Heterogenous-Agent Life-Cycle Models: Transitions

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(with thanks to Pavel Brendler and Kuan Liu)

November 9, 2020

Introduction

- We have so far learned how to compute stationary equilibria of life cycle models.
- However, comparing only outcomes of stationary equilibria would not provide us the full picture of the effects of a policy change (e.g. elimination of pay-as-you-go social security)
- Another important aspect of policy analysis is how the economy transits to a new stationary equilibrium consistent with the policy change, i.e. the transition path
- Here we learn how to compute the transition path from an equilibrium with social security to one without.

The Concept of Transition

- Imagine that we are in the initial steady state at $t = 0$ with social security tax rate $\theta_0 > 0$.

- The government makes an unexpected and credible announcement to eliminate the social security system from $t = 1$ onward. This is what is sometimes called an "MIT shock".

- This means that from $t = 1$ onward, workers do not need to pay social security taxes (i.e. $\theta_t = 0$ for $t \geq 1$) and retirees lose their social security benefits (i.e. $b_t = 0$ for $t \geq 1$).

- Unlike the stationary equilibrium, the decision rules, the value functions, the cross-sectional distribution, and prices become calendar time t dependent until reaching the new steady state.

- This means in solving for the transition path, we need to add one more discrete finite state variable t.

Preliminaries

- For illustration, assume an inelastic labor supply so $L = 1$ (while Conesa/Krueger (1999) correctly parameterize a leisure choice with distortionary taxes, this simple example generates an upper bound welfare calculation when substitution effects outweigh income effects).

- Let the initial $t = 0$ cross sectional distribution of agents and the initial value function be denoted by $\Gamma_0(z, a, n; K_0^{ss})$ and $V_0(z, a, n; K_0^{ss})$ which comes from the steady state with $\theta_0 > 0$.

- Assuming the economy reaches the new steady state at time T, denote the steady state cross sectional distn and value function by $\Gamma_T(z, a, n; K_T^{ss})$ and $V_T(z, a, n; K_T^{ss})$ with $\theta_T = 0$.

- Our job here is to find the equilibrium path of aggregate variables (e.g. $K_t \to (r(K_t), w(K_t))$) when endogenous variables transition between $t = 0$ and $t = T$ with $\theta_t = \theta_T = 0$ for $t \geq 1$.

- Note that we don't know how long it takes to get to T (in principle, it takes an infinite amount of time).

Basic Steps to Solve for the Transition Path

- Taking T as given, we need a path of aggregate capital $\{K_t^i\}_{t=0}^T$ where i denotes iteration number and $K_0^i = K_0^{ss}$ and $K_T^i = K_T^{ss}$
- With $L = 1$ for example, this induces a path for prices: $\{r_t^i, w_t^i\}_{t=0}^T$
- Since we know $V_T(z, a, n; K_T^{ss})$, we solve the household problem backwards from $t = T - 1$ to $t = 0$. This induces a set of savings decision rules $g_t^i(z, a, n; K_t^i)$ for $t = 0, ..., T - 1$ based on prices induced by $\{K_t^i\}_{t=0}^T$.
- Using $g_t^i(z, a, n; K_t^i)$ and $\Gamma_t^i(z, a, n; K_t^i)$, calculate forward a new path for capital $\{\hat{K}_t^{i+1}\}_{t=1}^T$ starting from $t = 0$ (noting that since $K_1^i \neq K_0^{ss} \rightarrow g_0^i(z, a, n; K_0^{ss}) \neq g_0^{ss}(z, a, n; K_0^{ss})$). Iterate on $\{\hat{K}_t^{i+1}\}_{t=1}^T$ until convergence.
- Shooting forward, there is nothing that ensures $\hat{K}_T^{i+1}$ is arbitrarily close to K_T^{ss}. Increase T until convergence.

