 + v^2} dv \\ &= \frac{1}{a} \arctan v \end{aligned}$$

Reverse the substitution of  $u$

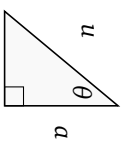
$$\begin{aligned} &= \frac{1}{a} \arctan \frac{u}{a} + C \end{aligned}$$

(17)

$$\int \frac{1}{a^2 + u^2} du = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$I_{18} = \int \frac{1}{u \sqrt{u^2 - a^2}} du$$

$$\text{Let } \sin \theta = \frac{\sqrt{u^2 - a^2}}{u}, \quad \cos \theta = \frac{a}{u} \quad \longrightarrow \quad \tan \theta = \frac{\sqrt{u^2 - a^2}}{a}$$



$$\text{and so } u = a \sec \theta \quad \longrightarrow \quad du = a \sec \theta \tan \theta \, d\theta$$

$$\text{and } \sqrt{u^2 - a^2} = a \tan \theta$$

$$I_{18} = \int \frac{a \sec \theta \tan \theta}{a \sec \theta \cdot a \tan \theta} d\theta$$

$$= \int \frac{1}{a} d\theta$$

$$= \frac{1}{a} \theta + C$$

Reverse the substitution, where  $u = a \sec \theta \quad \longrightarrow \quad \theta = \operatorname{arcsec} \frac{u}{a}$

$$= \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$$

18

$$\int \frac{1}{u \sqrt{u^2 - a^2}} du = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$$

$$I_{64} = \int \cos^2 u \, du$$

$$= \int \frac{1 + \cos 2u}{2} du$$

$$= \frac{1}{2} \cdot \int 1 + \cos 2u \, du$$

$$= \frac{1}{2} \cdot \left( u + \frac{1}{2} \sin 2u \right) + C$$

$$= \frac{1}{2} u + \frac{1}{4} \sin 2u + C$$

$$\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$$

64



$$\begin{aligned}
I_{63} &= \int \sin^2 u \, du \\
&= \int \frac{1 - \cos 2u}{2} \, du \\
&= \frac{1}{2} \cdot \int 1 - \cos 2u \, du \\
&= \frac{1}{2} \cdot \left( u - \frac{1}{2} \sin 2u \right) + C \\
&= \frac{1}{2} u - \frac{1}{4} \sin 2u + C
\end{aligned}$$

$$\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C \quad (63)$$

$$\begin{aligned}
I_{19} &= \int \frac{1}{a^2 - u^2} \, du \\
&= \int \frac{A}{a+u} + \frac{B}{a-u} \, du
\end{aligned}$$

$$\begin{aligned}
\text{Where } A(a-u) + B(a+u) &= 1 \longrightarrow A+B = \frac{1}{a} \text{ and } B-A=0 \\
\longrightarrow B=A \longrightarrow 2A &= \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = \frac{1}{2a}
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u} \, du \\
&= \frac{1}{2a} \cdot \int \frac{1}{a+u} + \frac{1}{a-u} \, du
\end{aligned}$$

Rearranging so  $u$  precedes  $a$  for aesthetic reasons

$$\begin{aligned}
&= \frac{1}{2a} \cdot \int \frac{1}{u+a} - \frac{1}{u-a} \, du \\
&= \frac{1}{2a} (\ln|u+a| - \ln|u-a|) + C \\
&= \frac{1}{2a} \cdot \ln \left| \frac{u+a}{u-a} \right| + C
\end{aligned}$$

$$\int \frac{1}{a^2 - u^2} \, du = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + C \quad (19)$$

$$I_{20} = \int \frac{1}{u^2 - a^2} du$$

$$= \int \frac{A}{u-a} + \frac{B}{u+a} du$$

Where  $A(u+a) + B(u-a) = 1 \longrightarrow A+B=0$  and  $A-B = \frac{1}{a}$   
 $\longrightarrow B = -A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = -\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a} du$$

$$= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a} du$$

$$= \frac{1}{2a} (\ln|u-a| - \ln|u+a| + C)$$

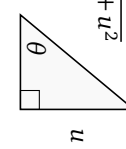
$$= \frac{1}{2a} \cdot \ln \left| \frac{u-a}{u+a} \right| + C$$

