Number of 0 swaps in derangement

Christopher He

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Let D be a derangement of length n with cycle decomposition $c_1, c_2, ..., c_k$. Let l_2 be the number of cycles with length 2. We know the number of 0-swaps in the derangement (or the degree) is

$$\sum_i (\binom{c_i}{2} - c_i) + \sum_{i,j} c_i c_j + l_2$$

$$= \sum_i \binom{c_i}{2} + \sum_{i,j} c_i c_j - n + l_2 \quad \text{(by linearity of expectation)}$$

Note that we add l_2 because if c_i is length 2 then $\binom{c_i}{2} - c_i = -1$ so we add l_2 to "ignore" the cycles of length 2.

I argue that the degree is $\binom{n}{2} - n + l_2$. That is,

$$\sum_i \binom{c_i}{2} + \sum_{i,j} c_i c_j - n + l_2 = \binom{n}{2} - n + l_2$$

or equivalently,

$$\sum_i \binom{c_i}{2} + \sum_{i,j} c_i c_j = \binom{n}{2}$$

Proof.

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\begin{aligned}
&\sum_i \binom{c_i}{2} + \sum_{i,j} c_i c_j \qquad (1)\\
&= \sum_i \binom{c_i}{2} + \frac{(\sum_i c_i)^2 - \sum_i {c_i}^2}{2}\\
&= \frac{\sum_i c_i(c_i - 1) + (\sum_i c_i)^2 - \sum_i {c_i}^2}{2}\\
&= \frac{n^2 + \sum_i c_i(c_i - 1) - \sum_i {c_i}^2}{2}\\
&= \frac{n^2 - \sum_i c_i}{2}\\
&= \frac{n^2 - n}{2}\\
&= \binom{n}{2}
\end{aligned}
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$\square$