Home Assignment 1

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Foreword

If there is something fishy about my calculations, you can probably and hopefully find something that looks somewhat like it in the Collection of Formulas(*CoF*).

Solutions

1. (a) *"Recursive systems are never stable"*
   False. Recursive systems are IIR systems e.g.

$$y(n) = \frac{1}{2}y(n-1) + x(n)$$

If we by stability mean *BIBO stability* we can simply prove that the system above is stable. The Z-transform of the system above is

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + X(z) \Rightarrow Y(z)(1 - \frac{1}{2}z^{-1}) = X(z) \Rightarrow$$

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}X(z) = H(z)X(z) \Rightarrow H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \longmapsto$$

$$h(n) = (\frac{1}{2})^n u(n)$$

By definition the system above is stable if $\sum_{n=0}^{\infty} |(\frac{1}{2})^n| < \infty$. This is true because $(\frac{1}{2})^n$ goes to zero when n goes to ∞, which means that the sum converges.

(b) *"A convolution of two sequences in the time domain corresponds to a multiplication of the Z transforms of the signals."*
True. If we look at the definitions(from the *CoF*) and the statement:

$$\begin{array}{rl} \text{Statement:} & a(n) = b(n) * c(n) \longmapsto A(z) = B(z)C(z) \\ \text{Defenitions:} & b(n) * c(n) = \sum_k b(k)c(n-k) \\ & A(z) = \sum_n a(n)z^{-n} \end{array}$$

Using the definitions to get confirm the statement:

$$A(z) = \sum_n a(n)z^{-n} = \sum_n \left[\sum_k b(k)c(n-k)\right] z^{-n} = \sum_k b(k) \left[\sum_n c(n-k)\right] z^{-n}z^k z^{-k}$$

$$= \sum_k b(k)z^{-k} \sum_n c(n-k)z^{-(n-k)} = B(z)C(z)$$

(c) *"An FIR filter is always stable."*
True. A FIR filter is always *BIBO stable* and can be discribed with a difference equation:

$$y(n) = \sum_{k=0}^{N} a_k x(n-k)$$

The system is always stable as every $a_n < \infty$ and $N < \infty$

(d) *"A first order IIR filter is stable iff the absolute value of the value of the factor in front of $y(n-1)$ is greater than 1."*
False. The opposite is true. We can prove this by contradiction: If we assume the statement above is true we get the following:

$$y(n) = Ay(n-1) + x(n), A > 1$$
$$and$$
$$\sum_k |h(k)| < \infty$$

Here A denotes the *"factor in front of $y(n-1)$"*.
Lets do the same steps as in *1a*:

$$y(n) = Ay(n-1) + x(n) \Rightarrow h(n) = A^n u(n)$$

At last we get

$$\sum_k |h(k)| < \infty \Leftrightarrow \sum_{n=0}^{\infty} |A^n| < \infty \Leftrightarrow \infty < \infty$$

Here we get a contradiction which means the inital statement is false and the opposite is true. In other words; **False.**

(e) *"An IIR-filter is never a linear-phase system."* **True.**
It is well known that only FIR systems have the ability to have linear phase. For a system to have linear phase it needs a symmetric impulse response, an attribute a IIR cannot posses.

2. *"A discrete-time system is described by the difference equation:"*

$$y(n) - y(n-1) + \frac{2}{9}y(n-2) = x(n)$$

(a) *"Determine the system function $H(z)$ and the impulse response $h(n)$ for the system and draw a pole-zero plot. Is the system stable?"*

$$y(n) - y(n-1) + \frac{2}{9}y(n-2) = x(n) \longmapsto Y(z) - Y(z)z^{-1} + \frac{2}{9}Y(z)z^{-2} = X(z) \Rightarrow$$
$$Y(z)(1 - z^1 + \frac{2}{9}z^{-2}) = X(z) \Rightarrow Y(z) = \frac{1}{1 - z^{-1} + \frac{2}{9}z^{-2}}X(z) = H(z)X(z)$$

Now we have the system function. To get the impulse response we just need to inverse transform the system function.

$$H(z) = \frac{1}{1 - z^{-1} + \frac{2}{9}z^{-2}} = \frac{z^2}{z^2 - z + \frac{2}{9}} = \frac{z^2}{(z - \frac{2}{3})(z - \frac{1}{3})} = 1 + \frac{z - \frac{2}{9}}{(z - \frac{2}{3})(z - \frac{1}{3})} =$$
$$\{PF\}\, 1 + \frac{\frac{4}{3}}{z - \frac{2}{3}} - \frac{\frac{1}{3}}{z - \frac{1}{3}} = 1 + z^{-1}\frac{\frac{4}{3}}{1 - \frac{2}{3}z^{-1}} - z^{-1}\frac{\frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$$

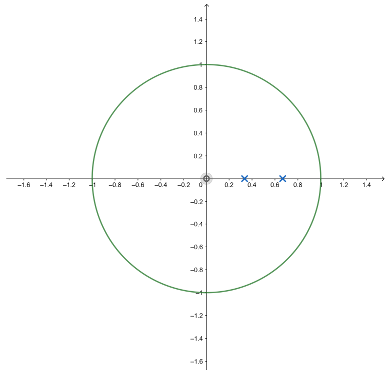

Figure 1: Poles and zeros of the system

$$\longmapsto h(n) = \delta(n) + \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} u(n-1) - \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u(n-1) = \delta(n) + \left(2\left(\frac{2}{3}\right)^{n} - \left(\frac{1}{3}\right)^{n}\right) u(n-1)$$

We can see that the system is stable in 2 ways: On the impulse response, where the potents of n is < 1 and in the graph, where the poles are inside the unit circle.

(b) *"The following signal is the input signal to the system"*

$$x(n) = 3(\frac{1}{2})^n u(n)$$

"where u(n) is the step function. Solve the difference equation by using the Z-transform, that is, determine a closed form expression for y(n)."

We know by definition that the output from the system is equal to the convolution of the input and the impulse response.

$$y(n) = h(n) * x(n) \Longleftrightarrow Y(z) = H(z)X(z)$$

When $Y(z)$ is calculated, we've got our answer. $X(z) = 3\frac{1}{1-\frac{1}{2}z^{-1}}$

$$H(z)X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})} * 3\frac{1}{1-\frac{1}{2}z^{-1}} = \{PF\}\left(-\frac{8}{5}\right)\frac{1}{1-\frac{1}{3}z^{-1}} + \frac{22}{5}\frac{1}{1-\frac{2}{3}z^{-1}} - \frac{14}{5}\frac{1}{1-\frac{1}{2}z^{-1}} \longmapsto$$

$$\longmapsto y(n) = \left(\frac{22}{5}\left(\frac{2}{3}\right)^{n} - \frac{8}{5}\left(\frac{1}{3}\right)^{n} - \frac{14}{5}\left(\frac{1}{2}\right)^{n}\right) u(n)$$

We can se that the system is stable as every base in the potents of n are less than 1.

3. *"The figures below show four pole-zero plots and four impulse responses."*

(a) *"Pair the correct plot A, B, C, D with the corresponding impulse response 1, 2, 3, 4."*

A	1
B	3
C	4
D	2

(b) *"Pair the correct plot A, B, C, D with the corresponding statement I, II, III, IV."*

A	I
B	III
C	II
D	IV