

GSR approach to Horndeski inflation

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Based on: “**Reconciling tensor and scalar observables in G-inflation**”

with

S. Passaglia, H. Motohashi, W. Hu, and O. Mena.



CSIC
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS



Outline:

1. Scalar - Tensor theories.
2. Generalized slow-roll (GSR) techniques.
3. G-inflation.

Scalar-tensor theories

Ostrogradsky Ghost

Healthy theories
with 3 DOF

beyond Horndeski: GLPV,
DHOST / EST

*Dark
energy*

Inflation

Horndeski theory
with 2nd order EL eq

$$f(\phi, X)R$$

Brans-Dicke

$$\square\phi$$

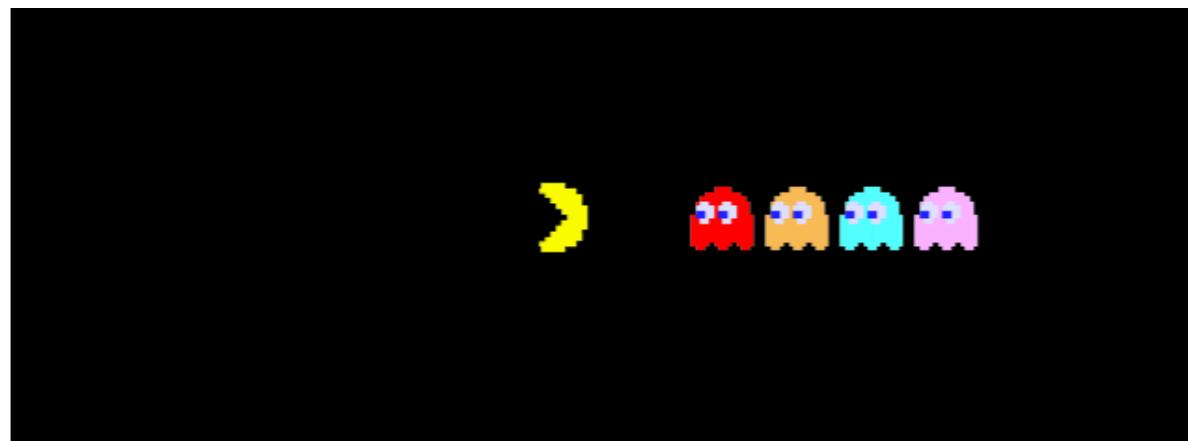
$$G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$$

$$f(R)$$

$$K(\phi, X)$$

Ostrogradski's theorem:

“Higher-derivative theories contain extra degrees of freedom, and are usually plagued by negative energies and related instabilities.”



Scalar - Tensor theories:

- G. Horndeski (1974)
- A. Nicolis *et al.*; 0811.2197
- C. Deffayet *et al.*; 1103.3260

$$S = \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i$$

$$\mathcal{L}_2 = G_2 ,$$

$$\mathcal{L}_3 = G_3 \square \phi ,$$

$$\mathcal{L}_4 = G_4 R - 2G_{4,X}[(\square \phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}] ,$$

$$\mathcal{L}_5 = G_5 G^{\mu\nu}\phi_{;\mu\nu} + \frac{G_{5,X}}{3}[(\square \phi)^3 - 3(\square \phi)\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}] .$$

$$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$G_i = G_i(\phi, X)$$

Applications
to:

- Late-time Cosmology (dark energy).
- Early-universe Cosmology (inflation).

Not only scalar-tensor theories, but also...

Vector - Tensor theories:

- G. Horndeski (1976)
- L. Heisenberg; 1402.7026

$$S = \int d^4x \sqrt{-g} \left[F + \sum_{i=2}^6 \mathcal{L}_i + \mathcal{L}_m \right]$$

$$\mathcal{L}_2 = G_2(X, F, Y),$$

$$\mathcal{L}_3 = G_3(X) \nabla_\mu A^\mu,$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) [(\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu],$$

$$\mathcal{L}_5 = G_5(X) G_{\mu\nu} \nabla^\mu A^\nu - \frac{1}{6} G_{5,X}(X) [(\nabla_\mu A^\mu)^3 - 3 \nabla_\mu A^\mu \nabla_\rho A_\sigma \nabla^\sigma A^\rho + 2 \nabla_\rho A_\sigma \nabla^\nu A^\rho \nabla^\sigma A_\nu]$$

$$- g_5(X) \tilde{F}^{\alpha\mu} \tilde{F}^\beta_\mu \nabla_\alpha A_\beta,$$

$$\mathcal{L}_6 = G_6(X) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2} G_{6,X}(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu,$$

$$X \equiv -\frac{1}{2} A_\mu A^\mu$$

■ $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

■ $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

■ $L \equiv \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}$

■ $F \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

■ $Y \equiv A^\mu A^\nu F_\mu^\alpha F_{\nu\alpha}$

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$$\mathcal{L}_6 = G_6(X) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2} G_{6,X}(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu,$$

$$X \equiv -\frac{1}{2} A_\mu A^\mu$$

Applications
to:

- Late-time Cosmology.
- Compact objects (hairy black holes, neutron stars,...)

And...

Scalar - Vector - Tensor theories:

• L. Heisenberg; 1801.01523

“The resulting Lagrangians consist of new genuine couplings...”

$$S = \int d^4x \sqrt{-g} \left(\underbrace{\sum_{i=3}^5 \mathcal{L}_{\text{ST}}^i + \sum_{i=2}^4 \mathcal{L}_{\text{SVT}}^i}_{\text{Horndeski Lagrangian}} \right)$$

$$\mathcal{L}_{\text{SVT}}^2 = f_2(\pi, X, F, \tilde{F}, Y) ,$$

$$\mathcal{L}_{\text{SVT}}^3 = \mathcal{M}_3^{\mu\nu} \nabla_\mu \partial_\nu \pi ,$$

$$\mathcal{L}_{\text{SVT}}^4 = \mathcal{M}_4^{\mu\nu\alpha\beta} \nabla_\mu \partial_\alpha \pi \nabla_\nu \partial_\beta \pi + f_4(\pi, X) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} .$$

$$X = -\frac{1}{2} (\partial\pi)^2$$

■ $Y \equiv \partial_\mu \pi \partial_\nu \pi F^{\mu\alpha} F_\alpha^\nu$ ■ $\mathcal{M}_4^{\mu\nu\alpha\beta} = \left(\frac{1}{2} f_{4,X} + \tilde{f}_4(\pi) \right) \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta}$

■ $F \equiv F_{\mu\nu} F^{\mu\nu}$

■ $\tilde{F} \equiv F_{\mu\nu} \tilde{F}^{\mu\nu}$ ■ $\mathcal{M}_3^{\mu\nu} = \left(f_3(\pi, X) g_{\rho\sigma} + \tilde{f}_3(\pi, X) \partial_\rho \pi \partial_\sigma \pi \right) \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma}$

And...

Scalar - Vector - Tensor theories:

- L. Heisenberg; 1801.01523

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$$X = -\frac{1}{2} (\partial\pi)^2$$

Applications
to:

- New black hole and neutron star solutions.
 - L. Heisenberg & S. Tsujikawa; 1802.07035
- Inflation and magnetogenesis.
 - H. Motohashi, **HR** and S. Tsujikawa; in progress.
- Dark energy and dark matter.

We focus on scalar-tensor theories...

