Heterogenous-Agent Life-Cycle Models: Steady States

Dean Corbae
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Introduction

- Two types of heterogenous-agent models constitute the workhorse in macroeconomists to study the wealth distribution.
  - *dynastic models*—infinite Horizon (i.e. agents live forever)
  - *life-cycle models*—finite Horizon (i.e. agents die at age N)
- The main mechanism to generate differences in assets is the same in both types of models
  - uninsurable idiosyncratic shocks to earnings
  - precautionary savings to self-insure

Comparing Models to Data

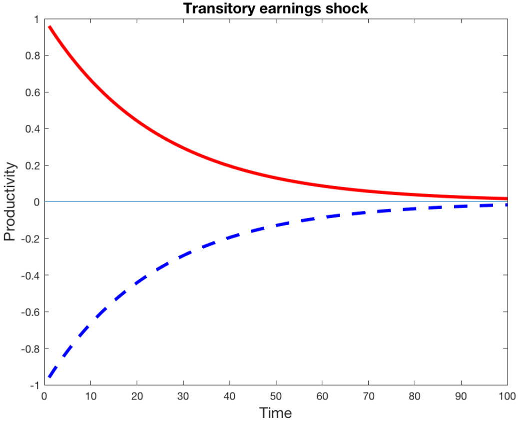
	Data	Dynastic Model	Life-Cycle Model
Earnings Gini	0.51	0.10	0.42
Wealth Gini	0.78	0.38	0.74

Source: Quadrini, V. and V. Rios-Rull (1997) "Understanding the U.S. Distribution of Wealth", Federal Reserve Bank of Minneapolis Quarterly Review, 21, p.22-36.

- A life cycle model helps explain the data better.

Why Life-Cycle Models match inequality better

- Consider the effect of a temporary earnings shock (pos. or neg.) over time:



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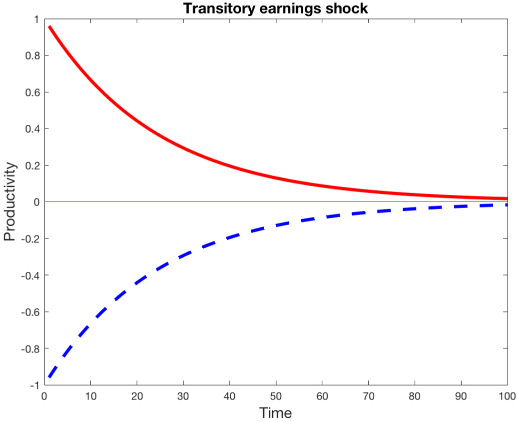

- The shorter the time horizon, the larger is the cross-sectional variance in earnings.
- More earnings variation will induce more wealth inequality.

Life-Cycle Models

References:

- M. Huggett (1996) "Wealth distribution in Life-Cycle Models" *Journal of Monetary Economics* Vol. 38, 469-494.

  Question: Can a life-cycle model match wealth inequality?

- J. Conesa and D. Krueger (1999) "Social Security Reform with Heterogeneous Agents, *Review of Economic Dynamics*, Vol. 2, 757-795.

  Question: Should the current Pay-as-you-go social security policy in the U.S. be abandoned?

Learning Objectives

- Finite horizon dynamic programming: $v_n = Tv_{n+1}$ backward iteration with known final period value function (i.e. 0).
- Cross-sectional distributions of each age cohort n solves $\psi_{n+1} = T^*\psi_n$, where initial distribution is known.
- While age adds another state variable (bad), both operators are applied a finite number of times (good) unlike until convergence in the dynastic model.
- This lecture, compute stationary equilibrium.
- Next lecture we will compute transition paths between stationary equilibria.

Environment

- Each period a continuum of agents is born.
- Agents live a maximum of N periods.
- Probability of surviving to age n conditional on having survived to age $n-1$ denoted s_n.
- Population grows at rate η.
- Assume that each age n cohort makes up a constant fraction μ_n of the population at any point in time where $\sum_{n=1}^{N} \mu_n = 1$.
- Under above assn, relative sizes of each cohort of age n is

$$\mu_{n+1} = \frac{s_{n+1}}{(1+\eta)} \mu_n$$

independent of calendar time t (like a balanced growth path).

▸ Derivation

Environment Cont'd

Technology

- Labor endowment $e(z, n)$ depends on age n and idiosyncratic labor productivity shock z. The shocks follow a finite state Markov chain and are iid across agents.
- Constant returns to scale production technology $Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$ with depreciation δ.
- Competitive labor and capital rental markets at prices $w = F_2(K, L)$ and $r = F_1(K, L) - \delta$.

Environment Cont'd

- Preferences of age 1 agents:

$$\mathbb{E}_0 \left[\sum_{n=1}^{N} \beta^n \left(\prod_{j=1}^{n} s_j \right) u(c_n, l_n) \right]$$

  where $u(c_n, l_n) = \frac{(c_n^{\gamma}(1-l_n)^{1-\gamma})^{1-\sigma}}{1-\sigma}$.

- One period bonds $a' \geq \underline{a}$ pay rate r.

- Agents cannot hold debt in the last period of their life and are endowed with zero asset holdings in the first period of life.

