Lecture 9 - Lagrangian Equations of Motion

Topics:

A) Lagrangian Mechanics: Brief Overview. (pages 2-5)

Theory: Short mathematical derivation of the Lagrangian approach.

This is just the application of "Newton's 2nd Law" (F=ma) for a system with various degrees of freedom and constraints between lumped masses or inertias. The insight in the approach is that you can write the equations of motion without including the internal forces, such as forces between gears or the force at the ground as a cart rolls back and forth on the ground.

B) Steps in solving for EOM's via the Lagrangian approach. (pages 6-19)

Practical Method: Sequential steps are presented to derive N equations of motion for a system with N independent and complete degrees of freedom, by hand.

Each equation consists of a "lefthand side" that contains only conservative terms, such as mass times acceleration and spring forces. This is very straightforward to calculate, so long as you have identified all components of the kinetic (co-)energy and the potential energy.

The "righthand side" is sometime more challenging to calculate. These notes provide a couple of intuitive approaches for doing so. A more formal (mathematical) approach will be presented in the Lecture 10 notes.

Examples are also given.

Lagrangian Mechanics: brief overview

The Lagrangian approach produces equations of motion that describe Newton's second law: $F = ma$, or more precisely, $F = \dot{p}$, where momentum p is mass times velocity: $p = mv$.

The approach exploits knowledge about particular (holonomic) constraints in the system, with the advantages of

- reducing the total number of equations required,
- avoiding the need for "free body diagrams", and
- eliminating tedious calculations involving internal constraint forces.

These internal constraint forces are very real and very important, but it turns out it is possible to avoid calculating them explicitly in order to determine the dynamics of the system. We already know certain directions are not allowed (i.e., constrained), and it turns out it is possible to write a set of n equations representing "$F = ma$" for each of n generalized coordinates (GC's), which are independent and complete degrees of freedom (DOF's) describing the possible motions of the (potentially constrained) full dynamic system.

So let's begin. For our equations, we will always be considering summations over ALL masses within the system, but to keep notation from getting too cluttered (while illustrating the general mathematical ideas), equations will avoid showing these summations. It may also be more intuitive to consider a single point mass, in following the derivation.

$$F = \dot{p} = \frac{d}{dt}(mv) = m\dot{v} + \dot{m}v = m\dot{v}, \tag{1}$$

if we assume mass is constant ($\dot{m} = 0$). Next, imagine a differential motion, δr, to turn this into a virtual work balance, i.e., for a teeny, tiny "virtual" motion, δr.

$$F\delta r = \dot{p}\delta r = \frac{d}{dt}(m\dot{r})\,\delta r = m\ddot{r}\delta r, \tag{2}$$

where r is position, $v = \dot{r}$ is velocity, and $a = \dot{v} = \ddot{r}$ is acceleration. This Principle of Virtual Work must hold for *any* differential motion direction. Say we have a generalized coordinate[1], q. We can rewrite the equation above in terms of a differential motion δq, by noting $\delta r = \frac{\partial r}{\partial q}\delta q$. For example, if r were aligned with the positive x axis, and q was the coordinate along a 60 degree uphill slop (to the right), moving some δq distance in the uphill direction can be broken down into x and y components; i.e., is $\hat{x}$: $r = q\cos(\pi/3)$, so $\frac{\partial r}{\partial q} = \cos(\pi/3) = 1/2$, and $\delta r = \frac{\partial r}{\partial q}\delta q = \frac{1}{2}\delta q$. Thus, we get:

$$F\frac{\partial r}{\partial q}\delta q = m\ddot{r}\frac{\partial r}{\partial q}\delta q \tag{3}$$

[1] ...or n such G.C.'s: q_1 through q_n. Obviously, we could instead use ξ as the variable for the G.C.'s, but q is more legible here, throughout! Take note that the G.C.'s might be actuator DOF's or end effector DOF's, so long as the set chosen is both independent and complete.

Next, we want to represent the term on the right (mass time acceleration) in a more tractable way, and we use the chain rule to get there. Via the chain rule:

$$\frac{d}{dt}\left(m\dot{r}\frac{\partial r}{\partial q}\right) = m\ddot{r}\frac{\partial r}{\partial q} + m\dot{r}\frac{d}{dt}\left(\frac{\partial r}{\partial q}\right). \tag{4}$$

Rearranging, we see:

$$m\ddot{r}\frac{\partial r}{\partial q} = \frac{d}{dt}\left(m\dot{r}\frac{\partial r}{\partial q}\right) - m\dot{r}\frac{d}{dt}\left(\frac{\partial r}{\partial q}\right), \tag{5}$$

so we can rewrite Equation 2 as:

$$F\frac{\partial r}{\partial q} = \frac{d}{dt}\left(m\dot{r}\frac{\partial r}{\partial q}\right) - m\dot{r}\frac{\partial v}{\partial q}, \tag{6}$$

where $\frac{d}{dt}\left(\frac{\partial r}{\partial q}\right) = \frac{\partial v}{\partial q}$. Note that Eq. 6 does not include the (δq) differential distances above in Eq. 3. We next aim to compactly represent just the righthand side of Eq. 6.