Horndeski theory:

$$S = \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i$$

$$\mathcal{L}_2 = G_2 ,$$

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$$\mathcal{L}_4 = G_4 R - 2G_{4,X}[(\square \phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}] ,$$

$$\mathcal{L}_5 = G_5 G^{\mu\nu}\phi_{;\mu\nu} + \frac{G_{5,X}}{3}[(\square \phi)^3 - 3(\square \phi)\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}] .$$

Canonical
inflation:

$$G_2 = -\frac{X}{2} - V(\phi) , \quad G_4 = \frac{M_{\text{pl}}^2}{2} ,$$
$$G_3 = 0 , \quad G_5 = 0 .$$

$$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$G_i = G_i(\phi, X)$$

Inflation

- T. Kobayashi *et al.*; 1105.5723
- H. Motohashi, W. Hu; 1704.01128

$$S_{\zeta}^{(2)} = \int d^4x \frac{a^3 b_s \epsilon_H}{c_s^2} \left(\dot{\zeta}^2 - \frac{c_s^2 k^2}{a^2} \zeta^2 \right)$$

$$S_{\gamma}^{(2)} = \sum_{\gamma=+,\times} \int d^4x \frac{a^3 b_t}{4c_t^2} \left(\dot{\gamma}_{\lambda}^2 - \frac{c_t^2 k^2}{a^2} \gamma_{\lambda}^2 \right)$$

In canonical inflation:

$$b_s = b_t = c_s = c_t = 1$$

Scalar parameters:

$$\epsilon_H = -\frac{\dot{H}}{H^2}$$

- $b_s = \frac{c_s^2}{\epsilon_H} \left[\frac{w_1(4w_1w_3 + 9w_2^2)}{3w_2^2} \right]$
- $c_s^2 = \frac{3(2w_1^2w_2H - w_2^2w_4 + 4w_1\dot{w}_1w_2 - 2w_1^2\dot{w}_2)}{w_1(4w_1w_3 + 9w_2^2)}$

Tensor parameters:

- $b_t = w_1 c_t^2$
- $c_t^2 = \frac{w_4}{w_1}$

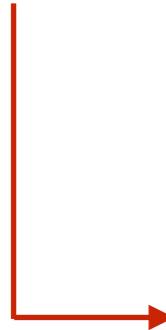
$$w_1 = M_{\text{pl}}^2 - 2 \left(3G_4 + 2HG_5\dot{\phi} \right) + 2G_{5,\phi}X ,$$

$$w_2 = 2M_{\text{pl}}^2H - 2G_3\dot{\phi} - 2 \left(30HG_4 - 5G_{4,\phi}\dot{\phi} + 14H^2G_5\dot{\phi} \right) + 28HG_{5,\phi}X ,$$

$$w_3 = -9M_{\text{pl}}^2H^2 + 3 \left(X + 12HG_3\dot{\phi} \right) + 6 \left(135H^2G_4 - 2G_{3,\phi}X - 45HG_{4,\phi}\dot{\phi} + 56H^3G_5\dot{\phi} \right) - 504H^2G_{5,\phi}X ,$$

$$w_4 = M_{\text{pl}}^2 + 2 \left(G_4 - 2G_5\ddot{\phi} \right) - 2G_{5,\phi}X .$$

$$S_{\zeta}^{(2)} = \int d^4x \frac{a^3 b_s \epsilon_H}{c_s^2} \left(\dot{\zeta}^2 - \frac{c_s^2 k^2}{a^2} \zeta^2 \right)$$



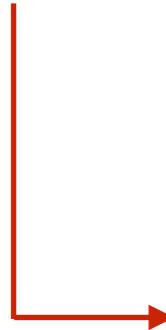
$$\frac{d^2v}{d\tau^2} + \left(c_s^2 k^2 - \frac{1}{z} \frac{d^2z}{d\tau^2} \right) v = 0$$

*Mukhanov - Sasaki equation

- $v = z\zeta$
 - Assume slow-roll approximation.

- $z = a \sqrt{\frac{2b_s \epsilon_H}{c_s^2}}$
 - Solve numerically.
 - Use GSR techniques.

$$S_{\zeta}^{(2)} = \int d^4x \frac{a^3 b_s \epsilon_H}{c_s^2} \left(\dot{\zeta}^2 - \frac{c_s^2 k^2}{a^2} \zeta^2 \right)$$



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3. G-inflation.

Generalized Slow-Roll

- E. Stewart; 0110322
- C. Dvorkin, W. Hu; 0910.2237
- W. Hu; 1104.4500
- W. Hu; 1405.2020
- H. Motohashi, W. Hu; 1503.04810
- H. Motohashi, W. Hu; 1704.01128

... and others.

Generalized slow-roll recipe:

1. Write down Mukhanov-Sasaki equation.

Generalized slow-roll recipe:

1. Write down Mukhanov-Sasaki equation.
2. Isolate deviations from de Sitter background.

$$\frac{d^2y}{dx^2} + \left(1 - \frac{2}{x^2}\right)y = \frac{f'' - 3f'}{f} \frac{y}{x^2}$$

$$\begin{array}{lll} \blacksquare \quad y \equiv \sqrt{2c_s k} v & \blacksquare \quad x \equiv ks_s & \blacksquare \quad s_s \equiv \int c_s d\tau \end{array}$$

$$\begin{array}{ll} \blacksquare \quad f = \sqrt{8\pi^2 \frac{b_s \epsilon_H c_s}{H^2}} \frac{a H s_s}{c_s} & \blacksquare \quad \Delta_\zeta^2(k) = \lim_{x \rightarrow 0} \left| \frac{xy}{f} \right|^2 \end{array}$$

Generalized slow-roll recipe:

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$$\frac{d^2y}{dx^2} + \left(1 - \frac{2}{x^2}\right)y = \frac{f'' - 3f'}{f} \frac{y}{x^2}$$

3. Apply Green function techniques (GSR).

$$\ln \Delta^{2(1)} = G(\ln x_m) + \int_{x_m}^{\infty} d(\ln x) W(kx) G'(\ln x)$$

- $G = -2\ln f + \frac{2}{3} (\ln f)'$
- $W(u) = \frac{3 \sin(2u)}{2u^3} - \frac{3 \cos(2u)}{u^2} - \frac{3 \sin(2u)}{2u}$

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3. Apply Green function techniques (GSR).

$$\ln \Delta^{2(1)} = G(\ln x_m) + \int_{x_m}^{\infty} d(\ln x) W(kx) G'(\ln x)$$

4. Taylor expand GSR formula and write down analytic equations (OSR).

Optimized SR for Horndeski (leading order):

- H. Motohashi, W. Hu; 1704.01128

$$\ln \Delta^2 \simeq G(\ln x_f) + \sum_{p=1}^{\infty} q_p(\ln x_f) G^{(p)}(\ln x_f)$$

$$G = -2\ln f + \frac{2}{3}(\ln f)'$$

$$f = \sqrt{8\pi^2 \frac{b_s \epsilon_H c_s}{H^2}} \frac{a H s_s}{c_s}$$

$$\ln \Delta_\zeta^2 \approx \ln \left(\frac{H^2}{8\pi^2 b_s c_s \epsilon_H} \right) - \frac{10}{3} \epsilon_H - \frac{2}{3} \delta_1 - \frac{7}{3} \sigma_{s1} - \frac{1}{3} \xi_{s1} \Big|_{x=x_1}$$

Scalars

$$n_s - 1 \approx -4\epsilon_H - 2\delta_1 - \sigma_{s1} - \xi_{s1} - \frac{2}{3}\delta_2 - \frac{7}{3}\sigma_{s2} - \frac{1}{3}\xi_{s2} \Big|_{x=x_1}$$

$$\alpha_s \approx -2\delta_2 - \sigma_{s2} - \xi_{s2} - \frac{2}{3}\delta_3 - \frac{7}{3}\sigma_{s3} - \frac{1}{3}\xi_{s3} - 8\epsilon_H^2 - 10\epsilon_H\delta_1 + 2\delta_1^2 \Big|_{x=x_1}$$

$$\ln \Delta_\gamma^2 \approx \ln \left(\frac{H^2}{2\pi^2 b_t c_t} \right) - \frac{8}{3} \epsilon_H - \frac{7}{3} \sigma_{t1} - \frac{1}{3} \xi_{t1} \Big|_{x=x_1}$$

$$n_t \approx -2\epsilon_H - \sigma_{t1} - \xi_{t1} - \frac{7}{3}\sigma_{t2} - \frac{1}{3}\xi_{t2} \Big|_{x=x_1}$$

Tensors

$$\alpha_t \approx -\sigma_{t2} - \xi_{t2} - \frac{7}{3}\sigma_{t3} - \frac{1}{3}\xi_{t3} - 4\epsilon_H^2 - 4\epsilon_H\delta_1 \Big|_{x=x_1}$$

$$\ln x_1 \equiv \frac{7}{3} - \ln 2 - \gamma_E$$

Optimized SR for Horndeski (leading order):

- H. Motohashi, W. Hu; 1704.01128

$$r \equiv \frac{4\Delta_\gamma^2}{\Delta_\zeta^2} \approx 16\epsilon_H \frac{b_s c_s}{b_t c_t} \approx -\frac{8b_s c_s}{b_t c_t} n_t,$$

Deviations from the standard consistency relation in the observations could be checked in this context.

