# Hopenhayn Model of Firm Heterogeneity

Hopenhayn, H. 1992. "Entry, Exit, and Firm Dynamics in Long Run Equilibrium", *Econometrica*, 60, p. 1127-50. (partial equilibrium, discrete choice problem to derive firm size distributions).

## 1 Environment

- Population: Industry composed of a continuum of firms (mass not necessarily 1) which produce a homogeneous product.
- Firms behave competitively, taking prices in the output ($p_t$) and labor input ($w_t$) markets as given.
- Aggregate Demand given by the *inverse demand* function $P(Q_t)$. Assumption A.1a: $P$ is continuous, strictly decreasing, and $\lim_{Q\to\infty} P(Q) = 0$.
  - For example, let $P = \frac{1}{d+Q}$ where $d = 0.1$.
- Assumption A.1.b: The input price $W(N_t)$,where $N_t$ is total industry labor demand, is assumed to satisfy properties: $W$ is continuous, nondecreasing and strictly bounded above zero.
  - For example, let $w = 1$.
- Technology: output of a given firm is $q_t = f(\varphi_t, n_t)$ where $\varphi_t \in [0,1]$ is a productivity shock which follows a Markov process, independent across firms with conditional distribution $F(\varphi_t|\varphi_{t-1})$.
  - For example, let $f(\varphi, n) = \varphi n^\alpha$ where $\alpha = \frac{1}{2}$. It is important that the production function exhibits decreasing returns to scale.
- A fixed cost $w_t c_f$ must be paid every period by an incumbent firm (for exit to take place). Here we will assume that the cost is associated with management.
- Profit function given by $\pi(\varphi_t, p_t, w_t) = max_n p_t q_t - w_t n_t - w_t c_f$.
- Firms discount profits at rate $0 < \beta < 1$.
  - For example, $\beta = .8$ (time period is 5 years at 4%).
- Assumption A.2.(a) $q$ and $n$ are single valued, increasing in $\varphi$, and continuous; (b) $\pi$ is continuous and strictly increasing in $\varphi$; (c) $\lim_{Q\to 0} \pi(\varepsilon, P(Q), w) > 0$.

- Assumption A.3(a) $F$ is continuous in $\varphi_{t-1}$ and $\varphi_t$, (b) $F$ is strictly decreasing in $\varphi_{t-1}$. (basically just persistence).

- Assumption A.4 (Recurrence) For any $\varepsilon > 0$, there exists an integer $k$ such that $F^k(\cdot|\varphi)$ gives the distribution of $\varphi_{t+k}$ given $\varphi_{t-1} = \varphi$ (a mixing condition).

  - For example, suppose there are two states. $\varphi \in \{\varphi_L, \varphi_H\}$ with $1 = \varphi_H > \varphi_L > 0$. $F(\varphi' = \varphi_{j\neq i}|\varphi = \varphi_i) = 1-\theta < \frac{1}{2}$, $i, j \in \{H, L\}$. Thus, the process displays persistence since $\theta > \frac{1}{2}$. We take $\theta = 0.9$.
  - New draws:

- A.2 and A.3 imply expected discounted profits are increasing in $\varphi$. Furthermore, if your profits are high (low) today, they are expected to stay high (low) in the near future. This is important for the exit decision.

- Each period, before $\varphi_t$ is realized, incumbent firms may exit. An exiting firm receives 0 present value.

- In equilibrium, firms exit whenever their state $\varphi_{t-1}$ falls below a reservation level $x_{t-1}$. Given this cutoff point, A.4 implies the life span of a firm is almost surely finite (i.e. there is a steady stream of exit).

- An entering firm must pay a fixed cost $w_t c_e$. Then it draws a productivity shock $\varphi_t$ from initial density function $\eta$. For example, let $\Pr\{\varphi = \varphi_H\} = \eta = \frac{1}{2}$.

- Assumption A.5 $\eta$ has a continuous distribution function $G$.

