Notes on Equilibrium Models of the Wealth Distribution

Reference: Huggett, M. 1993. "The risk-free rate in heterogeneous-agent incomplete-insurance economies", *Journal of Economic Dynamics and Control*, 17, p. 953-69.

1 Environment[1]

- Population: unit measure of agents.
- Preferences: $E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right]$. Assume $U:\mathbb{R}_+\rightarrow\mathbb{R}$ is ctsly differentiable, strictly increasing, strictly concave, and bounded (Huggett doesn't make the latter assumption).
- Endowments: In any period t, hhs face two earnings shocks $s_t\in S=\{e,u\}$ where e denotes employed and u denotes unemployed. These shocks are i.i.d. across agents. The employment process is Markov with transition matrix denoted $\pi(s'|s)=prob(s_{t+1}=s'|s_t=s)$. If employed, hh earns income $y(e)=1$. If unemployed, receives income $y(u)=b<1$.
- Market Structure: sequence of one-period, non-contingent discount bonds with borrowing constraint $\underline{a}\leq 0$. Define $A=\{a_t\in\mathbb{R}:a_t\geq\underline{a}\}$.Hhs enter period with assets a_t and purchase next period assets a_{t+1} at price q_t. Since there is no aggregate uncertainty and we will be looking for a steady state equilibrium, we will assume $q_t=q$. Assume that $\beta<q$ (something that must be verified in equilibrium). [2]
- More on borrowing constraints: Aiyagari (1993, QJE) sets $\underline{a}$ to be what he calls the "natural borrowing limit". It is the most that could be borrowed and repaid with probability one across all possible histories (including the worst possible earnings history where the agent receives b (in this case) each period. To see how this constraint is found, consider the agent's sequence of budget constraints

$$c_t+qa_{t+1}=y(s_t)+a_t,\forall t,s_t.$$

Recursively substituting for $a_{t+\eta}$ in the worst possible history yields

$$c_t+q\left(c_{t+1}+q\left(c_{t+2}+q\left(c_{t+3}+qa_{t+4}-b\right)-b\right)-b\right)=b+a_t$$

[1] An environment is a statement of population, preferences, technologies (production, matching, commitment, etc.), information structure. In some cases, we also take the market structure as given, though in more general market structure is actually endogenous.

[2] Since these are discount bonds, this amounts to the assumption that $\beta(1+r)<1$ in which case agents prefer not to accumulate an unbounded amount of assets. Again, since r is endogenous, this must be verified. But intuitively, if agents are risk averse then they will accumulate assets in order to avoid consumption fluctuations (i.e. this is a model of precautionary savings). In that case, the "excess demand" for bonds will raise the bond price (lower interest rates) consistent with the assumption $\beta(1+r)<1$.

or

$$\lim_{T\to\infty}\left[\sum_{\eta=0}^{T-1} q^\eta c_{t+\eta} - q^\eta b + q^T a_{t+T}\right] = a_t. \tag{1}$$

To rule out Ponzi schemes, $\lim_{T\to\infty} q^T a_{t+T} \geq 0$[3] and so even if $c_{t+\eta} = 0$, the most the household could borrow and still pay back is

$$\underline{a}^N \equiv -\sum_{\eta=0}^{\infty} q^\eta b = \frac{-b}{1-q} \leq a_t.$$

But if the price q is determined endogenously, then $\underline{a}^N$ is an endogenous object. So we could use a tighter bound based on the assumption that $\beta < q$ given by $\underline{a}^B = \frac{-b}{(1-\beta)}$ which will not vary with price.

- Parameters: $\beta, U(\cdot), b, \pi(e|e), \pi(u|u)$.
  - Suppose the model period is one quarter.
  - Data on real interest rates pins down $\beta^4 = 0.96$ on annual basis and CRRA=1.5.
  - Data on replacement rate pins down $b = 0.5$.
  - Data on duration of unemployment pins down $\pi(u|u)$ by $D = 1/(1-\pi(u|u))$ where D is in model units (e.g. 2 quarters) $\Longrightarrow \pi(u|u) = 1 - 1/D$. If $D = 2$ quarters, then $\pi(u|u) = \frac{1}{2}$.
  - Data on average unemployment $\overline{U}$ pins down $\pi(e|e)$. From $\overline{U}' = \pi(u|u)\overline{U} + \pi(u|e)(1-\overline{U}) \Longrightarrow$ in the long run where $\overline{U}' = \overline{U}$, then $\pi(u|e) = \frac{(1-\pi(u|u))\overline{U}}{1-\overline{U}}$. Thus if $\overline{U} = 6\%$, then $\pi(u|e) = \frac{0.06}{2*0.94} = 0.03 \Longrightarrow \pi(e|e) = 0.97$.

