$$I_{12} = \int \tan u \, du$$

$$= \int \frac{\sin u}{\cos u} \, du$$

Let $\quad v = \cos u \quad \longrightarrow \quad dv = -\sin u \, du \quad \longrightarrow \quad -dv = \sin u \, du$

$$= -\int \frac{1}{v} \, dv$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\cos u| + C$$

$$I_{13} = \int \cot u \, du$$

$$= \int \frac{\cos u}{\sin u} \, du$$

Let $\quad v = \sin u \quad \longrightarrow \quad dv = \cos u \, du$

$$= \int \frac{1}{v} \, dv$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sin u| + C$$

$$\int \tan u \, du = \ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$I_{14} = \int \sec u \, du$$

$$= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} \, du$$

$$= \int \frac{\sec^2 u + \sec u \cdot \tan u}{\sec u + \tan u} \, du$$

Let $\quad v = \sec u + \tan u \quad \longrightarrow \quad dv = \sec^2 u + \sec u \tan u \, du$

$$= \int \frac{1}{v}$$

$$= \ln|v| + C$$

Reverse the substitution

$$= \ln|\sec u + \tan u| + C$$

1. $\displaystyle\int u \, dv = uv - \int v \, du$

2. $\displaystyle\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \;\; n \neq -1$

3. $\displaystyle\int \frac{1}{u} \, du = \ln|u| + C$

4. $\displaystyle\int e^u \, du = e^u + C$

5. $\displaystyle\int a^u \, du = \frac{a^u}{\ln a} + C$

6. $\displaystyle\int \sin u \, du = -\cos u + C$

7. $\displaystyle\int \cos u \, du = \sin u + C$

8. $\displaystyle\int \sec^2 u \, du = \tan u + C$

9. $\displaystyle\int \csc^2 u \, du = -\cot u + C$

10. $\displaystyle\int \sec u \tan u \, du = \sec u + C$

11. $\displaystyle\int \csc u \cot u \, du = -\csc u + C$

103. $\displaystyle\int \sinh u \, du = \cosh u + C$

104. $\displaystyle\int \cosh u \, du = \sinh u + C$

These integrals can be used without explanation

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$I_{16} = \int \frac{1}{\sqrt{a^2 - u^2}} \, du$$

Let $\sin\theta = \dfrac{u}{a}$, $\cos\theta = \dfrac{\sqrt{a^2 - u^2}}{a}$ $\longrightarrow$ $\tan\theta = \dfrac{u}{\sqrt{a^2 - u^2}}$

and so $u = a\sin\theta$ $\longrightarrow$ $du = a\cos\theta\, d\theta$
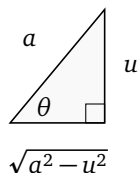
and $\sqrt{a^2 - u^2} = a\cos\theta$

$$= \int \frac{a\cos\theta}{a\cos\theta} \, d\theta$$

$$= \int 1 \, d\theta$$

$$= \theta + C$$

Reverse the substitution of u, where $\sin\theta = \dfrac{u}{a} \longrightarrow \theta = \arcsin\dfrac{u}{a}$

$$= \arcsin\frac{u}{a} + C$$

$$I_{17} = \int \frac{1}{a^2 + u^2} \, du$$

Let $v = \dfrac{u}{a}$ $\longrightarrow$ $av = u$ $\longrightarrow$ $a\, dv = du$

$$= \int \frac{a}{a^2 + (av)^2} \, dv$$

$$= \int \frac{a}{a^2(1 + v^2)} \, dv$$

$$= \frac{1}{a} \int \frac{1}{1 + v^2} \, dv$$

$$= \frac{1}{a} \arctan v$$

Reverse the substitution of u

$$= \frac{1}{a} \arctan\frac{u}{a} + C$$

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(16) $\quad \displaystyle\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \arcsin\frac{u}{a} + C$
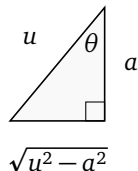
$\displaystyle\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a}\arctan\frac{u}{a} + C$ $\quad$ (17)

$$I_{18} = \int \frac{1}{u\sqrt{u^2 - a^2}}\, du$$

Let $\sin\theta = \dfrac{\sqrt{u^2 - a^2}}{u}$, $\cos\theta = \dfrac{a}{u}$ $\longrightarrow$ $\tan\theta = \dfrac{\sqrt{u^2 - a^2}}{a}$

and so $u = a\sec\theta$ $\longrightarrow$ $du = a\sec\theta \tan\theta\, d\theta$

and $\sqrt{u^2 - a^2} = a\tan\theta$

$$I_{18} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta \cdot a\tan\theta}\, d\theta$$

$$= \int \frac{1}{a}\, d\theta$$

$$= \frac{1}{a}\theta + C$$

Reverse the substitution, where $u = a\sec\theta$ $\longrightarrow$ $\theta = \operatorname{arcsec}\dfrac{u}{a}$

$$= \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

$$I_{15} = \int \csc u\, du$$

$$= \int \csc u \cdot \frac{\csc u + \cot u}{\csc u + \cot u}\, du$$

$$= \int \frac{\csc^2 u + \csc u \cdot \cot u}{\csc u + \cot u}\, du$$

Let $v = \csc u + \cot u$ $\longrightarrow$ $dv = -\csc^2 u + \csc u \cot u\, du$