- Summary of timing: Incumbent

1. enter period $t$ with last period's shock $\varphi_{t-1}$.
2. if $\varphi_{t-1} < x_{t-1}$,exit (to avoid future low productivity)
3. receive this period's shock $\varphi_t$ from $F(\varphi_t|\varphi_{t-1})$
4. choose labor demand $n_t = g(\varphi_t; p, w)$ and pay fixed cost $wc_f$
5. enter period $t+1$ with $\varphi_t$

- Summary of timing: Potential Entrant

1. Decide whether to pay fixed cost $wc_e$ and receive this period's shock $\varphi_t$ from $\eta(\varphi_t)$
2. Same as 4-5 above[1]

[1] Note that under this timing an entrant may immediately exit (which is actually consistent with (15) below.

- Let the measure $\mu_t(A)$ summarize the number of firms that have $\varphi_t \in A \subset S$. Due to entry and exit, total mass of firms $\mu_t(S)$ is not necessarily 1.
- Notice that under the timing above, even if firms with $\varphi_{t-1} \leq x_{t-1}$ are exiting, the distribution $\mu_t(\varphi_t)$ includes new entrants who may draw low $\varphi_t$ from the distribution function $G(\varphi_t)$.
- Aggregate output supply and input demand given by

$$Q_t^s(\mu_t; p, w) = \int q_t(\varphi_t; p, w)\mu_t(d\varphi_t) \tag{1}$$

$$N_t^d(\mu_t; p, w) = \int n_t(\varphi_t; p, w)\mu_t(d\varphi_t) \tag{2}$$

# 2 Equilibrium

- Given that all the uncertainty is idiosyncratic and there is a continuum of firms, we will be considering stationary equilibria where $\{p_t, w_t\} = \{p, w\}$.
- The problem of an incumbent firm at stage 4 in the above timing is really just a decision on exit next period (note that the static labor demand problem has already been solved implicitly in the profit function):

$$v(\varphi; p, w) = \pi(\varphi, p, w) + \beta \max\{0, E[v(\varphi'; p, w)|\varphi]\} \tag{3}$$

  where $E[g(\varphi')|\varphi] = \int_{\varphi' \in [0,1]} g(\varphi')F(d\varphi'|\varphi)$.[2]
- The solution to problem (3) is a cutoff rule. Let

$$x = \inf\{\varphi \in S : E[v(\varphi'; p, w)|\varphi] \geq 0\} \tag{4}$$

  Thus any firm with $\varphi < x$ will exit.
- Prop. 1. $v$ is a bounded, continuous, and strictly increasing function of $\varphi$. Note: present discounted profits are increasing in productivity follows from properties of $\pi$ and $F$.
- Since all firms are ex-ante identical, entry occurs provided

$$\int v(\varphi; p, w)\eta(d\varphi) \geq wc_e.$$

- Let $M_t$ denote the mass of entrants in period $t$. An equilibrium with free entry requires

$$\int v(\varphi; p, w)\eta(d\varphi) \leq wc_e, \tag{5}$$

  with equality if $M_t > 0$ (in which case we are free to "choose" any measure $M_t$ we want).

[2] In Hopenhayn's notation, $F(d\varphi'|\varphi)$ stands for the densitiy function for next period's shock $\varphi'$ conditional on $\varphi$.

- The entry and exit rules imply an evolution for the state of the industry

$$\mu_{t+1}([0,\overline{\varphi}']) = \int_{\varphi' \in [0,\overline{\varphi}']} \int_{\varphi \geq x} dF(\varphi'|\varphi)\mu_t(d\varphi)d\varphi' + M_{t+1}G(\overline{\varphi}') \tag{6}$$

  Note that the first term considers only those firms which do not exit.

- Could write this in terms of Transition function

$$T(\varphi, A) = \begin{cases} \int_{s' \in A} dF(s'|\varphi)ds' & \text{if } \varphi \geq x \\ 0 & \text{otherwise} \end{cases}$$

  so that (6) given by

$$\mu_{t+1} = T\mu_t + M_{t+1}\eta \tag{7}$$

- *Importantly*, it is simple to see that (7) $\mu_{t+1}$ is homogeneous of degree 1 in $\mu_t$ and $M_{t+1}$.