Kinetic energy is $T = \frac{1}{2}mv^2$. Note that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial v}\frac{\partial v}{\partial \dot{q}}\right) = \frac{d}{dt}\left(mv\frac{\partial v}{\partial \dot{q}}\right) = \frac{d}{dt}\left(m\dot{r}\frac{\partial r}{\partial q}\right), \tag{7}$$

since $\frac{\partial v}{\partial \dot{q}} = \frac{\partial r}{\partial q}$. Also,

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial v}\frac{\partial v}{\partial q} = mv\frac{\partial v}{\partial q} = m\dot{r}\frac{\partial v}{\partial q}. \tag{8}$$

Thus, if F is zero, since the righthand terms in Equations 7 and 8 can be combined to yield the righthand side of Equation 6, we can write:

$$0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}. \tag{9}$$

For conservative forces, $F_r = -\frac{\partial V}{\partial r}$, where $V = V(r)$ is the potential energy function, which depends on position **but not on velocity** for the mechanical systems we are analyzing. From Equation 3, let us derive $F\frac{\partial r}{\partial q}\delta q$:

$$F\frac{\partial r}{\partial q}\delta q = -\frac{\partial V}{\partial r}\frac{\partial r}{\partial q}\delta q = -\frac{\partial V}{\partial q}\delta q \tag{10}$$

and so:

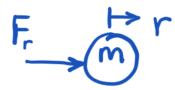
$$-\frac{\partial V}{\partial q} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}, \tag{11}$$

again now dropping the differential term δq to create the equation of motion. This means if we define $L = T - V$ (the Lagrangian), then a full force and momentum balance yields:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \Xi_i, \tag{12}$$

where Ξ represents non-conservative forces (or torques). If there are n generalized coordinates, there will be n equations of motion, with the i^{th} equation corresponding to coordinate q_i, with a corresponding Ξ_i. (See additional notes on calculating this "Big Xi" for each generalized coordinate.)

Lagrangian Explained



r : position
$v = \dot{r}$
$a = \dot{v} = \ddot{r}$

1. Force equals rate of change of momentum

$$p = mv$$

$$F_r = \frac{d}{dt}(mv) = \cancel{\dot{m}v} + m\dot{v}$$

$\dot{m} = 0$

$$\boxed{F_r = m\ddot{r}}$$

2. Virtual work

done by some virtual displacement: δ_r

$$\boxed{F_r \cdot \delta_r = m\ddot{r} \cdot \delta_r}$$

3. Generalized Coordinates

G.C.: q_i (or ξ_i)

$$\delta_r = \frac{\partial r}{\partial q}\delta_q$$

$$\boxed{F_r \frac{\partial r}{\partial q} \cdot \delta_q = m\ddot{r}\frac{\partial r}{\partial q} \cdot \delta_q}$$

$\dot{r} = v$

4. Chain Rule

$$\frac{d}{dt}\left(m\dot{r}\frac{\partial r}{\partial q}\right) = m\ddot{r}\frac{\partial r}{\partial q} + m\dot{r}\frac{d}{dt}\left(\frac{\partial r}{\partial q}\right)$$

$$\frac{\partial r}{\partial q} = \frac{\partial v}{\partial \dot{q}} \qquad \frac{d}{dt}\left(\frac{\partial r}{\partial q}\right) = \frac{\partial v}{\partial q}$$

$$F_r \frac{\partial r}{\partial q} \cdot \delta_q = \left(\frac{d}{dt}\left(\boxed{mv\frac{\partial v}{\partial \dot{q}}}\right) - \boxed{mv\frac{\partial v}{\partial q}}\right) \cdot \delta_q$$

5. Kinetic energy: T (really thinking about kinetic co-energy: T*)

$$T = \frac{1}{2}mv^2$$

$$\boxed{\frac{\partial T}{\partial \dot{q}}} = \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial \dot{q}} = \boxed{mv \cdot \frac{\partial v}{\partial \dot{q}}}$$

$$\boxed{\frac{\partial T}{\partial q}} = \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial q} = \boxed{mv \cdot \frac{\partial v}{\partial q}}$$

Rewriting:

$$F_r \frac{\partial r}{\partial q} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}$$

6. Potential energy, V

V (not $v = \dot{r}$)

F_c: conservative forces or torques

$$F_r = F_c + F_{nc}$$

$$F_c = \frac{-\partial V}{\partial r}$$

Note: $\frac{\partial V}{\partial \dot{q}} = 0$

$$\left(\frac{-\partial V}{\partial r} \cdot \frac{\partial r}{\partial q} + F_{nc} \cdot \frac{\partial r}{\partial q}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}$$

7. Lagrangian

$$\mathcal{L} \equiv T^* - V$$
$$= T - V$$

$$F_{nc} \cdot \frac{\partial r}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q}$$

8. Non-conservative forces and torques...

$$F_{nc} \cdot \frac{\partial r}{\partial q} = \Xi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \Xi_i$$

for $i = 1 \to n$,
for a system with
n G.C.'s

Practical Steps to Derive Lagrangian EOMs

The goal of this handout is to focus on the **"how"** (applying this method in specific cases) instead of the **"why"** (in giving theory to derive why the Lagrangian approach works at all).