2 Recursive Equilibrium

- The individual's problem can be written in terms of the (DP) operator $T : \mathcal{C}(S \times A) \to \mathcal{C}(S \times A)$ as:

$$(Tv)(s,a) = \max_{a' \in \Gamma(s,a)} U(y(s) + a - qa') + \beta \sum_{s' \in S} \pi(s'|s) v(s', a') \tag{2}$$

where

$$\Gamma(s,a) = \left\{ a' : \underline{a} \leq a' \leq \frac{y(s)+a}{q} \right\}$$

and $\mathcal{C}(S \times A)$ denotes the space of continuous, bounded functions.

- A solution to this problem is a decision rule $a' = g(s, a; q)$.

[3] A no-Ponzi game condition is a constraint that prevents overaccumulation of debt, while a transversality condition is an optimality condition that rules out overaccumulation of wealth.

- Since hhs differ in their employment histories, they will in general differ in their amount of assets.

- Describe economywide assets and employment via a probability measure, μ. Think of $\mu(S_0, A_0; q)$ as the fraction of the population with shocks in the set S_0 and asset holdings in the set A_0 when the price is q.

- The decision rule $g(s, a; q)$ and the shock process π induce a law of motion for the distribution of agents $\mu' = T^*\mu$ written in terms of the (TF) operator $T^* : \Xi(S \times A, \mathcal{P}(S \times A)) \to \Xi(S \times A, \mathcal{P}(S \times A))$ given by:

$$(T^*\mu)(S_0, A_0; q) = \int_{S_0, A_0} \left\{ \int_{S,A} \mathbf{1}_{\{a'=g(s,a;q)\in A_0\}}(s, a)\pi(s'|s)\mu(ds, da; q) \right\} ds'da' \tag{3}$$

  where:

  - $\Xi(S \times A, \mathcal{P}(S \times A))$ denotes the space of probability measures defined on the measurable space $(S \times A, \mathcal{P}(S \times A))$.
  - $\mathbf{1}_{\{a'=g(s,a;q)\in A_0\}}(s, a)$ is an indicator function that is 1 if the statement $\{a' = g(s, a; q) \in A_0\}$ is true for (s, a) and zero otherwise.

Definition 1 *A recursive steady state equilibrium is an allocation (c, a'), a price q^*, and an invariant distribution μ^* such that*

1. For given q^*, (c, a') solves hh optimization $v = Tv$ in (2).
2. Given μ^*, goods and asset markets clear

$$\int_{S,A} [c(s, a; q^*) - y(s)]\, d\mu^* = 0 \tag{4}$$

$$\int_{S,A} a'(s, a; q^*) d\mu^* = 0. \tag{5}$$

3. μ^* is a stationary probability measure (i.e. $\mu^* = T^*\mu^*$ in (3)).

- There are potentially many different (s, a) which map via T^* to the same a'. For example, you could have agents with low assets but high income (who save, thereby choosing $a' > a$) or agents with high assets and low income (who dissave, thereby choosing $a' < a$) both choosing the same a'). To see how the mapping (3) works, suppose that:

  - $A = \{-\alpha, 0, \alpha\}$ where α is sufficiently small.

– the decision rule is given by a nondecreasing function:

Table g

(s,a)	$g(s,a)$
$(e,-\alpha)$	0
$(e,0)$	α
(e,α)	α
$(u,-\alpha)$	$-\alpha$
$(u,0)$	$-\alpha$
(u,α)	0

that is, when unemployed, the person borrows or dissaves and when employed, the person saves.

