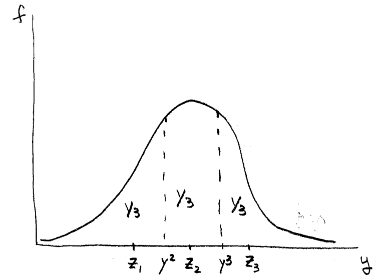
Handout for Finite Approximations of AR1

Suppose y_t follows an autoregressive AR1 process

$$y_t = \mu(1-\rho) + \rho y_{t-1} + \varepsilon_t$$

where ε_t is distributed $N(0, \sigma_\varepsilon^2)$.The unconditional mean of y_t is μ and the unconditional variance is $\sigma_y^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}$, both of which we can get from the data by running the AR1 regression. We can also always work with the standard normal through by appropriate normalization $x = (y-\mu)/\sigma_y$.

Since most of the problems we study are discretized anyway, here is a cookbook method to turn the continuous state variable y_t into a discrete variable z_t. The approximation we will use is based on Adda and Cooper (2003, p.56) who consider equal areas rather than Tauchen (1986) who considers equal interval lengths based on σ_y^2.[1]



[1] In particular, the equally spaced intervals in Tauchen are constructed as follows: let $z^1 < z^2 < ... < z^N$ denote the discrete values that $y_t = \rho y_{t-1} + \varepsilon_t$ can take on where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Let z^N be a multiple m of the unconditional standard deviation $\sigma_y = \sqrt{\sigma_\varepsilon^2/(1-\rho^2)}$. Then let $z^1 = -z^N$ and let the remaining be equispaced over the interval $[z^1, z^N]$.

The transition probabilities π_{jk} (from,to) in Tauchen are given as follows: Let $w = z^k - z^{k-1}$. For each j, if k is between 2 and $N-1$, set

$$\begin{aligned}\pi_{jk} &= \Pr[z^k - w/2 \le \rho z^j + \varepsilon_t \le z^k + w/2] \\ &= F\left(\frac{z^k + w/2 - \rho z^j}{\sigma_\varepsilon}\right) - F\left(\frac{z^k - w/2 - \rho z^j}{\sigma_\varepsilon}\right).\end{aligned}$$

- We will use the following cookbook recipe to approximate y_t into finite state markov process z^i, π_{ij}:
  - Find interval endpoints (denoted y^i, $i = 1, ..., N+1$)
  - Find conditional means of each interval (denoted z^i, $i = 1, ..., N$)
  - Find transition probabilities (denoted π_{ij})
- Specifically:
  1. Discretize y_t into N intervals.
     - Denote the limits of each of the N intervals of y_t as $y^1, y^2, ..., y^N, y^{N+1}$. Since y_t is unbounded, $y^1 = -\infty$ and $y^{N+1} = \infty$.
     - The intervals are constructed so that y_t has an equal probability $\frac{1}{N}$ of falling into them.
     - Given the normality assumption, the cutoff points are defined as

       $$F\left(\frac{y^{i+1} - \mu}{\sigma_y}\right) - F\left(\frac{y^i - \mu}{\sigma_y}\right) = \frac{1}{N}, i = 1, ..., N \tag{1}$$

       where F is the cumulative distribution function of the normal density.[2]
     - Working recursively, we have for $i = 2, ..., N$[3]

       $$y^i = \sigma_y F^{-1}\left(\frac{i-1}{N}\right) + \mu \tag{2}$$

     - Notice that all we need to calculate $\{y^i\}$ is (μ, σ_y) from our initial regression, the normal distribution F, and our choice of N.
  2. Compute the conditional mean of y_t within each interval. Call it z_i, $i = 1, ..., N$. These will be the support of the finite state Markov process.
     - That is,

       $$\begin{aligned} z_i &= E\left[y_t | y_t \in [y^i, y^{i+1}]\right] \\ &= \frac{\left[\frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{y^i}^{y^{i+1}} y e^{-(y-\mu)^2/(2\sigma_y^2)} dy\right]}{\frac{1}{N}} \end{aligned}$$

[2] The normalization $\left(\frac{y-\mu}{\sigma_y^2}\right)$ turns $N(\mu, \sigma_y^2)$ into $N(0,1)$.

[3] Equation (1) implies that for $i = 1$, we have
$F\left(\frac{y^2-\mu}{\sigma_y}\right) = \frac{1}{N}$ since $F\left(\frac{-\infty}{\sigma_y}\right) = 0$.
Then for $i = 2$, we have from (1) $F\left(\frac{y^3-\mu}{\sigma_y}\right) = F\left(\frac{y^i-\mu}{\sigma_y}\right) + \frac{1}{N} = 2 \cdot \frac{1}{N}$. By induction, for any i, (1) implies
$F\left(\frac{y^{i+1}-\mu}{\sigma_y}\right) = F\left(\frac{y^i-\mu}{\sigma_y}\right) + \frac{1}{N} = i \cdot \frac{1}{N} \Longleftrightarrow F\left(\frac{y^i-\mu}{\sigma_y}\right) = (i-1)\frac{1}{N}$
from which (2) follows..

