# Com S 4770/5770 Fall 2024
# Assignment 6 (80 pts)

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Due on Canvas by 11:59pm
Sunday, October 13

## Instructions

For the programming problem, you must use a general-purpose programming language. Do not use built-in routines from mathematical software such as Mathematica, Matlab, or Maple. You need to turn in your code, its runtime results, and your answers to the other problems.

Please submit these items in a zipped file named `Firstname_Lastname_HW6.zip` on Canvas.

## Problem 1 (8 pts)

Find three orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ that span the same space as the following three vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Construct the QR factorization of the matrix $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$.
Find the orthonormal vectors

$$u_1 = v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$e_1 = \frac{u_1}{|u_1|} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ \frac{8}{3} \end{pmatrix}$$

$$e_2 = \frac{u_2}{|u_2|} = \begin{pmatrix} -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{10}}{30} \\ \frac{4\sqrt{10}}{15} \end{pmatrix}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) = \begin{pmatrix} \frac{1}{9} \\ -\frac{7}{45} \\ \frac{4}{45} \end{pmatrix}$$

$$e_3 = \frac{u_3}{|u_3|} = \begin{pmatrix} \frac{\sqrt{10}}{6} \\ -\frac{7\sqrt{10}}{30} \\ \frac{2\sqrt{10}}{15} \end{pmatrix}$$

$$Q = (u_1, u_2, u_3) = (\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{10}}{30} \\ \frac{4\sqrt{10}}{15} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{10}}{6} \\ -\frac{7\sqrt{10}}{30} \\ \frac{2\sqrt{10}}{15} \end{pmatrix})$$

Find the matrix R

$$R = Q^T A$$

$$R = \begin{pmatrix} 3.0000 & 4.0000 & 1.6667 \\ 0 & 3.1623 & 0.4216 \\ 0 & 0 & 0.2108 \end{pmatrix}$$

Thus, the QR factorization of the matrix $A = (v_1\, v_2\, v_3)$ is:

$$A = QR$$

# Problem 2 (7 pts)

Construct the singular value decomposition of the matrix below:

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}.$$

Compute $A^T A$

$$A^T A = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$

The matrix $A^T A$ is symmetric, so find the eigenvalues of $A^T A$

$$\det(A^T A - \lambda I) = 0$$

$$\det \begin{pmatrix} 10-\lambda & 6 \\ 6 & 10-\lambda \end{pmatrix} = 0$$

$$(10-\lambda)^2 - 36 = 0$$

$$\lambda^2 - 20\lambda + 64 = 0$$

$$\lambda_1 = 16, \quad \lambda_2 = 4$$
$$\sigma_1 = \sqrt{16} = 4, \quad \sigma_2 = \sqrt{4} = 2$$

To find the right singular vectors, we solve $(A^T A - \lambda I)v = 0$.
For $\lambda_1 = 16$:

$$(A^T A - 16I) = \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}$$

Solving this gives $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 4$:

$$(A^T A - 4I) = \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$$

Solving this gives $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Normalize the vectors:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Find the left singular vectors $U$
The left singular vectors are given by $u_i = \frac{1}{\sigma_i} A v_i$.
For $u_1$ (corresponding to $\sigma_1 = 4$):

$$u_1 = \frac{1}{4} A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

Normalize $u_1$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $u_2$ (corresponding to $\sigma_2 = 2$):

$$u_2 = \frac{1}{2} A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Construct the matrices $U$, $\Sigma$, and $V$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The singular value decomposition of $A$ is:

$$A = U \Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$