1. $\int u \, dv = uv - \int v \, du$

2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{1}{u} \, du = \ln|u| + C$

4. $\int e^u \, du = e^u + C$

5. $\int a^u \, du = \frac{a^u}{\ln a} + C$

6. $\int \sin u \, du = -\cos u + C$

7. $\int \cos u \, du = \sin u + C$

8. $\int \sec^2 u \, du = \tan u + C$

9. $\int \csc^2 u \, du = -\cot u + C$

10. $\int \sec u \tan u \, du = \sec u + C$

11. $\int \csc u \cot u \, du = -\csc u + C$

103. $\int \sinh u \, du = \cosh u + C$

104. $\int \cosh u \, du = \sinh u + C$

These integrals can be used without explanation

$$
\begin{aligned}
I_{12} &= \int \tan u \, \mathrm{d}u \\
&= \int \frac{\sin u}{\cos u} \, \mathrm{d}u
\end{aligned}
$$

Let $\quad v = \cos u \quad \longrightarrow \quad \mathrm{d}v = -\sin u \, \mathrm{d}u \quad \longrightarrow \quad -\mathrm{d}v = \sin u \, \mathrm{d}u$

$$
\begin{aligned}
&= -\int \frac{1}{v} \, \mathrm{d}v \\
&= -\ln|v| + C
\end{aligned}
$$

Reverse the substitution

$$
= -\ln|\cos u| + C
$$

12 $\quad \int \tan u \, \mathrm{d}u = \ln|\cos u| + C$

$$I_{13} = \int \cot u \, \mathrm{d}u$$

$$= \int \frac{\cos u}{\sin u} \, \mathrm{d}u$$

Let $\quad v = \sin u \quad \longrightarrow \quad \mathrm{d}v = \cos u \, \mathrm{d}u$

$$= \int \frac{1}{v} \, \mathrm{d}v$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

$$\int \cot u \, \mathrm{d}u = \ln|\sin u| + C$$

$$
\begin{aligned}
I_{14} &= \int \sec u \, \mathrm{d}u \\
&= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \, \mathrm{d}u \\
&= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \, \mathrm{d}u
\end{aligned}
$$

Let $\quad v = \sec u + \tan u \quad \longrightarrow \quad \mathrm{d}v = \sec^2 u + \sec u \tan u \, \mathrm{d}u$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}v}{v} \\
&= \ln|v| + C
\end{aligned}
$$

Reverse the substitution

$$
= \ln|\sec u + \tan u| + C
$$

14 $\quad \int \sec u \, \mathrm{d}u = \ln|\sec u + \tan u| + C$

$$
\begin{aligned}
I_{15} &= \int \csc u \, \mathrm{d}u \\
&= \int \csc u \cdot \frac{\csc u - \cot u}{\csc u - \cot u} \, \mathrm{d}u \\
&= \int \frac{\csc^2 u - \csc u \cdot \cot u}{\csc u - \cot u} \, \mathrm{d}u
\end{aligned}
$$

Let $\quad v = \csc u - \cot u \quad \longrightarrow \quad \mathrm{d}v = \csc^2 u - \csc u \cot u \, \mathrm{d}u$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}v}{v} \\
&= \ln|v| + C
\end{aligned}
$$

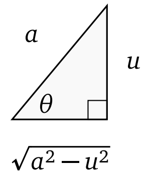
Reverse the substitution

$$
= \ln|\csc u - \cot u| + C
$$

$$
\int \csc u \, \mathrm{d}u = \ln|\csc u - \cot u| + C \tag{15}
$$

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}}\,\mathrm{d}u$$

Let $\sin\theta = \dfrac{u}{a}$, $\cos\theta = \dfrac{\sqrt{a^2-u^2}}{a} \longrightarrow \tan\theta = \dfrac{u}{\sqrt{a^2-u^2}}$



and so $u = a\sin\theta \longrightarrow \mathrm{d}u = a\cos\theta\,\mathrm{d}\theta$

and $\sqrt{a^2-u^2} = a\cos\theta$

$$= \int \frac{a\cos\theta}{a\cos\theta}\,\mathrm{d}\theta$$

$$= \int 1\,\mathrm{d}\theta$$

$$= \theta + C$$

Reverse the substitution of u, where $\sin\theta = \dfrac{u}{a} \longrightarrow \theta = \arcsin\dfrac{u}{a}$

