Hopenhayn Meets Pakes

Hopenhayn, H. 1992. "Entry, Exit, and Firm Dynamics in Long Run Equilibrium", *Econometrica*, 60, p. 1127-50. (partial equilibrium, discrete choice problem to derive firm size distributions).

Ericson, R. and A. Pakes. 1995. "Markov-Perfect Industry Dynamics: A Framework for Empirical Work", *Review of Economic Studies*, 62, p. 53-82 (see especially Section IV. An Example. Since this paper considers an environment with a homogeneous good, it is closer to Hopenhayn than the "seminal" paper Pakes, A. and P. McGuire (1994), "Computing Markov-Perfect Nash equilibria: Numerical implications of a dynamic differentiated-product model", RAND, 25, 555-89. See also http://www.economics.harvard.edu/faculty/pakes/program.html).

Gowrisankaran, G. and T. Holmes (2004) "Mergers and the Evolution of Industry Concentration: Results from the Dominant-Firm Model," *RAND Journal of Economics*, 35(3), p. 561-582. This paper considers a Stackelberg game which mixes the two; there is a monopolist that moves first and a competitive fringe that follows.

1 Environment

We will keep some aspects of the Hopenhayn environment and others from Ericson-Pakes. I will put a bold "different" where these notes differ from the earlier Hopenhayn notes.

- Population: Industry composed of different numbers of firms conditional on market structure considered - a continuum of firms (mass not necessarily 1) for Hopenhayn and a finite number for Ericson and Pakes.
- Each firm produces a homogeneous product.
- Firms face a linear inverse demand function $P(Q) = D - Q$, still decreasing in Q (**different**). We set $D = 5$. We normalize the wage rate to 1.
- Technology: output of a given firm is $q = \varphi n^{\alpha}$ where φ is a productivity shock which follows a Markov process, independent across firms and $\alpha = 1/2$.
- $\varphi \in \{\varphi_L, \varphi_H\}$ with $1 = \varphi_H > \varphi_L = 0.6$. $F(\varphi' = \varphi_j | \varphi = \varphi_i) = 1 - \theta < \frac{1}{2}$, $i, j \in \{H, L\}$. Displays persistence since $\theta > \frac{1}{2}$. We take $\theta = 0.9$.
- A fixed "managerial" cost c_f of must be paid every period by an incumbent firm (not in Ericson-Pakes, but they have scrap value or outside option which fills this role). However due to timing differences, here we will assume that you don't have to pay the cost the period you exit (**different**). As before, we are interpreting this is the fixed cost associated with a manager's wage bill (and wages are normalized to 1), the firm simply bears $1 \cdot c_f$. Assume $c_f = 1.9$.

- Profit function if produce given by $\pi(\varphi, P) = max_n Pq - n - c_f$.
- Firms discount profits at rate $0 < \beta < 1$ which we take to be 0.96.
- Exiting firm receives 0 present value.
- Summary of timing: Incumbent (same as Ericson and Pakes)

1. Shock is realized φ_t (**different -** shock after exit decision in earlier notes).
2. Incumbent firm may exit to avoid c_f.
3. If incumbent stays, pays fixed cost c_f and chooses labor demand $n = g(\varphi_t; P)$

- Entering firm must pay fixed cost c_e one period in advance (kind of like one period time to build) after stage 3 (**different**). For simplicity, we will assume the following cost structure. In the case of Hopenhayn, this is just a number. In the case of Ericson-Pakes,

$$c_e = \begin{cases} c & \text{if number of incumbent firms} \leq NF \\ \infty & \text{if number of incumbent firms} > NF \end{cases}$$

  We take $c_e = 3.0$. See Figure HMPcost.

- Then it draws a productivity shock φ_{t+1} at the beginning of next period from initial density function where $\Pr\{\varphi' = \varphi_H\} = \eta = \frac{1}{2}$. Can think of this as a draw from the invariant distribution associated with the Markov process F above for a symmetric case. After the draw at the beginning of next period, it is just like an incumbent firm.
- Let the measure $\mu_t(A)$ summarize the number of incumbent and entrant firms which have $\varphi \in A \subset S$. This can be thought of as being evaluated before the exit decision is made (stage 2) but after the realization of φ.

2 Hopenhayn Example (NF=∞)

In this case, there is no aggregate uncertainty due to the law of large numbers and firms take the price as given when maximizing profits since they are infinitesimal. Except for the **differences** noted above, the qualitative results are similar to what were in the previous notes.

- firm's static optimization problem:

$$\left[D - \int q(\varphi; P(\mu))\mu(d\varphi)\right] \alpha\varphi_i n^{\alpha-1} = 1 \Longleftrightarrow n_i^* = (P\alpha\varphi_i)^{1/(1-\alpha)}, i \in \{H, L\} \quad (1)$$

  In this specific case, $n_i^* = \left(\frac{P\varphi_i}{2}\right)^2$.

