Finite Approximation of AR1©

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Continuous AR1

- Suppose y_t follows an autoregressive AR1 process

$$y_t = \mu(1-\rho) + \rho y_{t-1} + \varepsilon_t$$

where ε_t is distributed $N(0, \sigma_\varepsilon^2)$.

- The unconditional mean of y_t is μ and the unconditional variance is $\sigma_y^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}$, both of which we can get from the data by running the AR1 regression.

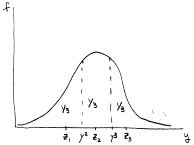
- We can also always work with the standard normal through by appropriate normalization $x = (y-\mu)/\sigma_y$.

- Since we want to keep our model state space small due to the curse of dimensionality, here is a cookbook method to turn the continuous AR1 state variable y_t into a discrete Markov process z_t.

Finite Markov Approximation

- The approximation we will use is based on Adda and Cooper (2003, p.56) who consider equal areas rather than Tauchen (1986) who considers equal interval lengths based on σ_y^2. ▸ Tauchen

- We will use the following cookbook recipe to approximate y_t into finite state markov process z^i, π_{ij}:
  - Find interval endpoints (denoted y^i, $i = 1, ..., N+1$)
  - Find conditional means of each interval (denoted z^i, $i = 1, ..., N$)
  - Find transition probabilities (denoted π_{ij})



Adda Cooper Cookbook

1. Discretize y_t into N intervals.
   - Denote the limits of each of the N intervals of y_t as $y^1, y^2, ..., y^N, y^{N+1}$ where $y^1 = -\infty$ and $y^{N+1} = \infty$.
   - The intervals are constructed so that y_t has an equal probability $\frac{1}{N}$ of falling into them.
   - Given the normality assumption, the cutoff points are:

$$F\left(\frac{y^{i+1}-\mu}{\sigma_y}\right) - F\left(\frac{y^i-\mu}{\sigma_y}\right) = \frac{1}{N}, i = 1, ..., N \quad (1)$$

     where F is the cum. distn. of the normal density using the normalization $\left(\frac{y-\mu}{\sigma_y^2}\right)$ to turn $N(\mu, \sigma_y^2)$ into $N(0,1)$.
   - Working recursively from $F\left(\frac{y^2-\mu}{\sigma_y}\right) = \frac{1}{N}$ since $F\left(\frac{-\infty}{\sigma_y}\right) = 0$ for $i = 1$, we have for $i = 2, ..., N$

$$y^i = \sigma_y F^{-1}\left(\frac{i-1}{N}\right) + \mu \quad (2)$$

   - All we need to calculate $\{y^i\}$ is (μ, σ_y) from our initial regression, the normal distribution F, and our choice of N.

Adda Cooper Cookbook - cont.

2. Compute the conditional means of y_t within each interval z_i, $i = 1, ..., N$. These are the support of the finite state Markov process.
   - That is, $z_i = E\left[y_t | y_t \in [y^i, y^{i+1}]\right]$

$$= \left[\frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{y^i}^{y^{i+1}} y e^{-(y-\mu)^2/(2\sigma_y^2)} dy\right] / \frac{1}{N}$$

   - Use the change of variable $x = \frac{y-\mu}{\sigma_y} \longrightarrow y = \sigma_y x + \mu$ and $dy = \sigma_y dx$. Then we can rewrite z_i using (1) as

$$z_i = N\sigma_y \left[f\left(\frac{y^i - \mu}{\sigma_y}\right) - f\left(\frac{y^{i+1} - \mu}{\sigma_y}\right)\right] + \mu.$$

   where f is the normal density function.
   - All we need to calculate $\{z_i\}$ is (μ, σ_y) from our initial regression, the normal distribution F, our choice of N, and a matlab function that numerically integrates functions.

Adda Cooper Cookbook - cont.

3. Compute the transition probabilities between any of these intervals $\pi_{j,i} = \Pr(y_t \in [y^j, y^{j+1}] | y_{t-1} \in [y^i, y^{i+1}])$

$$= \frac{\Pr(y_t \in [y^j, y^{j+1}], y_{t-1} \in [y^i, y^{i+1}])}{\Pr(y_{t-1} \in [y^i, y^{i+1}])}$$

After some manipulation, we have (see p.58 of A-C):

$$\pi_{j,i} = \frac{N}{\sqrt{2\pi\sigma_y^2}} \left[\int_{y^i}^{y^{i+1}} e^{-(y_{t-1}-\mu)^2/(2\sigma_y^2)} \left[\begin{array}{c} F\left(\frac{y^{j+1}-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \\ -F\left(\frac{y^j-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \end{array} \right] dy_{t-1} \right].$$

- All we need to calculate $\{\pi_{j,i}\}$ is (μ, σ_y) from our initial regression, the normal distribution F, our choice of N, and a matlab function that numerically integrates functions.

Tauchen Method

The equally spaced intervals in Tauchen are constructed as follows:

- let $z^1 < z^2 < ... < z^N$ denote the discrete values that $y_t = \rho y_{t-1} + \varepsilon_t$ can take on where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$.
- Let z^N be a multiple m of the unconditional standard deviation $\sigma_y = \sqrt{\sigma_\varepsilon^2/(1-\rho^2)}$. Then let $z^1 = -z^N$ and let the remaining be equispaced over the interval $[z^1, z^N]$.
- The transition probabilities π_{jk} (from,to) in Tauchen are given as follows: Let $w = z^k - z^{k-1}$. For each j, if k is between 2 and $N-1$, set

$$\begin{aligned}\pi_{jk} &= \Pr[z^k - w/2 \leq \rho z^j + \varepsilon_t \leq z^k + w/2] \\ &= F\left(\frac{z^k + w/2 - \rho z^j}{\sigma_\varepsilon}\right) - F\left(\frac{z^k - w/2 - \rho z^j}{\sigma_\varepsilon}\right).\end{aligned}$$

- The code for this is https://quanteconpy.readthedocs.io/en/latest/modules/quantecon/markov/approximation.html