$$= \arcsin\frac{u}{a} + C$$

16 $\int \frac{1}{\sqrt{a^2-u^2}}\,\mathrm{d}u = \arcsin\frac{u}{a} + C$

$$I_{17} = \int \frac{1}{a^2 + u^2} \, \mathrm{d}u$$

$$\text{Let} v = \frac{u}{a} \quad \longrightarrow \quad av = u \quad \longrightarrow \quad a \, \mathrm{d}v = \mathrm{d}u$$

$$= \int \frac{a}{a^2 + (av)^2} \, \mathrm{d}v$$

$$= \int \frac{a}{a^2 (1 + v^2)} \, \mathrm{d}v$$

$$= \frac{1}{a} \int \frac{1}{1 + v^2} \, \mathrm{d}v$$

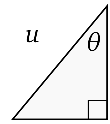
$$= \frac{1}{a} \arctan v$$

Reverse the substitution of u

$$= \frac{1}{a} \arctan \frac{u}{a} + C$$

$$I_{18} = \int \frac{1}{u\sqrt{u^2 - a^2}}\,\mathrm{d}u$$

Let $\sin\theta = \dfrac{\sqrt{u^2 - a^2}}{u}$, $\cos\theta = \dfrac{a}{u}$ $\longrightarrow$ $\tan\theta = \dfrac{\sqrt{u^2 - a^2}}{a}$



and so $u = a\sec\theta$ $\longrightarrow$ $\mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2 - a^2} = a\tan\theta$

$$\begin{aligned} I_{18} &= \int \frac{a\sec\theta\tan\theta}{a\sec\theta \cdot a\tan\theta}\,\mathrm{d}\theta \\ &= \int \frac{1}{a}\,\mathrm{d}\theta \\ &= \frac{1}{a}\theta + C \end{aligned}$$

Reverse the substitution, where $u = a\sec\theta$ $\longrightarrow$ $\theta = \operatorname{arcsec}\dfrac{u}{a}$

$$= \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

18 $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,\mathrm{d}u = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$

$$I_{19} = \int \frac{1}{a^2 - u^2}\, \mathrm{d}u$$
$$= \int \frac{A}{a+u} + \frac{B}{a-u}\, \mathrm{d}u$$

Where $\quad A(a-u)+B(a+u)=1 \quad \longrightarrow \quad A+B=\frac{1}{a} \quad$ and $\quad B-A=0$
$\longrightarrow \quad B=A \quad \longrightarrow \quad 2A=\frac{1}{a} \quad \longrightarrow \quad A=\frac{1}{2a} \quad \longrightarrow \quad B=\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u}\, \mathrm{d}u$$
$$= \frac{1}{2a} \cdot \int \frac{1}{a+u} + \frac{1}{a-u}\, \mathrm{d}u$$

Rearranging so u preceeds a for aesthetic reasons

$$= \frac{1}{2a} \cdot \int \frac{1}{u+a} - \frac{1}{u-a}\, \mathrm{d}u$$
$$= \frac{1}{2a}\left(\ln|u+a| - \ln|u-a|\right) + C$$
$$= \frac{1}{2a} \cdot \ln\left|\frac{u+a}{u-a}\right| + C$$

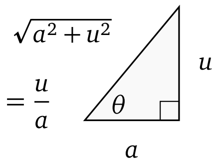
$$
\begin{aligned}
I_{20} &= \int \frac{1}{u^2 - a^2}\,\mathrm{d}u \\
&= \int \frac{A}{u-a} + \frac{B}{u+a}\,\mathrm{d}u
\end{aligned}
$$

Where $\quad A(u+a) + B(u-a) = 1 \quad \longrightarrow \quad A + B = 0 \quad \text{and} \quad A - B = \frac{1}{a}$
$\longrightarrow \quad B = -A \quad \longrightarrow \quad 2A = \frac{1}{a} \quad \longrightarrow \quad A = \frac{1}{2a} \quad \longrightarrow \quad B = -\frac{1}{2a}$

$$
\begin{aligned}
&= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a}\,\mathrm{d}u \\
&= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a}\,\mathrm{d}u \\
&= \frac{1}{2a}\left(\ln|u-a| - \ln|u+a| + C\right) \\
&= \frac{1}{2a} \cdot \ln\left|\frac{u-a}{u+a}\right| + C
\end{aligned}
$$