Entry Cost Function

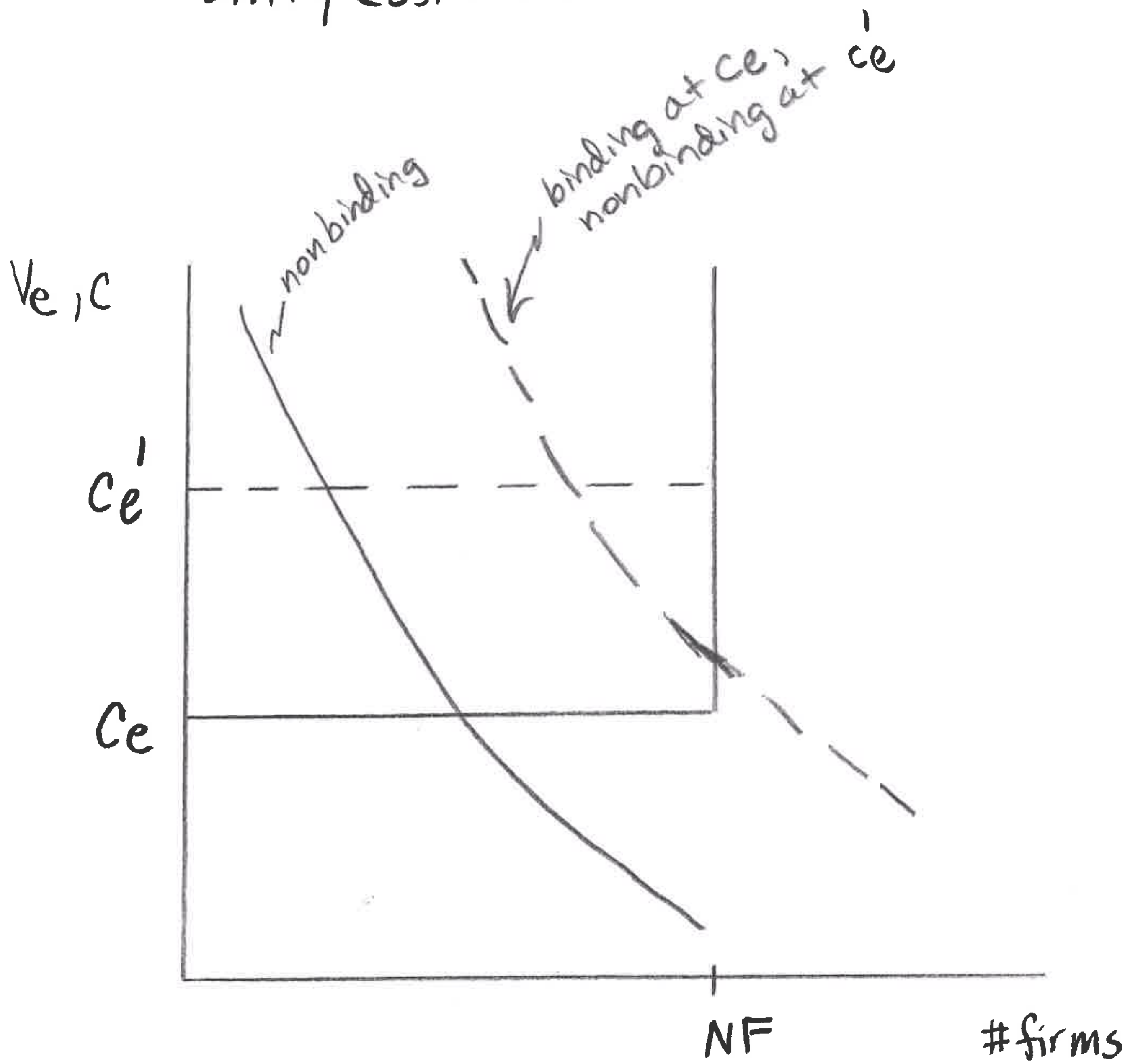


Figure HMPcost

- Thus

$$\begin{aligned}\pi(\varphi_i, P, 1) &= P\varphi_i\left(n_i^*\right)^{\alpha} - n_i^* - c_f \\ &= \left(P\alpha\varphi_i\right)^{1/(1-\alpha)}\left[\frac{1}{\alpha} - 1\right] - c_f\end{aligned}$$

In this specific case, $\pi_i^* = \left(\frac{P\varphi_i}{2}\right)^2 - c_f$.

- The two steady state value functions indexed by φ_i are

$$\begin{aligned}v(\varphi_i; P) &= \max\left\{0, \pi(\varphi_i, P) + \beta\left[\theta v(\varphi_i; P) + (1-\theta)v(\varphi_{j\neq i}; P)\right]\right\} \\ &= \max_{x\in\{0,1\}}(1-x)\left\{\pi(\varphi_i, P) + \beta\left[\theta v(\varphi_i; P) + (1-\theta)v(\varphi_{j\neq i}; P)\right]\right\}\end{aligned}$$

where the decision rule $x(\varphi_i; P) = 1$ denotes exit.

- Specifically, $x = 1$ if $\pi(\varphi_i, P) + \beta\left[\theta v(\varphi_i; P) + (1-\theta)v(\varphi_{j\neq i}; P)\right] < 0$.
- Here we will "construct" an equilibrium where if a firm receives a low shock, it will exit (i.e. $\pi(\varphi_L, P) + \beta\left[(1-\theta)v(\varphi_H; P)\right] < 0$) but if it receives a high shock, it will not exit (i.e. $\pi(\varphi_H, P) + \beta\left[\theta v(\varphi_H; P) + (1-\theta)v(\varphi_L; P)\right] > 0$).
- Conjecture this exit strategy is optimal and then check under what conditions it is optimal. Under the construction,

$$v(\varphi_L; P) = 0 \tag{2}$$

$$v(\varphi_H; P) = \frac{\pi(\varphi_H, P)}{1-\beta\theta} \tag{3}$$

Note that (2) is **different** from before due to the timing difference.

