Integral infeasibility and testing total dual integrality

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Structure

Introduction of the problem

- Let s be a rational linear system $Ax \leqslant b$
  s is called totally dual integral if :
- $\forall$ integral vector w such that there is an optima for the following equation :
  $max\{wx : Ax \leqslant b\} = min\{yb : yA = w, y \geqslant 0\}$, there is an integral solution to the minimum equation.
- If b is integral aswell, there is also an inegral solution for the maximum in the equation.

Integral Infeasibility

Variations of the Linear System

For a system $Ax \leq b$ of m linear inequalities and a set $T \subset \{1, \ldots, m\}$ and $\bar{T} = \{1, \ldots, m\} \setminus T$, we let

$$A_T x = b_T, A_{\bar{T}} x \leq b_{\bar{T}} \tag{1}$$

denote the system obtained by settig each inequality in T to equality.

Farkas Lemma

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ then exactly one of the following is true:

1. There exists $x \in \mathbb{R}^n$ such that $Ax \leq b$.
2. There exists $y \in \mathbb{R}^m$ such that $yA = 0, yb < 0$ and $y \geq 0$.

Note

Furthermore, in relation to Farkas Lemma, we have that exactly one of the following is true:

1. There exists $x \in \mathbb{R}^n$ such that $Ax = b$.
2. There exists $y \in \mathbb{R}^m$ such that $yA = 0$ and $by \neq 0$.

Infeasibility realtion

With the use of Farkas lemma for $A_{\bar{T}}x \leq b_{\bar{T}}$ and the observation for $A_T x = b_T$ we see that the system $A_{\bar{T}}x \leq b_{\bar{T}}$, $A_T x = b_T$ is not feasible if there exists a vector $(y_T, y_{\bar{T}})$ such that

$$\begin{aligned} y_T b_T + y_{\bar{T}} b_{\bar{T}} &< 0 \\ y_T A_T + y_{\bar{T}} A_{\bar{T}} &= 0 \\ y_{\bar{T}} \geq 0. \end{aligned} \tag{2}$$

Values of y_T can possibly be negative due to equality constraints. Due to scaling we can still pose the constraints

$$\begin{aligned} y_T b_T + y_{\bar{T}} b_{\bar{T}} &< 0 \\ y_T A_T + y_{\bar{T}} A_{\bar{T}} &= 0 \\ y_T \geq -1 \quad y_{\bar{T}} &\geq 0. \end{aligned} \tag{3}$$

By construction of the linear system, this now gives us, with the use of Farkas Lemma and the observation, that if such a vector exists the system $A_{\bar{T}} x \leq b_{\bar{T}}$, $A_T x = b_T$ has not feasible solution. This result is used in the upcomming Theorem.

Definitions

We say that the **infeasibility** of (1) can be **proven integrally** if (3) does in fact have an integral solution.

Hilbert basis

A set of vectors $\{h_1, \ldots, h_k\}$ is called Hilbert basis if each integral vector in the cone $C(\{h_1, \ldots, h_k\}) := \{\sum_{i \in \{1,\ldots,k\}} \lambda_i h_i; \lambda_i \geq 0 \text{ for all } i \in \{1, \ldots, k\}\}$ can be written as integral combination of $h_1, \ldots, h_k$.

Theorem 1

Let $A \in \mathbb{Z}^{m \times n}$ and b a rational vector such that the linear system $Ax \leq b$ has at least one solution. Then $Ax \leq b$ is totally dual integral **if and only if**

1. the rows of A form a Hilbert basis, and
2. for each subset T of inequalities from $Ax \leq b$, if (1) is infeasible, then this can be proven interally.

Reminder - (1)

For $T \subset \{1, \dots, m\}$, $\bar{T} = \{1, \dots, m\} \setminus T$ the system (1) was given by $A_T x = b_T$ and $A_{\bar{T}} x \leq b_{\bar{T}}$.

Testing for total dual inegrality

Test by Cook, Lovász and Schrijver

Cook, Lovász and Schrijver (source) developed a polynomial-time test (for fixed dimension) for the total dual integrality.

Based on the fact that
$Ax \leq b$ is totally dual integral **if and only if** for each minimal face F_J the set of active rows form a Hilbert basis.

Can be checked with Lenstra's integer programming algorithm. Number of times is exponential in with respect to the dimension. Not praktical for application.

Faces; minimal Faces

For $P \subset \mathbb{R}^n$ a polyhedron given by $Ax \leq b$, $J \subset \{1, \dots, m\}$, define $F_J := \{x \in P | a_i x = b_i \text{ for } i \in J\}$ as a Face of P.
A minimal Face of P does not contain another Face.

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Example

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1 + x_2 + x_3 = 1$$

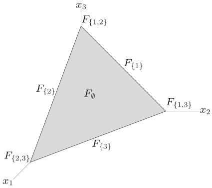

Figure:
http://www.seas.ucla.edu/ vandenbe/ee236a/lectures/polyhedra.pdf

It is of practical interest to avoid Hilber basis tests whenever possible. Thus Theorem 2 helps in certain cases.

