ECON899, Computational Economics
Fall 2021, University of Wisconsin
Instructor: Dean Corbae

Problem Set #5- Goal Due Date 10/13/21

You are to compute an approximate equilibrium of an Aiyagari (1994) paper with aggregate uncertainty using the techniques in Krusell and Smith (1998). As discussed in class, there is a unit measure of agents, the time period is one quarter, preferences are given by

$$\sum_{\tau=0}^{\infty} \beta^t \ln(c_t)$$

where $\beta = 0.99$.The production technology is given by

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

where $\alpha = 0.36$,and aggregate technology shocks $z_t \in \{z_g = 1.01, z_b = 0.99\}$ are drawn from a markov process to be described more fully below. Capital depreciates at rate $\delta = 0.025$.Agents have 1 unit of time and face idiosyncratic employment opportunities $\varepsilon_t \in \{0,1\}$ where $\varepsilon_t = 1$ means the agent is employed an receives wage $w_t \overline{e}$ (where $\overline{e} = 0.3271$ denotes labor efficiency per unit of time worked) and $\varepsilon_t = 0$ means he is unemployed. The probability of transition from state (z, ε) to (z', ε'), denoted $\pi_{zz'\varepsilon\varepsilon'}$ must statisfy certain conditions:

$$\pi_{zz'00} + \pi_{zz'01} = \pi_{zz'10} + \pi_{zz'11} = \pi_{zz'}$$

and

$$u_z \frac{\pi_{zz'00}}{\pi_{zz'}} + (1-u_z)\frac{\pi_{zz'10}}{\pi_{zz'}} = u_{z'}$$

where u_z denotes the fraction of those unemployed in state z with $u_g = 4\%$ and $u_b = 10\%$. The other restrictions on $\pi_{zz'\varepsilon\varepsilon'}$ necessary to pin down the transition matrix are that: the average duration of good and bad times is 8 quarters; the average duration of unemployment spells is 1.5 quarters in good times and 2.5 quarters in bad times; and

$$\frac{\pi_{gb00}}{\pi_{gb}} = 1.25 \cdot \frac{\pi_{bb00}}{\pi_{bb}} \text{ and } \frac{\pi_{bg00}}{\pi_{bg}} 0.75 \cdot \frac{\pi_{gg00}}{\pi_{gg}}.$$

See my website for the file transmatrix.m which actually computes the transition matrix for you. Capital is the only asset to self insure fluctuations; households rent their capital $k_t \in [0, \infty)$ to firms and receive rate of return r_t. Without loss of generality, we can consider one firm which hires L_t units of labor efficiency units (so that $L_t = \overline{e}(1-u_t)$) and rents capital K so that wages and rental rates are given by their marginal products:

$$\begin{aligned} w_t &\equiv w(K_t, L_t, z_t) = (1-\alpha) z_t \left(\frac{K_t}{L_t}\right)^{\alpha} \qquad (1) \\ r_t &\equiv r(K_t, L_t, z_t) = \alpha z_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} \end{aligned}$$

As in Krusell and Smith, approximate the true distribution Γ_t over (k_t, ε_t) in state z_t by I moments and let the law of motion for the moment be $m' = h_I(m, z, z')$.

To start the Krusell-Smith algorithm, we need initial conditions. There's only

2 possibilities for (z_t, ε_t) so choose the ones that are most likely (i.e. z_g and use $L_g = 1 - u_g = 0.96$ to generate $\varepsilon_{t=0}$). But to speed things along, we would like to have a good starting point for (k_t, K_t). To that end, we can solve for a steady state of the complete markets (representative agent) version of the model. Specifically we let $z = 1$, $L^{ss} = \pi L_g + (1-\pi)L_b$ where π is the long run probability of state g induced by $\pi_{zz'}$ and $L_g = 1 - u_g = 0.96$ and $L_b = 1 - u_b = 0.9$. The steady state solves the Euler equation

$$u'(c) = \beta u'(c)(r(K^{ss}, L^{ss}) + 1 - \delta) \iff \frac{1}{\beta} = \left(\alpha\left(\frac{K^{ss}}{L^{ss}}\right)^{\alpha-1} + 1 - \delta\right) \iff K^{ss} = \left(\frac{\alpha}{1/\beta + \delta - 1}\right)^{\frac{1}{1-\alpha}} L^{ss}.$$

