Errors of Observation

extracted from the Physics Department undergraduate laboratory handbooks

This section will give you a basic overview of measurement error. A summary of how to treat errors can be found in section 6. Many of the examples are taken from 'Practical Physics' by G.L. Squires, a reference book for laboratory work [1].

1. Importance of Errors

All physical measurements (and also some theoretical calculations) of a quantity are subject to errors. We cannot make an 'exact' measurement since even the most accurate instruments have a limit to their precision. It is therefore important to give an idea of how close we believe our measured result is likely to be to the true value. We do this by including an estimate of the error along with the result. For example, we would quote the diameter, d of a ball bearing as d = (0.30 ± 0.01) cm. This means that we *expect* the diameter to be somewhere in the range 0.29-0.31 cm.

An estimation of the error is essential if we intend to draw any proper conclusions from our results. For example, consider the following measured values of the resistance of a coil at different temperatures:

200.025Ω at 10ºC and 200.034Ω at 20ºC.

We need to know whether temperature has an effect on the resistance of the coil, in other words, is the difference between the two values significant? We cannot say unless we know the error. If the error in each measurement is 0.001Ω then the difference is significant, but if it is 0.01Ω, it is not.

You might think that every measurement should be made as precisely as possible but constraints such as time and resources need to be taken into consideration (in both experimental and theoretical work). It is therefore important to plan an experiment to obtain a measurement precision that is appropriate to the final purpose. Few quantities can be measured directly so we usually have to measure several primary quantities and combine them in some way to find the final value. In this case, the errors in the primary quantities determine the error in the final result. In general, each quantity contributes different amounts to the final error and so in order to reduce the error in the final amount, it is important to plan the experiment so that time and resources are concentrated on reducing those errors that contribute most to the final error. This will be discussed further in section 5.

2. Systematic and Random Errors

There are two kinds of measurement error: systematic and random:

Systematic Errors are (generally) constant throughout a set of readings and are caused by factors which are not accounted for in the measurements or the theory. Examples are:

- A voltmeter which is not zeroed or calibrated properly so there is a constant offset or percentage error on all readings from the meter;
- heat losses in a calorimetric experiment
- poor calibration of a measuring instrument e.g. an ammeter shows a value which is 10% smaller than the value of the actual current.
- a clock running fast or slow.

Random Errors vary arbitrarily from one observation to another and are likely to be positive or negative. For this reason, a single measurement of a quantity (e.g. length or time interval) should not be relied upon. Random errors are always present in an experiment and, in the absence of systematic errors cause successive readings to be distributed about the true value (see Fig. 1a). In addition if some systematic error is present, the readings are spread about some displaced value (see Fig 1b).

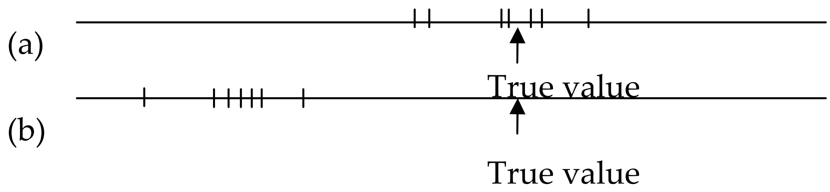

Fig 1 Set of Measurements with (a) random errors only and (b) systematic plus random errors. Each point indicates the result of a measurement. (after [1]).

A result is said to be **precise** if the random error is small and **accurate** if it is free from systematic and random error. Random errors can be detected and reduced by repeating the measurements and taking the mean of all the readings as your value. The more readings you take, the closer the mean value will approach the true value. (Note: In some cases a spread of values can represent real differences e.g. when the diameter of a wire is measured in several places) The magnitude of the random error in a value can be estimated using statistics (see section 4). However there is no easy way to detect or estimate the magnitude of systematic errors and they can only be eliminated by good experimental design. Always think about the particular method you are using to perform the experiment and be suspicious of your apparatus.

3 Error in a Single Reading

You should use your common sense to estimate the error in a single reading. Sometimes the accuracy of the instrument is known from the manufacturer or can be determined in a calibration experiment, otherwise we estimate the precision of the measurement from the markings on the scale. You can normally assume that the **precision** of your reading is to the nearest interval of the least significant digit on the display (i.e. ± ½ smallest digit on the display). However, its **accuracy** is unlikely to be that good (probably around 0.5-1%) unless it is a high quality instrument which has recently been recalibrated.