Outline:

1. Scalar - Tensor theories.
2. Generalized slow-roll (GSR) techniques.
3. G-inflation.

$G_3 + \text{chaotic inflation} = G\text{-inflation}$

J. Ohashi, S. Tsujikawa; 1207.4879

$$\mathcal{L}_2 = X - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_3 = M^{-3}X\square\phi$$

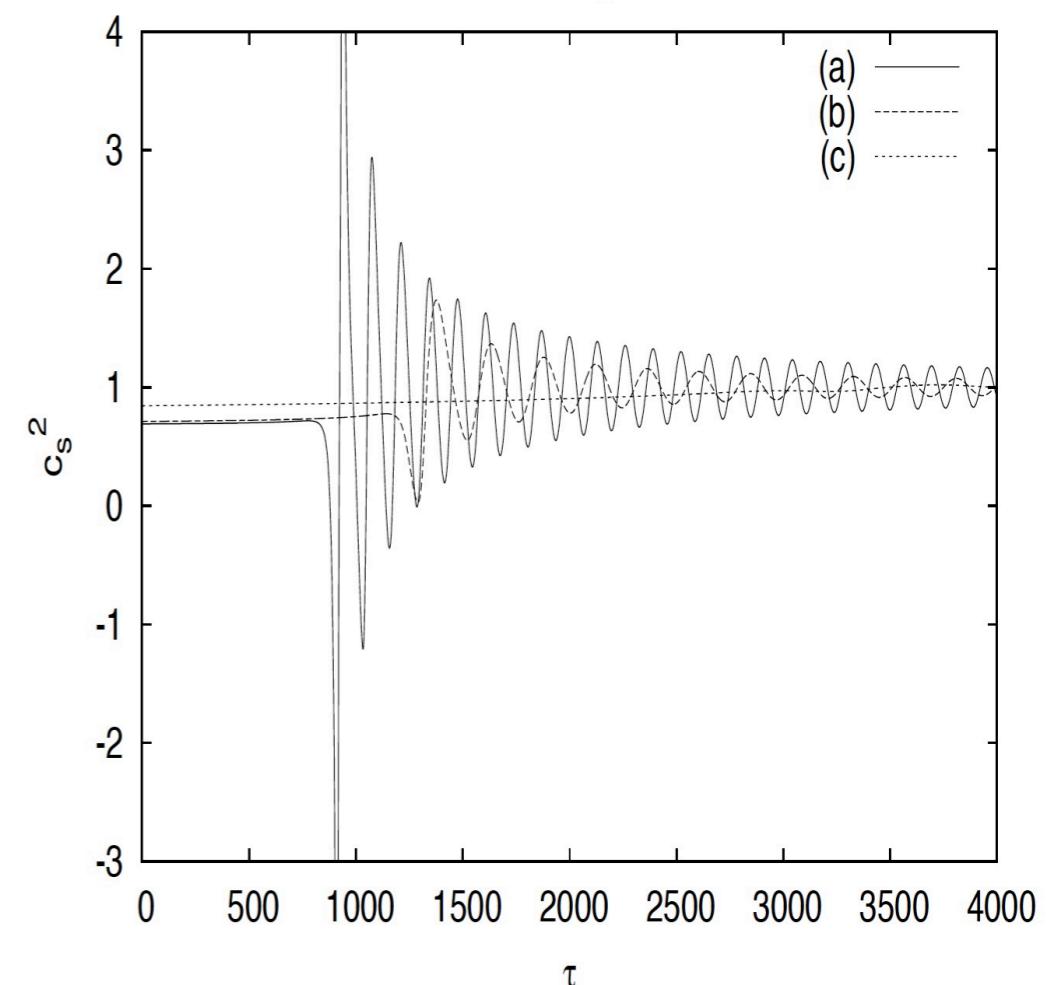
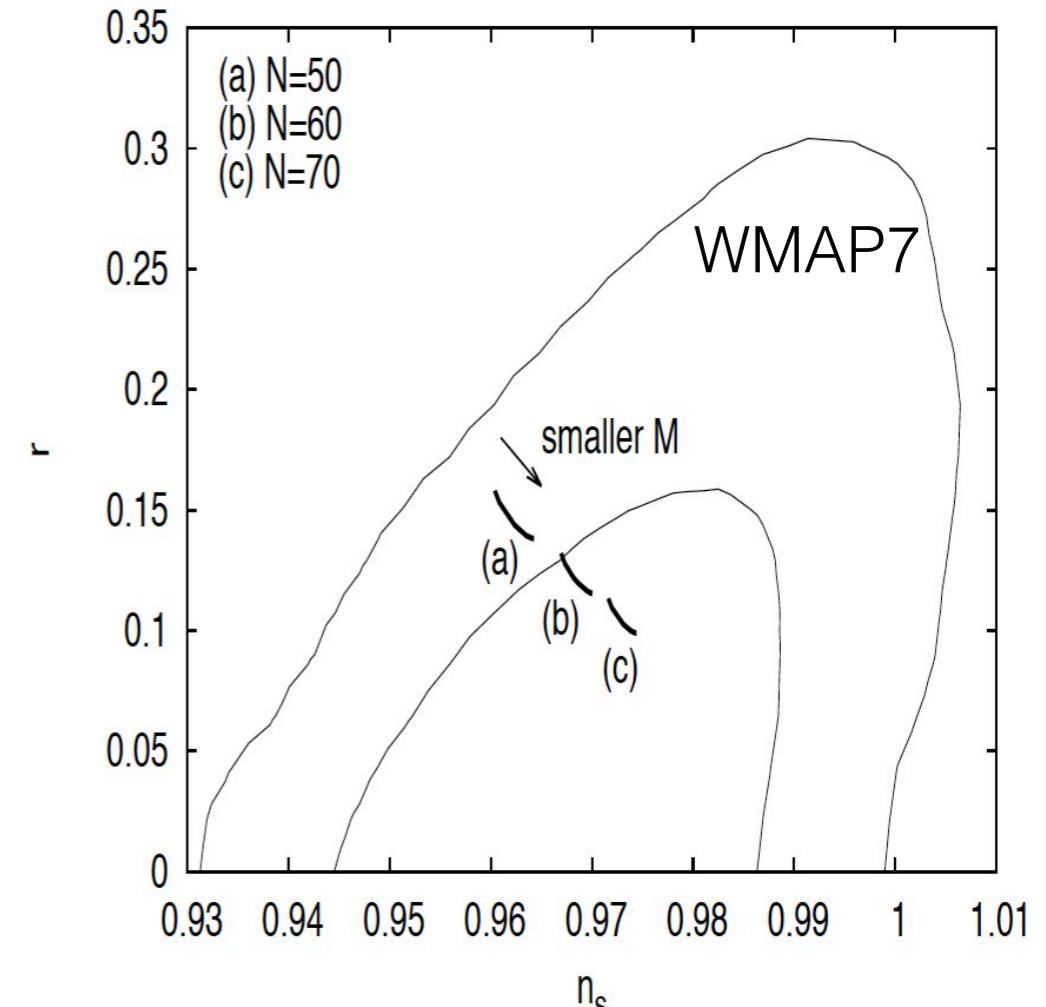
$$\mathcal{L}_4 = \frac{1}{2}M_{\text{pl}}^2\mathcal{R}$$