– then the law of motion for the distribution is given by

Table μ'

(s',a')	$(T^*\mu)(s',a')$
$(e,-\alpha)$	$\pi(e\vert u)\mathbf{1}_{\{-\alpha=g(u,-\alpha;q)\}}\mu(u,-\alpha)+\pi(e\vert u)\mathbf{1}_{\{-\alpha=g(u,0;q)\}}\mu(u,0)$
$(e,0)$	$\pi(e\vert e)\mathbf{1}_{\{0=g(e,-\alpha;q)\}}\mu(e,-\alpha)+\pi(e\vert u)\mathbf{1}_{\{0=g(u,\alpha;q)\}}\mu(u,\alpha)$
(e,α)	$\pi(e\vert e)\mathbf{1}_{\{\alpha=g(e,0;q)\}}\mu(e,0)+\pi(e\vert e)\mathbf{1}_{\{\alpha=g(e,\alpha;q)\}}\mu(e,\alpha)$
$(u,-\alpha)$	$\pi(u\vert u)\mathbf{1}_{\{-\alpha=g(u,-\alpha;q)\}}\mu(u,-\alpha)+\pi(u\vert u)\mathbf{1}_{\{-\alpha=g(u,0;q)\}}\mu(u,0)$
$(u,0)$	$\pi(u\vert e)\mathbf{1}_{\{0=g(e,-\alpha;q)\}}\mu(e,-\alpha)+\pi(u\vert u)\mathbf{1}_{\{0=g(u,\alpha;q)\}}\mu(u,\alpha)$
(u,α)	$\pi(u\vert e)\mathbf{1}_{\{\alpha=g(e,0;q)\}}\mu(e,0)+\pi(u\vert e)\mathbf{1}_{\{\alpha=g(e,\alpha;q)\}}\mu(e,\alpha)$

So for example (the 2nd row of Table μ'), the mass of people employed at the beginning of next period with zero assets ($\mu'(e,0)$) is the mass employed this period with borrowings ($\mu(e,-\alpha)$) who stay employed ($\pi(e|e)$) and save ($\mathbf{1}_{\{0=g(e,-\alpha;q)\}}$) (i.e. the first row in Table g) as well as the mass of people who are unemployed this period with positive assets ($\mu(u,\alpha)$) who become employed ($\pi(e|u)$) and dissave ($\mathbf{1}_{\{0=g(u,\alpha;q)\}}$) in order to consumption smooth (the last row in Table g).

3 Existence

3.1 Existence of Decision Rules

- A standard reference for the existence of a unique solution to the fixed point problem $v^*=Tv^*$ in (2) and associated policy function $g^*(s,a)$ via a contraction mapping argument is given by Lucas and Stokey when $U(\cdot)$ is bounded. Huggett proves:

Theorem 1 (Huggett). For $q>0$, $\underline{a}+b-q\underline{a}>0$, $T^n v_0$ converges uniformly to a unique solution $v^*=Tv^*$ in (2). $v^*(s,a;q)$ is strictly increasing, strictly concave, and continuously differentiable in a. The decision rule $g^*(s,a)$ is continuous, nondecreasing in a, and strictly increasing in a for $g(s,a;q)>\underline{a}$.

- Except for showing boundedness of Tv, existence follows as direct consequence of S-L Theorem 9.6, while the property that v is increasing (concave, differentiable) follows from Theorem 9.7 (Theorem 9.8, 9.10). Singlevaluedness and continuity of g follows from Theorem 9.8, while the fact that $g(s,a)$ is increasing in a follows from concavity of v (which itself follows from concavity of U).
- To see this latter assertion, Suppose not. Then $g(s,a_1) < g(s,a_0)$ for $a_1 > a_0$. In that case,

$$\begin{aligned} c(s,a_1) &= y(s) + a_1 - q \cdot g(s,a_1) \\ &> c(s,a_0) = y(s) + a_0 - q \cdot g(s,a_0) \end{aligned}$$

since on the lhs you are subtracting something smaller from something larger. In that case, concavity of U implies

$$U'(c(s,a_1)) < U'(c(s,a_0)).$$

But then by the Euler equation, this implies

$$\frac{\beta}{q}\sum_{s'\in S}\pi(s'|s)v_2(s',g(s,a_1)) = U'(c(s,a_1)) < U'(c(s,a_0)) = \frac{\beta}{q}\sum_{s'\in S}\pi(s'|s)v_2(s',g(s,a_0)).$$

But this yields a contradiction to the fact that v is concave in a. In particular, since $g(s,a_1) < g(s,a_0)$, $v_2(s',g(s,a_0)) < v_2(s',g(s,a_1))$ since the derivative is decreasing for a concave function.