As engineers, and as students stressing about assignments and exams, you arguably can get away with just understanding the **"how"** (…if you trust professors like me, that is). For those who are interested, there is another reading for today's lecture on **"why"** the approach works. In short, the Lagrangian method is a clever way to avoid having to write out the internal forces and torques that occur due to particular constraints.

The steps that follow are first outlined one-at-a-time and then presented all on a single page, **as a visual "checklist"**.

Holonomic / Nonholonomic

Recall: Nonholonomic when

- Accessible configuration space has higher dimension than the accessible velocity space.

i.e.,

- **More Generalized Coordinates** are required to describe the long-term configurations that are achievable…
- **…than Differential Degrees of Freedom** (DDOFs) for local motion, instantaneously

$$\boxed{\#GC = \xi_j > \delta\xi_k = \#DDOF \longleftrightarrow \text{Nonholonomic}}$$

$$\boxed{\#GC = \xi_j = \delta\xi_k = \#DDOF \longleftrightarrow \text{Holonomic}}$$

Lagrangian Approach: Newton's 2nd Law

A. Define generalized coordinates that are:

- Complete
- Independent

ξ_j (j = 1, 2, 3, …)

- **(For now, we will also assume system is holonomic…)**

B. Force-dynamics (F=ma) relationships require:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

where:

$$\mathcal{L} = T^* - V$$

The information above summarizes everything you need to know, to calculate the LEFTHAND SIDE of each of n equations of motion (EOMs) for a system with n independent and complete degrees of freedom (DOFs). Simplify perform the differentiation shown above with respect to the ith DOF to get the lefthand side of the ith EOM. (We will focus on the righthand side in more detail in the next lecture…)

Keep in mind that for this class:

Kinetic energy includes only mechanical kinetic energy, and kinetic co-energy will be identical to kinetic energy. (This is not necessarily so, more generally.)

Potential energy will include just "mgh", due to gravity, and "(1/2)*k*(delta_x)^2", from springs, as contributions to V. (This is not necessarily so, more generally.)

By identifying the system as "holonomic", we know we can use the Lagrangian approach to derive EOM's. However, in this class we not focus on identifying a system as either holonomic or nonholonomic in determining Lagrangian EOMs. This "step 0" will be assumed to have been done when you are asked to derive EOMs.

Lagrangian Approach

ξ_j (j = 1, 2, 3, …)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

$$\mathcal{L} = T^* - V$$

- Be sure you can identify all variables and expressions above…
- They are described below:

Lagrangian Approach

Little "xi" is the set of DOF. Since we assume system is holonomic, there are the same number of GCs and DOFs.

ξ_j (j = 1, 2, 3, …)

Big "Xi" represents all non-conservative forces, for each DOF. This includes both the ACTUATION and LOSSES (due to damping, most typically).

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

Fancy "L" is the "Lagrangian"

V is potential energy

$$\mathcal{L} = T^* - V$$

T* is kinetic CO-ENERGY

Lagrangian Approach: Full Procedure

0. *Assume* system is **holonomic**:

$$\boxed{\#GC = \xi_j = \delta\xi_k = \#DOF \longleftrightarrow \text{Holonomic}}$$

1. Pick **generalized coordinates** (DOFs) that are:
   - **Complete**
   - **Independent**
     (j = 1, 2, 3, …)

$\boxed{\xi_j}$ ← Little xi: GCs (gen. coord's)

- Possible GC set(s) ***might not be unique***!
- **Any** independent and complete set is OK.
- Either **absolute or relative** coords is OK.

2. Define kinetic (co-)energy, T. (*Technically*, T*)

Kinetic energy includes the sum of "one-half mass time velocity-squared" for every particle:

$$\boxed{T = \sum_i \frac{1}{2} m_i v_i^2}$$

For a **rigid rotating body**, we can use moment of inertia, J:

$$\boxed{T = \frac{1}{2} J\dot{\theta}^2}$$

J is the moment of inertia ABOUT THE CENTER OF MASS (COM)!

(Note: We are **ignoring** electrical kinetic or potential energy here, for now and focusing on **MECHANICAL energy**.)

Lagrangian Approach: Full Procedure

3. Define potential energy, V.

Potential energy can be stored in a **spring**:

$$\boxed{V_{spring} = \tfrac{1}{2}k\left(x - x_o\right)^2}$$

…and it also includes any "mgh" of particles in a gravitational field:

$$\boxed{V_{gravity} = mgh}$$

Note: "h" is relative to any arbitrary (but fixed) inertial reference frame.