a) $M = 3 \times 10^{-4} M_{\text{pl}}$

b) $M = 4.2 \times 10^{-4} M_{\text{pl}}$

c) $M = 1 \times 10^{-3} M_{\text{pl}}$

For $c_s^2 > 0$, $M > 4.2 \times 10^{-4} M_{\text{pl}}$



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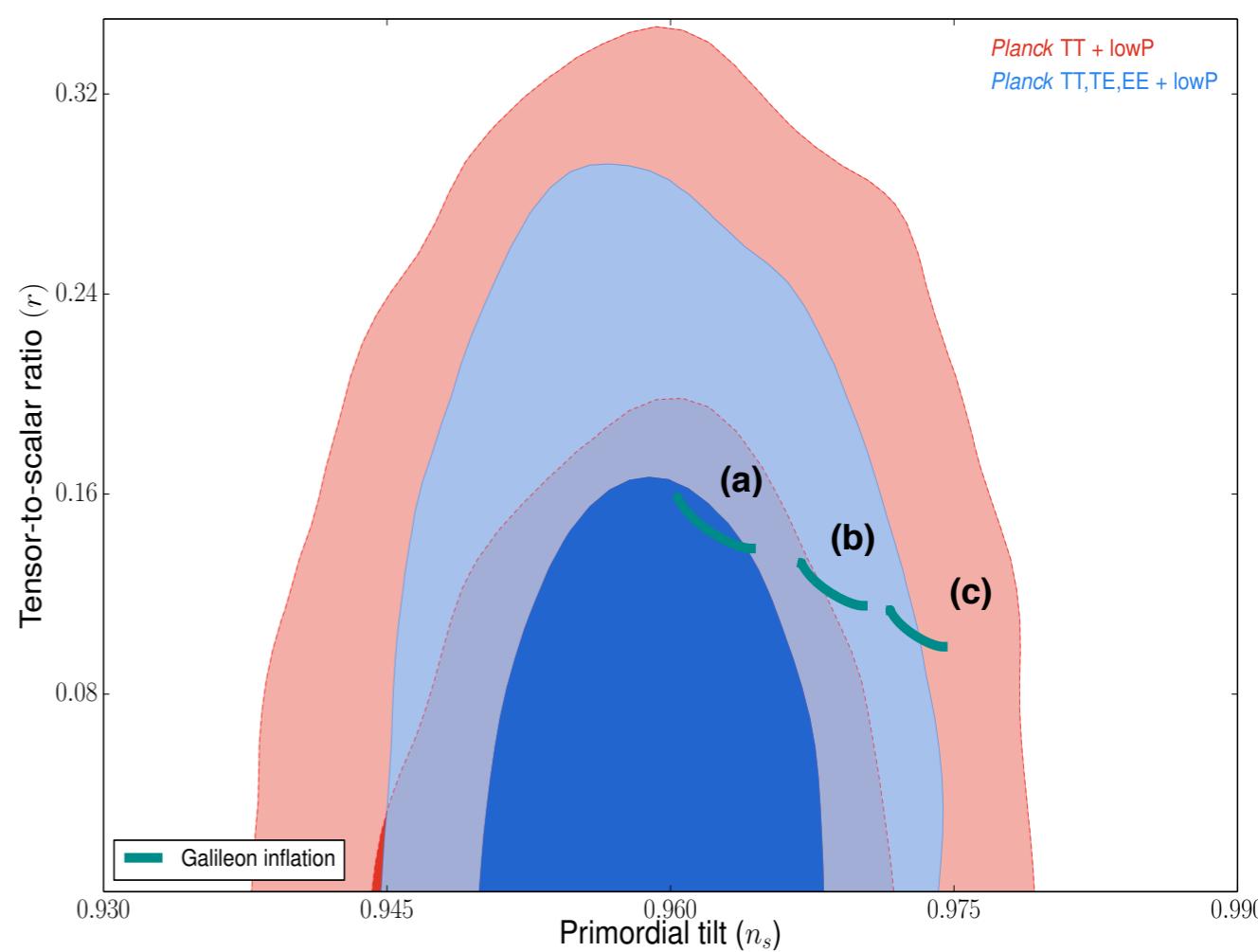
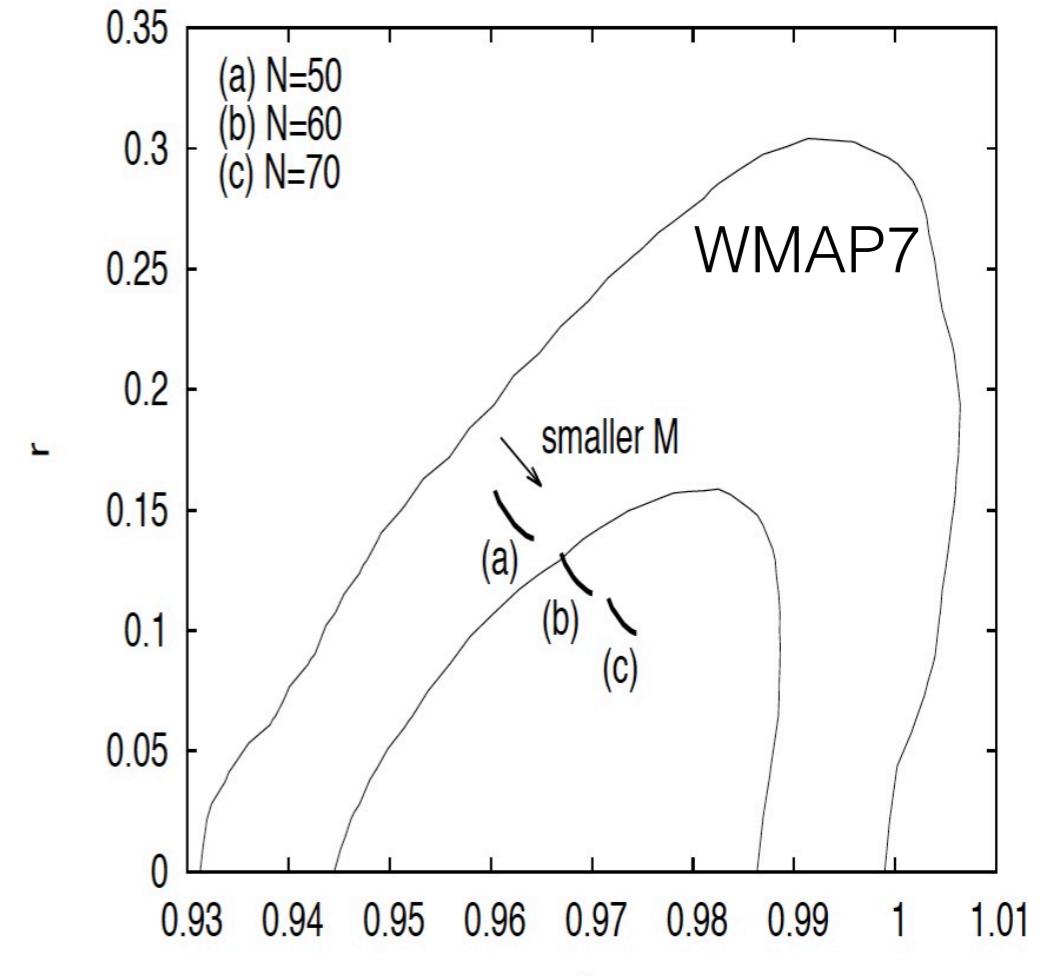
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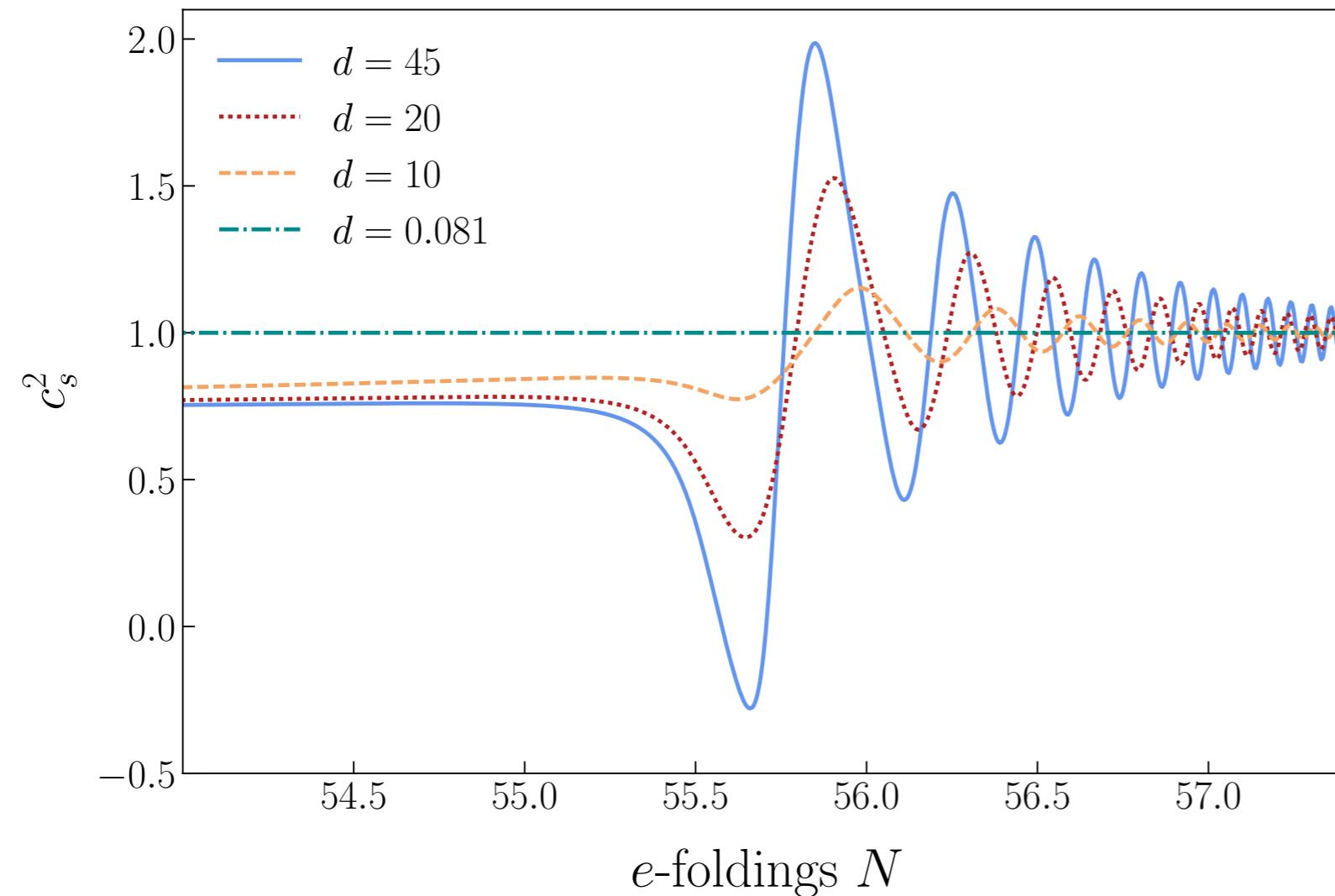
c) $M = 1 \times 10^{-3} M_{\text{pl}}$