Plugging the parameter values into the above expression, we have $K^{ss} = 11.55$. As you will see below we set the lower bound of the interval of possible K to 11 but the upper bound all the way to 15. Why? Think like economists: complete markets means that households do not have to precautionarily save for an unemployed, low consumption state. Hence one should expect K^{ss} to be a lower bound.

Algorithm

1. Let $I = 1$ (which means only average capital holdings matter).

2. Generate a $T = 11,000$ sequence of z_t using the $\pi_{zz'}$ markov matrix starting state z_g and for each generate z_t generate the $N = 5000$ of ε_t shocks using $\pi_{zz'\varepsilon\varepsilon'}$. Save these $Tx1$ vector of z_t (call it Z) and NxT matrix (where the row is a person's employment status in a given aggregate state calling it $\mathcal{E}$).

3. Conjecture a log linear functional form for h_1;Specifically let
$$\log K' = \begin{cases} a_0 + a_1 \log K & \text{if } z = z_g \\ b_0 + b_1 \log K & \text{if } z = z_b \end{cases} \tag{2}$$
As an initial guess, one could simply start with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$.To speed things along, choose $a_0 = 0.095, b_0 = 0.085$ and $a_1 = b_1 = 0.999$.

4. Given h_I, solve the conumers problem. For the above example
$$v(k, \varepsilon; K, z) = \max_{c,k'} u(c) + \beta E_t\left[v(k', \varepsilon'; K', z')\right]$$
s.t.
$$c + k' = r(K, L, z)k + w(K, L, z)\varepsilon + (1-\delta)k$$
as well as (1) and (2). Let k lie in $[0, 15]$ and K lie in $[11, 15]$ and use bilinear interpolation over these two dimensions of the state vector (the other 4 states are discrete so simply index 4 different value functions by $(g, 0), (b, 0), (g, 1), (b, 1)$). On my website is code written by Phil Coyle to do bilinear interpolation (this can be found in Numerical Recipes, Section 3.6).

5. Use the decision rules generated in step 4 and the $\mathcal{E}$ matrix in step 3 to simulate the savings behavior of N households starting from an initial condition $K^{ss} = 11.55$, discarding the first 1000 periods (to deal with initial condition dependence).[1] This generates a huge $N \times \widetilde{T}$ matrix where each row is a different agent's k_{t+1} choice in state

[1] The initial capital stock "guess" is just the average capital stock in a version of the model without aggregate uncertainty (i.e. Aiyagari) described above.

(ε_t, z_t). Call it V.

6. Use the simulated data in step 5 to (re-)estimate a set of parameters for the functional form conjectured in step 3. That is, average over all agents in each period (i.e. down a column of V). The resulting $\widetilde{T} \times 1$ vector of aggregate capital holdings is K. Run the (auto)regression in (2) using the information in Z to know which branch to run. Obtain a measure of "goodness of fit" (e.g. R^2 of regression in (2)).

7. If the new parameter vector (a_0', a_1', b_0', b_1') is sufficiently close to the original parameter vector (a_0, a_1, b_0, b_1) and the "goodness of fit" (e.g. R^2 of regression in (2)), is sufficiently high, stop. If the new parameter vector (a_0', a_1', b_0', b_1') is not sufficiently close to the original parameter vector (a_0, a_1, b_0, b_1), go back to step 3 with the new parameter vector. If the parameter values have converged, but the goodness of fit is not high enough, increase I in step 1 or try a different functional form in step 3.