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Theorem 2

Let $A \in \mathbb{Z}^{m \times n}$ and b a rational vector such that the linear system $Ax \leq b$ has at least one solution. Then $Ax \leq b$ is totally dual integral **if and only if**

1. the rows of A form a Hilbert basis, and
2. for each subset T of at most n inequalities from $Ax \leq b$, the linear programming problem $min\{yb : yA = 1 \cdot A_T, y \geq 0\}$ has an integral solution.

Improvements

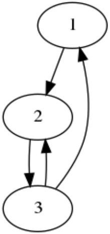
- integer programming problems of the form

$$min\{yb : yA = w, y \geq 0\}$$

  for totally dual integral systems $Ax \leq b$ (A, w integral) are solvable in polynomial time (Chandrasekaran, Schrijver)
- sometimes checking of condition (i) is possible without using the test of Cook, Lovász and Schrijver

Linear system of Barahona and Mahjoub problem

- Let s be the following linear system for a graph D :
  $\forall$ directed circuit C of D, $\sum\{x_e : e \in C\} \geqslant 1$
  $x_e \geqslant 0$ for each arc e of D
- Example :

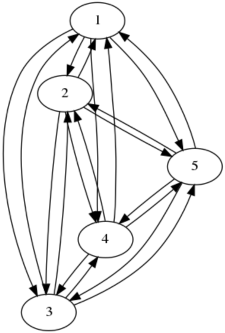


The associated linear system may be written:

$$\underbrace{\begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_{1->2} \\ x_{2->3} \\ x_{3->1} \\ x_{3->2} \end{pmatrix}}_{x} \leqslant \underbrace{\begin{pmatrix} -1 & -1 \end{pmatrix}}_{b}$$

Barahona and Mahjoub problem

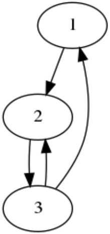
Let D_5, the complete symetric directed graph on 5 nodes :



Is s for D_5 totally dual integral?

Feedback sets

- A 0-1 solution is a solution such that : $\forall e \in D, x_e \in \{0,1\}$
- Such a solution corresponds to a subset of arcs $S \subseteq D$ which meets every circuit in D
- S is called a *feedback set*
- Example

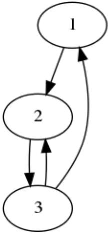


In this example, $\{2- > 3\}$ is a feedback set.

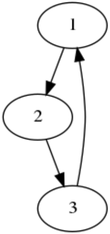
Lemma3

- D_5 has 84 directed circuits, so we want to reduce that number before computing anything. For that we will need the lemma 3.
- Lemma 3 : Suppose that D is a directed graph with arcs ij and ji, that D_1 is D with ji deleted, and D_2 is D with ij deleted. If s is totally dual integral for both D_1 and D_2, it is totally dual integral for D.
- Example :

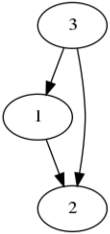
D :



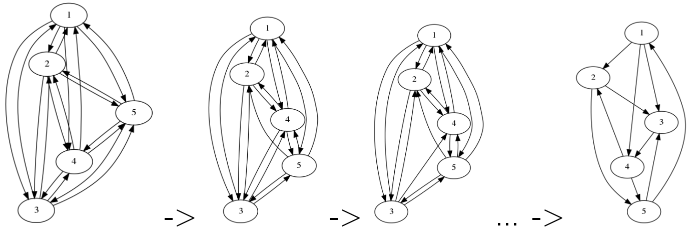
D_1 :



D_2 :



From D_5 to K_5

- Thanks to lemma 3, we can conclude that if for each orientation of K_5, s is totally dual integral, then it is totally dual integral for D_5
- 

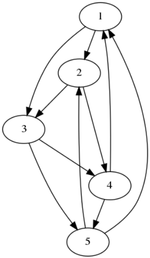
- This reduces the maximum number of circuits for each orientation to 12.

Barahona and Mahjoub lemma

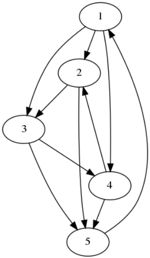
- We now want to reduce the number of orientation to fully treat. For that we need the lemma 4, from Barahona and Mahjoub.
- Lemma 4 : Let D be an orientation of K_5. Then if either some node of D meets all directed circuit or some arc is in no directed circuit, s is totally dual integral for D.
- By checking for lemma 4 and isomorphism, there only remain 3 distinct orientation of K_5 to treat.

The final computation

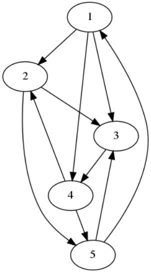
$D(12)$:



$D(10)$:



$D(9)$:



Compute times for each run :

- $D(12)$: 14h 33mn 21s
- $D(10)$: 2h 52mn 7s
- $D(9)$: 1h 6mn 3s