$G_3 + \tanh +$ chaotic inflation
= transient G-inflation

HR, S. Passaglia, H. Motohashi, W. Hu, O. Mena; 1207.4879

$$\mathcal{L}_3 = M^{-3} \left[1 + \tanh \left(\frac{\phi - \phi_r}{d} \right) \right] X \square \phi$$



$$\phi_r = 13$$

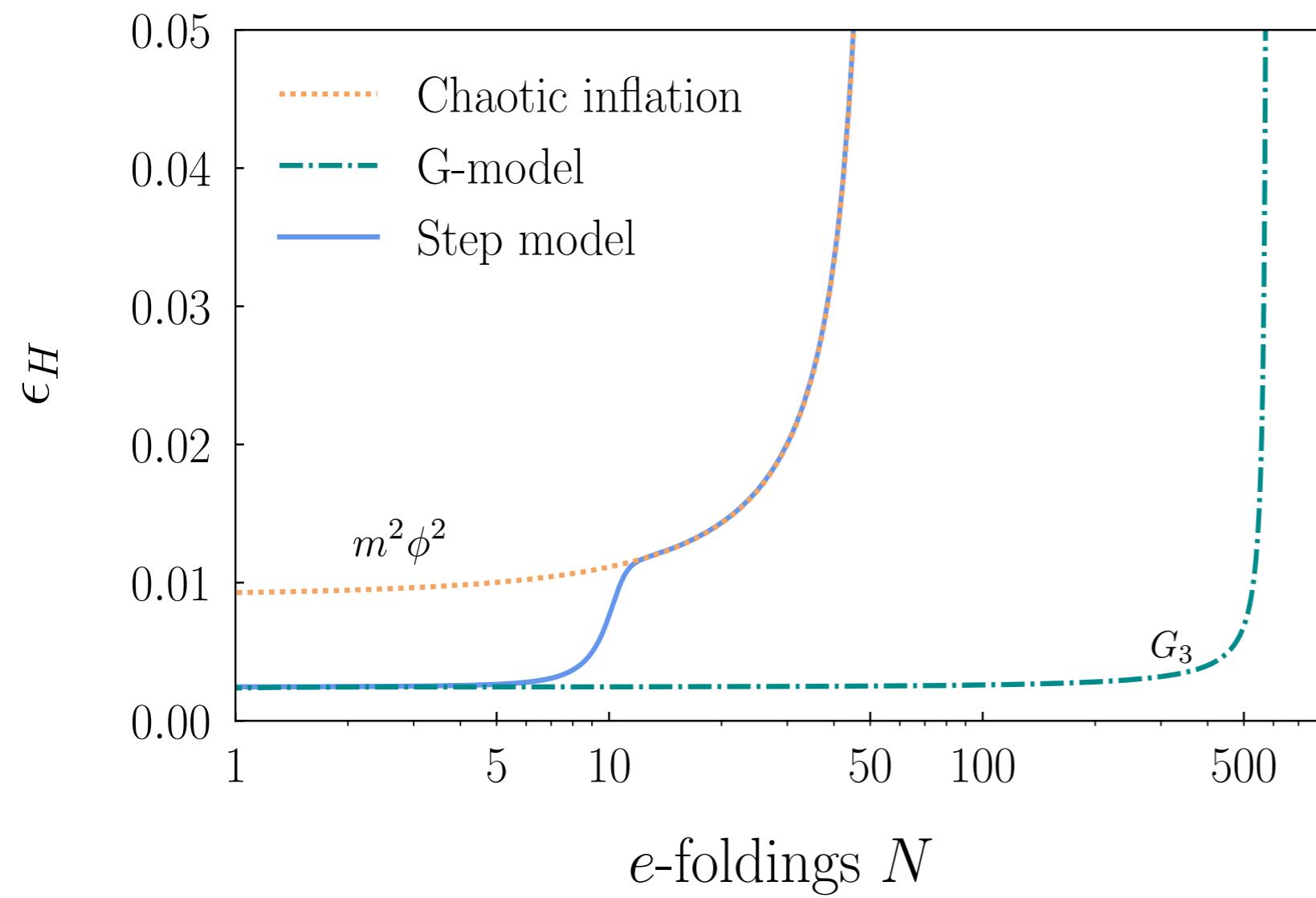
$$M = 1.3 \times 10^{-4}$$

No gradient instabilities for any mass scale M .