HH Problem at t=T-1 at iteration i

$$V^i_{T-1}(z,a,n;K^i_{T-1}) = \max_{\{c\geq 0, a'\geq \underline{a}\}} u(c) + \beta s_{n+1}\mathbb{E}_{T-1}\left[V_T(z',a',n+1;K^{ss}_T)\right]$$

s.t.

$$c + a' = a(1+r(K^i_{T-1})) + (1-\theta_{T-1})e(z,n)w(K^i_{T-1}) + T + b_{n,T-1}(K^i_{T-1})$$
$$a = 0 \text{ if } n = 1$$
$$a' \geq 0 \text{ if } n = N.$$

- Assuming a steady state at time T, $K^i_T = K^{ss}_T$ with $\theta_T = 0$ and no matter what iteration i we are on, $V_T(z',a',n+1;K^{ss}_T)$ is the same.
- However, if $K^i_{T-1} \neq K^{ss}_T$ (i.e. we are not already at the steady state), the solution to this problem induces

$$k^i_T = g^i_{T-1}(z,a,n;K^i_{T-1}) \neq g_T(z,a,n;K^{ss}_T). \quad (1)$$

HH Problem between t=T-2 and t=1 at iteration i

$$V_t^i(z,a,n;K_t^i) = \max_{\{c\geq 0, a'\geq \underline{a}\}} u(c) + \beta s_{n+1}\mathbb{E}_t\left[V_{t+1}^i(z',a',n+1;K_{t+1}^i)\right]$$

s.t.

$$c + a' = a(1+r(K_t^i)) + (1-\theta_t)e(z,n)w(K_t^i) + T + b_{n,t}(K_t^i)$$
$$a = 0 \text{ if } n = 1$$
$$a' \geq 0 \text{ if } n = N.$$

- The solution to this problem induces $\{g_t^i(z,a,n;K_t^i)\}_{t=1}^{t=T-2}$.

HH Problem at t=0 at iteration i

$$V_0^i(z,a,n;K_0^{ss},K_T^{ss}) = \max_{\{c\geq 0, a'\geq \underline{a}\}} u(c) + \beta s_{n+1}\mathbb{E}_0\left[V_1^i(z',a',n+1;K_1^i)\right]$$

s.t.

$$\begin{aligned} c + a' = a(1+r(K_0^{ss})) &+ (1-\theta_0)e(z,n)w(K_0^{ss}) + T + b_{n,t}(K_0^{ss}) \\ a &= 0 \text{ if } n = 1 \\ a' &\geq 0 \text{ if } n = N. \end{aligned}$$

- Note that even though $\theta_0 > 0$ and K_0^{ss} at $t = 0$, since $\theta_1 = 0$ and $K_1^i \neq K^{ss}$, $V_0^i(z,a,j;K_0^{ss},K_T^{ss})$ can be different from the steady state $V_0(z,a,j;K_0^{ss})$ inducing $g_0^i(z,a,n;K_0^{ss}) \neq g_0(z,a,n;K_0^{ss})$.

Algorithm

1. Choose the number of periods to reach the final steady state, T. For example, start with $T = 20$
2. Given T, make an initial guess for the sequence $\{K_t^{i=1}\}_{t=0}^{T}$ with $K_0^1 = K_0^{SS}$ and $K_T^1 = K_T^{SS}$.
   - A good guess could simply be a linear function from K_0^{SS} to K_T^{SS} given by $K_t^1 = K_0^{SS} + \triangle$, where $\triangle = [K_T^{SS} - K_0^{SS}]/T$
3. Given $\{K_t^i\}_{t=0}^{T}$, solve HH's dynamic programming problem backwards starting with $t = T - 1$ obtaining decision rules $g_t^i(z, a, n; K_t^i)$ for $t = 0, \ldots T - 1$.

Algorithm - cont'd

4. Update the path of capital as follows: start with $\Gamma_0(z, a, n; K_0^{ss})$ and use the decision rules $g_t^i(z, a, n; K_t^i)$ to obtain Γ_t^{i+1} and $\hat{K}_t^{i+1}$ for $t = 0 \ldots T$. That is:

$$\Gamma_{t+1}^{i+1}(z', a', n; K_{t+1}^i) = H(\Gamma_t^{i+1}(z, a, n; K_t^i))$$

   and

$$\hat{K}_{t+1}^{i+1} = \sum_{n=1}^{N} \sum_{z} \int_a g_t^i(z, a, n; K_t^i) \Gamma_t^i(z, a, n; K_t^i) da. \tag{2}$$

5. If $\left\| \hat{K}_t^{i+1} - K_t^i \right\|_\infty < \epsilon_K$ for $t = 0, \ldots, T$ where ϵ_K is your tolerance level, go to step 6. If not, update K_t^{i+1} as follows:

$$K_t^{i+1} = \rho K_t^i + (1-\rho)\hat{K}_t^{i+1} \;\; \forall t \text{ with some } \rho \in (0,1)$$

   and go to step 3 using the updated $\{K_t^{i+1}\}_{t=0}^{T-1}$.