$\longrightarrow$ $-dv = \csc^2 u + \csc u \cot u\, du$

$$= \int \frac{-dv}{v}$$

$$= -\int \frac{1}{v}\, dv$$

$$= -\ln|v| + C$$

Reverse the substitution

$$= -\ln|\csc u + \cot u| + C$$

$$I_{20} = \int \frac{1}{u^2 - a^2}\, \mathrm{d}u$$

$$= \int \frac{A}{u-a} + \frac{B}{u+a}\, \mathrm{d}u$$

Where $A(u+a) + B(u-a) = 1 \longrightarrow A+B=0$ and $A-B=\frac{1}{a}$
$\longrightarrow B = -A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = -\frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{u-a} + \frac{-\frac{1}{2a}}{u+a}\, \mathrm{d}u$$

$$= \frac{1}{2a} \cdot \int \frac{1}{u-a} - \frac{1}{u+a}\, \mathrm{d}u$$

$$= \frac{1}{2a} \left(\ln|u-a| - \ln|u+a| + C \right)$$

$$= \frac{1}{2a} \cdot \ln\left| \frac{u-a}{u+a} \right| + C$$

$$I_{21} = \int \sqrt{a^2 + u^2}\, \mathrm{d}u$$

Let $\sin\theta = \frac{u}{\sqrt{a^2+u^2}}, \cos\theta = \frac{a}{\sqrt{a^2+u^2}} \longrightarrow \tan\theta = \frac{u}{a}$
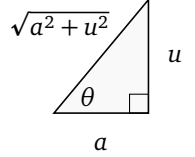
and so $u = a\tan\theta \longrightarrow \mathrm{d}u = a\sec\theta\tan\theta\, \mathrm{d}\theta$
and $\sqrt{u^2 + a^2} = a\sec\theta$

$$= a\sec\theta\, a\sec^2\theta\, \mathrm{d}\theta$$

$$= a^2 \int \sec^3\theta\, \mathrm{d}\theta$$

Apply integal identity $\boxed{71}$: $\int \sec^3 u\, \mathrm{d}u = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

$$= a^2 \left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C \right)$$

$$= \frac{a^2}{2}\frac{\sqrt{u^2+a^2}}{a} \cdot \frac{u}{a} + \frac{a^2}{2}\ln\left| \frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a} \right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left| \frac{u + \sqrt{u^2+a^2}}{a} \right| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left| u + \sqrt{u^2+a^2} \right| + \frac{a^2}{2}\ln|a| + C$$

$$= \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left| u + \sqrt{u^2+a^2} \right| + C$$

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$$I_{19} = \int \frac{1}{a^2 - u^2} \, du$$

$$= \int \frac{A}{a+u} + \frac{B}{a-u} \, du$$

Where $A(a-u) + B(a+u) = 1 \longrightarrow A+B = \frac{1}{a}$ and $B-A=0$

$\longrightarrow B=A \longrightarrow 2A = \frac{1}{a} \longrightarrow A = \frac{1}{2a} \longrightarrow B = \frac{1}{2a}$

$$= \int \frac{\frac{1}{2a}}{a+u} + \frac{\frac{-1}{2a}}{a-u} \, du$$

$$= \frac{1}{2a} \cdot \int \frac{1}{a+u} + \frac{1}{a-u} \, du$$

Rearranging so u preceeds a for aesthetic reasons

$$= \frac{1}{2a} \cdot \int \frac{1}{u+a} - \frac{1}{u-a} \, du$$

$$= \frac{1}{2a} \left(\ln|u+a| - \ln|u-a| \right) + C$$

$$= \frac{1}{2a} \cdot \ln\left|\frac{u+a}{u-a}\right| + C$$

$$\int \frac{1}{a^2 - u^2} \, du = \frac{1}{2a} \ln\left|\frac{u+a}{u-a}\right| + C \qquad \text{(19)}$$

$$I_{64} = \int \cos^2 u \, du$$

$$= \int \frac{1 + \cos u}{2} \, du$$

$$= \frac{1}{2} \cdot \int 1 + \cos u \, du$$

$$= \frac{1}{2} \cdot \left(u + \frac{1}{2} \sin u \right) + C$$

$$= \tfrac{1}{2} u + \tfrac{1}{4} \sin 2u + C$$

$$I_{65} = \int \tan^2 u \, du$$

Rewrite using trigonometric identity $\tan^2 u = \sec^2 u - 1$

$$= \int \sec^2 u - 1 \, du$$

$$= \tan u - u + C$$

(64) $\qquad \int \cos^2 u \, du \;=\; \tfrac{1}{2} u \,+\, \tfrac{1}{4} \sin 2u \,+\, C$