4. Lagrangian is **kinetic minus potential** energy:

$$\boxed{\mathcal{L} = T^* - V}$$

5. For each GC (DOF), solve lefthand side (LHS) for an equation of motion:

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i}$$

Note: A system with **N generalized coordinates** (GCs) will have **N equations of motion.**

6. For each GC (DOF), determine non-conservative forces, for righthand side (RHS).

We'll cover step 6, above, in more detail in the NEXT lecture…

Below is a one-page summary of the last few slides:

Lagrangian Approach: Summary

Little "xi" is the set of DOF. Since we assume system is holonomic, there are the same number of GCs and DOFs.

ξ_j (j = 1, 2, 3, …)

Big "Xi" represents all non-conservative forces, for each DOF. This includes both the ACTUATION and LOSSES (due to damping, most typically).

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

Fancy "L" is the "Lagrangian"

V is potential energy

$$\mathcal{L} = T^* - V$$

T* is kinetic CO-ENERGY

Lagrangian Approach: Procedure

0. *Assume* system is **holonomic**:

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   - **Complete**
   - **Independent**

   (j = 1, 2, 3, …) $\boxed{\xi_j}$ Little xi: GCs (gen. coord's)

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Lagrangian Approach: Procedure

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Kinetic energy includes the sum of "one-half mass time velocity-squared" for every particle:

$$\boxed{T = \sum_i \tfrac{1}{2} m_i v_i^2}$$

For a **rigid rotating body**, we can use moment of inertia, J:

$$\boxed{T = \tfrac{1}{2} J\dot{\theta}^2}$$

(Note: We are **ignoring** electrical kinetic or potential energy here, for now and focusing on **MECHANICAL energy**.)

Lagrangian Approach: Procedure

3. Define potential energy, V.

Potential energy can be stored in a **spring**:

$$\boxed{V_{spring} = \tfrac{1}{2} k\left(x - x_o\right)^2}$$

…and also includes any "mgh" of particles in a gravitational field:

$$\boxed{V_{gravity} = mgh}$$

Note: "h" is relative to any arbitrary (but fixed) inertial reference frame.

Lagrangian Approach: Procedure

4. Lagrangian is **kinetic minus potential** energy:

$$\boxed{\mathcal{L} = T^* - V}$$

5. For each GC (DOF), solve lefthand side for an equation of motion:

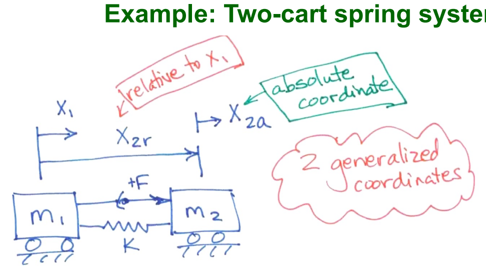
$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i}$$

Note: A system with **N generalized coordinates** (GCs) will have **N equations of motion.**

6. For each GC (DOF), determine non-conservative forces (righthand side).

More on Step "6" in next lecture…

Example: Two-cart spring system



V

Absolute: Ⓐ $\boxed{\xi_1 = x_1,\ \xi_2 = x_{2a}}$

$x_1 = x_1$
$x_2 = x_{2a}$

Relative: Ⓑ $\boxed{\xi_1 = x_1,\ \xi_2 = x_{2r}}$

$x_1 = x_1$
$x_2 = x_1 + x_{2r}$

Choice of GCs is NOT UNIQUE.

(Just stay consistent!)

More on "absolute" vs "relative" coordinates in next lecture…

Lagrangian Approach: Summary

Little "xi" is the set of DOF. Since we assume system is holonomic, there are the same number of GCs and DOFs.

ξ_j (j = 1, 2, 3, …)

Big "Xi" represents all non-conservative forces, for each DOF. This includes both the ACTUATION and LOSSES (due to damping, most typically).

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

Fancy "L" is the "Lagrangian"

V is potential energy

$$\mathcal{L} = T^* - V$$

T* is kinetic CO-ENERGY

In this class, T* will always equal T. (Kinetic energy will be equal to the "kinetic co-energy".)

Lagrangian Approach: Procedure

0. *Assume* system is **holonomic**:

$$\# GC = \xi_j = \delta\xi_k = \# DOF \longleftrightarrow \text{Holonomic}$$

1. Pick **generalized coordinates** (DOFs) that are:
   - **Complete**
   - **Independent**

   (j = 1, 2, 3, …) $\boxed{\xi_j}$ ← Little xi: GCs (gen. coord's)

- Possible GC set(s) ***might not be unique***!
- **Any** independent and complete set is OK.
- Either **absolute or relative** coords is OK.

The generalized coordinates must be chosen so that:
- They fully describe positions of all particles.
- Any one can be imagined to "wiggle", while the others remain fixed.

Lagrangian Approach: Procedure

2. Define kinetic (co-)energy, T. (*Technically*, T*)

Kinetic energy includes the sum of "one-half mass time velocity-squared" for every particle:

$$\boxed{T = \sum_i \tfrac{1}{2} m_i v_i^2}$$

For a **rigid rotating body**, we can use moment of inertia, J:

$$\boxed{T = \tfrac{1}{2} J \dot{\theta}^2}$$

(Note: We are **ignoring** electrical kinetic or potential energy here, for now and focusing on **MECHANICAL energy**.)