3.2 Existence of Stationary Distributions

- Next turn to existence of a solution to the fixed point problem $\mu^* = T^*\mu^*$ where T^* is defined in (3).

3.2.1 Language of Measure Theory and Markov Processes:

This material draws from Stokey and Lucas Chapters 7,8,11, and 12.

Definition 2 *Let X be a set and $\mathcal{X}$ a collection of subsets of X. Then $\mathcal{X}$ is a $\underline{\sigma-algebra}$ if (i) $\varnothing, X \in \mathcal{X}$; (ii)$\mathcal{X}$ is closed under complementation (i.e.$X_0 \in \mathcal{X} \implies X_0^C \in \mathcal{X}$); (iii) $\mathcal{X}$ is closed under countable union (as well as countable intersection since $\{X_n\}_{n=1}^{\infty} \in \mathcal{X} \implies \cup_{n=1}^{\infty} X_n \in \mathcal{X}$ and $\cap_{n=1}^{\infty} X_n = \left(\cup_{n=1}^{\infty} X_n^C\right)^C$).*[4]

Definition 3 *Let $X \subset \mathbb{R}^n$ be any set and $\mathcal{X}$ a Borel sigma algebra (the smallest collection of open/closed,etc.subsets of X). A $\underline{\text{measure}}$ is a function $\mu : \mathcal{X} \to \mathbb{R}$ such that: (i)$\mu(\varnothing) = 0$; (ii) $\mu(X_0) \geq 0, \forall X_0 \in \mathcal{X}$; (iii) if $\{X_n\}_{n=1}^{\infty}$ is a countable, disjoint collection of subsets of $\mathcal{X}$, then $\mu\left(\cup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mu(X_n)$.*

[4] In mathematics, a set is said to be **closed** under some operation if performance of that operation on members of the set always produces a unique member of the set. For example, the real numbers are closed under subtraction, but the natural numbers are not: 3 and 8 are both natural numbers, but the result of 3 - 8 is not a natural number.

Definition 4 *$(X, \mathcal{X}, \mu)$ is a measure space.*

Definition 5 *If $\mu(X) = 1$, then μ is a probability measure and $(X, \mathcal{X}, \mu)$ is a probability space.*

Definition 6 *Given a measurable space $(X, \mathcal{X})$, a function $f : X \to \mathbb{R}$ is measurable with respect to X if $\{x \in X : f(x) \leq k\} \in \mathcal{X}, \forall k \in \mathbb{R}$ (that is, the inverse image of the function is an element of the measurable space).*

- Note that continuous functions (as in the Huggett model) are measurable.

Definition 7 *Let $(X, \mathcal{X})$ be a measurable space. A transition function is a function $Q : X \times \mathcal{X} \to [0, 1]$ such that: (i) for each $x \in X$, $Q(x, \cdot)$ is a probability measure on $(X, \mathcal{X})$; (ii) for each $X_0 \in \mathcal{X}$, $Q(\cdot, X_0)$ is a measurable function.*

- The interpretation is that $Q(x, X_0)$ is the probability that next period's state is in the set X_0 given the current state is x. In terms of the Huggett model, you can think of the transition function as

  $$Q(x, X_0) = \int_{S_0, A_0} \mathbf{1}_{\{a'=g(s,a;q)\in A_0\}}(s, a)\pi(s'|s)ds'da'$$

  from (3) where $X = S \times A$ which is composed of both the exogenous law of motion given by π and the endogenous law of motion given by g.

- For any probability measure μ on $(X, \mathcal{X})$, define the operator T^* by

  $$(T^*\mu)(X_0) = \int_X Q(x, X_0)\mu(dx), \forall X_0 \in \mathcal{X}. \tag{6}$$

  Since for each $X_0 \in \mathcal{X}$, $Q(\cdot, X_0)$ is bounded (in $[0, 1]$) and measurable, then $T^*\mu$ is well defined (i.e. the "integral" exists) .

- Interpret that $(T^*\mu)(X_0)$ is the probability that the state next period lies in the set X_0 if the current state is drawn from the probability measure μ. That is, $T^*\mu$ is the probability measure over the state next period if μ is the probability measure over the current state.

