Exploration Lab 1

July 11, 2017

1 Linear Diophantine of two variables

Given $ax + by = n$ where $a, b, n \in \mathbb{Z}$

- When $x, y \in \mathbb{Z}$, then there exists solution if and only if $gcd(a, b)|n$. In addition, if it has one, it has infinitely many.
- When $x, y \in \mathbb{N}$ and all $a, b, n \in \mathbb{N}$, there does not exists any solution if the same formula is not solvable in $\mathbb{Z}$. Otherwise, given a solution $x_0, y_0 \in \mathbb{Z}$, there exists solution in the first quadrant if and only if ??? and we can actually compute the actual number, where there is ??? numbers of solutions that are in the first quadrant.

2 Induction and WOP

Notice the fact that 3, 4, 5 are equivalent, that is, you can prove any one of them from another.

1 A set S is an induction set if and only if $1 \in S$ and $\forall s \in S$, $s + 1 \in S$.

2 The intersection of all induction sets exists and is called Natural numbers. (The axiom of the naturals)

3 The theorem of induction: if $S \subseteq \mathbb{N}$ and S is an inductive set, then $S = N$.

4 The theorem of strong induction: if $S \subseteq \mathbb{N}$ and $1 \in S$ and if $\exists s \in S$ such that $\forall i \leq s$, $i \in S$, then $s + 1 \in S$

5 The well ordering principle (WOP): if $S \subseteq \mathbb{N}$, $\exists s \in S$ such that $\forall k \in S$, $s \leq k$.

3 Gomory's Cut

Basic theorem: if $ax + by = n$ and $a \neq 0$ and $x, y \geqslant 0$, then

$$x + \left\lfloor \frac{b}{a} \right\rfloor y \leq \left\lfloor \frac{n}{a} \right\rfloor$$

Extension: if $\sum_{i=0}^{k} a_i x_i = n$ and $a_0 \neq 0$ and $x_i \geqslant 0,\ \forall i \in \mathbb{N}$, then

$$x_0 + \sum_{i=1}^{k} \left\lfloor \frac{a_i}{a_0} \right\rfloor x_i \leq \left\lfloor \frac{n}{a_0} \right\rfloor$$

Conjecture: if $\sum_{i=0}^{\infty} a_i x_i = n$ and $a_0 \neq 0$ and $x_i \geqslant 0,\ \forall i \in \mathbb{N}$, then $\exists k \in \mathbb{N}$ such that $\forall k' > k$,

$$x_0 + \sum_{i=1}^{k'} \left\lfloor \frac{a_i}{a_0} \right\rfloor x_i \leq \left\lfloor \frac{n}{a_0} \right\rfloor$$

If this is not true then what about this one?
Conjecture+: $\forall \epsilon > 0$, if $\sum_{i=0}^{\infty} a_i x_i = n$ and $a_0 \neq 0$ and $x_i \geqslant 0,\ \forall i \in \mathbb{N}$, then $\exists k \in \mathbb{N}$ such that $\forall k' > k$,

$$x_0 + \sum_{i=1}^{k'} \left\lfloor \frac{a_i}{a_0} \right\rfloor x_i \leq \left\lfloor \frac{n}{a_0} \right\rfloor + \epsilon$$