**Definition 1** *A **stationary competitive industry equilibrium** is a list* $\{p^*, w^*, Q^*, N^*, M^*, x^*, \mu^*\}$ *such that*

1. $p^* = D(Q^*)$, $w^* = W(N^*), Q^* = Q^s(\mu^*, p^*, w^*)$, $N^* = N^d(\mu^*, p^*, w^*)$ (markets clear)
2. $x^*$ satisfies (4) (optimal exit)
3. (5) satisfied at $p^*$ and $w^*$ with equality if $M^* > 0$ (free entry)
4. $\mu_t = \mu^*$ solves (7)

- To prove existence, find a fixed point of $\mu \to (p(\mu), w(\mu)) \to (v, g) \to \mu$.
- Theorem 2. A SS equilibrium exists (there are many possible equilibria). e.g. no trade is always an equilibrium.
- Theorem 3. A SS equilibrium with entry/exit exists iff entry costs are sufficiently low.
- Theorem 4. Conditions for uniqueness include production functions of the form $f(\varphi, n) = \varphi n^\alpha$, $\alpha \in (0, 1)$.This is condition U.2 on p. 1137.
- An important step in Theorem 2 is existence of a unique invariant measure $\mu^*$ which follows from the recurrence or mixing condition of Ass4.

# 3 Example

- Given only two states for the productivity shock, the example is not rich enough to generate a size distribution over firms. However, it does illustrate entry and exit.

- Firm's static optimization problem:

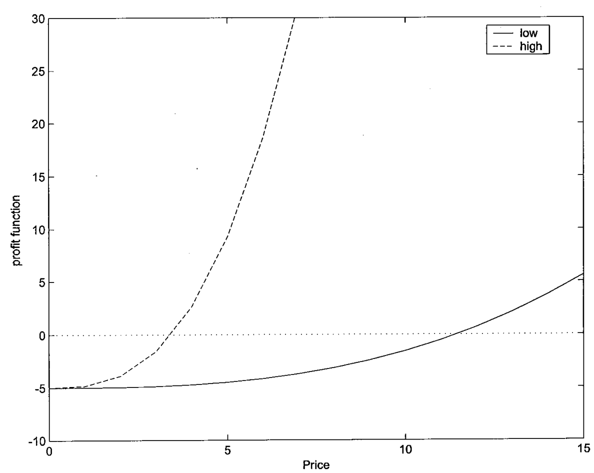
$$P\alpha\varphi_i n^{\alpha-1} = 1 \Longleftrightarrow n_i^* = (P\alpha\varphi_i)^{1/(1-\alpha)}, i \in \{H, L\}$$

In this specific case, $n_i^* = \left(\frac{P\varphi_i}{2}\right)^2$.Hence firm size, measured by number of workers, is increasing in productivity.

- Thus

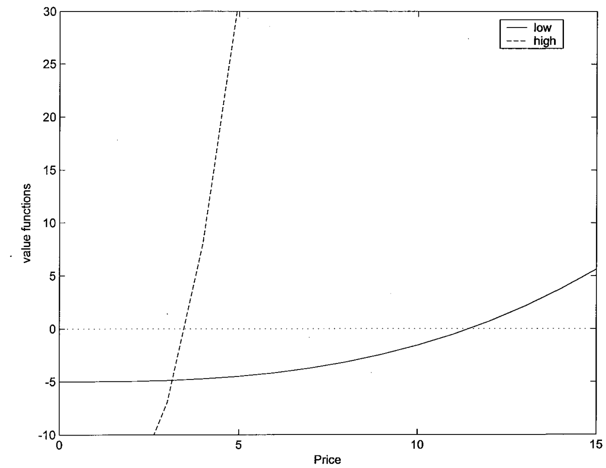
$$\begin{aligned} \pi(\varphi_i, P, 1) &= P\varphi_i \left(n_i^*\right)^{\alpha} - n_i^* - c_f \\ &= (P\alpha\varphi_i)^{1/(1-\alpha)} \left[\frac{1}{\alpha} - 1\right] - c_f \end{aligned}$$

Note that with $\alpha = 1/2$, the static profit function is increasing and convex in price (as in Varian) and productivity.