- In terms of the Huggett model, (6) is just another way to write (3). To see this,

  $$\begin{aligned}(T^*\mu)(S_0, A_0; q) &= \int_{S_0,A_0} \left\{ \int_{S,A} \mathbf{1}_{\{a'=g(s,a;q)\in A_0\}}(s, a)\pi(s'|s)\mu(ds, da; q) \right\} ds'da' \\ &= \int_{S,A} \left\{ \int_{S_0,A_0} \mathbf{1}_{\{a'=g(s,a;q)\in A_0\}}(s, a)\pi(s'|s)ds'da' \right\} \mu(ds, da; q)\end{aligned}$$

  where the second equality simply rearranges the integrals.

Theorem 8 *(S-L, 8.2) Let $\Xi(X, \mathcal{X})$ be the space of probability measures. The operator T^* defined in (6) maps the space of probability measures on $\Xi(X, \mathcal{X})$ into itself (i.e. $T^* : \Xi(X, \mathcal{X}) \to \Xi(X, \mathcal{X})$).*

Proof. Must show that for any $\mu \in \Xi(X, \mathcal{X}), T^*\mu$ satisfies the properties of a probability measure. (i) Since $Q(x, \varnothing) = 0$, then $(T^*\mu)(\varnothing) = 0$ and since $Q(x, X) = 1$, then $(T^*\mu)(X) = 1$; (ii) Since $Q \geq 0$, $T^*\mu \geq 0$; (iii) Let $\{X_n\}$ be a disjoint sequence in $\mathcal{X}$ with $X = \cup_{n=1}^{\infty} X_n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty}(T^*\mu)(X_n) &= \sum_{n=1}^{\infty}\left[\int_X Q(x, X_n)\mu(dx)\right] \\
&= \int_X \left[\sum_{n=1}^{\infty} Q(x, X_n)\right]\mu(dx) \\
&= \int_X Q(x, X)\mu(dx) \\
&= (T^*\mu)(X)
\end{aligned}$$

where the 2nd line uses Monotone Convergence Theorem and 3rd uses the fact that $Q(x, \cdot)$ is a probability measure. ■

- Now we are prepared to define our object of interest (see page 317 of S-L).

Definition 9 *A probability measure μ^* is invariant under T^* if it is a fixed point $\mu^* = T^*\mu^*$ of the T^* operator.*

- Sufficient conditions to ensure that an invariant distribution exists can be established using the following definitions (this is in Section 12. 4 of S-L).

- Define another operator $\widetilde{T}$ associated with Q which takes bounded, measurable functions $f : X \to \mathbb{R}$ given by:

$$(\widetilde{T}f)(x) = \int_X f(x')Q(x, dx'), \forall x \in X \tag{7}$$

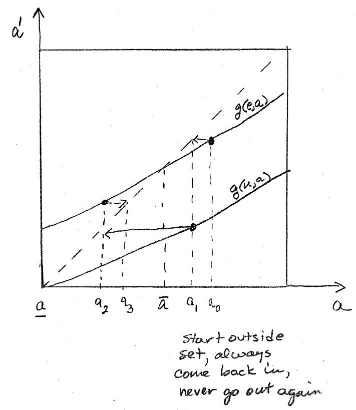
  with the interpretation as being the expected value of the function next period given the current state x.

- Theorem 8.1. of S-L establishes a result similar to above for this operator (i.e. $\widetilde{T}[B(X)] \subset B(X)$).

Definition 10 *A transition function Q on $(X, \mathcal{X})$ has the Feller property if the operator $\widetilde{T}$ in (7) maps the space of of bounded, continuous functions on X into itself (i.e. $\widetilde{T}[C(X)] \subset C(X)$).*

- This is a stronger property (due to requirement of continuity) than in Theorem 8.1 since $C(X) \subset B(X)$ and is associated with stability.

Definition 11 *A transition function Q on $(X, \mathcal{X})$ is monotone if the operator $\widetilde{T}$in (7) has the property that for every nondecreasing function $f : X \to \mathbb{R}$, the function $\widetilde{T}f$ is also nondecreasing.*



3.2.2 Application to the Wealth Distribution

- One of the big issues is that for μ to be invariant, the distribution cannot fan out. While asset holdings are bounded below by $\underline{a}$, they are not necessarily bounded above. It is impossible for μ to be invariant if there is mass being put on higher and higher a. One of the important parts of Huggett proofs is to show that agents never accumulate savings beyond an endogenously determined upper bound $\overline{a}$.