$G_3 + \tanh +$ chaotic inflation
= transient G-inflation

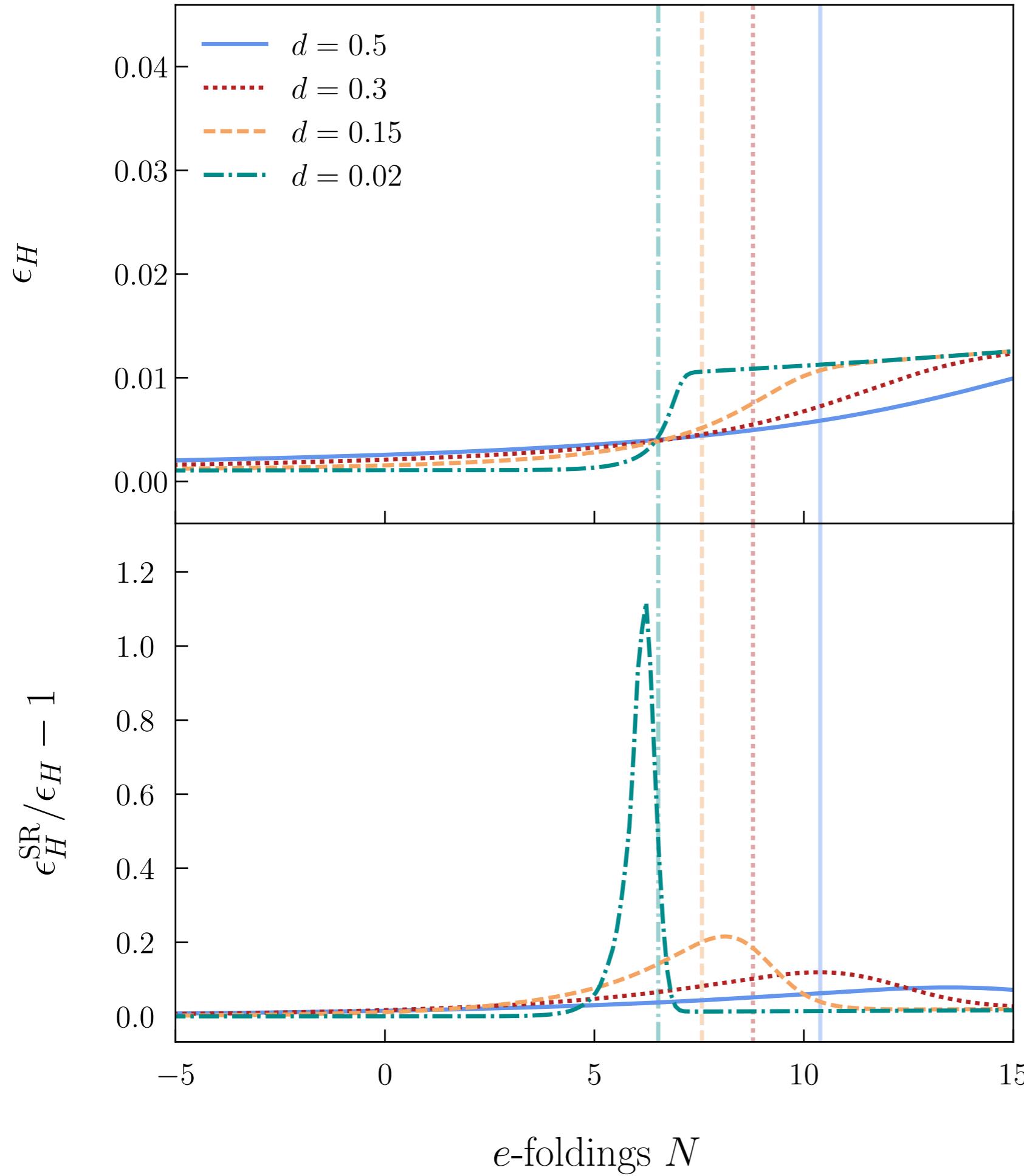
HR, S. Passaglia, H. Motohashi, W. Hu, O. Mena; 1207.4879

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Step size ~ 4 e-folds

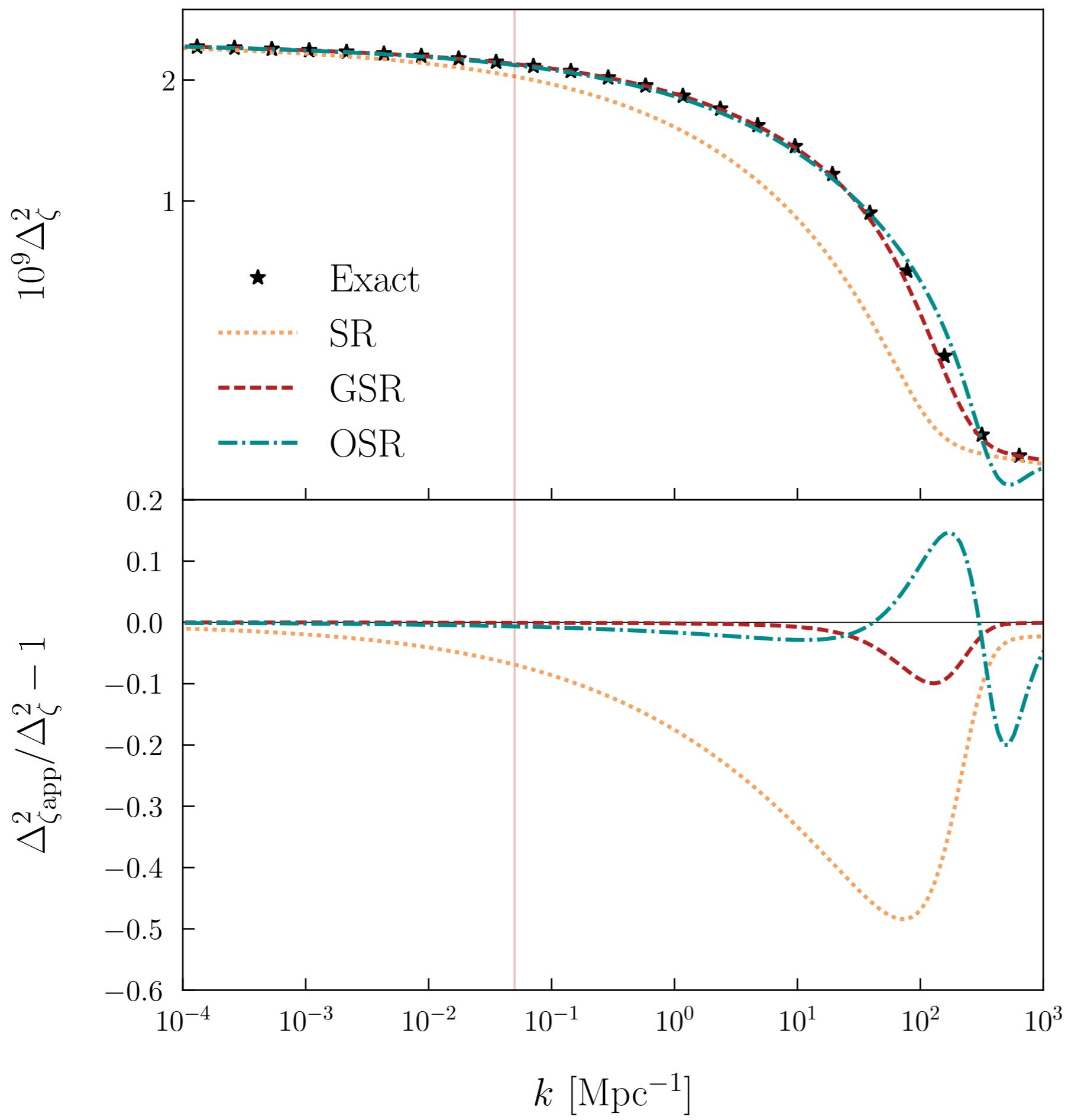
Ideal for OSR



Slow - roll violation:

$$\epsilon_H = -\frac{\dot{H}}{H^2}$$

- $N = 0$: CMB scales.
- Vertical lines: where the transition occurs.
- SR violation is maximal around the transition.