- Since we want positive entry, we must satisfy $\eta\frac{\pi(\varphi_H; P)}{1-\beta\theta} = c_e$. Again, under our conjecture, we have

$$\begin{aligned}c_e &= \eta\frac{\left(P\alpha\varphi_H\right)^{1/(1-\alpha)}\left[\frac{1}{\alpha} - 1\right] - c_f}{1-\beta\theta} \\ \Rightarrow P &= \frac{\left\{\left[c_e\left(\frac{1-\beta\theta}{\eta}\right) + c_f\right]\left(\frac{\alpha}{1-\alpha}\right)\right\}^{(1-\alpha)}}{\alpha\varphi_H}.\end{aligned}$$

- Now we need to pin down the law of motion for the distribution (which will give us Q) and ultimately M.[1]

$$\begin{aligned}\mu_{t+1}(\varphi_H) &= \mu_t(\varphi_H)\theta + M_t\eta \\ \mu_{t+1}(\varphi_L) &= \mu_t(\varphi_H)(1-\theta) + (1 - x(\varphi_L; P))\mu_t(\varphi_L) + M_t(1-\eta)\end{aligned} \tag{4}$$

Under the conjectured equilibrium $x(\varphi_L; P) = 1$.

[1] Here I write the distribution as a pdf rather than a cdf.

- From (4), we can solve for the invariant distribution

$$\begin{aligned}\mu(\varphi_H) &= \frac{\eta M}{(1-\theta)} \\ \mu(\varphi_L) &= \eta M + M(1-\eta) = M\end{aligned}$$

- Output given by

$$\begin{aligned}Q(M) &= \mu(\varphi_H)\varphi_H\left(n_H^*\right)^\alpha \\ &= \frac{M\eta}{1-\theta}\varphi_H\left(n_H^*\right)^\alpha\end{aligned}$$

  That is there are some incumbent firms and new entrants who produce in a given period t after receiving the high shock (if they receive a bad shock in t they exit and they don't produce). Now we can pin down M when we combine this equation with the demand function and the value of P we derived above.

- We need to check that a one period deviation is not profitable, that is, at the equilibrium price, the firms with the low shock prefer to exit (i.e. $\pi(\varphi_L;P)+\beta\left[\theta v(\varphi_L;P)+(1-\theta)v(\varphi_H;P)\right]<0$). At the above parameter values, this condition holds.

- The next table provides the equilibrium values for this example. Note that aggregate output and prices are state invariant (which will be very different in the next examples).

Competitive Example
$P = 3.296$, $Q = 1.704$, $M = 0.301$
$n(0.6) = 0.994$, $\pi(0.6) = -0.922, v(0.6) = 0$ (off-the-equilibrium path)
$n(1.0) = 1.284$, $\pi(1.0) = 0.551, v(1.0) = 4.0498$

3 Monopoly Example (NF=1)

In this case, there is "trivial" aggregate uncertainty due to the idiosyncratic shocks the monopolist receives and the monopolist recognizes that its quantity choice will affect the price it faces.

- firm's static optimization problem:

$$n_i^m = \arg\max_n \left[D-\varphi_i n^\alpha\right]\varphi_i n^\alpha - n - c_f \tag{5}$$

  The FOC is

$$P\alpha\varphi_i\left(n^m\right)^{\alpha-1} - \varphi_i^2\alpha(n^m)^{2\alpha-1} = 1 \tag{6}$$

  Using this equation we can solve for the optimal labor choice n_i^m.

- Notice that for a given price, the marginal benefit of adding labor for a monopolist (which is just the left hand side of (6)) is lower than the marginal benefit for a competitive firm (1)) - compare $-\varphi_i^2\alpha(n^m)^{2\alpha-1}$ to 0 - while the marginal cost (which is just the right hand side of the equations) is identical. This is the key factor why the monopoly quantity will be lower than that with perfect competition. Of course, the prices will differ too so this is just the intuition.

- Thus
$$\pi^m(\varphi_i) = [D - \varphi_i(n_i^m)^\alpha]\,\varphi_i(n_i^m)^\alpha - n_i^m - c_f$$

- The two steady state value functions indexed by φ_i are
$$\begin{aligned} v^m(\varphi_i) &= \max\Big\{0, \pi^m(\varphi_i) + \beta\big[\theta v^m(\varphi_i) + (1-\theta)v^m(\varphi_{j\neq i})\big]\Big\} \\ &= \max_{x\in\{0,1\}}(1-x)\Big\{\pi^m(\varphi_i) + \beta\big[\theta v^m(\varphi_i) + (1-\theta)v^m(\varphi_{j\neq i})\big]\Big\} \end{aligned}$$

- The monopolist will choose $x(\varphi_i) = 1$ if $\pi^m(\varphi_i) + \beta\big[\theta v^m(\varphi_i) + (1-\theta)v^m(\varphi_{j\neq i})\big] < 0$.

- Here we will “construct" an equilibrium where the monopolist always stays in the market for the same parameter values as the competitive case. This is possible because the monopoly profits are uniformly higher than with perfect competition (where firms with φ_L exit).

