Approximating the Wealth Distribution with Aggregate Uncertainty

Krusell and Smith (1998, JPE) consider an Aiyagari economy with aggregate uncertainty (where the lower bound on capital holdings is assumed to be zero) and argue that it may be sufficient to characterize the wealth distribution simply by its first moment.

Note that there are many similarities between this approximation methodology in macro and the approximation methodology used in IO called "oblivious equilibrium" by Weintraub, G.Y., C.L. Benkard, and B. Van Roy (2008, Econometrica). In that paper, the distribution of investment strategies of other firms are approximated by the industry "long run" or "average" strategy. Intuitively aggregate shocks have a "big" impact on prices/distributions in the macro literature in the same way that a finite number of strategic firms have a "big" impact on prices in the IO literature.

Methodologically, this will be the first time you generate a pseudo panel of data. The pseudo panel can be used to calculate moments of interest in the model (here the conditional mean) to be compared to the associated moment in the data. This will prove useful when doing simulated method of moments or indirect inference.

1 Environment

- Unit measure of agents.
- Time period is one quarter.
- Preferences
$$\sum_{\tau=0}^{\infty} \beta^t \ln(c_t)$$
where $\beta = 0.99$.
- Production technology
$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$$
where $\alpha = 0.36$,and aggregate technology shocks $z_t \in \{z_g = 1.01, z_b = 0.99\}$ are drawn from a markov process with transition matrix probabilities $\pi_{zz'} = prob(z_{t+1} = z' | z_t = z)$.
- Capital depreciates at rate $\delta = 0.025$.
- Idiosyncratic employment opportunities ε_t. Since agents derive no utility from leisure and have 1 unit time, labor input $\varepsilon_t = 1$ means the agent is employed with efficiency $\overline{e}$ receiving wage w_t per unit of labor efficient time and $\varepsilon_t = 0$ means he is unemployed.

- Employment opportunities are correlated with the aggregate state of the economy. By virtue of the law of large numbers and iid employment opportunities across agents, the only source of aggregate uncertainty are aggregate shocks. More specifically, the number of agents who are unemployed in the good state is always u_g and the number unemployed in the bad state is always u_b (i.e. when you control for z, individual shocks are uncorrelated).

- In terms of exogenous uncertainty, the markov transition matrix from state (z, ε) to (z', ε') is denoted $\pi_{zz'\varepsilon\varepsilon'}$. In sum, there are 12 parameters in the transition matrix (which takes into account the adding up constraint across each row) K-S use a set of restrictions/assumptions to pin down $\pi_{zz'\varepsilon\varepsilon'}$.

- Incomplete Asset Markets: Households rent their capital $k_t \in [0, \infty)$ to firms and receive rate of return r_t (strict borrowing constraint).

2 Equilibrium

- Given CRS and no firm level uncertainty, WLOG can consider one firm which hires L_t units of labor and capital K_t so that wages and rental rates are given by their marginal products:

$$\begin{aligned} w_t &\equiv w(K_t, L_t, z_t) = (1-\alpha) z_t \left(\frac{K_t}{L_t}\right)^{\alpha} \quad (1) \\ r_t &\equiv r(K_t, L_t, z_t) = \alpha z_t \left(\frac{K_t}{L_t}\right)^{\alpha - 1} \end{aligned}$$

- In each aggregate state z_t, letting Γ_t denote the distribution of agents over (k_t, ε_t),the aggregate state of the economy at time t can be summarized by (Γ_t, z_t) and the individual's state is $(k_t, \varepsilon_t; \Gamma_t, z_t)$. In that case, the agent's optimization problem can be expressed as:

$$v(k_t, \varepsilon_t; \Gamma_t, z_t) = \max_{c_t, k_{t+1} \geq 0} u(c_t) + \beta E_t \left[v(k_{t+1}, \varepsilon_{t+1}; \Gamma_{t+1}, z_{t+1}) \right] \quad (2)$$

s.t.

$$c_t + k_{t+1} = r(K_t, L_t, z_t) k_t + w(K_t, L_t, z_t) \bar{e} \varepsilon_t + (1-\delta) k_t \quad (3)$$

Let the decision rule that solves this problem be given by $k_{t+1} = g(k_t, \varepsilon_t; \Gamma_t, z_t)$.