Different approximations

$$d = 0.086$$

$$M = 1.3 \times 10^{-4}$$

SR

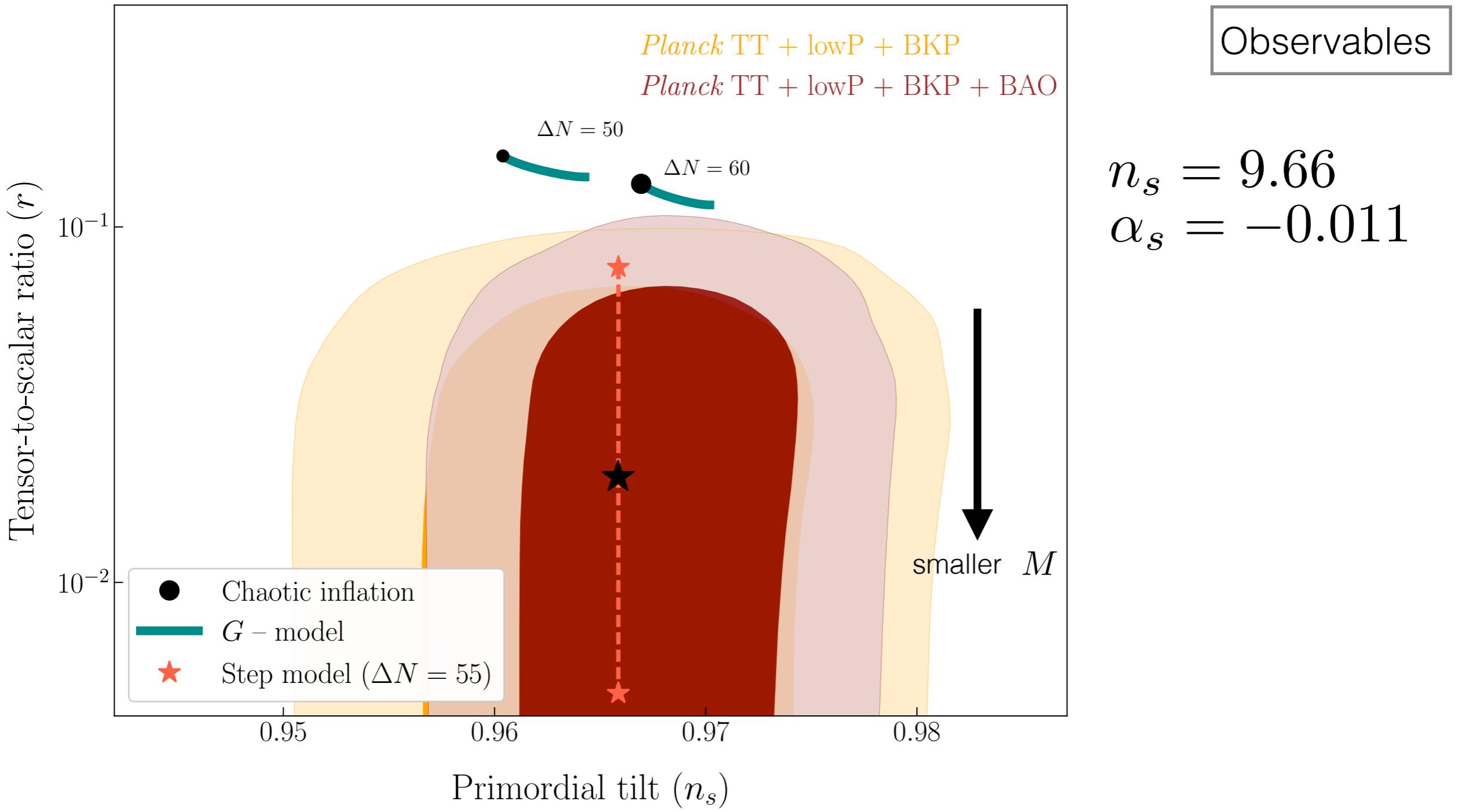


GSR

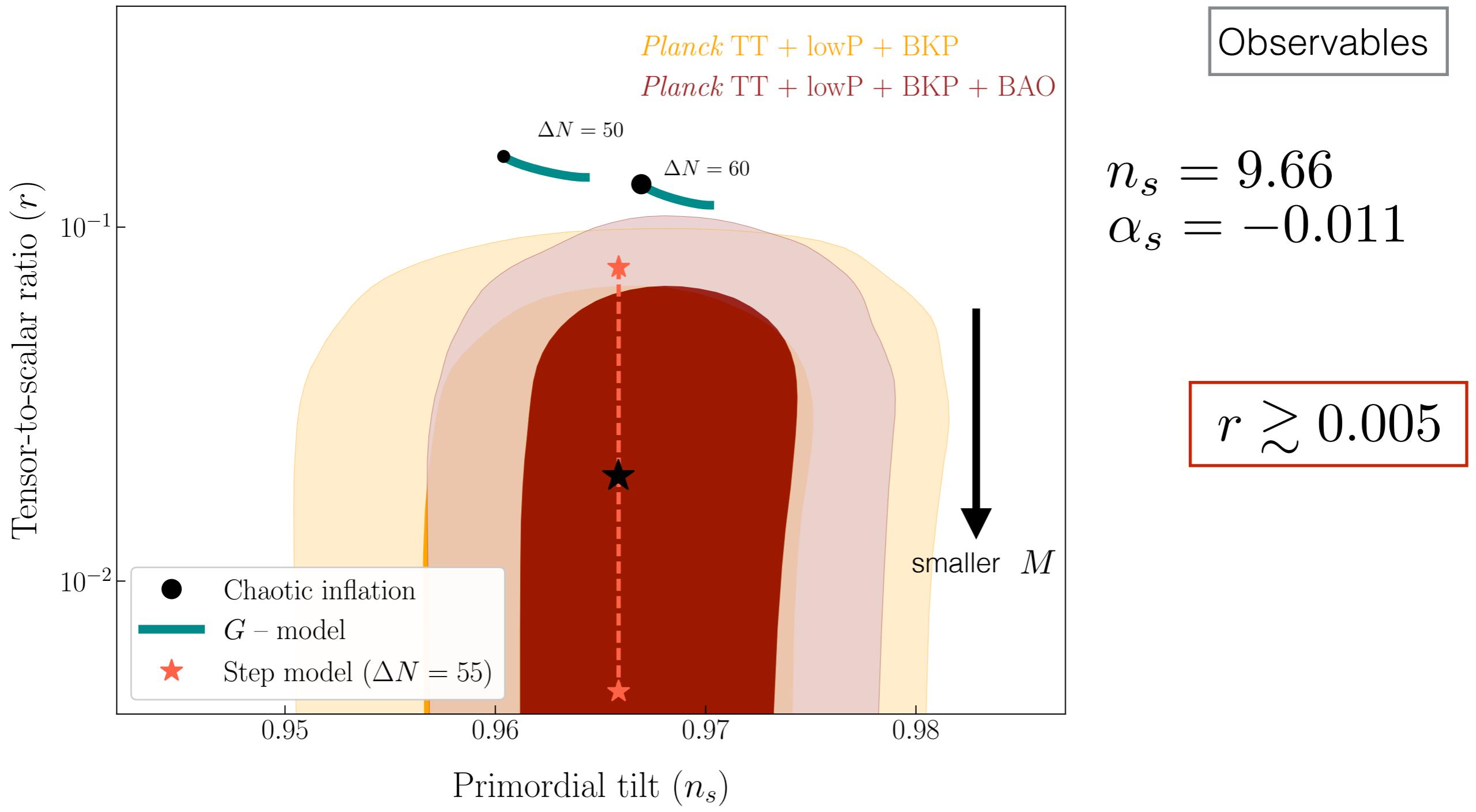


OSR

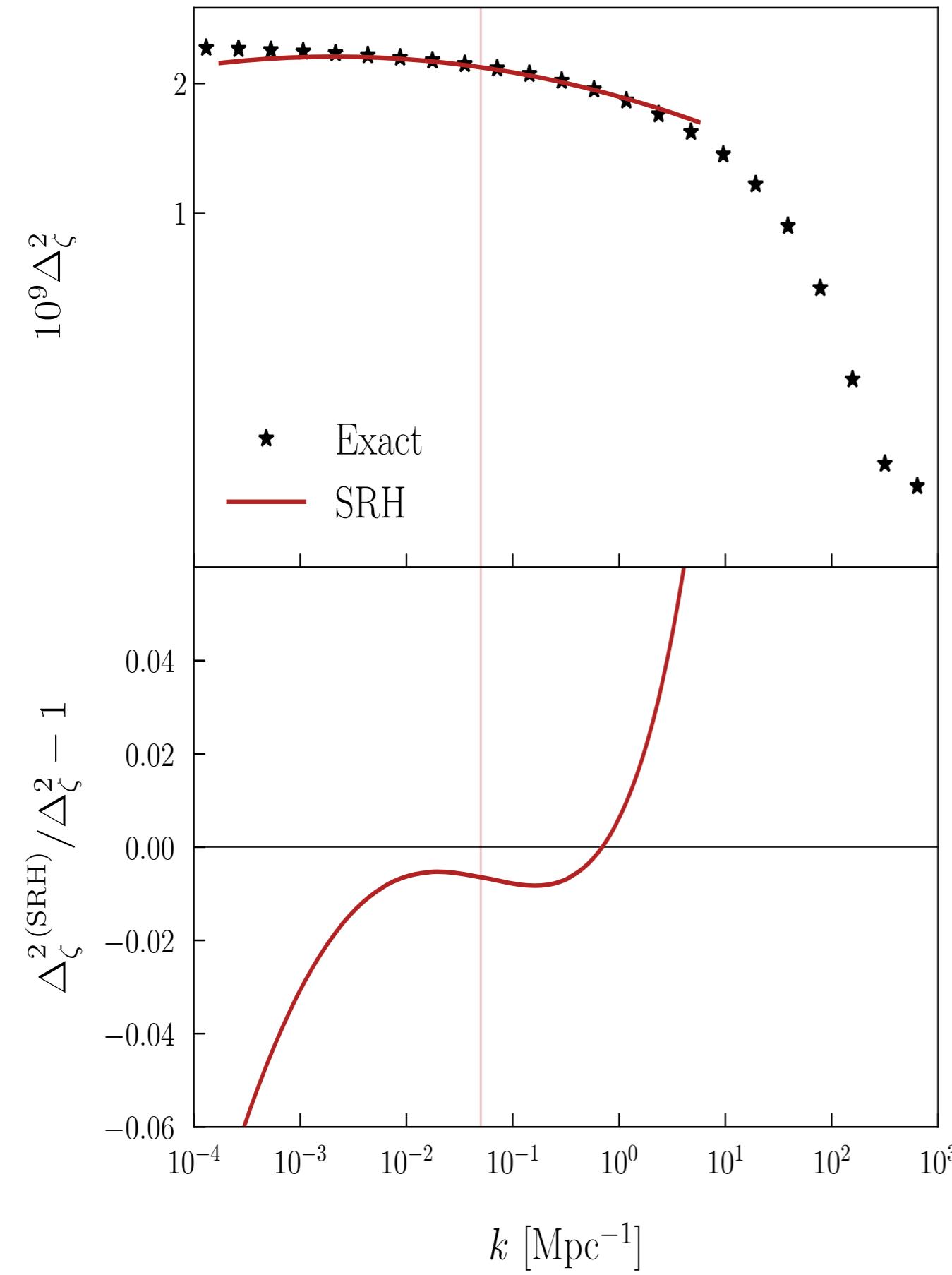




- n_s and α_s fixed.
- Find a set of values for d and ϕ_r .
- This places lower and upper bounds on r .



- A smaller α_s would shift the line upwards because the step gets wider.
- A larger α_s would be in tension with measurements.



Slow - roll hierarchy formula:

$$\Delta_\zeta^{2(\text{SRH})}(k) = A_s \left(\frac{k}{k_*} \right)^{n_s - 1 + \frac{1}{2} \alpha_s \ln(k/k_*)}$$

■ Using OSR parameters

Deviations of less
than 1%

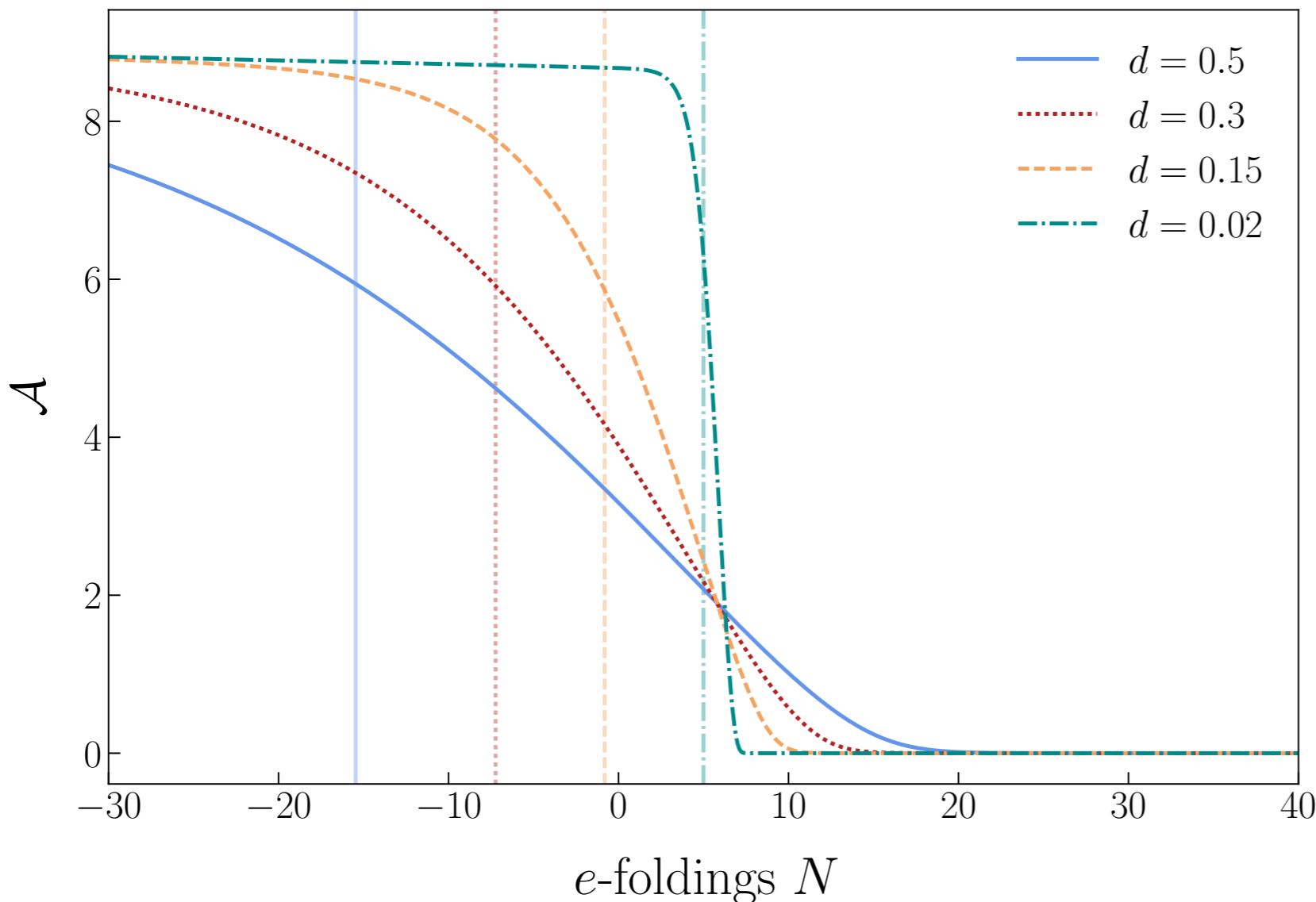
Summary

- ★ Inflation in the Horndeski framework is viable and can *cure* some popular models.
- ★ The **transient G-inflation** model allows us to compute observables during a G-inflation period and to end inflation as canonical.
- ★ Generalized slow-roll and Optimized slow-roll techniques are efficient tools for this type of models to compute the power spectra and also the bispectrum.

Back-up slides

Transition parameter

- $3H^2 \approx V,$
- $\epsilon_H \equiv -\frac{H'}{H} \approx \frac{1}{2(1+\mathcal{A})} \left(\frac{V_{,\phi}}{V}\right)^2.$
- $3H^2\phi'(1+\mathcal{A}) + V_{,\phi} \approx 0 ,$



- Vertical lines: the position of the step.