Algorithm to find number of derangements of length N with K good swaps

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1 Introduction

We consider a derangement of size N to be an array D that is a permutation of $0, \ldots, N-1$ where no element is in its correct sorted position. That is, for all $i, 0 <= i < N, D[i] \neq i$. Note that in our definition of derangement, all elements are distinct.

A good swap in a derangement D of size N is a pair $(i, j), 0 <= i < j < N$ such that $D[i] = j$ or $D[j] = i$.

Let s be the number of good swaps in a derangement D of size N. First observe that for every element x in D, there is at most one good swap that places x in its correct position, so $s <= N$. Also, $s > \lfloor N/2 \rfloor$, this follows from the fact that a good swap, in the best case, places two elements in their correct positions (i.e $D[i] = j$ AND $D[j] = i$).

So we know that $\lfloor N/2 \rfloor < s <= N$. using these bounds, we can show that the expected number of random swaps to sort D is between $N^2/8$ and $N^2/2$. To get a more exact running time, it could be helpful to find the expected number of good swaps. This is defined as

$$E[s] = \sum_{k=\lfloor N/2 \rfloor}^{N} p_k \times k$$

Where p_k is the probability of a derangement having k good swaps. We can define this as

$$\frac{\text{number of derangements with } k \text{ good swaps}}{\text{total number of derangements}}$$

It can be shown that the number of derangements of size N is exactly $\lfloor \frac{N!+1}{e} \rfloor$. But how many derangements of size N have K good swaps?

2 Model as Graph

Given a derangement D of size N we construct a directed graph $G(V, E)$ where $V = \{0, ..., N-1\}$ and $(i, j) \in E$ if $D[i] = j$. First observe that there are no self-loops in this graph because in a derangement there is no element in its correct position. Secondly, observe that each vertex i has an outdegree of exactly 1, that is, $D[i] = j$ for some j and if $D[i] = j$ and $D[i] = k$ then $j = k$. A similar argument shows that each vertex has an indegree of 1. With these observations, the graph must be a connected cycle graph, or if disconnected, decomposed into multiple cycle graphs each with 2 or more nodes. Let's call a graph with these properties a **derangement graph**. The number of good swaps in D is almost the number of edges in its derangement graph, except if we have a cycle of 2 nodes, that represents 1 good swap, whereas if we have a cycle of m nodes where $m > 2$ then that cycle contributes m good swaps (m nodes implies m edges, each being an instance where $D[i] = j$ so one of (i, j) or (j, i) is a good swap).

We then rephrase our original problem to the following: how many ways can we make a graph with vertices $\{0, ..., N-1\}$ such that it is a derangement graph with k good swaps?

3 Naive Recurrence

Let T(N, k) be the number of derangement graphs with vertices $\{0, ..., N-1\}$ with k good swaps where $k <= N$. We define a recurrence for T(N, K). Let's first look at the base cases:

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4 Conclusions

We worked hard, and achieved very little.