- The two steady state value functions indexed by $\varphi_i$ are

$$v(\varphi_i; P) = \frac{(P\varphi_i)^2}{4} - c_f + \beta \max\left\{0, \theta v(\varphi_i; P) + (1-\theta) v(\varphi_{j\neq i}; P)\right\}$$




- A firm will exit if $\theta v(\varphi_i; P) + (1-\theta) v(\varphi_{j\neq i}; P) < 0$.

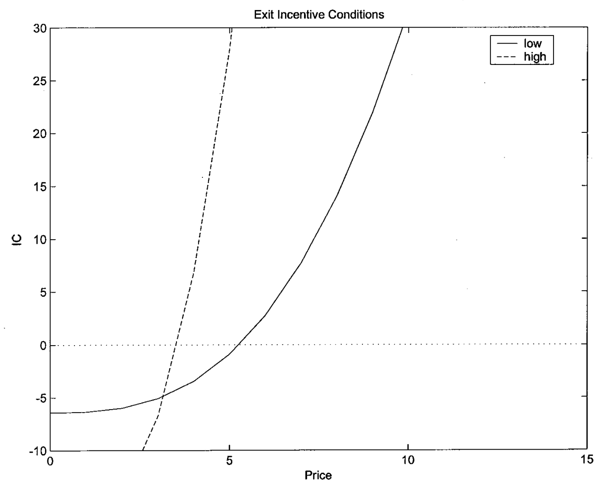



Figure 3

- Here we will "construct" an equilibrium where if a firm receives a low shock, it will exit if a one shot deviation is suboptimal or

$$0 > \theta v(\varphi_L; P) + (1-\theta) v(1; P) \tag{8}$$

  but if it receives a high shock, it will not exit if a one shot deviation is suboptimal or

$$\theta v(1; P) + (1-\theta) v(\varphi_L; P) > 0. \tag{9}$$

  It may be possible to satisfy these one shot deviation conditions ((8) and (9)) since $v(1; P) > v(\varphi_L; P)$ and $\theta > \frac{1}{2}$.

- Conjecture this exit strategy is optimal and then check under what conditions on parameters it is optimal. Under the construction,

$$v(\varphi_L; P) = \frac{(P\varphi_L)^2}{4} - c_f + \beta \cdot 0 \tag{10}$$

$$v(1; P) = \frac{P^2}{4} - c_f + \beta \left\{ \theta v(1; P) + (1-\theta) v(\varphi_L; P) \right\} \tag{11}$$

  That is, 2 linear equations in 2 unknowns.

- Since we have exit, in order to ensure a stationary equilibrium we need positive entry. If

$$\eta v(1,P) + (1-\eta) v(\varphi_L, P) = c_e \tag{12}$$

then since firms are indifferent between entering and staying out of the market, we are free to choose how many firms enter (i.e. $M$). From (10) and (11), it is clear that (12) is one equation in one unknown $P$, so that entry pins down prices. Then what pins down the amount of entry?