20 $\quad \int \frac{1}{u^2-a^2}\,\mathrm{d}u = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

$$I_{21} = \int \sqrt{a^2+u^2}\,\mathrm{d}u$$



Let $\sin\theta = \dfrac{u}{\sqrt{a^2+u^2}}$, $\cos\theta = \dfrac{a}{\sqrt{a^2+u^2}} \longrightarrow \tan\theta = \dfrac{u}{a}$

and so $u = a\tan\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\,\mathrm{d}\theta$

and $\sqrt{u^2+a^2} = a\sec\theta$

$$=a\sec\theta\, a\sec^2\theta\,\mathrm{d}\theta$$

$$=a^2\int \sec^3\theta\,\mathrm{d}\theta$$

Apply integal identity (71) :

$$\int \sec^3 u\,\mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$$

$$=a^2\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| + C\right)$$

$$=\frac{a^2}{2}\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \frac{a^2}{2}\ln\left|\frac{\sqrt{u^2+a^2}}{a}+\frac{u}{a}\right| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|\frac{u+\sqrt{u^2+a^2}}{a}\right| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| - \frac{a^2}{2}\ln|a| + C$$

$$=\frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C$$

$$\int\sqrt{a^2+u^2}\,\mathrm{d}u = \tfrac{u}{2}\sqrt{u^2+a^2} + \tfrac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C \qquad (21)$$

$$
\begin{aligned}
I_{63} &= \int \sin^2 u \, \mathrm{d}u \\
&= \int \frac{1-\cos 2u}{2} \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \int 1 - \cos 2u \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \left(u - \frac{1}{2} \sin 2u \right) + C \\
&= \tfrac{1}{2} u - \tfrac{1}{4} \sin 2u + C
\end{aligned}
$$

(63) $$\int \sin^2 u \, \mathrm{d}u = \tfrac{1}{2} u - \tfrac{1}{4} \sin 2u + C$$

$$
\begin{aligned}
I_{64} &= \int \cos^2 u \, \mathrm{d}u \\
&= \int \frac{1+\cos 2u}{2} \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \int 1 + \cos 2u \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \left(u + \frac{1}{2} \sin 2u \right) + C \\
&= \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C
\end{aligned}
$$

$$\int \cos^2 u \ \mathrm{d}u \ = \ \tfrac{1}{2} u \ + \ \tfrac{1}{4} \sin 2u \ + \ C$$

$$I_{65} = \int \tan^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \sec^2 u - 1$

$$= \int \sec^2 u - 1 \, \mathrm{d}u$$

$$= \tan u - u + C$$

65 $\int \tan^2 u \ \mathrm{d}u \ = \ \tan u \ - \ u \ + \ C$

$$I_{66} = \int \cot^2 u \, \mathrm{d}u$$

Rewrite using trigonometric identity $\tan^2 u = \csc^2 u - 1$

$$= \int \csc^2 u - 1 \, \mathrm{d}u$$

$$= -\cot u - u + C$$

$$\int \cot^2 u \ \mathrm{d}u \ = \ -\cot u \ - \ u \ + \ C$$

$$I_{71} = \int \sec^3 u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} (\sec u)' = \sec u \tan u \\ \sec^2 u = (\tan u)' \end{cases}$

$$= (\sec u)(\tan u) - \int (\sec u \tan u)(\tan u) \, \mathrm{d}u$$

$$= \sec u \tan u - \int \sec u \tan^2 \mathrm{d}u$$

$$= \sec u \tan u - \int \sec u(\sec^2 u - 1) \, \mathrm{d}u$$

$$= \sec u \tan u - \int \sec^3 u \, \mathrm{d}u + \int \sec u \, \mathrm{d}u$$

$$2\int \sec^3 u \, \mathrm{d}u = \sec u \tan u + \int \sec u \, \mathrm{d}u$$

$$\int \sec^3 u \, \mathrm{d}u = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u \, \mathrm{d}u$$

$$= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$$

71 $\int \sec^3 u \, \mathrm{d}u = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$

$$I_{72} = \int \csc^3 u \,\mathrm{d}u$$

$$\text{Integrate by parts} \quad \begin{cases} (\csc u)' &= -\csc u \cot u \\ \csc^2 u &= (-\cot u)' \end{cases}$$