Algorithm Cont'd

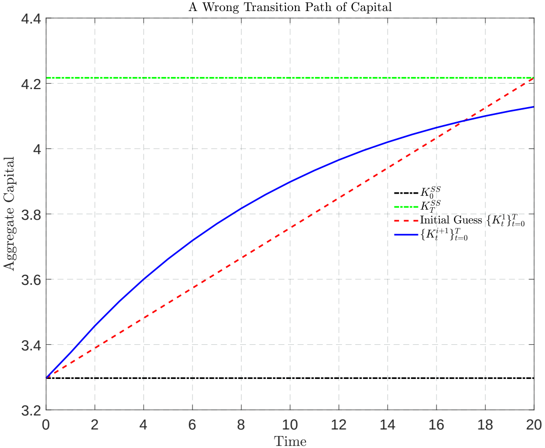
6. By (1) and (2), there is nothing to ensure that after step 5, $\hat{K}_T^{i+1} = K_T^{SS}$. If there is a $t^* \leq T$ in the sequence such that:

$$\left\| \hat{K}_t^{i+1} - K_T^{SS} \right\| < \epsilon_T \ \forall t \geq t^*$$

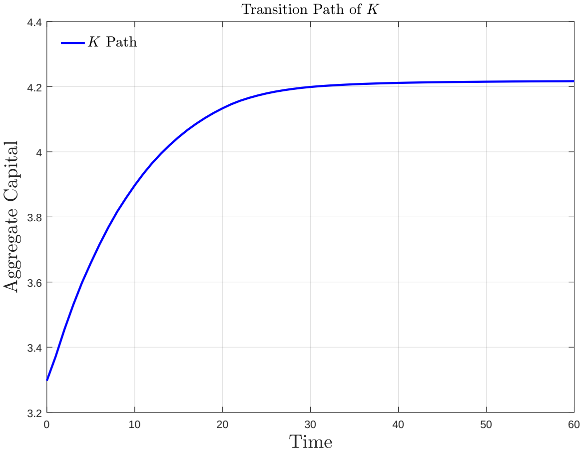
   where ϵ_T is your tolerance level, you have approximated the equilibrium path. If not, increase T and go back to step 2.

A Wrong Transition Path of Capital

- Suppose we set $T = 20$
- The resulting transition path shown below satisfies the criterion of step 5 in the algorithm; however it fails the criterion of step 6.



The Equilibrium Transition Path of Capital



- If you get rid of social security, people need to save privately for retirement (i.e. K rises).

Computing Welfare Gains/Losses with Transitions

We quantify the welfare change of the policy reform for an agent in state (z, a, n) relative to the initial steady state by asking:

- What fraction of per-period consumption would an agent in the steady state (with $b > 0$) be willing to pay (if positive) or have to be paid (if negative) in all future periods to achieve the utility level associated with a transition to the final steady state (with $b = 0$)?
- Conesa/Krueger use the following utility function (we set $\gamma = 1$ instead of $\gamma = 0.42$ and $\sigma = 2$):

$$u(c, l) = \frac{(c^{\gamma}(1-l)^{1-\gamma})^{1-\sigma}}{1-\sigma}$$

- We compute for each (z, a, n) the consumption equivalent $\lambda(z, a, n)$ such that:

$$V_0(z, a, n; K_0^{SS}, K_T^{SS}) =$$

$$\mathbb{E}\Big[\sum_{t=n}^{N} \beta^t \frac{[((1+\lambda(z,a,n))c_t(z,a,n))]^{1-\sigma}}{1-\sigma} |(z, a, n)\Big]$$

where the lhs is the $b = 0$ value and the rhs is the $b > 0$ ss value.