  Under the construction,
$$\begin{aligned} v^m(\varphi_L) &= \pi^m(\varphi_L) + \beta\Big[\theta v^m(\varphi_L) + (1-\theta)v^m(\varphi_H)\Big] \\ v^m(\varphi_H) &= \pi^m(\varphi_H) + \beta\Big[\theta v^m(\varphi_H) + (1-\theta)v^m(\varphi_L)\Big] \end{aligned}$$

  Solving the system of equations we find that
$$\begin{aligned} v^m(\varphi_H) &= \Big[\pi^m(\varphi_H) + \frac{\beta(1-\theta)}{1-\beta\theta}\pi^m(\varphi_L)\Big]\Big(\frac{1-\beta\theta}{1-2\beta\theta-\beta^2(1-2\theta)}\Big) \\ v^m(\varphi_L) &= \frac{\pi^m(\varphi_L) + \beta(1-\theta)v^m(\varphi_H)}{1-\beta\theta} \end{aligned}$$

  Note that $v^m(\varphi_H)$ is in $v^m(\varphi_L)$.

- The total output produced will be $Q^m(\varphi_i) = \varphi_i(n_i^*)^\alpha$.

- The only condition we need to check is that exit is not profitable in any state. It is sufficient to simply check $v^m(\varphi_L) \geq 0$.

- The following table provides the equilibrium values for this example. Note that aggregate output and prices are now state dependent.

Monopoly Example
$P^m(0.6) = 4.338$, $Q^m(0.6) = 0.662$
$P^m(1.0) = 3.750$, $Q^m(1.0) = 1.250$
$n^m(0.6) = 1.217$, $\pi^m(0.6) = -0.246, v^m(0.6) = 9.073$
$n^m(1.0) = 1.563$, $\pi^m(1.0) = 1.225, v^m(1.0) = 15.412$

4 Duopoly Example (NF=2)

In this case the price clears the market between firms playing a non-cooperative Cournot game. In particular, when there is more than one firm, each firm chooses the quantity of goods produced taking into account the production and exit strategies of other firms. In this case total output is given by $Q^d = \sum_{i=1}^{2} q^i$.

In this type of industry the state space is doubled. We need to consider all possible combinations of φ between the two possible firms induced by the idiosyncratic uncertainty. In fact the set of all possible states is given by $\{(\varphi_L, \varphi_L), (\varphi_L, \varphi_H), (\varphi_H, \varphi_L), (\varphi_H, \varphi_H), (\varphi_L, \varnothing), (\varphi_H, \varnothing), (\varnothing, \varnothing)\}$. Notice that in a symmetric duopoly equilibrium there are 3 aggregate states (i.e. boom, bust, and middle road).

We will use the following notation. Let any function f (which could be a value function or decision rule or distribution of firms) be written $f(\varphi_s^i, \overline{\varphi}_{-i})$ where $\varphi_s^i \in \{\varnothing, \varphi_L, \varphi_H\}$ is the productivity shock of firm i (and if $\varphi_s^i = \varnothing$ it is understood that firm i is contemplating entry) and $\overline{\varphi}_{-i} \in \{\varnothing, \varphi_L, \varphi_H\}$ is the vector of all other possible productivity shocks $\overline{\varphi}_{-i}$ of potential competitors (with $NF = 2$, $\overline{\varphi}_{-i}$ is simply a scalar, not a vector). For example, $\pi^d(\varphi_H, \varphi_L)$ is the profit of a firm with own productivity φ_H whose competitor received φ_L. As another example, $\pi^d(\varphi_H, \varnothing)$ is the profit of the firm with own productivity φ_H currently in a monopoly state. Finally, $e^d(\varnothing, \varphi_s)$ denotes the entry decision of a potential firm when there is an incumbent firm with productivity φ_s.

- If only one firm produces the optimal labor choice is given by the solution to the monopolist problem. That is $\pi^d(\varphi_i, \varnothing) = \pi^m(\varphi_i)$. Note that the value of the other firm's productivity is irrelevant because it is not active.

- If the two firms produce, firm i solves

$$\pi^d(\varphi_i, \varphi_j) = \max_{n_i} \Big[D - \varphi_i(n_i)^\alpha - \varphi_j(n_j)^\alpha\Big]\varphi_i(n_i)^\alpha - n_i - c_f \qquad (7)$$

The FOC is

$$D\alpha\varphi_i(n_i^d)^{\alpha-1} - 2\varphi_i^2\alpha\left(n_i^d\right)^{2\alpha-1} - \varphi_j n_j^\alpha \alpha\varphi_i\left(n_i^d\right)^{\alpha-1} = 1 \qquad (8)$$

This gives the labor demand policy function (given the second firm labor choice) of firm i, $n_i^d(\varphi_i, \varphi_j|n_j)$.

- Specifically, suppose $\alpha = 1/2$. Then (8) can be written

$$n_i^d = \left(\frac{\varphi_i \left[D - \varphi_j n_j^{1/2} \right]}{2 \left[1 + \varphi_i^2 \right]} \right)^2$$

- Note that in the foc (8), firm i takes as given its competitor's actions n_j. A Cournot-Nash equilibrium requires consistency; that is, firm $i's$ belief about n_j in (8) is consistent with optimal actions taken by firm j. Specifically, in equilibrium $n_j = n_j^d(\varphi_j, \varphi_i | n_i)$.