- Let the law of motion for the distribution be given by

$$\Gamma_{t+1} = H(\Gamma_t, z_t, z_{t+1}) \quad (4)$$

where H maps distributions to distributions (via a transition function). The reason why z_{t+1} is included in $H(\cdot)$ is because even though k_{t+1} is chosen at t, ε_{t+1} in Γ_{t+1} is drawn from a matrix which depends on z_{t+1}(i.e. the current aggregate state affects employment opportunities and hence the current cross sectional distribution).

- Notice that in general Γ_{t+1} depends not only on z_{t+1} but also on where you came from z_t. That is, $\Gamma_{t+1}(\cdot; z_{t+1} = z_g)$ would in general be different depending on whether last period $z_t = z_g$ or $z_t = z_b$. Even more generally, using (4) we see that Γ_{t+1} depends on the entire history of aggregate shocks since

$$\Gamma_{t+1} = H(H(\Gamma_{t-1}, z_{t-1}, z_t), z_t, z_{t+1}) = H(H(H(\Gamma_{t-2}, z_{t-2}, z_{t-1}), z_{t-1}, z_t), z_t, z_{t+1}).$$

  Note that if there were no $z's$ and if we start with a steady state distribution, then we maintain the steady state according to $\Gamma = H(H(H(\Gamma)))$.

- Market clearing (i.e. the demand for capital and labor by the firm equals the supply of capital and labor by households):[1]

$$K_{t+1} = \int g(k_t, \varepsilon_t; \Gamma_t, z_t)\Gamma_t(dk_t, d\varepsilon_t; \Gamma_t, z_t), \tag{5}$$

$$L_t = \overline{e}(1 - u_t) \tag{6}$$

Definition 1 *A recursive competitive equilibrium is a law of motion H, a pair of functions (v, g), and pricing functions (r, w) such that: (i) (v, g) solve (2); (ii) (r, w) solve (1); (iii) H in (4) is generated by g and $\pi_{zz'\varepsilon\varepsilon'}$; and (iv) markets clear (5)-(6).*

- Miao (2006, JET) proves existence of a recursive competitive equilibrium for this problem. He does so in the following steps:
  - Lemma 1. Existence of a unique sequence of policy functions applying the T operator.
  - Lemma 2. The sequence of equilibrium distributions evolve according to the T^* operator (a closedness result on H).
  - Theorem 1. There exists a sequential competitive equilibrium and the set of equilibrium distributions are compact.
  - Theorem 2. A recursive competitive equilibrium generates a sequential competitive equilibrium.
  - Theorem 3. A recursive competitive equilibrium exists.

3 Computation

- Since Γ is a high dimensional object, the state space in the dynamic programming problem makes computation difficult.

[1] Note that with a specification of preferences which don't depend on leisure, L is exogenously determined independent of the cross-sectional wealth distribution. Appendix B of the paper studies the elastic labor supply decision.

- The role of the aggregate state of the economy (Γ_t, z_t) is to help households predict future prices (interest rates and wages), which they need to know in order to choose savings k_{t+1}. That is, substituting the budget constraint into the objective, a necessary condition for household optimization is:

$$\begin{aligned} u'(c_t) &= \beta E_t \left[\frac{\partial v(k_{t+1}, \varepsilon_{t+1}; \Gamma_{t+1}, z_{t+1})}{\partial k_{t+1}} \right] \\ &= \beta E_t \left[u'(c_{t+1}) \cdot \{ r(K_{t+1}, L_{t+1}, z_{t+1}) + (1-\delta) \} \right] \end{aligned}$$

  where the second equality follows from the envelope condition (where c_{t+1} depends on $w(K_{t+1}, L_{t+1}, z_{t+1})$).

- To figure out (r_{t+1}, w_{t+1}), all we need to know is K_{t+1} in (5) for each state of the world z_t given that $L_{t+1} = \bar{e}(1-u_{t+1})$ is exogenous. However, $K_{t+1} \neq g(K_t, \varepsilon_t; \Gamma_t, z_t)$. That is, the average capital stock at $t+1$ is not in general equal to the savings function of the representative agent evaluated at the time t average capital stock. This is because consumers with different (k_t, ε_t) generally have different propensities to save out of current wealth.

