Lie Groups and Lie Algebras

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Chapter 1

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1.1 1B

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1.2 The Matrix Exponential

1.2.1 Matrix group as a metric space

Definition 1.1. Given a matrix A, a matrix norm $\|A\|$ is a nonnegative number such that

1. $\|A\| \geq 0$
2. $\|A\| = 0 \iff A = 0$
3. $\|k \cdot A\| = |k| \cdot \|A\|$ for any scalar k
4. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)
5. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Example 1.2. For a matrix $A \in M_n(\mathbb{F})$ we can define the matrix norm

$$\|A\| = \sum_{i,j=1}^{n} |A_{ij}|$$

This allows us to view $M_n(\mathbb{F})$ as a metric space. Now, just as on $\mathbb{R}^n$, we can apply notions of topology, completeness, continuity and differentiability.

Definition 1.3. A sequences of matrices $\{A_n\}_{n\in\mathbb{N}}$ in $M_n(\mathbb{F})$ *converges* to A, for which we write $\lim_{n\to\infty} A_n = A$, if

$$\lim_{n\to\infty} \|A_n - A\| = 0$$

It can be shown that this occurs if and only if $\lim_{n\to\infty}(A_n)_{ij} = A_{ij}$

Definition 1.4. A series $\sum_{n=0}^{\infty} A_n$ *converges absolutely* if $\sum_{n=0}^{\infty}\|A_n\|$ converges. When this occurs, terms in $\sum_n A_n$ can be rearranged, and we can take derivatives with respect to any parameters if present.

Definition 1.5. For a square matrix A we can define its *matrix exponential*

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Proposition 1.6. $\exp(A)$ satisfies

1. $\exp(0) = \mathbb{1}$
2. $\exp(A)^{-1} = \exp(-A)$
3. If $AB = BA$ then $\exp(A+B) = \exp(A)\exp(B)$
4. $Ce^AC^{-1} = e^{CAC^{-1}}$ for any invertible matrix C

Proposition 1.7. For any matrix $A \in M_n(\mathbb{F})$, the map $t \mapsto e^{tA}$ is a smooth curve through $\mathbb{1}$ in $M_n(\mathbb{F})$. Moreover, since $\exp(A)$ converges absolutely, we can differentiate with respect to t

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} = A \cdot e^{tA} = e^{tA} \cdot A$$

In particular

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA}\bigg|_{t=0} = A$$

Definition 1.8. Given a matrix A, another matrix B is said to be a *matrix logarithm* of A if $e^B = A$.

Proposition 1.9. For $\|A - \mathbb{1}_N\| < 1$ we have

$$\log(A) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m}(A - \mathbb{1}_N)$$

Chapter 2

Lie algebra - general structure

The main aim of this chapter will be to develop some contrasting notions of *solvable* and *semisimple* Lie algebras. Informally, solvable Lie algebras are those that are "nearly abelian", which have uninteresting and unwieldy representations. semisimple Lie algebras are those which are "far from abelian", which have interesting representations. We will see later in chapter 4 that this second type are classifiable.

Note that Representation theory will always be in sight, as this is how Lie algebras show up in perspective, and that also in a sense, solvable and semisimple Lie algebras can cover everything as any Lie algebra can be viewed as a formation of solvable and semisimple building blocks.

2.1 Basic definitions, subalgebras, ideals

Definition 2.1. A *Lie algebra* L is a vector space with a bilinear map $[\cdot,\cdot] : L \times L \to L$, satisfying

1. The anti-symmetric property

$$[x, y] = -[y, x] \qquad \forall x, y, z \in L$$

2. The Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \qquad \forall x, y, z \in L$$

Here we shall always take L to be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. For now we will consider the Lie algebras $\mathrm{Lie}(G)$ of Lie groups as $\mathbb{R}$-vector spaces.

Definition 2.2. We say that a Lie algebra L is *abelian* if $[\cdot, \cdot] = 0$

Note that if L has dimension 1 then L is abelian, due to the property of anti-symmetry.

Definition 2.3. For Lie algebras L_1 and L_2, a *Lie algebra homomorphism* over $\mathbb{F}$ is an $\mathbb{F}$-linear map $\varphi : L_1 \to L_2$ such that

$$[\varphi(x), \varphi(y)]_{L_2} = \varphi([x, y]_{L_1}) \qquad \forall x, y \in L_1$$

Definition 2.4. We call φ an *isomorphism* if is invertible and an *automorphism* if it is an isomorphism and $L_1 = L_2$

Example 2.5. The endomorphism group of a vector space V

$$\mathrm{End}(V) := \{f : V \to V \mid V \text{ is linear } \}$$

is a Lie algebra over $\mathbb{F}$ with bracket equal to the commutator of linear maps, for any $\mathbb{F}$-vector space V. We also denote this $\mathrm{gl}(V) = \mathrm{End}(V)$

A special case for this occurs when $V = \mathbb{F}^n$, where $\mathrm{gl}(V) = \mathrm{Mat}(n, \mathbb{F})$

Example 2.6. A *representation* of a Lie algebra L on a vector space V is a morphism $\varphi : L \to \mathrm{End}(V)$

Definition 2.7. For a Lie algebra L, we call a subvector space $H \subset L$ a *Lie-subalgebra* (or 'sub-Lie algebra' or 'subalgebra') if H is closed under the bracket, i.e.

$$[x, y] \in H \qquad \forall x, y \in H$$

For which we write $[H, H] \subset H$.

Definition 2.8. An *Ideal* I of a Lie algebra L is a subspace $I \subset L$ such that $[L, I] \subset I$. That is

$$[x, y] \in I \qquad \forall x \in L, \forall y \in I$$

Note that any ideal is a subalgebra.

Lemma 2.9. If $I \subset L$ is an ideal; then the quotient space

$$L/I := \{x + I \mid x \in L\}$$

carries a canonical bracket

$$[x + I, y + I] := [x, y] + I$$

and $\pi : L \to L/I$ is surjective morphism of Lie algebras. Note that $x + I := \{x + z \mid z \in I\} \subset L)$

Lemma 2.10. If $\varphi : L_1 \to L_2$ is a morphism of Lie algebras then

1. Kerφ is an ideal in L_1
2. Imφ is a subalgebra of L_2
3. $L_1/Ker\varphi$ is isomorphic to Imφ

Lemma 2.11. if I, J ideals in L $\implies I, J, I+J, [I,J]$ are also ideals in L.

Lemma 2.12. $(I+J)/J$ isomorphic $I/(I,J)$

See problem sheets for proof