- Conditional on holding firm j's output constant (i.e. n_j and φ_j), labor at firm i rises if productivity there rises (of course with a finite set of φ we can't literally take derivatives, but let's be a bit sloppy here):

$$\begin{aligned} \frac{dn_i}{d\varphi_i} &= 2 \left(\frac{\varphi_i \left[D - \varphi_j n_j^{1/2} \right]}{2 \left[1 + \varphi_i^2 \right]} \right) \cdot \left[\frac{2 \left[1 + \varphi_i^2 \right] \left[D - \varphi_j n_j^{1/2} \right] - \varphi_i \left[D - \varphi_j n_j^{1/2} \right] 4\varphi_i}{4 \left[1 + \varphi_i^2 \right]^2} \right] \\ &= 2 \left(\frac{\varphi_i \left[D - \varphi_j n_j^{1/2} \right]}{2 \left[1 + \varphi_i^2 \right]} \right) \cdot \left[\frac{\left[D - \varphi_j n_j^{1/2} \right] \left[1 - \varphi_i^2 \right]}{2 \left[1 + \varphi_i^2 \right]^2} \right] > 0. \end{aligned}$$

- The decision rules $n_i^d(\varphi_i, \varphi_j | n_j^d)$ and $n_j^d(\varphi_j, \varphi_i | n_i^d)$ can also be thought of as reaction functions. The slope of the reaction function of firm i is given by

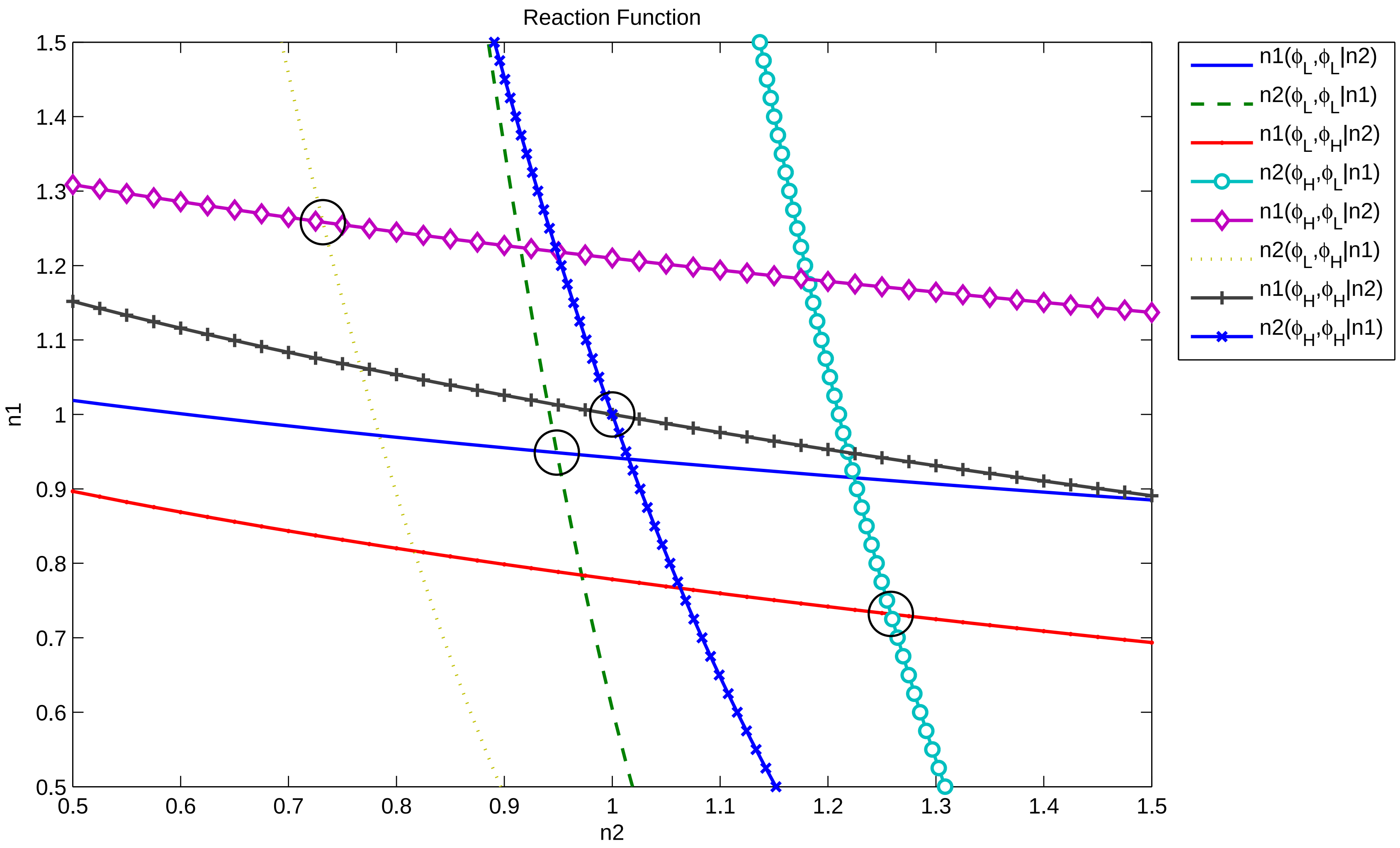
$$\frac{dn_i^d}{dn_j} = - \left(\frac{\varphi_i \left[D - \varphi_j n_j^{1/2} \right]}{2 \left[1 + \varphi_i^2 \right]} \right) \frac{\varphi_i \left[\varphi_j n_j^{-1/2} \right]}{2 \left[1 + \varphi_i^2 \right]} < 0.$$

- We plot reaction functions in Figure 1 for a particular equilibrium where firms with a low productivity shock exit (hence any reaction function that has a vector where $\varphi_i \neq \varphi_j$ is not consistent with behavior on-the-equilibrium path, but of course we still need to solve for best responses off-the-equilibrium path). As can be seen in the figure, the higher the common shocks on-the-equilibrium path, the higher demand for labor. Also, in the asymmetric off-the-equilibrium path case where there are two incumbents with $\varphi_i \neq \varphi_j$,the figure shows that the firm with the high productivity shock demands more labor than the symmetric case.