$$\begin{aligned}
&= (\csc u)(-\cot u) - \int (-\csc u \cot u)(-\cot u)\,\mathrm{d}u \\
&= -\csc u \cot u - \int \csc u \cot^2 \mathrm{d}u \\
&= -\csc u \cot u - \int \csc u(\csc^2 u - 1)\,\mathrm{d}u \\
&= -\csc u \cot u - \int \csc^3 u \,\mathrm{d}u + \int \csc u \,\mathrm{d}u \\
2\int \csc^3 u \,\mathrm{d}u &= -\csc u \cot u + \int \csc u \,\mathrm{d}u \\
\int \csc^3 u \,\mathrm{d}u &= -\frac{1}{2}\csc u \cot u + \frac{1}{2}\int \csc u \,\mathrm{d}u \\
&= \frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C
\end{aligned}$$

$$\int \csc^3 u \,\mathrm{d}u = -\tfrac{1}{2}\csc u \cot u + \tfrac{1}{2}\ln|\csc u - \cot u| + C \quad 72$$

$$I_{82} = \int u \sin u \, \mathrm{d}u$$

Integrate by parts $\left\{ \begin{array}{rcl} (u)' & = & 1 \\ \sin u & = & (-\cos u)' \end{array} \right.$

$$= (u)(-\cos u) - \int (1)(-\cos u) \, \mathrm{d}u$$

$$= \int \cos u \, \mathrm{d}u - u \cos u$$

$$= \sin u - u \cos u$$

82 $\int u \sin u \; \mathrm{d}u \; = \; \sin u \; - \; u \; \cos u \; + \; C$

$$I_{83} = \int u \cos u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' & = 1 \\ \cos u & = (\sin u)' \end{cases}$

$$\begin{aligned} &= (u)(\sin u) - \int (1)(\sin u) \, \mathrm{d}u \\ &= u \sin u - (-\cos u) \\ &= \cos u + u \sin u \end{aligned}$$

$$\int u \cos u \ \mathrm{d}u \ = \ \cos u \ + \ u \sin u \ + \ C$$

$$I_{92} = \int u \arctan u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} u &= \left(\tfrac{1}{2}u^2\right)' \\ (\arctan u)' &= \dfrac{1}{1+u^2} \end{cases}$

$$= \left(\tfrac{1}{2}u^2\right)(\arctan u) - \int \left(\tfrac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right) \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int \frac{u^2+1-1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int 1 - \frac{1}{1+u^2} \, \mathrm{d}u$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2}(u - \arctan u) + C$$

$$= \frac{u^2+1}{2} \arctan u - \frac{u}{2} + C$$

92 $\int u \arctan u \;\mathrm{d}u \;=\; \frac{u^2+1}{2} \arctan u \;-\; \frac{u}{2} \;+\; C$

$$I_{96} = \int ue^{au}\,\mathrm{d}u$$

Integrate by parts $\begin{cases} (u)' = 1 \\ e^{au} = \left(\dfrac{e^{au}}{a}\right)' \end{cases}$

$$= (u)\left(\frac{e^{au}}{a}\right) - \int (1)\left(\frac{e^{au}}{a}\right)\mathrm{d}u$$

$$= \frac{1}{a}ue^{au} - \frac{1}{a}\int e^{au}\,\mathrm{d}u$$

$$= \frac{1}{a}ue^{au} - \frac{1}{a^2}e^{au} + C$$

$$= \left(\frac{1}{a}u - \frac{1}{a^2}\right)e^{au} + C$$

Simplify

$$= \frac{1}{a^2}(au-1)\,e^{au} + C$$

$$\int ue^{au}\ \mathrm{d}u \ = \ \tfrac{1}{a^2}(au-1)e^{au} \ + \ C$$

$$I_{100} = \int \ln u \, \mathrm{d}u$$

Integrate by parts $\begin{cases} 1 & = & (u)' \\ (\ln u)' & = & \dfrac{1}{u} \end{cases}$

$$= (u)(\ln u) - \int (u)\left(\frac{1}{u}\right) \mathrm{d}u$$

$$= u \ln u - \int 1 \, \mathrm{d}u$$

$$= u \ln u - u + C$$

100 $\quad \int \ln u \ \mathrm{d}u \ = \ u \ln u - u + C$

$$I_{102} = \int \frac{1}{u \ln u}\, \mathrm{d}u$$

Let $\quad v = \ln u \quad \longrightarrow \quad \mathrm{d}v = \frac{1}{u}\, \mathrm{d}u$

$$= \int \frac{1}{v}\, \mathrm{d}v$$

$$= \ln v + C$$

Reverse the substitution

$$= \ln|\ln u|$$

$$\int \tfrac{1}{u \ln u}\, \mathrm{d}u \;=\; \ln|\ln u| \;+\; C$$