Blind selection sort analysis

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1 Exact running time

Let a be a permutation of $1, 2, ..., N$.

Definition 1. *Let* $inv(a) = \{(i, j) \mid 1 \leq i < j \leq N, a_i > a_j\}$ *be the set of inversions of* a*.*

Definition 2. *If* $a_j > j$*, we say* a_j *starts* ***before*** *its position. If* $a_j < j$*, we say* a_j *starts* ***after*** *its position.*

Definition 3. *Let* $ginv(a) = \{(i, j) \in inv(a) \mid a_j$ *starts* ***before*** *its position*$\}$ *be the set of good inversions of* a*.*

Definition 4. *Let* $inv_k(a) = \{(i, j) \in inv(a) \mid i = k\}$ *denote inversions where the left is* k*, and similarly define* $ginv_k(a)$*.*

First we proved that if $(i, j) \in inv(a)$ and a_j starts before its correct position j, then so does a_i.

Lemma 1. *If* $(i, j) \in ginv(a)$ *then* $a_i > i$*.*

Proof. If $(i, j) \in ginv(a)$ then $a_i > a_j > j > i$, so $a_i > i$. □

We then proved that if an element starts before its position i.e $a_i > i$, the distance $a_i - i$ is at most $|inv_i(a)|$.

Lemma 2. *If* $a_i > i$ *then* $a_i - i \leq |inv_i(a)|$*.*

Proof. There are $a_i - 1$ elements smaller than a_i, but only $i - 1$ available positions to its left. That means that at least $a_i - 1 - (i - 1) = a_i - i$ are on its right, forming that many inversions. So $|inv_i(a)| \geq a_i - i$. □

Conjecture 1. *The number of swaps* $s(a)$ *performed by blind selection sort on permutation* a *is*

$$s(a) = \sum_{i < a_i} a_i - i + \sum_{i > a_i} |inv_i(a)| - \sum_{i} |ginv_i(a)| \tag{1}$$

Proof. In general, the number of swaps involving a_i (where a_i is the left element) is equal to $|inv_i(a)|$. For example, let $a = [6, 5, 4, 3, 2, 1]$. The number of swaps involving 3 is 2. When the algorithm searches for 1, it will generate one swap for 3. Another one will be generated when 2 is being searched.

But notice that a swap with a_i has a net effect of shifting a_i to the right. If an element starts before its position ($a_i > i$), then by Lemma 2, the element shifts at most $a_i - i$ times before it is frozen.

Lastly, we subtract $\sum_i |ginv_i(a)|$ to account for that fact that elements may be frozen "accidentally", saving one swap for every unfrozen element before it. Note that by Lemma 1, $\sum_i |ginv_i(a)| = \sum_{i<a_i} |ginv_i(a)|$. $\square$

Conjecture 2. *The number of swaps $s(a)$ performed by blind selection sort on permutation a is*

$$s(a) = \sum_i inv_i(a) - 2\sum_i |ginv_i(a)| \tag{2}$$

Proof. From Conjecture 1 we know the number of swaps is

$$s(a) = \sum_{i<a_i} a_i - i + \sum_{i>a_i} |inv_i(a)| - \sum_i |ginv_i(a)| \tag{3}$$

If the following equality is proven (TODO: prove this by proving that good inversions balance the distance)

$$\sum_{i<a_i} a_i - i = \sum_{i\leq a_i} inv_i(a) - \sum_i ginv_i(a) \tag{4}$$

then this statement is proven.

$$\begin{aligned} s(a) &= \sum_{i<a_i} a_i - i + \sum_{i>a_i} |inv_i(a)| - \sum_i |ginv_i(a)| \qquad (5)\\ &= \sum_{i\leq a_i} inv_i(a) - \sum_i |ginv_i(a)| + \sum_{i>a_i} |inv_i(a)| - \sum_i |ginv_i(a)| \\ &= \sum_i inv_i(a) - 2\sum_i |ginv_i(a)| \end{aligned}$$

$\square$

2 Structure of the worst case input

Theorem 1. *If a is the worst case permutation of $1, 2, ..., N$ for blind selection sort, then $ginv_i(a) = \emptyset$.*

Proof. Assume $ginv_i(a) \neq \emptyset$, then pick $(i, j) \in ginv_i(a)$ such that $\forall k, i < k < j, a_k < k$. So we start with $a = [a_1, ..., a_i, ..., a_j, ..., a_N]$. We can construct a new permutation by swapping a_i and a_j i.e $a' = [a_1, ..., a_j, ..., a_i, ..., a_N]$. This effectively removes the good inversion (i, j). Furthermore, let $k, i < k < j$.

The number of inversions involving k remains the same after the swap. TODO: finish. By Conjecture 2, $s(a') = s(a) + 1$, thus contradicting that a is the worst case. $\square$

This gives us the immediate result that in the worst input a, if an element starts before its position, all elements before it are smaller (otherwise $ginv(a) \neq \emptyset$).

Corollary 1. *In the worst case permutation of $1, 2, ..., N$ for blind selection sort, if $a_j > j$ then $\forall i < j, a_i < a_j$.*

Theorem 2. *The worst case permutation of $1, 2, ..., N$ for blind selection sort starts with an increasing, consecutive sequence up to N.*

Lemma 3. *If a is the worst input for blind selection sort and $a_j > j$, then $\forall i < j, a_i > i$.*

Proof. We will prove the equivalent statement: if $j > 1, a_j > j$, then $a_{j-1} > j - 1$. Assume for the sake of contradition that $a_{j-1} < j - 1$. By Corollary 1, we know $a_{j-1} < a_j$. So switch the position of a_{j-1}, and a_j. We created an inversion, and we know this is not a good inversion since $a_{j-1} < j - 1 < j$ (in other words, after moving a_{j-1} to the right, it is still after its correct position.) So by Conjecture 2, we made an even worse input. $\square$

Definition 5. *If $a_k = x$ then $idx(x) = k$.*

Lemma 4. *If a is the worst input for blind selection sort and $\exists j > 1$ such that $a_j > j$, then $a_{j-1} = a_j - 1$.*

Proof. By Corollary 1, $a_{j-1} < a_j$. Assume for the sake of contradiction that $a_{j-1} \neq a_j - 1$, so $a_{j-1} < a_j - 1$. Let $k = idx(a_j - 1)$, we know that $k > j - 1$, otherwise $(k, j - 1) \in ginv(a)$. Switch a_{j-1} with a_k. By doing this we create an inversion $(j - 1, k)$, and we know this is not a good inversion since $a_{j-1} < a_k < k$. TODO: show that elements between j and k were unaffected. Thus by Conjecture 2, we made an even worse input. $\square$

Proof. Since $idx(N) < N$ (otherwise it is frozen), then by Lemma 2 and 3, the statement is proven. $\square$