- If two firms are in the market there are four possible states of the world (φ_L, φ_L), (φ_L, φ_H), (φ_H, φ_L) and (φ_H, φ_H).

- The value function is given by

$$v(\varphi_i, \varphi_{-i}) = \max \left\{ 0, \pi(\varphi_i, \varphi_{-i}) + \beta E \left[v(\varphi_i', \varphi_{-i}') \right] \right\}.$$



Reaction Function
n1
n2
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
n1(ϕL,ϕL|n2)
n2(ϕL,ϕL|n1)
n1(ϕL,ϕH|n2)
n2(ϕH,ϕL|n1)
n1(ϕH,ϕL|n2)
n2(ϕL,ϕH|n1)
n1(ϕH,ϕH|n2)
n2(ϕH,ϕH|n1)

- As before we can write this expression as:

$$v(\varphi_i, \varphi_{-i}) = \max_{x \in \{0,1\}} (1-x)\Big\{\pi(\varphi_i, \varphi_{-i}) + \beta E\big[v(\varphi_i', \varphi_{-i}')\big]\Big\}$$

  which yields exit decision rules $x(\varphi_i, \varphi_{-i})$.

4.1 An Equilibrium where NF=2 binds

- We will construct an equilibrium with the following characteristics:
  - If two firms are active (i.e. $NF = 2$), Table 1

(φ_i, φ_j)	Exit Decision
(φ_L, φ_L)	$(stay, stay)$
(φ_L, φ_H)	$(exit, stay)$
(φ_H, φ_L)	$(stay, exit)$
(φ_H, φ_H)	$(stay, stay)$

Table 1:

  - If only one firm is active it always stays in the market (i.e. $x(\varphi_i, \varnothing) = 0$ for all $\varphi_i \in \{\varphi_L, \varphi_H\}$).
  - As for entry, a firm enters only if a monopolist receives φ_L. Since $NF = 2$, it would be infinitely costly to enter with two active incumbents. If there are no incumbents (which never happens on-the-equilibrium-path of this construction but could happen off the equilibrium path), then two potential entrants are randomly selected to enter.
- Note that in this environment the economy will fluctuate between duopoly and monopoly of random lengths. For instance, if there is a realization of shocks such that there is asymmety in the duopoly case, then we enter a monopoly phase until the monopolist recieves a low shock then a duopoly will arise in the following period.
- The 6 value functions become

$$
\begin{aligned}
v(\varphi_L,\varphi_L) &= \pi(\varphi_L,\varphi_L)+\beta\Big\{\theta\big[\theta v(\varphi_L,\varphi_L)+(1-\theta)\cdot 0\big]+\\
&\quad (1-\theta)\big[\theta v(\varphi_H,\varphi_L)+(1-\theta)v(\varphi_H,\varphi_H)\big]\Big\}\\
v(\varphi_L,\varphi_H) &= 0\\
v(\varphi_H,\varphi_L) &= \pi(\varphi_H,\varnothing)+\beta\Big\{\theta v(\varphi_H,\varnothing)+(1-\theta)v(\varphi_L,\varnothing)\Big\}\\
v(\varphi_H,\varphi_H) &= \pi(\varphi_H,\varphi_H)+\beta\Big\{(1-\theta)\big[\theta\cdot 0+(1-\theta)v(\varphi_L,\varphi_L)\big]+\\
&\quad \theta\big[\theta v(\varphi_H,\varphi_H)+(1-\theta)v(\varphi_H,\varphi_L)\big]\Big\}\\
v(\varphi_H,\varnothing) &= \pi(\varphi_H,\varnothing)+\beta\Big\{\theta v(\varphi_H,\varnothing)+(1-\theta)v(\varphi_L,\varnothing)\Big\}\\
v(\varphi_L,\varnothing) &= \pi(\varphi_L,\varnothing)+\beta\Big\{\theta\big[(1-\eta)v(\varphi_L,\varphi_e=\varphi_L)+\eta\cdot 0\big]+\\
&\quad (1-\theta)\big[(1-\eta)v(\varphi_H,\varphi_e=\varphi_L)+\eta v(\varphi_H,\varphi_e=\varphi_H)\big]\Big\}
\end{aligned}
$$

This is a system of 6 equations in 6 unknowns that you can solve with the help of the computer.

