

# Home Assignment 1

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## Foreword

If there is something fishy about my calculations, you can probably and hopefully find it in the Collection of Formulas.

## Solutions

1. (a) *"Recursive systems are never stable"*

**False.** Recursive systems are IIR systems e.g.

$$y(n) = \frac{1}{2}y(n-1) + x(n)$$

If we by stability mean BIBO-stability we can simply prove that the system above is stable. The Z-transform of the system above is

$$\begin{aligned} Y(z) &= \frac{1}{2}z^{-1}Y(z) + X(z) \Rightarrow Y(z)(1 - \frac{1}{2}z^{-1}) = X(z) \Rightarrow \\ Y(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}}X(z) = H(z)X(z) \Rightarrow H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \mapsto \\ h(n) &= (\frac{1}{2})^n u(n) \end{aligned}$$

By definition the system above is stable if  $\sum_{n=0}^{\infty} |(\frac{1}{2})^n| < \infty$ . This is true because  $(\frac{1}{2})^n$  goes to zero when  $n$  goes to  $\infty$ , which means that the sum converges.

- (b) *"A convolution of two sequences in the time domain corresponds to a multiplication of the Z transforms of the signals."*

**True.** If we look at the definition and the statement:

$$\begin{aligned} \text{Statement: } & a(n) = b(n) * c(n) \mapsto A(z) = B(z)C(z) \\ \text{Defenitions: } & b(n) * c(n) = \sum_k b(k)c(n-k) \\ & A(z) = \sum_n a(n)z^{-n} \end{aligned}$$

Lets start with the last defenition:

$$\begin{aligned} A(z) &= \sum_n a(n)z^{-n} = \sum_n \left[ \sum_k b(k)c(n-k) \right] z^{-n} = \sum_k b(k) \left[ \sum_n c(n-k) \right] z^{-n} z^k z^{-k} \\ &= \sum_k b(k)z^{-k} \sum_n c(n-k)z^{-(n-k)} = B(z)C(z) \end{aligned}$$

- (c) "An FIR filter is always stable."

**True.** A FIR filter is always *BIBO stable* and can be described with a difference equation:

$$y(n) = \sum_{k=0}^N a_k x(n-k)$$

The system is always stable as every  $a_n < \infty$  and  $N < \infty$

- (d) "A first order IIR filter is stable iff the absolute value of the value of the factor in front of  $y(n-1)$  is greater than 1."

**False.** The opposite is true. We can prove this by contradiction: If we assume the statement above is true we get the following:

$$y(n) = Ay(n-1) + x(n), A > 1$$

and

$$\sum_k |h(k)| < \infty$$

Here  $A$  denotes the "factor in front of  $y(n-1)$ ".

Lets do the same steps as in 1a):

$$y(n) = Ay(n-1) + x(n) \Rightarrow h(n) = A^n u(n)$$

At last we get

$$\sum_k |h(k)| < \infty \Leftrightarrow \sum_{n=0}^{\infty} |A^n| < \infty, (A > 1)$$

Which can't be true as the sum  $\sum_{n=0}^{\infty} |A^n| \rightarrow \infty$  as  $A > 1$ .

- (e) "An IIR-filter is never a linear-phase system."

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2. "A discrete-time system is described by the difference equation:"

$$y(n) - y(n-1) + \frac{2}{9}y(n-2) = x(n)$$

- (a) "Determine the system function  $H(z)$  and the impulse response  $h(n)$  for the system and draw a pole-zero plot. Is the system stable?"

$$y(n) - y(n-1) + \frac{2}{9}y(n-2) = x(n) \mapsto Y(z) - Y(z)z^{-1} + \frac{2}{9}Y(z)z^{-2} = X(z) \Rightarrow$$

$$Y(z)(1 - z^{-1} + \frac{2}{9}z^{-2}) = X(z) \Rightarrow Y(z) = \frac{1}{1 - z^{-1} + \frac{2}{9}z^{-2}}X(z) = H(z)X(z)$$

Now we have the system function. To get the impulse response we just need to inverse transform the system function.

