COMS$_4770_H W8$

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October 2024

1 (20 pts)

What is the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $a, b, c > 0$?

The volume of the rectangular parallelepiped is $V = 8xyz$. The vertices of the parallelepiped lie on the ellipsoid, so we have the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The Lagrangian function is:

$$\mathcal{L}(x, y, z, \lambda) = 8xyz + \lambda \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right).$$

Taking partial derivatives of $\mathcal{L}$ with respect to x, y, z, and λ, and setting them equal to zero:

$$\frac{\partial \mathcal{L}}{\partial x} = 8yz - \frac{2\lambda x}{a^2} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = 8xz - \frac{2\lambda y}{b^2} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial z} = 8xy - \frac{2\lambda z}{c^2} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

From the first three equations, we get:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

Let k be a proportionality constant such that:

$$x = ka, \quad y = kb, \quad z = kc.$$

Substituting these into the constraint equation gives:

$$\frac{(ka)^2}{a^2} + \frac{(kb)^2}{b^2} + \frac{(kc)^2}{c^2} = 1,$$

$$k^2 \left(\frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2} \right) = 1,$$

$$k^2(1 + 1 + 1) = 1.$$

Thus, $k^2 = \frac{1}{3}$, giving $k = \frac{1}{\sqrt{3}}$.

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.$$

$$V = 8xyz = 8 \left(\frac{a}{\sqrt{3}} \right) \left(\frac{b}{\sqrt{3}} \right) \left(\frac{c}{\sqrt{3}} \right) = 8 \cdot \frac{abc}{3\sqrt{3}} = \frac{8abc}{3\sqrt{3}}.$$

The maximum volume of the rectangular parallelepiped that can be inscribed in the ellipsoid is:

$$\frac{8abc}{3\sqrt{3}}$$