$\int \tan^2 u \, du \;=\; \tan u \,-\, u \,+\, C$ (65)

$$I_{71} = \int \sec^3 u \, du$$

$$\text{Integrate by parts} \quad \begin{cases} (\sec u)' &= \sec u \tan u \\ \sec^2 u &= (\tan u)' \end{cases}$$

$$= (\sec u)(\tan u) - \int (\sec u \tan u)(\tan u) \, du$$

$$= \sec u \tan u - \int \sec u \tan^2 du$$

$$= \sec u \tan u - \int \sec u (\sec^2 u - 1) \, du$$

$$= \sec u \tan u - \int \sec^3 u \, du + \int \sec u \, du$$

$$2 \int \sec^3 u \, du = \sec u \tan u + \int \sec u \, du$$

$$\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \int \sec u \, du$$

$$= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C$$

$$\int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln|\sec u + \tan u| + C \quad \text{(71)}$$

$$I_{63} = \int \sin^2 u \, du$$

$$= \int \frac{1 - \cos 2u}{2} \, du$$

$$= \frac{1}{2} \cdot \int 1 - \cos 2u \, du$$

$$= \frac{1}{2} \cdot \left(u - \frac{1}{2} \sin 2u \right) + C$$

$$= \frac{1}{2} u - \frac{1}{4} \sin 2u + C$$

$$\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C \quad \text{(63)}$$

$$I_{96} = \int u e^{au} \, du$$

Integrate by parts
$$\begin{cases} (u)' &= 1 \\ e^{au} &= \left(\dfrac{e^{au}}{a} \right)' \end{cases}$$

$$= (u) \left(\frac{e^{au}}{a} \right) - \int (1) \left(\frac{e^{au}}{a} \right) du$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a} \int e^{au} \, du$$

$$= \frac{1}{a} u e^{au} - \frac{1}{a^2} e^{au} + C$$

$$= \left(\frac{1}{a} u - \frac{1}{a^2} \right) e^{au} + C$$

Simplify

$$= \frac{1}{a^2} (au - 1) e^{au} + C$$

$$I_{100} = \int \ln u \, du$$

Integrate by parts
$$\begin{cases} 1 &= (u)' \\ (\ln u)' &= \dfrac{1}{u} \end{cases}$$

$$= (u)(\ln u) - \int (u) \left(\frac{1}{u} \right) du$$

$$= u \ln u - \int 1 \, du$$

$$= u \ln u - u + C$$

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(96) $$\int u e^{au} \, du \;=\; \frac{1}{a^2}(au - 1) e^{au} \;+\; C$$

(100) $$\int \ln u \, du \;=\; u \ln u - u + C$$

$$I_{102} = \int \frac{1}{u \ln u}\, du$$

Let $v = \ln u \longrightarrow dv = \frac{1}{u}\, du$

$$= \int \frac{1}{v}\, dv$$

$$= \ln v + C$$

Reverse the substitution

$$= \ln|\ln u|$$

$$I_{92} = \int u \arctan u\, du$$

Integrate by parts
$$\begin{cases} u = \left(\frac{1}{2}u^2\right)' \\ (\arctan u)' = \frac{1}{1+u^2} \end{cases}$$

$$= \left(\tfrac{1}{2}u^2\right)(\arctan u) - \int \left(\tfrac{1}{2}u^2\right)\left(\frac{1}{1+u^2}\right) du$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int \frac{u^2 + 1 - 1}{1+u^2}\, du$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2} \int 1 - \frac{1}{1+u^2}\, du$$

$$= \frac{u^2}{2} \arctan u - \frac{1}{2}(u - \arctan u) + C$$

$$= \frac{u^2 + 1}{2} \arctan u - \frac{u}{2} + C$$

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(102) $\displaystyle\int \frac{1}{u \ln u}\, du = \ln|\ln u| + C$

(92) $\displaystyle\int u \arctan u\, du = \frac{u^2 + 1}{2} \arctan u - \frac{u}{2} + C$