Here, J is the MOMENT OF INERTIA ABOUT THE CENTER OF MASS (COM) of a rigid body, always.

- Track COM linear velocity AS WELL AS the angular velocity, when a rigid, rotating body is in the system.

Lagrangian Approach: Procedure

3. Define potential energy, V.

Potential energy can be stored in a **spring**:

$$\boxed{V_{spring} = \tfrac{1}{2} k\left(x - x_o\right)^2}$$

…and also includes any "mgh" of particles in a gravitational field:

$$\boxed{V_{gravity} = mgh}$$

Note: "h" is relative to any arbitrary (but fixed) inertial reference frame.

- Potential energy of a spring is proportional to the square of its deflection from its neutral point.

Lagrangian Approach: Procedure

4. Lagrangian is **kinetic minus potential** energy:

$$\boxed{\mathcal{L} = T^* - V}$$

5. For each GC (DOF), solve lefthand side for an equation of motion:

v

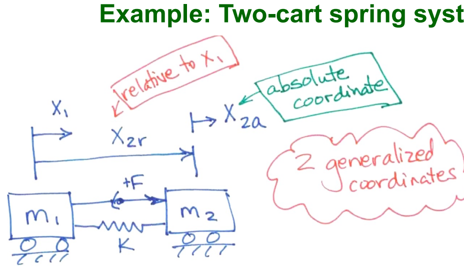
$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i}$$

Note: A system with **N generalized coordinates** (GCs) will have **N equations of motion.**

6. For each GC (DOF), determine non-conservative forces (righthand side).

More on Step "6" in next lecture…

Example: Two-cart spring system



v

Absolute:

(A) $\boxed{\begin{aligned}\xi_1 &= x_1 \\ \xi_2 &= x_{2a}\end{aligned}}$

$x_1 = x_1$
$x_2 = x_{2a}$

Relative:

(B) $\boxed{\begin{aligned}\xi_1 &= x_1 \\ \xi_2 &= x_{2r}\end{aligned}}$

$x_1 = x_1$
$x_2 = x_1 + x_{2r}$

Choice of GCs is NOT UNIQUE.

(Just stay consistent!)

See next page for "absolute" vs "relative" cases…

More on "absolute" vs "relative" coordinates in next lecture…

Below is a summary of "EXAMPLE 3" ahead.
(See more detailed, numbered steps in pages ahead…)

(See diagram at bottom right of previous page…)

Example: Two-cart spring system.

Absolute Coord's

$$T^* = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_{2a}^2$$

$$V = \tfrac{1}{2} k (x_1 - x_{2a})^2$$

$$\mathcal{L} = T^* - V$$

$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_{2a}^2 + \ldots$$

$$- \tfrac{1}{2} k x_1^2 - \tfrac{1}{2} k x_{2a}^2 + k x_1 x_{2a}$$

$$\boxed{\begin{aligned} \Xi_{1a} &= -F \\ \Xi_{2a} &= +F \end{aligned}}$$

Note the difference

① $-F = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \dfrac{\partial \mathcal{L}}{\partial x_1}$

$$-F = \frac{d}{dt}(m_1 \dot{x}_1) - (-k x_1 + k x_{2a})$$

$\boxed{1a}$ $\boxed{-F = m_1 \ddot{x}_1 + k x_1 - k x_{2a}}$

② $F = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_{2a}}\right) - \dfrac{\partial \mathcal{L}}{\partial x_{2a}}$

$$F = \frac{d}{dt}(m_2 \dot{x}_{2a}) - (-k x_{2a} + k x_1)$$

$\boxed{2a}$ $\boxed{F = m_2 \ddot{x}_{2a} + k x_{2a} - k x_1}$

Relative Coord's

$$T^* = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$$

$$V = \tfrac{1}{2} k x_{2r}^2$$

$$\mathcal{L} = T^* - V$$

$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \ldots$$

$$\tfrac{1}{2} m_2 (\dot{x}_1^2 + 2 \dot{x}_1 \dot{x}_{2r} + \dot{x}_{2r}^2) + \ldots$$

$$- \tfrac{1}{2} k x_{2r}^2$$

$$\boxed{\begin{aligned} \Xi_{1r} &= 0 \\ \Xi_{2r} &= +F \end{aligned}}$$

① $0 = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \dfrac{\partial \mathcal{L}}{\partial x_1}$

$$0 = \frac{d}{dt}(m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 \dot{x}_{2r})$$

$\boxed{1r}$ $\boxed{0 = (m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_{2r}}$

② $F = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_{2r}}\right) - \dfrac{\partial \mathcal{L}}{\partial x_{2r}}$

$$F = \frac{d}{dt}(m_2 \dot{x}_1 + m_2 \dot{x}_{2r}) + \ldots$$

$$-(-k x_{2r})$$

$\boxed{2r}$ $\boxed{F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + k x_{2r}}$

Let's compare!!!