Theorem 2 (Huggett). If the conditions of Theorem 1 hold, $\beta < q$ (ie. people are impatient and borrowing rates are not too high), and $\pi(e|e) > \pi(e|u)$ (i.e. the probability of staying employed is higher than becoming employed), then there exists a unique invariant measure given q.

- That the distribution will stay in a compact set on $S \times A$ is proven in three steps. The basic idea of the result comes from the fact that since v is concave, then additions to assets add less and less future utility but come at the cost of more lost current utility. The next result simply establishes the hh dissaves when unemployed.

Lemma 1 (Huggett). Under the conditions of Theorem 2, $g(u, a) < a$ for $a > \underline{a}$.

Proof (Sketch). Let $v_2^n(s, a) \equiv \frac{\partial v^n(s,a)}{\partial a}$ denote the partial of the value function at iteration n where $v^{n+1} = Tv^n$. Start by showing by induction that the marginal utility of wealth is higher when unemployed than employed:

$$v_2^n(e,a) \leq v_2^n(u,a). \tag{8}$$

For $n=0$, let $v^0 = 0$ so this is trivial. Suppose it holds for n. The f.o.c. characterizing Tv^n is given by

$$U'(y(s)+a-qa') \geq \frac{\beta}{q}\sum_{s'}\pi(s'|s)v_2^n(s',a'), \text{ with } = \text{ if } a' > \underline{a}. \tag{9}$$

For simplicity consider the case where the constraint does not bind. Then rewriting (9) and the envelope condition imply:

$$v_2^{n+1}(e,a) = U'(y(e)+a-qa') = \frac{\beta}{q}\left[\pi(e|e)v_2^n(e,a') + \pi(u|e)v_2^n(u,a')\right] \tag{10}$$

and

$$v_2^{n+1}(u,a) = U'(y(u)+a-qa') = \frac{\beta}{q}\left[\pi(e|u)v_2^n(e,a') + \pi(u|u)v_2^n(u,a')\right]. \tag{11}$$

Note that the reason we can use $v_2^{n+1}(\cdot,\cdot)$ is because we are using the updated optimal choice. Re-writing the rhs of (10) and (11), using $\pi(u|e) = 1-\pi(e|e)$, $\pi(u|u) = 1-\pi(e|u)$, $\pi(e|e) > \pi(e|u)$, and (8) which states $v_2^n(u,a') - v_2^n(e,a') \geq 0$ implies

$$\begin{aligned} v_2^{n+1}(e,a) &= \frac{\beta}{q}\left\{v_2^n(u,a') - \pi(e|e)\left[v_2^n(u,a') - v_2^n(e,a')\right]\right\} \\ &\leq \frac{\beta}{q}\left\{v_2^n(u,a') - \pi(e|u)\left[v_2^n(u,a') - v_2^n(e,a')\right]\right\} = v_2^{n+1}(u,a) \end{aligned}$$

which completes the induction (so that assumption (8) holds). But letting $n\to\infty$ in (11) implies

$$\begin{aligned} v_2(u,a) &= \frac{\beta}{q}\left\{v_2(u,g(u,a)) - \pi(e|u)\left[v_2(u,g(u,a)) - v_2(e,g(u,a))\right]\right\} \\ &< \left\{v_2(u,g(u,a)) - \pi(e|u)\left[v_2(u,g(u,a)) - v_2(e,g(u,a))\right]\right\} \end{aligned} \tag{12}$$

where the strict inequality follows since $\frac{\beta}{q} < 1$. Rewriting (12)

$$0 \leq \pi(e|u)\left[v_2(u,g(u,a)) - v_2(e,g(u,a))\right] < v_2(u,g(u,a)) - v_2(u,a). \tag{13}$$

The only way that (13) can hold, given Theorem 1 states that v is strictly concave in a, is if $g(u,a) < a$. Our intuition that concavity matters for this result is verified in condition (13).

- The next lemma says that eventually every person (even the employed) dissave if they have enough assets.

Lemma 2 (Huggett). If $\frac{\partial v(s,a)}{\partial a} > \frac{\beta}{q} \sum_{s'} \pi(s'|s) \frac{\partial v(s',a)}{\partial a}$ for $a \geq a^* > \underline{a}$, then $g(s,a) < a$ for $a \geq a^*$.

Similar proof to above.