4 Error in Repeated Readings

As discussed in section 2, it is always advisable to take repeated readings of a quantity. Assuming that they are free from systematic error, the individual values (x_1, x_2 etc) will vary due to random errors and the mean value, $\bar{x}$ (i.e. arithmetic average) should be taken as the best estimate of the measured value. However, $\bar{x}$ will not necessarily be equal to the true value of the quantity and we therefore need to estimate how close we expect it to be to the true value. Common sense tells us that it depends on the spread of the individual readings. For instance if successive readings are close together then we would expect $\bar{x}$ to be close to the true value but if the readings are spread over a large range then we might expect the error in $\bar{x}$ to be large. Also the more measurements you take, the better your estimate of the mean. In fact statistical analysis is used to estimate the expected error in the mean. You will meet the detailed derivations in the Statistics course in the Summer Term but for the moment we will simply quote the formulae so that you can use them in your error analysis in the laboratory. Further details can also be found in Chapter 3 of Squires [1].

Suppose you have a set of n successive measurements of the same quantity:

$$x_1, x_2, x_3, x_4 \ldots\ldots\ldots\ldots, x_n.$$

Best estimate of the measured value is the arithmetic mean or average value of the n readings, $\bar{x}$:

$$\bar{x} = \frac{1}{n}\sum_{i}^{n} x_i = \frac{(x_1 + x_2 + x_3 + \ldots\ldots x_n)}{n} \qquad [1]$$

The error in the mean, Δx is called the **standard error**, and is derived from the **standard deviation, σ**, which is the root mean square deviation from the mean value of all the individual readings:

Standard Error in the Mean, Δx: $\Delta x = \dfrac{\sigma}{\sqrt{n}}$ where

Standard Deviation, σ: $\sigma = \sqrt{\dfrac{1}{n}\sum (x_i - \bar{x})^2}$

The standard deviation, σ represents the error in a single measurement. The standard error in the mean, Δx, represents the error in the mean of n measurements and is 1/√n times the error in a single reading. The easiest way to calculate σ is to use your scientific calculator in the statistics mode and to enter your individual readings into the memory. Use the σ_n and $\bar{x}$ buttons which will display the standard deviation and mean of your values respectively. Alternatively if you tabulate readings of x and x^2, σ can be calculated easily from the **variance, σ²**:

Variance, σ^2: σ^2 = (Mean of the Squares) – (Square of the Mean)

$$\sigma^2 = \overline{x^2} - \bar{x}^2$$

Example

In an experiment the value for the acceleration due to gravity, g, was measured 11 times. The best estimate of the measured value of g is the mean. The error in the mean is the standard error, Δx, which in the absence of a scientific calculator can be calculated from the variance as described above:

x	9.9	9.8	10.2	9.8	10	9.9	10.5	11.4	10.2	8.4	11.2
x^2	98.01	96.04	104.04	96.04	100.00	98.01	110.25	129.96	104.04	70.56	125.44

Mean value of g = $\sum x/N$ = 111.32/11 = 10.12 ms^{-2} and Square of Mean =10.12^2 = 102.4

Mean of the squares = $\sum(x^2)/N$ =1133.4/11 = 103.0 so variance, σ^2 = 103.0-102.4 = 0.6

Therefore the standard deviation, σ = 0.8 and the standard error in the mean, $\Delta x = 0.8/\sqrt{11} = 0.2$

Finally we would quote our measured value as **g = (10.1± 0.2)⚜ms^{-2} (2% error)**

What does the standard error really tell us about where the true value lies?

The statistical interpretation of the standard error is that there is ~67% chance of the true value lying within 1 standard error of the measured mean value and ~95% chance of it lying within 2 standard errors. ***Note:*** increasing the readings reduces the standard error but only at the rate of $1/\sqrt{n}$ so time and resources often dictate the optimum number of readings to take.

5 Combining Errors

When several different quantities are measured, each with its associated error, and combined to calculate a final result, then we need to know how to combine the individual errors to obtain the error in the final result. If we simply added the errors together then this would produce an overestimate of the actual error because errors can be positive and negative and therefore some errors will reduce or cancel the effect of other errors. This can be taken into account by adding the errors in **quadrature** i.e. add squared terms.