- From the solution to this system of equations, you can pin down the highest and the lowest value of the entry cost, $(\underline{c},\overline{c})$, such that this equilibrium exists. That is, if $N=1$, it must be profitable to enter if the incumbent firm currently has φ_L and not to enter if the incumbent firm has φ_H. To check this (recalling that entry comes one period in advance (which explains β)):
  - If incumbent has φ_L, expected value of entry $\geq c_e$ or

    $$\beta\left\{\begin{array}{c}(1-\eta)\left[(\theta v(\varphi_e=\varphi_L,\varphi_L)+(1-\theta)\cdot 0\right]\\ +\eta\left[\theta v(\varphi_e=\varphi_H,\varphi_L)+(1-\theta)v(\varphi_e=\varphi_H,\varphi_H)\right]\end{array}\right\}\geq c_e. \quad (9)$$

    Let $\overline{c}$ denote the value of the left hand side of (9). Then entry occurs provided costs are below that number (i.e. $\overline{c}\geq c_e$).
  - If incumbent has φ_H, expected value of entry $< c_e$ or

    $$\beta\left\{\begin{array}{c}(1-\eta)\left[(\theta\cdot 0+(1-\theta)v(\varphi_e=\varphi_L,\varphi_L)\right]\\ +\eta\left[\theta v(\varphi_e=\varphi_H,\varphi_H)+(1-\theta)v(\varphi_e=\varphi_H,\varphi_L)\right]\end{array}\right\}< c_e \quad (10)$$

    Let $\underline{c}$ denote the value of the left hand side of (10). Then entry does not occur if costs are above that number (i.e. $\underline{c}<c_e$).
  - Notice that if $v(\varphi_L,\varphi_L)<v(\varphi_H,\varphi_H)<v(\varphi_H,\varphi_L)$, then it is possible that the lhs of (9) exceeds the lhs of (10) since $\theta>(1-\theta)$.
- Hence costs which satisfy the condition for entry when the incumbent has φ_L and no entry when the incumbent has φ_H must satisfy $c_e\in(\underline{c},\overline{c}]$. Generally we should prove that this set is non-empty. In the particular parameterization above, $\underline{c}=2.36$ and $\overline{c}=4.90$.

- Also, we need to verify that in the off-the-equilibrium path event that there are no incumbents at all, two randomly selected entrants would choose to enter at $c_e \in (\underline{c}, \overline{c}]$. That is

$$\beta\left\{(1-\eta)^2 v(\varphi_L,\varphi_L) + (1-\eta)\eta v(\varphi_L,\varphi_H) + \eta(1-\eta)v(\varphi_H,\varphi_L) + \eta^2 v(\varphi_H,\varphi_H)\right\} \geq c_e. \tag{11}$$

  In this particular case, the left hand side equals 3.63, so there will be entry by two firms.

- Next, we must check that no firm has an incentive to deviate (for one period) from what we conjectured in the above table.

  - Row 1: $v(\varphi_L,\varphi_L) \geq 0$.
  - Row 2: This is the most important condition - we need to check that exit is profitable for an incumbent firm with φ_L when the other incumbent firm has φ_H, i.e. $v(\varphi_L,\varphi_H) = 0$. Hence, we must check that the incumbent firm with φ_L does not deviate from exiting in this state. Then since the value from such a deviation is given by

    $$v^{dev}(\varphi_L,\varphi_H) = \pi(\varphi_L,\varphi_H) + \beta\left\{\begin{array}{c} \theta\left[\theta\cdot 0 + (1-\theta)v(\varphi_L,\varphi_L)\right] \\ +(1-\theta)\left[\theta v(\varphi_H,\varphi_H) + (1-\theta)v(\varphi_H,\varphi_L)\right] \end{array}\right\}$$

    we need $v^{dev}(\varphi_L,\varphi_H) < 0 = v(\varphi_L,\varphi_H)$. Note the 0 in v^{dev} occurs since the firm returns to being on the equilibrium path in state (φ_L,φ_H) and that $v(\varphi_H,\varphi_L)$ ensures the deviating firm enjoys monopoly profits in that period.
  - Row 3: $v(\varphi_H,\varphi_L) \geq 0$.
  - Row 4: $v(\varphi_H,\varphi_H) \geq 0$

- Finally, to check that a monopolist does not deviate from what we conjectured above and exit, we need:

$$\begin{aligned} v(\varphi_H,\varnothing) &\geq 0 \\ v(\varphi_L,\varnothing) &\geq 0. \end{aligned}$$

- The following table provides the equilibrium values for this example. Note that aggregate output and prices are now state dependent.

Duopoly Example
$P^d(0.6,0.6) = 3.831$, $Q^d(0.6) = 1.169$
$P^d(1.0,1.0) = 3.000$, $Q^d(1.0,1.0) = 2.000$
$n^d(0.6,0.6) = 0.949$, $\pi^d(0.6,0.6) = -0.610$, $v^d(0.6,0.6) = 1.200$
$n^d(1.0,1.0) = 1.0$, $\pi^d(1.0,1.0) = 0.100$, $v^d(1.0,1.0) = 4.259$

- Comparing all cases we have:

	competitive	monopoly	duopoly
prices	$P = 3.296$	$P^m(0.6) = 4.338$	$P^d(0.6, 0.6) = 3.831$
		$P^m(1.0) = 3.750$	$P^d(1.0, 1.0) = 3.000$
quantities	$Q = 1.704$	$Q^m(0.6) = 0.662$	$Q^d(0.6) = 1.169$
		$Q^m(1.0) = 1.250$	$Q^d(1.0, 1.0) = 2.000$
profits	$\pi(0.6) = -0.922$(o-e-p)	$\pi^m(0.6) = -0.246$	$\pi^d(0.6, 0.6) = -0.610$
	$\pi(1.0) = 0.551$	$\pi^m(1.0) = 1.225$	$\pi^d(1.0, 1.0) = 0.100$
value of firm	$v(0.6) = 0$ (o-e-p)	$v^m(0.6) = 9.073$	$v^d(0.6, 0.6) = 1.200$
	$v(1.0) = 4.0498$	$v^m(1.0) = 15.412$	$v^d(1.0, 1.0) = 4.259$

