COMS 4770 - Homework 3

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1. (20 pts)

Consider the projection of the tetrahedron with vertices $(0, 1, 0)$, $(3, 1, 1)$, $(-1, -1, 1)$, $(0, -2, -1)$ onto the viewplane $5x - 3z + 2 = 0$ from the viewpoint $(1, 4, -1)$. Let a viewplane coordinate system be defined by origin $(-1, 1, -1)$, x-axis direction $(3, 0, 5)$, and y-axis direction $(0, -1, 0)$.

(a) (3 pts)

Construct the projection matrix M.

$$M = vl^T - (l \cdot v)I_3$$

$$v = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

$$l = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & -3 & 2 \end{bmatrix} - (\begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ -3 \\ 2 \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 5 & 0 & -3 & 2 \\ 20 & 0 & -12 & 8 \\ -5 & 0 & 3 & -2 \\ 5 & 0 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$M = \begin{bmatrix} -5 & 0 & -3 & 2 \\ 20 & -10 & -12 & 8 \\ -5 & 0 & -7 & -2 \\ 5 & 0 & -3 & -8 \end{bmatrix}$$

(b) (9 pts)

Determine the viewplane coordinate matrix V. (You may use a mathematical software such as Matlab or Mathematica to calculate a matrix inverse if necessary. Exact expressions and numerical values are both allowed, as for part (c).)

$$r = (3, 0, 5, 0)$$

$$s = (0, -1, 0, 0$$

$$q = (-1, 1, -1, 1)$$

$$K = \begin{bmatrix} r_1 & s_1 & q_1 \\ r_2 & s_2 & q_2 \\ r_3 & s_3 & q_3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & 1 \\ 5 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V = (K^T K)^{-1}$$

$$V = (\begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & 1 \\ 5 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 34 & 0 & -8 \\ 0 & 1 & -1 \\ -8 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{3}{38} & \frac{4}{19} & \frac{4}{19} \\ \frac{4}{19} & \frac{36}{19} & \frac{17}{19} \\ \frac{4}{19} & \frac{17}{19} & \frac{17}{19} \end{bmatrix} \begin{bmatrix} 3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{38} & 0 & \frac{7}{38} & \frac{4}{19} \\ \frac{-5}{19} & -1 & \frac{3}{19} & \frac{17}{19} \\ \frac{-5}{19} & 0 & \frac{3}{19} & \frac{17}{19} \end{bmatrix}$$

(c) (8 pts)

Apply M and V sequentially to the vertices of the tetrahedron.
Multiply projection matrix M by the matrix of the points:

$$\begin{bmatrix} -5 & 0 & -3 & 2 \\ 20 & -10 & -12 & 8 \\ -5 & 0 & -7 & -2 \\ 5 & 0 & -3 & -8 \end{bmatrix} \begin{bmatrix} 0 & 3 & -1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -16 & 4 & 5 \\ -2 & 46 & -14 & 40 \\ -2 & -24 & -4 & 5 \\ -8 & 4 & -16 & -5 \end{bmatrix}$$

Multiply the coordinate matrix V my the result of the last result:

$$\begin{bmatrix} \frac{1}{38} & 0 & \frac{7}{38} & \frac{4}{19} \\ \frac{-5}{19} & -1 & \frac{3}{19} & \frac{17}{19} \\ \frac{-5}{19} & 0 & \frac{3}{19} & \frac{17}{19} \end{bmatrix} \begin{bmatrix} 2 & -16 & 4 & 5 \\ -2 & 46 & -14 & 40 \\ -2 & -24 & -4 & 5 \\ -8 & 4 & -16 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -4 & 0 \\ -6 & -42 & -2 & -45 \\ -8 & 4 & -16 & -5 \end{bmatrix}$$

The resulting matrix are points in homogeneous coordinates. The points in Cartesian coordinates are:

$$(\frac{1}{4}, \frac{3}{4}), (-1, \frac{21}{2}), (\frac{1}{4}, \frac{1}{8}), (0, 9)$$