The most important piece of advice is to **use your 'common sense' when combining errors**! Often we end up measuring a large number of primary quantities in order to calculate a final result which can result in a very laborious error calculation. Always remember *why* you are estimating the error – in order to provide a measure of the precision of the final result so you usually only need to find the error to within a factor of 2. Since errors are added in quadrature, i.e. add squared terms, one error is often negligible compared to another and so can be ignored. We will illustrate what this means by example in the following set of rules (look out for the ⓘ symbol):

a. Sum and Difference Z = A + B or Z = A – B

We add the squares of the '***absolute errors***':

The absolute error in Z, ΔZ is: $\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$

where ΔA and ΔB are the errors in A and B.

ⓘ ***For a difference, the error can be very big*** e.g. if A=10±1 and B=8±1, then Z=2±1.4.

ⓘ ***You can nearly always ignore errors that are less than 1/3 of the largest error as it will only affect the 2nd significant figure of the error.***

e.g. suppose ΔA=2 and ΔB=1. $\Delta Z = \sqrt{(2^2+1^2)} = 2.24$. So even though ΔB is as much as 1.5 times ΔA, ignoring B and putting ΔZ ~ΔA =2 makes a difference of only 1 part in 8 to the final error! We can therefore safely ignore the error in B.

ⓘ ***Also notice what happens when the quantities themselves differ greatly in magnitude*** e.g. if Z=A+B where A=100±6 and B=5±ΔB. The error in B, ΔB will be negligible unless it is as much as 3 (i.e. 60% of B!). In other words you do not need to measure B very precisely.

b. Products and Quotients Z = AB or Z = A/B

We add the squares of the '***fractional errors***':

The fractional error in Z, $\frac{\Delta Z}{Z}$ is: $\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$

ⓘ ***Any constants in your expression have negligible errors and can therefore be ignored.***

ⓘ ***Remember that your errors are estimates so do not calculate to high precision. It is easiest to quote each fractional error as a percentage.***

e.g. suppose R=10±1Ω and I=8±1A, then V=IR=80 Volts and

$$\frac{\Delta V}{V} = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{8}\right)^2} = \sqrt{(10\%)^2 + (12.5\%)^2} \therefore \frac{\Delta V}{V} = 16\% \quad \therefore \text{V} = 80 \pm 13 \text{ Volts.}$$

ⓘ ***You can nearly always ignore fractional (or percentage) errors that are less than 1/3 of the largest fractional error as it will only affect the 2nd significant figure of the error.***

e.g. if the error in A were 3% and the error in B were 1%, then

$\Delta Z\% = \sqrt{(3\%)^2 + (1\%)^2} = \sqrt{10}\%$. So ΔZ = 3% to 1 s.f. and thus we could have dropped the 1% error as it has a negligible effect on the final error.

3. Power Z = A^n

We multiply the '***fractional error***' in A by the power.

So the fractional error in Z, $\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$

ⓘ ***Since the error is multiplied by the power it is often dominant and you usually therefore need to measure any quantity raised to a power to a higher precision than the other quantities.***

e.g. Suppose Z=A^3B and both ΔA and ΔB = 1%.

$\frac{\Delta Z}{Z} = \sqrt{\left(3.\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2} = \sqrt{(3\%)^2 + (1\%)^2} = \sqrt{10}$. So $\frac{\Delta Z}{Z}$=3% to 1 s.f. and it is clear that the error in A has dominated the error.

4. General function $Z = f(A, B......)$

If the function is a combination of the different types mentioned above (e.g. a combination of a sum and a product), do the calculation in stages:

e.g. Consider the function $Z = \frac{A - B}{C.D}$. Let $Z = \frac{P}{C.D}$ where P = A- B.

Work out the value of ΔP first using $\Delta P = \sqrt{(\Delta A)^2 + (\Delta B)^2}$

and then calculate $\frac{\Delta Z}{Z}$ using $\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta P}{P}\right)^2 + \left(\frac{\Delta C}{C}\right)^2 + \left(\frac{\Delta D}{D}\right)^2}$

ⓘ **Remember to ignore all errors whose contribution to the final error is less than 1/3 of largest contribution.** For instance, suppose that A = 50±5, B=5±1, C=4.05±0.05 and D=10±1. ΔB can be ignored since ΔA is five times larger than ΔB and $\Delta P = \sqrt{(\Delta A)^2 + (\Delta B)^2}$. So ΔP = ΔA (to 1 s.f.) = 5 and P=A-B=45. So $\frac{\Delta P}{P}$ = 0.1, $\frac{\Delta C}{C}$=1% and $\frac{\Delta D}{D}$=10% and so we can ignore the % error in C as it is less than 1/3 of the largest % error. So $\frac{\Delta Z}{Z} = \sqrt{(0.1)^2 + (0.1)^2}$ = 14%.