- Write $\boxed{1a}$ in terms of x_{1r} & x_{2r}:

$$-F = m_1 \ddot{x}_1 + k x_1 - k(x_1 + x_{2r})$$

$$\therefore \boxed{-F = m_1 \ddot{x}_1 - k x_{2r}} \quad \boxed{B\text{-}1}$$

- Write $\boxed{2a}$ in terms of x_{1r} & x_{2r}:

$$F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + (k x_1 + k x_{2r}) - k x_1$$

$$\therefore \boxed{F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + k x_{2r}} \quad \boxed{B\text{-}2}$$

- Equation $\boxed{B\text{-}2}$ matches $\boxed{2r}$.

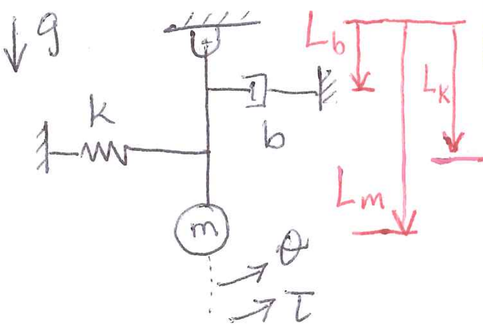
→ Now, add $\boxed{B\text{-}1}$ and $\boxed{B\text{-}2}$

$$\boxed{0 = (m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_{2r}}$$

Matches $\boxed{1r}$!

$\boxed{1a}$ & $\boxed{2a}$ can be written as $\boxed{1r}$ & $\boxed{2r}$!

Lagrangian E.O.M.s : Example 1



① G.C.s ("generalized coordinates")

only one degree-of-freedom (DOF),

$$\boxed{\xi_1 = \theta}$$ ← N=1 total GC's here...

② Kinetic Co-energy, T^*

represent all "$\frac{1}{2}mv^2$", as a function of G.C. velocities & positions...

$$\boxed{T^* = \frac{1}{2}m(L_m\dot{\theta})^2}$$

③ Potential Energy, V

for our systems, V may include conservative spring &/or gravitational terms: "$\frac{1}{2}k(\Delta x)^2$" & "mgh"...

$$V = \frac{1}{2}k(L_k\theta)^2 - mgL_m\cos\theta$$

④ $\mathcal{L} = T^* - V$, where $\mathcal{L}$ is the "Lagrangian"

(fancy L)

$$\boxed{\mathcal{L} = \frac{1}{2}mL_m^2\dot{\theta}_m^2 - \frac{1}{2}kL_k^2\theta^2 + mgL_m\cos\theta}$$

note damper effect mimics spring, w "bL^2" impedance.

$$\boxed{\Xi_1 = \tau - bL_b^2\dot{\theta}}$$

⑤ EOM, LHS : $\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_i}\right) - \left(\frac{\partial\mathcal{L}}{\partial\xi_i}\right)}$

"lefthand side" of the i^{th} EOM (i.e., for GC i of N)

$$\frac{d}{dt}(mL_m^2\dot{\theta}) - [-kL_k^2\theta - mgL_m\sin\theta] = \Xi_i$$

$$\boxed{mL_m^2\ddot{\theta} + kL_k^2\theta + mgL_m\sin\theta = \Xi_i}$$

⑥ EOM, RHS : $\boxed{\Xi_i}$

"righthand side"

Represent all non-conservative forces or torques for the i^{th} GC, {eg., τ_i or $-B\dot{\theta}$ or $-F\dot{x}$}

Combining ⑤ (LHS) & ⑥ (RHS): $\boxed{(mL_m^2)\ddot{\theta} + (bL_b^2)\dot{\theta} + (kL_k^2)\theta + mgL_m\sin\theta = \tau}$

Some notes on **non-conservative forces**.

1. The non-conservative forces obviously depend on your choice of generalized coordinates.

   We will go over this in detail next lecture, and in Homework. A general rule, however, is to draw a box around all elements that would move if you change just one, particular DOF. For example, if x1 is an absolute position, and x2 is the displacement RELATIVE to x1, then the non-conservative forces for the equation regarding x1 are forces acting on a box including elements moving with x1 and also the "children" moving with x2. (See bottom of last page of this handout, on "Example 3B"...)

2. Damping is another type of non-conservative force. It seems frustrating that the lefthand side of each equation of motion comes out "automatically", by following simply steps, while the righthand side seemingly does not.

   To cope with a standard, linear damper (either translational or rotational), one "trick" is to replace the damper with a spring element. This "fake spring" will result in additional terms in some of the equations of motion, which can then be replaced with analogous damper terms.

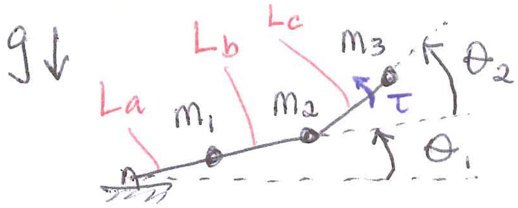
   For example, if an equation has a term $kL^2 * x_3$, we would then replace this with $bL^2 * (dx_3/dt)$.