Computing CE with Transitions - cont.

- We can re-arrange the previous equation defining $\lambda(z,a,n)$ as

$$
\begin{aligned}
&V_0(z,a,n;K_0^{SS},K_T^{SS}) = \\
&(1+\lambda(z,a,n))^{(1-\sigma)}\mathbb{E}\Big[\sum_{t=n}^{N}\beta^t\frac{c_t(z,a,n)^{1-\sigma}}{1-\sigma}|(z,a,n)\Big] \\
&\qquad = (1+\lambda(z,a,n))^{(1-\sigma)}V_0(z,a,n;K_0^{SS})
\end{aligned}
$$

- Hence we have

$$
(1+\lambda(z,a,n))^{(1-\sigma)} = \frac{V_0(z,a,n;K_0^{SS},K_T^{SS})}{V_0(z,a,n;K_0^{SS})} \Rightarrow
$$

$$
\lambda_{tran}(z,a,n) = \left[\frac{V_0(z,a,n;K_0^{SS},K_T^{SS})}{V_0(z,a,n;K_0^{SS})}\right]^{\frac{1}{(1-\sigma)}} - 1
$$

Interpreting CE

- For example, $\lambda_{tran}(z, a, n) = 0.1$ implies that eliminating social security increases the welfare of an individual in state (z, a, n) by an amount equivalent to receiving 10% higher consumption per period in the initial steady state for all future periods.
- Therefore $\lambda_{tran}(z, a, n) \geq 0$ means that the agent is made better off by the reform and will thus vote for it; otherwise, the agent is made worse off by the reform and will vote against it.
- The total mass of population in the initial steady state who vote for the reform is given by

$$\sum_{n=1}^{N} \sum_{z} \int_{a} \Gamma_0^{SS}(z, a, n; K_0^{SS}) \times 1_{(\lambda_{tran}(z,a,n) \geq 0)} da$$

CE across Steady States

- We can also compute CE simply across the initial and the new steady states (i.e. without accounting for the transition path).
- This can be thought of CE in the counter-factual case where the economy jumps to the new steady state immediately after social security is abolished, unrealistically skipping the transition.
- Using the results for the initial and the new steady states and following the definition of CE given above, it is computed by:

$$\lambda_{SS}(z,a,n) = \left[\frac{V_T(z,a,n;K_T^{SS})}{V_0(z,a,n;K_0^{SS})} \right]^{\frac{1}{(1-\sigma)}} - 1$$

The Welfare Effect of the Transition Path I

- Let's first compare the average welfare gain/loss within each age cohorts with and without transition path.
- We use CE_{type} for $type \in \{tran, SS\}$ as our measure of welfare gain/loss and its age cohort average given by

$$CE_{type}(n) = \sum_{z} \int_{a} \lambda_{type}(z, a, n) \frac{\Gamma_0^{SS}(z, a, n; K_0^{SS})}{\mu_n} da, \forall n$$

What accounts for the welfare differences across age and SS versus Transition?

- The above figure establishes that there is:
  - Across age: the reform only makes agents under 25 years old better off ($CE_{type}(n) > 0$).
  - Across type:
    - less (more) of a welfare gain (loss) accounting for the transition for younger agents.
    - less of a welfare loss accounting for the transition for the old.
- Why?
  - There are higher wages in the final steady state than in the transition boosting new steady state income for the young. ▸ Wage Path
  - There are lower interest rates in final new steady state than in the transition boosting transition interest income for the old. ▸ Interest Rate Path ▸ Decision Rules

Aggregate Voting Differences

- Now we compare the total mass of population who vote for the reform with and without considering the transition path as given by:

$$\sum_{n=1}^{N} \sum_{z} \int_{a} \Gamma_0^{SS}(z, a, n; K_0^{SS}) \times 1_{(\lambda_{type}(z,a,n) \geq 0)} da$$

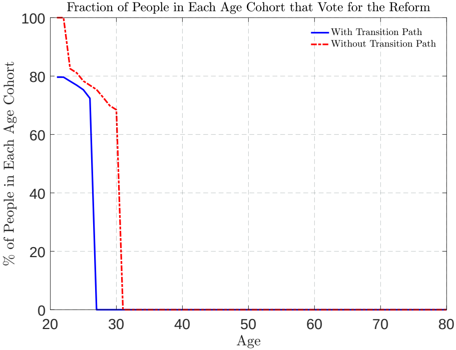
- Results:

	Fraction of Population that Vote for the Reform
With Transition	10.56%
Without Transition	16.29%

- This implies that we will *over-estimate* the welfare gain of the reform if we do not consider transition path.