- Only if g is a linear function in k_t with the same slope for everyone can we obtain a simple aggregation result. This is the key observation - since savings functions are approximately linear for most people (i.e. those not near the borrowing constraint), it may be a good approximation to use the mean. That is because if g is linear in (5), the right hand side is effectively the definition of the mean capital stock.

- Krusell and Smith explore whether it is possible to approximate the distribution Γ_t with a finite set of M moments, call it $m = \{m_1, m_2, ..., m_M\}$ with law of motion for the moments $m' = h_M(m, z, z')$.[2]

- Given the perceived law of motion h_M, the agents' optimal decision rules are given by $k' = g_M(k, \varepsilon; m, z)$. This methodology yields an approximate equilibrium, and the objective is to find a parsimonious but accurate (in the sense of forecasting prices) perceived law of motion h_M (which means both choosing M and functional form h correctly).

- In summary, while the law of motion of the wealth distribution for the "true" economy is $\Gamma' = H(\Gamma, z, z')$, the law of motion for a "mean approximate" economy is given by $\overline{K}' = h_1(\overline{K}, z)$ as in (7) below. Specifically, we approximate the true programming problem of

$$\begin{aligned} g(k, \varepsilon; \Gamma, z) &= \arg\max_{k' \geq 0} u(r(K, L, z)k + w(K, L, z)\bar{e}\varepsilon + (1-\delta)k - k') + \beta E\left[v(k', \varepsilon'; H(\Gamma, z, z'), z') | z, \varepsilon \right] \\ s.t. K' &= \int g(k, \varepsilon; \Gamma, z) \Gamma(dk, d\varepsilon; \Gamma, z), L = \bar{e}(1-u) \end{aligned}$$

[2] Just as a distribution function is an equally informative function of a given density function, so too is the moment generating function (when it exists). That is, there is a one-to-one correspondence with the density.

by

$$g_1(k,\varepsilon;\overline{K},z) = \arg\max_{k'\geq 0} u(r(\overline{K},\overline{e}(1-u),z)k+w(\overline{K},\overline{e}(1-u),z)\overline{e}\varepsilon+(1-\delta)k-k')+\beta E\left[v(k',\varepsilon';h_1(\overline{K},z),z')|z,\right.$$

3.1 Algorithm Steps

1. Select M. For example, $M = 1$. Then only the mean matters (i.e. $m = \{\overline{K}\}$).

2. Conjecture a parameterized functional form for h_M. For example, h_1 is log linear. Specifically, for iteration i, let

$$\log \overline{K}' = \begin{cases} a_0^i + a_1^i \log \overline{K} & \text{if } z = z_g \\ b_0^i + b_1^i \log \overline{K} & \text{if } z = z_b \end{cases} \tag{7}$$

   Notice that they do not conjecture that it depends on z'. If so, it would have more parameters.

3. Given h_M^i, solve the consumers problem. For above example

$$v^i(k,\varepsilon;\overline{K},z) = \max u(c) + \beta E_t\left[v^i(k',\varepsilon';\overline{K}',z')\right]$$

   s.t. (3) and (7). This yields a decision rule for savings $k' = g^i(k,\varepsilon;\overline{K},z)$. It depends on i since the law of motion you assign for $\overline{K}$ affects forecasts of future prices which affect savings decisions. Note that since "prices" vary with the state, we now have 2 continuous state variables $(k,\overline{K})$ to compute value functions for so we will learn interpolation methods in the next set of notes).

4. Use the decision rules generated in step 3 and the transition function $\pi_{zz'\varepsilon\varepsilon'}$ to simulate the behavior of N households, where $N = 5000$, for $T = 11,000$ periods, discarding the first 1000 periods (to deal with initial condition dependence).

   (a) Starting from any $z_{t=1}$ (say z_g) simulate a path of technology shocks (this will be the first row of the matrix V below. Since Krusell and Smith say (p. 877) that the average duration of both good and bad times is 8 quarters, this implies that the transition matrix for aggregate shocks is given by $\pi(z' = g|z = g) = \pi(z' = b|z = b) = 1-1/8 = 0.875$ So just run a uniform [0,1] random number generator in Matlab and if the number is in $[0, 0.875]$ stay in the same state and if it is in $(0.875, 1]$ switch states. Alternatively you can just use the appropriate parts of the transition matrix $\pi_{zz'\varepsilon\varepsilon'}$ I provide on my website: http://www.eco.utexas.edu/~corbae/transmat.m.