Government

The government is involved in 3 activities:

- It imposes a linear social security tax θ to finance age-dependent social security benefits b_n, where

$$b_n = \begin{cases} 0 & \text{if } n < R \\ b & \text{if } n \geq R \end{cases}$$

  given retirement age R.

- It imposes a liner capital and labor tax τ to finance government spending, G, which is unproductive (i.e. wasted).

- It collects accidental bequests and redistributes them as lump-sum transfers, Υ, across all agents.

Household Problem

- Household's problem is

$$V(z,a,n) = \max_{\{c,l,a'\}} u(c,l) + \beta s_{n+1} \mathbb{E}\left[V(z',a',n+1) \mid (z,a,n) \right] \quad (1)$$

s.t.

$$\begin{aligned} c + a' &= a(1 + r(1-\tau)) + (1-\theta-\tau)e(z,n)wl + \Upsilon + b_n \\ c &\geq 0 \\ 0 &\leq l \leq 1 \\ a' &\geq \underline{a} \\ a = 0 &\quad \text{if} \quad n = 1 \\ a' \geq 0 &\quad \text{if} \quad n = N. \end{aligned}$$

The Cross-Sectional Distribution

- Distribution of agents in the population defined over age(n), asset(a), and earnings status(z).
- Specifically, let $x = (z, a)$ and let $(X, \mathcal{B}(X), \psi_n)$ be a probability space
- $\psi_n(B_0)$ is the fraction of age n agents whose state x lies in set B_0 given initial distribution ψ_1.
- With zero initial wealth, ψ_1 is just the cross-sectional initial distribution of earnings in the first period of life.
- The overall fraction of age n agents whose state x lies in set B_0 in the whole population is then $\psi_n(B_0)\mu_n$

Law of Motion of the Cross-Sectional Distribution

- The distribution across agents at age $n = 1, ..., N-1$ is given recursively as

$$\begin{aligned}\psi_{n+1}(B_0) &= (T^*\psi_N)(B_0) = \int_X P(x, n, B_0)\psi_n(dx), \forall B_0 \in \mathcal{B}(X) \\ &= \int_{Z_0, A_0} \left\{ \int_{Z,A} \chi_{\{a'=g(z,a,n)\}}\pi(z'|z)\psi_n(dz, da) \right\} dz' da'\end{aligned}$$

- $P(x, n, B_0)$ is a transition function which gives the probability that an age n agent transits to the set B_0 next period given the agent's current state is x.

Equilibrium Definition

A stationary equilibrium is $(c(x,n), g(x,n), l(x,n), r, w, K, L, \Upsilon, G, \tau, \theta, b)$ and distributions $\{\psi_1, \psi_2, ..., \psi_N\}$ such that:

- $c(x,n)$, $l(x,n)$ and $g(x,n)$ solve the HH decision problem in (1)
- $w = F_2(K,L)$ and $r = F_1(K,L) - \delta$ in competitive input markets
- Markets clear:
  1. goods:
  $$\sum_n \mu_n \int_X [c(x,n) + g(x,n)] \, d\psi_n = F(K,L) + (1-\delta)K - G$$
  2. capital
  $$K' = \sum_n \mu_n \int_X g(x,n) d\psi_n$$
  3. labor
  $$\sum_n \mu_n \int_X l(x,n) e(z,n) d\psi_n = L$$

Equilibrium Definition Cont'd

- $\{\psi_1, \psi_2, ..., \psi_N\}$ is consistent with individual decision rules implied by (1)
- the govt budget constraint is satisfied: $G = \tau(rK + wL)$
- social security (pay-as-you-go) feasibility:

$$\theta wL = b\left(\sum_{n=R}^{N} \mu_n\right)$$

- transfer wealth equals accidental bequests:

$$\Upsilon = \left[\sum_{n=1}^{N-1} \frac{\mu_n(1-s_{n+1})}{(1+\eta)} \int_X g(x,n)(1+r(1-\tau))d\psi_n\right]$$

Note that $\mu_n(1-s_{n+1})/(1+\eta)$ is the fraction of people who do not survive to the next period.

Computation of the Stationary Equilibrium

1. Make initial guesses of the steady state values of aggregate K, aggregate L and government transfers Υ
2. Compute social security benefits b.
3. Compute the prices w and r, which solve the firm's problem.
4. Given w and r, compute the household's decision functions by backward iteration.
5. Using decision rules of HH to compute the cross sectional distribution of each age n cohort.
6. Compute the aggregate capital stock, aggregate labor supply and government transfers.
7. Update K, L and Υ and return to step 2 until convergence.

Deriving μ

- Let calendar time indexed by t.
- Let p_n^t denote the total number of age n agents alive at time t.
- Then the total population at time t is $P^t = \sum_{n=1}^{N} p_n^t$.
- Letting $\mu_n^t = \frac{p_n^t}{P^t}$, we have

$$1 = \sum_{n=1}^{N} \frac{p_n^t}{P^t} = \sum_{n=1}^{N} \mu_n^t$$

- In steady state per capita terms, μ_n^t is independent of calendar time t.
- This normalization makes the model stationary with population growth.

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