– Use the change of variable $x = \frac{y-\mu}{\sigma_y}$ which implies $y = \sigma_y x + \mu$ and $dy = \sigma_y dx$. In this case we can rewrite z_i as

$$\begin{aligned} z_i &= \frac{N}{\sqrt{2\pi\sigma_y^2}} \left[\int_{\frac{y^i-\mu}{\sigma_y}}^{\frac{y^{i+1}-\mu}{\sigma_y}} \sigma_y^2 x e^{-x^2/2} dx + \int_{\frac{y^i-\mu}{\sigma_y}}^{\frac{y^{i+1}-\mu}{\sigma_y}} \mu\sigma_y e^{-x^2/2} dx \right] \\ &= N\sigma_y \left[-f\left(\frac{y^{i+1}-\mu}{\sigma_y}\right) - \left(-f\left(\frac{y^i-\mu}{\sigma_y}\right)\right)\right] \\ &\quad + N\mu \left[F\left(\frac{y^{i+1}-\mu}{\sigma_y}\right) - F\left(\frac{y^i-\mu}{\sigma_y}\right)\right] \end{aligned}$$

where f is the normal density function.

– From (1) we know this expression reduces to

$$z_i = N\sigma_y \left[f\left(\frac{y^i-\mu}{\sigma_y}\right) - f\left(\frac{y^{i+1}-\mu}{\sigma_y}\right)\right] + \mu.$$

– Notice that all we need to calculate $\{z_i\}$ is (μ, σ_y) from our initial regression, the normal distribution F, our choice of N, and a matlab function that numerically integrates functions.

3. Compute the transition probabilities between any of these intervals.

$$\begin{aligned} \pi_{j,i} &= \Pr(y_t \in [y^j, y^{j+1}] | y_{t-1} \in [y^i, y^{i+1}]) \\ &= \frac{\Pr(y_t \in [y^j, y^{j+1}], y_{t-1} \in [y^i, y^{i+1}])}{\Pr(y_{t-1} \in [y^i, y^{i+1}])} \end{aligned}$$

But the numerator can be manipulated to yield:

$$\begin{aligned} \Pr(y_t &\in [y^j, y^{j+1}], y_{t-1} \in [y^i, y^{i+1}]) \\ &= \Pr(\varepsilon_t \in [y^j - \mu(1-\rho) - \rho y_{t-1}, y^{j+1} - \mu(1-\rho) - \rho y_{t-1}], y_{t-1} \in [y^i, y^{i+1}]) \\ &= \int_{y^i}^{y^{i+1}} \int_{y^j-\mu(1-\rho)-\rho y_{t-1}}^{y^{j+1}-\mu(1-\rho)-\rho y_{t-1}} f(\varepsilon_t) f(y_{t-1}) d\varepsilon_t dy_{t-1} \text{by independence in } f(\varepsilon_t, y_{t-1}) \\ &= \frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{y^i}^{y^{i+1}} e^{-(y_{t-1}-\mu)^2/(2\sigma_y^2)} \begin{bmatrix} F\left(\frac{y^{j+1}-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \\ -F\left(\frac{y^j-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \end{bmatrix} dy_{t-1}. \end{aligned}$$

Hence, since $\Pr(y_{t-1} \in [y^i, y^{i+1}]) = 1/N$, we have the following result (analogous to page 58 of A-C):

$$\pi_{j,i} = \frac{N}{\sqrt{2\pi\sigma_y^2}} \left[\int_{y^i}^{y^{i+1}} e^{-(y_{t-1}-\mu)^2/(2\sigma_y^2)} \begin{bmatrix} F\left(\frac{y^{j+1}-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \\ -F\left(\frac{y^j-\mu(1-\rho)-\rho y_{t-1}}{\sigma_\varepsilon}\right) \end{bmatrix} dy_{t-1} \right].$$

You just need to numerically integrate (using a matlab function) this integral to get $\pi_{j,i}$. Notice that all we need to calculate $\{\pi_{j,i}\}$ is

(μ, σ_y) from our initial regression, the normal distribution F, our choice of N, and a matlab function that numerically integrates functions.

- While many of the calculations above are specific to the Normal distribution, it seems possible to generalize it since the basic idea is based on chopping up distribution F (so there's probably a paper about the generalization).