- One can then check whether the model generates procylical entry, countercylical exit, and countercylical markups (see Jaimovich and Floetotto (2008, Journal of Monetary Economics) for a DSGE model with imperfect competition with these properties).
  - Obviously, the competitive case cannot generate this unless we add aggregate uncertainty.
  - The monopoly case does not have entry and exit, but the markup $P^m(\varphi) - 1$ is countercyclical.
  - The duopoly case also has countercyclical markups, countercyclical exit, but predicts countercylical entry.

4.2 An Equilibrium where NF=2 doesn't bind

Here we check whether it is possible to find a monopoly equilibrium even though the costs are not infinite to enter if there is already an incumbent. We will show that there is an equilibrium with only one firm operating (a monopolist) under this structure and he always stays. This means that $NF = 2$ does not constrain entry. Basically, we simply need that c_e is high enough to deter entry if there is already an incumbent.

- We construct an equilibrium almost identical to what we had before except that if there is one firm, there is no entry for all $\varphi_i \in \{\varphi_L, \varphi_H\}$ of the incumbent. That is, we keep table 1 as a conjecture for the off-the-equilibrium path node where two firms are active.
- The 6 value functions become

$$
\begin{aligned}
v(\varphi_L,\varphi_L) &= \pi(\varphi_L,\varphi_L)+\beta\Big\{\theta\big[\theta v(\varphi_L,\varphi_L)+(1-\theta)\cdot 0\big]+ \\
&\quad (1-\theta)\big[\theta v(\varphi_H,\varphi_L)+(1-\theta)v(\varphi_H,\varphi_H)\big]\Big\} \\
v(\varphi_L,\varphi_H) &= 0 \\
v(\varphi_H,\varphi_L) &= \pi(\varphi_H,\varnothing)+\beta\Big\{\theta v(\varphi_H,\varnothing)+(1-\theta)v(\varphi_L,\varnothing)\Big\} \\
v(\varphi_H,\varphi_H) &= \pi(\varphi_H,\varphi_H)+\beta\Big\{(1-\theta)\big[\theta\cdot 0+(1-\theta)v(\varphi_L,\varphi_L)\big] \\
&\quad \theta\big[\theta v(\varphi_H,\varphi_H)+(1-\theta)v(\varphi_H,\varphi_L)\big]\Big\} \\
v(\varphi_H,\varnothing) &= \pi(\varphi_H,\varnothing)+\beta\Big\{\theta v(\varphi_H,\varnothing)+(1-\theta)v(\varphi_L,\varnothing)\Big\} \\
v(\varphi_L,\varnothing) &= \pi(\varphi_L,\varnothing)+\beta\Big\{\theta v(\varphi_L,\varnothing)+(1-\theta)v(\varphi_H,\varnothing)\Big\}
\end{aligned}
$$

Note the only difference from the previous one is the last line.

- Next, we must check that no firm has an incentive to deviate from what we conjectured in the above table.
  - Row 1: $v(\varphi_L,\varphi_L)\geq 0$.
  - Row 2: Since the value from such a deviation is given by

    $$
    \begin{aligned}
    v^d(\varphi_L,\varphi_H) &= \pi(\varphi_L,\varphi_H)+\beta\Big\{\theta\left[\theta\cdot 0+(1-\theta)v(\varphi_L,\varphi_L)\right] \\
    &\quad +(1-\theta)\left[\theta v(\varphi_H,\varphi_H)+(1-\theta)v(\varphi_H,\varphi_L)\right]\Big\}
    \end{aligned}
    $$

    we need $v^d(\varphi_L,\varphi_H)<0=v(\varphi_L,\varphi_H)$.
  - Row 3: $v(\varphi_H,\varphi_L)\geq 0$.
  - Row 4: $v(\varphi_H,\varphi_H)\geq 0$

- To check that a monopolist does not deviate from what we conjectured above and exit, we need:

  $$
  \begin{aligned}
  v(\varphi_H,\varnothing) &\geq 0 \\
  v(\varphi_L,\varnothing) &\geq 0.
  \end{aligned}
  $$

- Also, we must check that no one enters to compete with the monopolist with φ_L (it is sufficient to check when the monopolist has a low shock, since if this condition holds, then a potential entrant would surely not enter if the monopolist has the high shock.):

  $$
  \beta\left\{\begin{array}{c}(1-\eta)\left[(\theta v(\varphi_e=\varphi_L,\varphi_L)+(1-\theta)\cdot 0\right] \\ +\eta\left[\theta v(\varphi_e=\varphi_H,\varphi_L)+(1-\theta)v(\varphi_e=\varphi_H,\varphi_H)\right]\end{array}\right\}<c_e.
  $$

  If you kept all the parameters the same, except let $c_e=5$ instead of $c_e=3$, then you would get this type of equilibrium since the value on the lhs is 4.91.