20

$$\int \frac{1}{u^2 - a^2} du = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C$$

$$I_{21} = \int \sqrt{a^2 + u^2} du$$

Let  $\sin \theta = \frac{u}{\sqrt{a^2 + u^2}}$ ,  $\cos \theta = \frac{a}{\sqrt{a^2 + u^2}} \longrightarrow \tan \theta = \frac{u}{a}$



and so  $u = a \tan \theta \longrightarrow du = a \sec \theta \tan \theta d\theta$

and  $\sqrt{u^2 + a^2} = a \sec \theta$

$$= a \sec \theta a \sec^2 \theta d\theta$$

$$= a^2 \int \sec^3 \theta d\theta$$

Apply integral identity (71) :

$$\int \sec^3 u du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$= a^2 \left( \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \right)$$

$$= \frac{a^2}{2} \frac{\sqrt{u^2 + a^2}}{a} \cdot \frac{u}{a} + \frac{a^2}{2} \ln \left| \frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a} \right| + C$$

$$= \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln \left| \frac{u + \sqrt{u^2 + a^2}}{a} \right| + C$$

$$= \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 + a^2}| - \frac{a^2}{2} \ln |a| + C$$

$$= \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 + a^2}| + C$$

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 + a^2}| + C$$

21

$$I_{96} = \int u e^{au} \, du$$

$$\text{Integrate by parts} \quad \left\{ \begin{array}{l} (u)' = 1 \\ e^{au} = \left( \frac{e^{au}}{a} \right)' \end{array} \right.$$

$$= (u) \left( \frac{e^{au}}{a} \right) - \int (1) \left( \frac{e^{au}}{a} \right) du$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a} \int e^{au} \, du$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a^2} e^{au} + C$$

$$= \left( \frac{1}{a} u - \frac{1}{a^2} \right) e^{au} + C$$

Simplify

$$= \frac{1}{a^2} (au - 1) e^{au} + C$$

$$\int u e^{au} \, du = \frac{1}{a^2} (au - 1) e^{au} + C$$

$$I_{100} = \int \ln u \, du$$

$$\text{Integrate by parts} \quad \left\{ \begin{array}{l} 1 = (u)' \\ (\ln u)' = \frac{1}{u} \end{array} \right.$$

$$= (u) (\ln u) - \int (u) \left( \frac{1}{u} \right) du$$

$$= u \ln u - \int 1 \, du$$

$$= u \ln u - u + C$$

$$\int \ln u \, du = u \ln u - u + C$$

$$I_{102} = \int \frac{1}{u \ln u} du$$

$$\text{Let } v = \ln u \longrightarrow dv = \frac{1}{u} du$$

$$\begin{aligned} &= \int \frac{1}{v} dv \\ &= \ln v + C \end{aligned}$$

Reverse the substitution

$$= \ln |\ln u|$$

$$\int \frac{1}{u \ln u} du = \ln |\ln u| + C$$

$$12: \tan u$$

$$20: \frac{1}{u^2 - a^2}$$

$$82: u \sin u$$

$$13: \cot u$$

$$21: \sqrt{a^2 + u^2}$$

$$83: u \cos u$$

$$14: \sec u$$

$$63: \sin^2 u$$

$$92: u \arctan u$$

$$15: \csc u$$

$$64: \cos^2 u$$

$$96: ue^{au}$$

$$16: \frac{1}{\sqrt{a^2 - u^2}}$$

$$65: \tan^2 u$$

$$97: ue^{au}$$

$$17: \frac{1}{a^2 + u^2}$$

$$66: \cot^2 u$$

$$100: \ln u$$

$$18: \frac{1}{u\sqrt{u^2 - a^2}}$$

$$71: \sec^3 u$$

$$102: \frac{1}{u \ln u}$$

$$19: \frac{1}{a^2 - u^2}$$

$$72: \csc^3 u$$

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