2. (10 pts)

Show that an affine mapping preserves collinearity of points in the plane. More specifically, the image points of any three distinct and collinear points p_1, p_2, and p_3 under the mapping are collinear.
Hint: These three points are collinear if and only if

$$p_2 - p_1 = \lambda(p_3 - p_1), \quad \text{for some } \lambda \neq 0.$$

The affine mapping of the points p_1, p_2, p_3:

$$q_1 = Ap_1 + b$$

$$q_2 = Ap_2 + b$$

$$q_3 = AP_3 + b$$

In order to prove collinearity the following must be true:

$$q_2 - q_1 = \lambda(q_3 - q_1)$$

$$q_2 - q_1 = (Ap_2 + b) - (Ap_1 + b) = A(p_2 - p_1)$$

$$q_3 - q_1 = (Ap_3 + b) - (Ap_1 + b) = A(p_3 - p_1)$$

Apply the affine mapping:

$$A(p_2 - p_1) = \lambda A(p_3 - p_1)$$

Substitute the $A(p - p)$ expressions with their equivalent $(q - q)$ expressions:

$$(q_2 - q_1) = \lambda(q_3 - q_1)$$

Since the previous equation is true after applying the affine mapping. The statement holds that affine mapping preserves collinearity of points in the plane.

Programming Problem

3. (25 pts)

Implement the three-dimensional projection procedure with the viewpoint and viewplane as input, and the computed projection matrix as output. In addition, output the projected images of a number of input data points. All the inputs and outputs should be in homogeneous coordinates. Determine the projection matrix for each of the following transformations, and apply the matrix to the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$.

(a) (4 pts)

Perspective projection onto the viewplane $-x + 3y + 2z - 4 = 0$ from the viewpoint $(2, -1, 1)$.

```
Runtime: 13039845982 ns

Projection Matrix:
[[ 5.  6.  4. -8.]
 [ 1.  4. -2.  4.]
 [-1.  3.  9. -4.]
 [-1.  3.  2.  3.]]

Projected Vertices:
[[-8.  4. -4.  3.]
 [-3.  5. -5.  2.]
 [-2.  8. -1.  6.]
 [ 7.  7.  7.  7.]]
```

(b) (4 pts)

Perspective projection onto the viewplane $5x - 3z + 2 = 0$ from the viewpoint $(1, 4, -1)$.

```
Runtime: 12730047674 ns

Projection Matrix:
[[ -5.   0.  -3.   2.]
 [ 20. -10. -12.   8.]
 [ -5.   0.  -7.  -2.]
 [  5.   0.  -3.  -8.]]

Projected Vertices:
[[  2.   8.  -2.  -8.]
 [ -3.  28.  -7.  -3.]
 [  2.  -2.  -2.  -8.]
```

```
 [ -6.   6. -14.  -6.]]
```

(c) (4 pts)

Parallel projection onto the viewplane $2y + 3z + 4 = 0$ in the direction of the vector $(1, -2, 3)$.

```
Runtime: 16348924073 ns

Projection Matrix:
[[ -9.   2.   3.   4.]
 [  0. -13.  -6.  -8.]
 [  0.   6.   0.  12.]
 [  0.   2.   3.  -5.]]

Projected Vertices:
[[  4.  -8.  12.  -5.]
 [ -5.  -8.  12.  -5.]
 [  6. -21.  18.  -3.]
 [  0. -27.  18.   0.]]
```

(d) (4 pts)

Parallel projection onto the viewplane $7x - 8y + 5 = 0$ in the direction of the vector $(0, 4, 9)$.

```
Runtime: 13640190593 ns

Projection Matrix:
[[ 27.   0.   0.   0.]
 [ 28.  -5.   0.  20.]
 [ 63. -72.  27.  45.]
 [  7.  -8.   0.  32.]]

Projected Vertices:
[[  0.  20.  45.  32.]
 [ 27.  48. 108.  39.]
 [  0.  15. -27.  24.]
 [ 27.  43.  63.  31.]]
```