   (b) Starting from any $\varepsilon_{t=1}$ (say $\varepsilon = 1$) and using the realization of the aggregate shock, use Krusell and Smith's transition matrix $\pi_{zz'\varepsilon\varepsilon'}$ to simulate N sequences of ε_t^n shocks, $n = 1, ..., N$ and $t = 2, ...T$. .

(c) Solve for the steady state value of the capital stock for a complete markets version of the Krusell-Smith economy. Call it K^{ss}. Let that value be the initial average capital stock for the simulation and endow each agent with that amount of capital (i.e. $k_1^n = \overline{K}_1 = K^{ss}$).

(d) Using the decision rules from step 3, calculate $k_2^n = g^i(k_1 = K^{ss}, \varepsilon_1^n; \overline{K}_1 = K^{ss}, z_1)$ for each person $n = 1, ..., N$. Calculate the mean of rows 2 to N+1 (i.e.$\overline{K}_2 = \frac{1}{N}\sum_{n=1}^{N} k_2^n$) This fills up the first column of V.

(e) Using the values you have for k_{t+1}^n and $\overline{K}_{t+1}$ in the previous column $t = 1, ..., T-1$, calculate the next column of matrix V.

person/time	z_1	z_2	...
$n=1$	$k_2^1 = g^i(K^{ss}, \varepsilon_1^1, K^{ss}, z_1)$	$k_3^1 = g^i(k_2^1, \varepsilon_2^1, \overline{K}_2, z_2)$	...
$n=2$	$k_2^2 = g^i(K^{ss}, \varepsilon_1^2, K^{ss}, z_1)$	$k_3^2 = g^i(k_2^2, \varepsilon_2^2, \overline{K}_2, z_2)$	...
...	...	...	...
$n=N$	$k_2^N = g^i(K^{ss}, \varepsilon_1^N, K^{ss}, z_1)$	$k_3^N = g^i(k_2^N, \varepsilon_2^N, \overline{K}_2, z_2)$	...
Agg. Capital	$\overline{K}_2 = \frac{1}{N}\sum_{n=1}^{N} k_2^n$	$\overline{K}_3 = \frac{1}{N}\sum_{n=1}^{N} k_3^n$	

5. Dropping the first 1000 observations of the matrix, (re)estimate the autoregression in step 2 by sorting the data on the basis of whether z_t is good or bad. That is, if $z_t = g$, place $\overline{K}_{t+1}$ and $\overline{K}_t$ into one matrix and if $z_t = b$ place them into another. Run the two regressions to obtain the a^{i+1} and b^{i+1} coefficients.

6. If the new parameter vector $(a_0^{i+1}, a_1^{i+1}, b_0^{i+1}, b_1^{i+1})$ is sufficiently close to the original parameter vector $(a_0^i, a_1^i, b_0^i, b_1^i)$ and the "goodness of fit" (e.g. R^2 of regression in (7)), is sufficiently high, stop. If the parameter values have converged, but the goodness of fit is not high enough, increase M in step 1 or try a different functional form in step 2.

- Note that unlike the earlier papers without aggregate uncertainty, here the state vector depends on k and K (in the earlier ones it really only depended on k since agents faced the same price in the future whereas now they face different prices depending on whether there is a good or bad aggregate shock). Thus you need to have 2 continuous state variables here instead of one, which is why we will use bilinear interpolation. Linear interpolation along the K dimension works well.

3.2 Results

- If $z_t = z_g$, $\log \overline{K}_{t+1} = 0.095 + 0.962 \log \overline{K}_t$ with $R^2 = 0.99998$ and standard deviation of the regression error $\widehat{\sigma} = 0.0028\%$.

- If $z_t = z_b$, $\log \overline{K}_{t+1} = 0.085 + 0.965 \log \overline{K}_t$ with $R^2 = 0.99998$ and standard deviation of the regression error $\widehat{\sigma} = 0.0036\%$.

- Including higher moments doesn't improve fit.