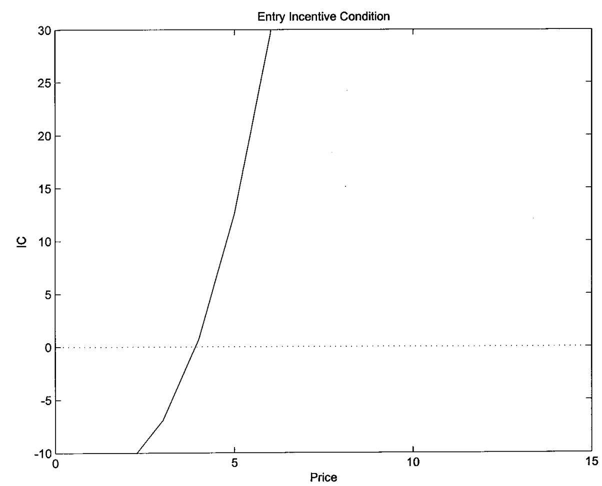



Figure 4

- Under our conjecture, this entry condition can be written

$$\eta \left[ \frac{\frac{(P^*)^2}{4}\left[1+\beta(1-\theta)\varphi_L^2\right] - c_f\left[1+\beta(1-\theta)\right]}{(1-\beta\theta)} \right] + (1-\eta)\left[ \frac{(P^*\varphi_L)^2}{4} - c_f \right] = c_e \tag{13}$$

which is one equation in one unknown $P$. The lhs is continuously increasing in $P$,is negative if $P = 0$, is $\infty$ as $P \to \infty$ and is independent of $c_e$. Thus a solution $P^*$ exists. Note from the previous figure on exit incentive conditions that a necessary condition for the conjectured equilibrium to exist is that $P^*$ lies somewhere between 4 and 5. We can choose $c_e$ to make that happen. Another way to think of this is that if we have data on the equilibrium price in the market $P^*$, this can help us identify $c_e$which we may not have data on.

- Now we need to pin down the law of motion for the distribution (which will give us $Q$ ) and ultimately $M$. Abusing notation somewhat (i.e. writing the distribution as a pdf rather than a cdf), we have

$$\begin{aligned} \mu_{t+1}(1) &= \mu_t(1)\theta + M_{t+1}\eta \\ \mu_{t+1}(\varphi_L) &= \mu_t(1)(1-\theta) + M_{t+1}(1-\eta) \end{aligned} \tag{14}$$

  That is, the mass of firms at the beginning of next period at the high productivity level are incumbents who remain high and new entrants who receive a high productivity level. Further, the mass of firms at the beginning of next period (before exit) at the low productivity level are incumbents who were high but received a low productivity draw and new entrants who receive a low productivity draw.

- From (14), we can solve for the invariant distribution

$$\begin{aligned} \mu(1) &= \frac{\eta M}{(1-\theta)} \\ \mu(\varphi_L) &= \eta M + M(1-\eta) = M \end{aligned} \tag{15}$$

  which makes clear the dependence of the distribution on the mass of new entrants $M$ and $\theta$ through $x$ (so that the notation $m(x, M)$ makes sense). Note that under the conjecture that all firms with the low productivity $\mu(\varphi_L)$ exit, then we will need to have entry by $M$ equal that measure if there is to be a stationary distribution (i.e. $\mu(\varphi_L) = M$ is rationalized by the fact that exit has to equal entry).

  – Note further from (15) it is easy to see the homogeneity property. If $M$ doubles, so does $\mu$.

- Output given by

$$Q^s = \mu(1)\left(n_H^*\right)^\alpha + \mu(\varphi_L)\varphi_L\left(n_L^*\right)^\alpha \tag{16}$$

  That is there are some incumbent firms and new entrants who produce in a given period $t$ before they exit in $t+1$ (if they receive a bad shock in $t$).

- In that case, goods market clearing implies $Q^s = Q$ from (16) and the inverse demand function

$$P = 1/(d + Q^s) \tag{17}$$

  implies 2 equations (13), (17) in 2 unknowns $P$ and $M$. Note that once we know $P^*$ from (13) and substitute it in $n^*$, then the rhs of (17) only depends on $M$ and the lhs is a number, so we can use it to determine $M^*$.

- Finally, we need to verify that at $P^*$, the one shot deviation conditions (8) and (9) are satisfied (which they are).
- In summary, it may seem counterintuitive, but the free entry condition effectively pins down prices ($P$) and the market clearing condition pins down the mass of new entrants ($M$).