   On the previous page, look at how the spring and damper appear in the equation of motion as one illustration of this.

   previous

   (The sketch on the ~~last~~ page is meant to "look intuitive", but to be more exact, just imagine the spring and damper are designed to travel along an arc, instead of along a straight line...)

Lagrangian EOM's : Example 2



little xi ("zye")

① GC's $\boxed{\xi_1 = \theta_1 \,,\; \xi_2 = \theta_2}$

where θ_2 is RELATIVE to θ_1...

$L_{ab} \equiv L_a + L_b$

② T^*, kinetic co-energy

$$\boxed{T^* = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \tfrac{1}{2} m_3 (\dot{x}_3^2 + \dot{y}_3^2)}$$

Then, just represent "$\dot{x}_1, \dot{y}_1, \dot{x}_2, \ldots$" as a function of $\xi_1, \xi_2, \dot{\xi}_1$ & $\dot{\xi}_2$

$$x_1 = L_a \cos\theta_1 \rightarrow \dot{x}_1 = -L_a \sin\theta_1 \cdot \dot{\theta}_1$$
$$y_1 = L_a \sin\theta_1 \rightarrow \dot{y}_1 = L_a \cos\theta_1 \cdot \dot{\theta}_1$$
$$x_2 = (L_a + L_b)\cos\theta_1 \rightarrow \dot{x}_2 = -(L_a + L_b)\sin\theta_1 \cdot \dot{\theta}_1$$
$$y_2 = (L_a + L_b)\sin\theta_1 \rightarrow \dot{y}_2 = (L_a + L_b)\cos\theta_1 \cdot \dot{\theta}_1$$

* Can be calculated in MATLAB, using symbolic toolbox...

$$x_3 = x_2 + L_c \cos(\theta_1 + \theta_2) \rightarrow \dot{x}_3 = \dot{x}_2 - L_c \sin(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2)$$
$$y_3 = y_2 + L_c \sin(\theta_1 + \theta_2) \rightarrow \dot{y}_3 = \dot{y}_2 + L_c \cos(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2)$$

③ V, potential energy

Then, just represent y_1, y_2 & y_3 using $\xi_1, \xi_2, \dot{\xi}_1$ & $\dot{\xi}_2$ (see above)

$$\boxed{V = m_1 g y_1 + m_2 g y_2 + m_3 g y_3}$$

④ $\boxed{\mathcal{L} = T^* - V}$ $\mathcal{L} = \tfrac{1}{2} m_1 \left(L_a^2 \sin^2\theta_1 \cdot \dot{\theta}_1^2 + L_a^2 \cos^2\theta_1 \cdot \dot{\theta}_1^2\right) + \ldots$

$$\ldots \tfrac{1}{2} m_2 \left(L_{ab}^2 \sin^2\theta_1 \cdot \dot{\theta}_1^2 + L_{ab}^2 \cos^2\theta_1 \cdot \dot{\theta}_1^2\right) + \tfrac{1}{2} m_3 \left(-L_{ab} \sin\theta_1 \cdot \dot{\theta}_1 - L_c \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)\right)^2$$

$$\ldots + \tfrac{1}{2} m_3 \left(L_{ab} \cos\theta_1 \cdot \dot{\theta}_1 + L_c \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \theta_2)\right)^2 + \ldots$$

$$- m_1 g L_a \sin\theta_1 - m_2 g L_{ab} \sin\theta_1 - m_3 g \left(L_{ab} \sin\theta_1 + L_c \sin(\theta_1 + \theta_2)\right)$$

(EOM, Example 2, continued!)

* → You can do all the "messy math" symbolically, in MATLAB!

(4) simplifying... $\boxed{\text{note: } (\sin^2\theta + \cos^2\theta = 1)}$ → i.e. $L_a^2 \sin^2\theta_1 \dot{\theta}_1^2 + L_a^2 \cos^2\theta_1 \dot{\theta}_1^2 = L_a^2 \dot{\theta}_1^2$

$$\mathcal{L} = \frac{1}{2} m_1 L_a^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_{ab}^2 \dot{\theta}_1^2 + \dots$$

$$\frac{1}{2} m_3 \Big(L_{ab}^2 \sin^2\theta_1 \dot{\theta}_1^2 + 2 L_{ab} L_c \sin\theta_1 \sin(\theta_1 + \theta_2) \cdot \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + \dots$$

$$L_c^2 \sin^2(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 + \{\text{etc., etc.}\}$$

(5) $\boxed{\text{LHS of each of N EOM's}}$ ← $N = 2$ GC's here...

#1 → $\boxed{\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \dfrac{\partial \mathcal{L}}{\partial \theta_1} = \Xi_1}$ ← big Xi ("zye")

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = \Xi_2}$$

(6) $\boxed{\text{RHS of each EOM:}}$

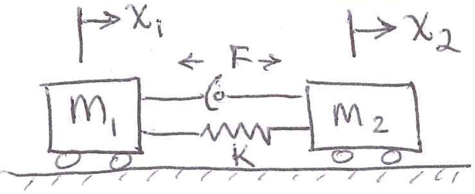
$\boxed{\Xi_1 = 0}$ ← no torque @ first joint

$\boxed{\Xi_2 = \tau}$ ← since θ_2 is RELATIVE, Ξ_2 is the torque on θ_2, relative to θ_1.