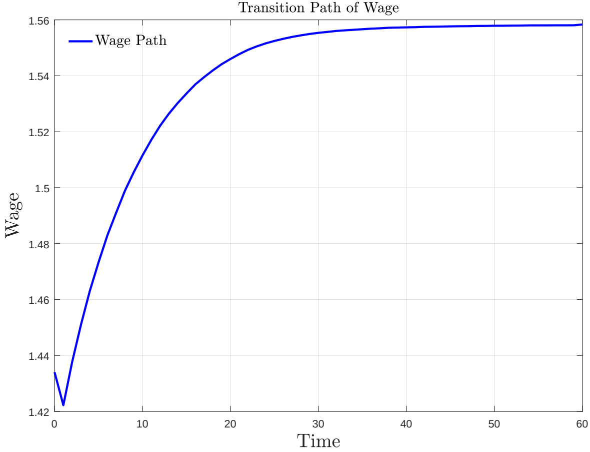
Voting by Age Differences

- Finally, we compare the fraction of people in each age cohort that would vote for the reform with and without considering the transition path given by: $\sum_z \int_a \frac{\Gamma_0^{SS}(z,a,n;K_0^{SS})}{\mu_n} \times 1_{(\lambda_{type}(z,a,n)\geq 0)} da, \forall\, n$



- As implied by the above figure, this measure suggests that by not considering the transition path, we (weakly) over-estimate welfare gains of the reform for every age group.

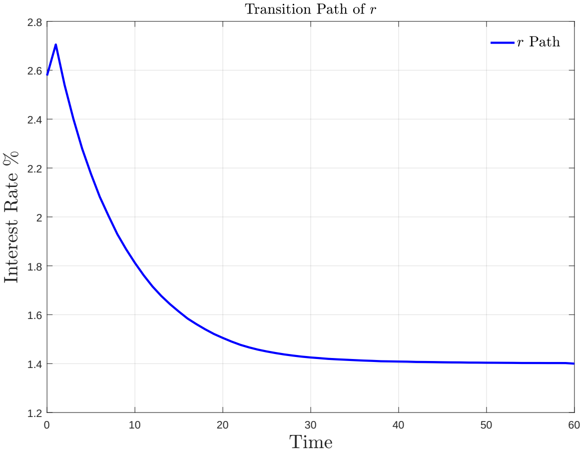
The Equilibrium Transition Path of Wages



- Since $w = (1-\alpha)(\frac{K}{L})^{\alpha}$ and $K \uparrow$, wages rise.

Back

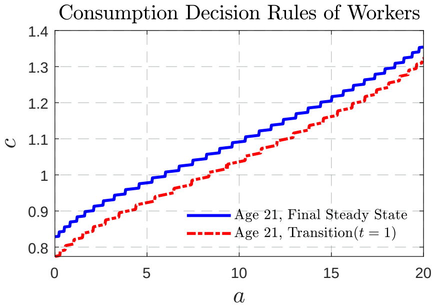
The Equilibrium Transition Path of Interest Rates

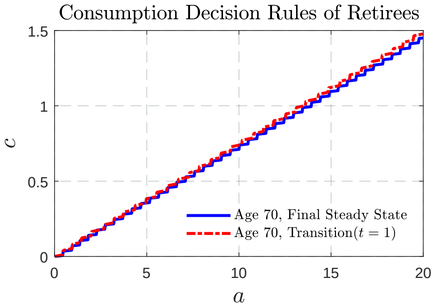


- Since $r = \alpha(\frac{L}{K})^{1-\alpha}$ and $K \uparrow$, interest rates fall.

Back

Decision Rules Consumption for Young and Old





- 21 year olds consume less (save more) at start of transition than in steady state, the opposite of retirees.

Back