- The next lemma establishes the highest asset level $\overline{a}$ beyond which no one would save. This will form the upper bound of the support of the invariant measure.

Lemma 3 (Huggett). Under the conditions of Theorem 2, there exists an $\overline{a}$ such that $g(e, \overline{a}) = \overline{a}$.

Proof (Sketch). Like an intermediate value theorem result; that is, we know that $g(s,a)$ is continuous, single valued, and increasing by Theorem 1. We know by Lemma 2, that eventually it falls below the $45°$ line. All we have to establish is that it starts above the $45°$ line (for $s = e$), which he does by a proof by contradiction.

- Let $\overline{a}$ be the above fixed point in Lemma 3 and follow Figure 1.

- An alternative to using Hopenhayn and Prescott's Fixed Point results to finish proof of Theorem 2 (what Huggett does) is to apply Theorems S-L 12.12 (Existence of a unique invariant measure) and S-L 12.13 (Continuity of the measure w.r.t. parameters, in this case q). The latter result is important for establishing the existence of an equilibrium price q^* such that the asset market clears (i.e. necessary for continuity of the excess demand function).

- To apply Theorem 12.12, we need to establish that the transition function Q satisfies the Feller property (which basically says that the operator associated with Q maps the space of continuous functions into itself), is monotone (the associated operator maps the space of monotone functions into itself), and satisfies the following "mixing" condition. In particular, there exists $z \in [z_{\min}, z_{\max}]$, $\varepsilon > 0$, and $N \geq 1$ such that $Q^N(z_{\min}, [z, z_{\max}]) \geq \varepsilon$ and $Q^N(z_{\max}, [z_{\min}, z]) \geq \varepsilon$ where $Q^1(z, \widetilde{Z}) = Q(z, \widetilde{Z})$ and $Q^{i+1}(z, \widetilde{Z}) = \int_Z Q^i(z', \widetilde{Z}) Q(z, dz')$, $i = 1, 2, ...$ That is, $Q^i(z, \widetilde{Z})$ is the probability of going from z to $\widetilde{Z}$ in n periods. This mixing condition can be thought of, loosely, as the existence in this economy of both the American Dream and the American Nightmare. The condition requires that no matter how poor (rich) a household is, the probability that it becomes one of the richest (poorest) is arbitrarily close to one provided enough time passes. The Figure suggests that this is the case.

- To apply Theorem 12.13, we need to establish that the transition function $Q_\theta\ (X, \mathcal{X})$ for parameter vector $\theta \in \Theta$ on a compact set X, satisfies: (i) if $\{(x_n, \theta_n)\}$ is a sequence in $X \times \Theta$ converging to (x_0, θ_0), the sequence $\{Q_{\theta_n}(x_n, \cdot)\}$ in the space of probability measures on $(X, \mathcal{X})$ converges weakly to $Q_\theta(x_0, \cdot)$ (this is where continuity of $g(s, a; q)$ in q is important);

and (ii) for each θ, T_θ^* has a unique fixed point (this was the object of Theorem 12.12) μ_θ^*. Then if $\{\theta_n\}$ is a sequence in Θ converging to θ_0, then the sequence $\{\mu_{\theta_n}^*\}$ converges weakly to μ_θ^*.

Summary There are three important theorems in S-L used in this section.

- Existence, uniqueness, and properties of value function, decision rules. Theorem 9.10 in S-L (p. 266).
- Existence and uniqueness of invariant distribution. Theorem 12.12 in S-L (p. 382).
- Continuity of invariant distribution with respect to parameters. Theorem 12.13 in S-L (p. 384).

All three are important for computation. In particular, if you didn't know conditions for existence, your program might never converge. If you didn't know about uniqueness, your program might end, but not necessarily at the "likely" decision rules or invariant distribution. If you didn't know about continuity, you wouldn't necessarily find equilibrium prices (i.e. if the aggregate excess demand function was not continuous, you might never find market clearing). Furthermore, if you know something about the value functions (i.e. that they are strictly concave from Theorem 9.10), then you may be able to speed up the computational algorithm. That is, even when you are discretely trying to find the a' on the grid for a given (a, s) which maximizes utility, you can use that fact that if a_0' yields higher utility than a_1', then $a' > a_1'$ is suboptimal by strict concavity so that we do not need to evaluate all points on the grid.