$$H(z) = \frac{1}{1 - z^{-1} + \frac{2}{9}z^{-2}} = \frac{z^2}{z^2 - z + \frac{2}{9}} = \frac{z^2}{(z - \frac{2}{3})(z - \frac{1}{3})} = 1 + \frac{z - \frac{2}{9}}{(z - \frac{2}{3})(z - \frac{1}{3})} =$$

$$\{PF\} 1 + \frac{\frac{4}{3}}{z - \frac{2}{3}} - \frac{\frac{1}{3}}{z - \frac{1}{3}} = 1 + z^{-1} \frac{\frac{4}{3}}{1 - \frac{2}{3}z^{-1}} - z^{-1} \frac{\frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$$

$$\mapsto h(n) = \delta(n) + \frac{4}{3} \left(\frac{2}{3}\right)^{n-1} u(n-1) - \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u(n-1) = \delta(n) + \left(2 \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n\right) u(n-1)$$

We can see that the system is stable in 2 ways: On the impulse response, where the potents of  $n$  is  $< 1$  and in the graph, where the poles are inside the unit circle.

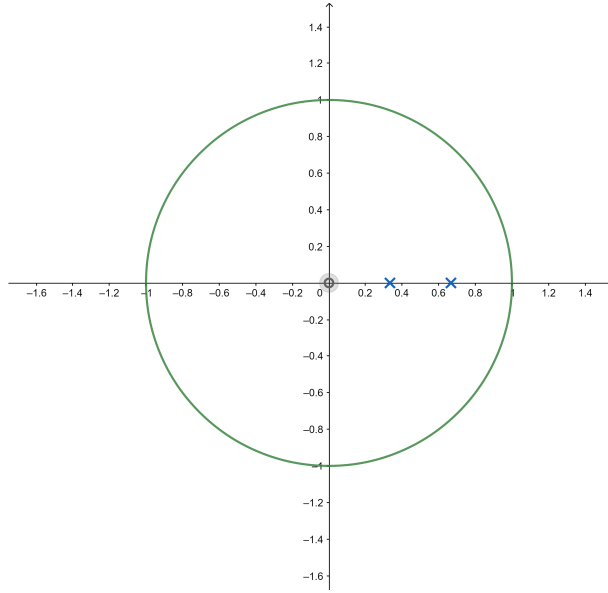


Figure 1: Poles and zeros of the system

(b) "The following signal is the input signal to the system"

$$x(n) = 3\left(\frac{1}{2}\right)^n u(n)$$

"where  $u(n)$  is the step function. Solve the difference equation by using the Z-transform, that is, determine a closed form expression for  $y(n)$ ."

We know by definition that the output from the system is equal to the convolution of the input and the impulse response.

$$y(n) = h(n) * x(n) \iff Y(z) = H(z)X(z)$$

When  $Y(z)$  is calculated, we've got our answer.  $X(z) = 3 \frac{1}{1 - \frac{1}{2}z^{-1}}$

$$\begin{aligned} H(z)X(z) &= \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})} * 3 \frac{1}{1 - \frac{1}{2}z^{-1}} = \{PF\} \left( -\frac{8}{5} \right) \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{22}{5} \frac{1}{1 - \frac{2}{3}z^{-1}} - \frac{14}{5} \frac{1}{1 - \frac{1}{2}z^{-1}} \mapsto \\ &\mapsto y(n) = \left( \frac{22}{5} \left( \frac{2}{3} \right)^n - \frac{8}{5} \left( \frac{1}{3} \right)^n - \frac{14}{5} \left( \frac{1}{2} \right)^n \right) u(n) \end{aligned}$$

We can see that the system is stable as every base in the potents of  $n$  are less than 1.

3. "The figures below show four pole-zero plots and four impulse responses."

(a) "Pair the correct plot A, B, C, D with the corresponding impulse response 1, 2, 3, 4."

A	1
B	3
C	4
D	2

(b) "Pair the correct plot A, B, C, D with the corresponding statement I, II, III, IV."

A	I
B	III
C	II
D	IV