ⓘ **Another example is given in table 1.**

Otherwise use the **general rule**, from which all the examples in section 5 have been derived. If Z is some function of A, B, etc, and the standard error in A is ΔA etc, then the standard error in Z, ΔZ is:

$$\Delta Z = \sqrt{(\Delta Z_A)^2 + (\Delta Z_B)^2 + (\Delta Z_B)^2 + } \ldots\text{etc}$$

where $\Delta Z_A = \left(\frac{\partial f}{\partial A}\right)\Delta A$ and so on….

ⓘ ***Note that this can produce very complicated expressions and it is often quicker to calculate the error (ΔZ$_A$ etc) due to each variable (A, etc) in the final quantity.*** To do this calculation Z using the mean values of each of the variables ($\overline{A}, \overline{B}$ etc), and then calculate Z again for A=$\overline{A} + \Delta A$, B, C etc...and the difference will give you ΔZ$_A$.

Calculate ΔZ_B etc similarly and then calculate the final error using the expression for ΔZ above. See Squires, Section 5.2 for an full explanation of this method [1].

6 Summary of Treatment of Errors

1. Strategy – Identify the quantities contributing most to the final error by performing a rough error calculation ***before beginning your measurements.*** Try to reduce them, either by making more measurements or by using a different method to measure each quantity.

> **Error in Single Reading** – take precision of reading to be ±½ smallest division or digit on instrument or the manufacturer's quoted accuracy of the instrument, whichever is larger.
>
> **Error in Repeated Readings –** take mean value as best value and use **standard error ($\sigma/\sqrt{n}$)** as error unless readings show little or no spread in values compared to smallest division on instrument. If so, take error as for single reading if you cannot find a more precise way of measuring the quantity.

2. Calculation

a. An estimate of the error in the final result to 1 part in 4 is adequate so do the error calculation to **1 significant figure** (or at most 2).
b. Ignore all errors whose contribution to the final error is less than 1/3 of largest contribution.
c. Combine all errors that remain using table 1 as a guide. Use percentages.
d. The final error should be equal to, or larger than, the largest contribution. If it is less or much large than this, you have probably made a mistake.

3. Final Result

a. Quote the result and error to the same number of decimal places, **with units**.
b. Only quote your estimate of the error to 1 significant figure (at most 2 but this is usually only done when the error is '1' or '2' units
c. In addition, quote the error as a percentage of the result in brackets.

e.g.

Measured value of Young's Modulus, E = (2.03 ± 0.02) $*10^{11}$ Nm (1% error)

References

[1] G. L. Squires, "Practical Physics", 4th edition (Cambridge University Press) 2001.

Table 1 Combination of Errors (each can be derived using general rule)

Relation between Z and A,B	**Relation between Standard Errors**
Z = Ca + Kb Z = Ca – Kb (c,k are constants)	$\Delta Z = \sqrt{(c\Delta A)^2 + (k\Delta B)^2}$ Ignore any error less than 1/3 of largest error.
Z = Cab Z = Ca/B where c is a constant	$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$ Ignore any fractional error <1/3 of largest fractional error.
Z = c A^n where c is a constant	$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$ Usually dominates error so measure precisely.
Z = c ln A where c is a constant Z = c exp(Ka) (c,k are constants)	$\Delta Z = c\frac{\Delta A}{A}$ $\frac{\Delta Z}{Z} = k\Delta A$
Z = c sin(Ka) Z = c cos(Ka) Z = c tan(Ka) (c,k are constants)	ΔZ = c k cos(Ka)ΔA ΔZ = c k sin(Ka)ΔA **Angles MUST be in radians** ΔZ = c k $\sec^2$(Ka)ΔA
A combination of the above: e.g. Z = 5A^n (4B^2+ C^3)	Do in stages, ignoring "non-contributing" errors as above: 1. Let Y = 4B^2 + C^3 so Z = 5A^nY. Work out Y and ΔY first using $(\Delta Y)^2 = (4 \times 2\Delta B)^2 + (3\Delta C)^2$. Then: 2. $\frac{\Delta Z}{Z} = \sqrt{\left(n\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2}$
General function: $Z = f(A, B, C......)$	$\Delta Z = \sqrt{(\Delta Z_A)^2 + (\Delta Z_B)^2 + (\Delta Z_B)^2 +}$ ….etc where $\Delta Z_A = \left(\frac{\partial f}{\partial A}\right)\Delta A$ and so on….