* Note if θ_2 had been ABSOLUTE, then $\boxed{\Xi_1 = -\tau}$, $\boxed{\Xi_2 = \tau}$

to give the <u>total</u> (net) torque on joint ξ_1...)

Lagrangian EOMs, Example 3 ← 3A



① **GC's** $\boxed{\xi_1 = x_1 \,,\ \xi_2 = x_2}$

Both ABSOLUTE coordinates.

② $\boxed{T^*}$ (kinetic en.)

$$\boxed{T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2}$$

③ $\boxed{V}$ (potential en.)

$$\boxed{V = \frac{1}{2} k (x_1 - x_2)^2}$$

④ $\boxed{\mathcal{L} = T^* - V}$

$$\boxed{\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_1^2 - 2x_1 x_2 + x_2^2)}$$

⑤ EOM, LHS: $\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = \Xi_1 \ ,\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = \Xi_2}$

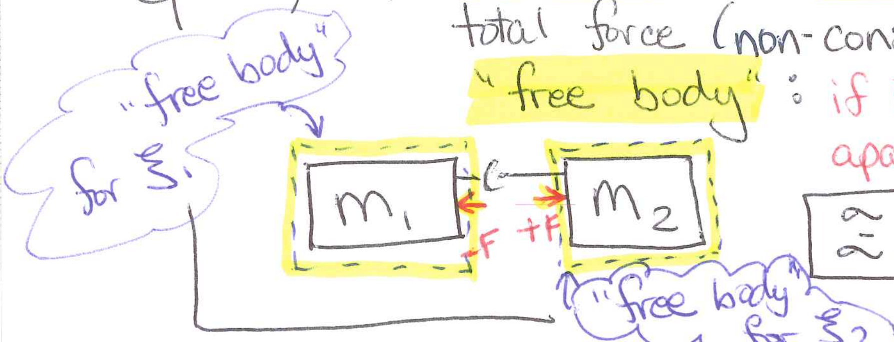
eq1 $\frac{d}{dt}(m_1 \dot{x}_1) - \left(-\frac{1}{2} k (2x_1 - 2x_2)\right) = \Xi_1$

$$\boxed{m_1 \ddot{x}_1 + k(x_1 - x_2) = \Xi_1}$$

eq2 $\frac{d}{dt}(m_2 \dot{x}_2 - (-k(-x_1 + x_2)) = \Xi_2$

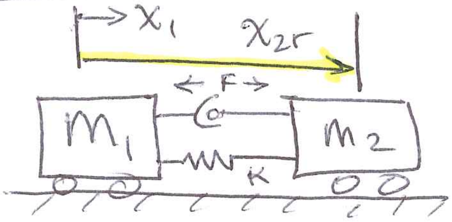
$$\boxed{m_2 \ddot{x}_2 - k(x_1 - x_2) = \Xi_2}$$

⑥ EOM, RHS. Since all GC's are ABSOLUTE coordinates, using total force (non-conservative) on each "free body": if F defined positive when "pushing apart" m_1 & m_2, then:



$\boxed{\Xi_1 = -F_1}$ $\boxed{\Xi_2 = +F}$

$\mathcal{L}$'n EOM, Example 3 ← 3B



① GC's $\quad \xi_1 = x_1 \quad , \quad \xi_2 = x_{2r}$

now, ξ_2 is RELATIVE to ξ_1

② T^* Be careful! The total velocity of m_2 is: $\dot{x}_1 + \dot{x}_{2r}$

$$\boxed{T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2}$$

③ V Be careful! "Δx" for the spring is $\equiv$ x_{2r} (relative)

$$\boxed{V = \frac{1}{2} k x_2^2}$$

④ $\mathcal{L} = T^* - V$

$$\boxed{\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) - \frac{1}{2} k x_2^2}$$

⑤ EOM, LHS: $\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = \Xi_1 \quad , \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = \Xi_2}$

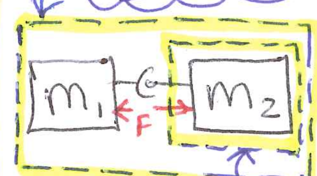
eq 1 $\quad \frac{d}{dt}(m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 \dot{x}_2) - (-k x_2) = \Xi_1$

$$\boxed{(m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_2 = \Xi_1}$$

eq 2 $\quad \frac{d}{dt}(m_2 \dot{x}_1 + m_2 \dot{x}_2) + k x_2 = \Xi_2$

$$\boxed{m_1 \ddot{x}_1 + m_2 \ddot{x}_2 + k x_2 = \Xi_2}$$

ξ_1 "free body" includes all children...



ξ_2 "free body" has no child so only m_2

⑥ EOM RHS: Each "free body" must include all children GC's bodies that are dependent i.e. RELATIVE, to this "parent"!

$\Xi_1 = 0$

$\Xi_2 = +F$