Algorithm to find number of derangements of length N with K good swaps

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1 Introduction

We consider a derangement of size N to be an array D that is a permutation of $0, \ldots, N-1$ where no element is in its correct sorted position. That is, for all $i, 0 <= i < N, D[i] \neq i$. Note that in our definition of derangement, all elements are distinct.

A good swap in a derangement D of size N is a pair $(i, j), 0 <= i < j < N$ such that $D[i] = j$ or $D[j] = i$.

Let s be the number of good swaps in a derangement D of size N. First observe that for every element x in D, there is at most one good swap that places x in its correct position, so $s <= N$. Also, $s > \lfloor N/2 \rfloor$, this follows from the fact that a good swap, in the best case, places two elements in their correct positions (i.e $D[i] = j$ AND $D[j] = i$).

So we know that $\lfloor N/2 \rfloor < s <= N$. using these bounds, we can show that the expected number of random swaps to sort D is between $N^2/8$ and $N^2/2$. To get a more exact running time, it could be helpful to find the expected number of good swaps. This is defined as

$$E[s] = \sum_{k=\lfloor N/2 \rfloor}^{N} p_k \times k$$

Where p_k is the probability of a derangement having k good swaps. We can define this as

$$\frac{\text{number of derangements with } k \text{ good swaps}}{\text{total number of derangements}}$$

It can be shown that the number of derangements of size N is exactly $\lfloor \frac{N!+1}{e} \rfloor$. But how many derangements of size N have K good swaps?

2 Model as Graph

Given a derangement D of size N we construct a directed graph $G(V, E)$ where $V = \{0, ..., N-1\}$ and $(i, j) \in E$ if $D[i] = j$. First observe that there are no self-loops in this graph because in a derangement there is no element in its correct position. Secondly, observe that each vertex i has an outdegree of exactly 1, that is, $D[i] = j$ for some j and if $D[i] = j$ and $D[i] = k$ then $j = k$. A similar argument shows that each vertex has an indegree of 1. With these observations, the graph must be a connected cycle graph, or if disconnected, decomposed into multiple cycle graphs each with 2 or more nodes. Let's call a graph with these properties a **derangement graph**. The number of good swaps in D is almost the number of edges in its derangement graph, except if we have a cycle of 2 nodes, that represents 1 good swap, whereas if we have a cycle of m nodes where $m > 2$ then that cycle contributes m good swaps (m nodes implies m edges, each being an instance where $D[i] = j$ so one of (i, j) or (j, i) is a good swap).

We then rephrase our original problem to the following: how many derangement graphs with vertices $\{0, ..., N-1\}$ are there with k good swaps?

3 Recurrence

We start with a naive recurrence: Let T(N, k) be the number of derangement graphs with vertices $\{0, ..., N-1\}$ with k good swaps where $k <= N$. We define a recurrence for T(N, K) based on the following idea:

Pick i vertices from the N vertices. With these i vertices, we will make a cycle of length i, so i must be greater than 1.

If we make a cycle of length 2, that contributes 1 good swap. We would like the rest of the $n-2$ vertices to contribute $k-1$ good swaps. Assume we have $T(n-2, k-1)$

If we make a cycle of length i where $i > 2$, that contributes i good swaps. Assume we have $T(n-i, k-i)$.

Let C_i be the number of cycles we can make of length i. There are $\binom{i}{2}$ ways to pick i vertices. With these i vertices, there are $i!$ cycles we can make (fix a starting vertex, there are $(i-1)$ choices for the second vertex in the cycle, then $(i-2)$, and so on).

So $C_i = (i-1)! * \binom{i}{2}$.

Putting this all together, we have the following recurrence:

$$T(N,k) = C_2 \times T(N-2,k-1) + \sum_{i=2}^{k} C_i \times T(N-i,k-i)$$

This recurrence is naive because of the following observation: Take for example a graph of 8 vertices. Let's say we split this graph into cycles of 4, 3, 2, and 1. We counted this arrangement 4! times since there are that many ways to permute these cycles. Thus we define another recurrence $T'(N,K,S)$, the number of ways to make a derangement graph of N vertices with K good swaps with S connected components. The naive reccurence is slightly modified:

$$T'(N,k,s) = C_2 \times T'(N-2,k-1,s-1) + \sum_{i=2}^{k} C_i \times T'(N-i,k-i,s-1)$$

Thus our final solution is
$T(N,K) = \sum_{s=1}^{N-1} \frac{T'(N,K,s)}{s!}$

4 Conclusions

Running this algorithm shows that the expected number of good swaps in a derangement of size N is almost exactly $N-0.5$.