Lecture 4: Dynamic discrete choice models

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Introduction: Dynamic Discrete Choices

- Machine replacement and investment decisions: Rust (1987)
- Renewal or retirement decisions: Pakes (1986), Berkovec and Stern (1991)
- Inventory control: Erdem, Imai, and Keane (2003), Hendel and Nevo (2006)
- Bayesian learning: Erdem and Keane (1996), Ackerberg (2003), Crawford and Shum (2005)
- Demand for durable goods: Gordon (2009), Gowrisankaran and Rysman (2012)
- Migration: Kennan and Walker (2011)
- Education/Occupation: Keane and Wolpin (1997)

Machine replacement and investment decisions

- Consider a firm producing a good at N plants (indexed by i) that operate independently.
- Each plant has a machine.
- Examples:
  - Rust (1987): Each plant is a Madison WI bus, and Harold Zucher is the plant operator.
  - Das (1992): Consider cement plants, where the machines are cement kiln.
  - Rust and Rothwell (1995): Study the maintenance of nuclear power plants.
- Related applications: Export decisions (Das et al. (2007)), replacement of durable goods (Adda and Cooper (2000), Gowrisankaran and Rysman (2012)).

Bus Replacement: Rust (1987)

- Investment decision (a_{it}): Replace ($a = 1$) or not ($a = 0$)
- State variables: machine age x_{it} (miles), choice-specific profit shock $\{\epsilon_{it}(0), \epsilon_{it}(1)\}$.
- Variable profit for bus i:

$$\pi_{it} = \begin{cases} Y(0, \epsilon_{it}(1)) - RC(x_{it}) & \text{If } a_{it} = 1 \\ Y(x_{it}, \epsilon_{it}(0)) & \text{Otherwise.} \end{cases}$$

- Aging/depreciation process:

$$\begin{aligned} &\text{Deterministic:} && x_{it+1} = (1 - a_{it})x_{it} + 1 \\ &\text{Stochastic (Rust):} && x_{it+1} = (1 - a_{it})x_{it} + \xi_{t+1} \end{aligned}$$

Profits and Depreciation: Assumptions

1. Additive separable (AS) profit shock:

$$Y((1-a)x, \epsilon(a)) = \theta_{Y_0} + \theta_{Y_1}(1-a)x + \epsilon(a)$$

2. Conditional independence (CI): $f(\epsilon_{t+1}|\epsilon_t, x_t) = f(\epsilon_{t+1})$
3. Aging follows is a discrete random-walk process: $x_{it} \in \{0, 1, ..., M\}$ and matrix $F(x'|x, a)$ characterizes its controlled Markov transition process.

Dynamic Optimization

- Harold Zucher maximizes expected future profits:

$$V(a_{it}|x_{it},\epsilon_{it}) = E\left(\sum_{\tau=0}^{\infty}\beta^{\tau}\pi_{it+\tau}\middle|x_{it},\epsilon_{it},a_{it}\right)$$

- **Recursive formulation:** Bellman equation

$$\begin{aligned} V(a|x,\epsilon) &= Y((1-a)\cdot x) - a\cdot RC(x) + \epsilon(a) \\ &\quad +\beta\sum_{x'} E_{\epsilon'}\left(V(x',\epsilon')\right) F(x'|x,a) \\ &= v(a,x) + \epsilon(a) \end{aligned}$$

where $V(x,\epsilon) \equiv \max_{a\in\{0,1\}} V(a|x,\epsilon)$.

Discrete Choice

- Optimal replacement decision:

$$a^* = \begin{cases} 1 & \text{If } v(1,x) - v(0,x) = \tilde{v}(x) > \epsilon(0) - \epsilon(1) = \tilde{\epsilon} \\ 0 & \text{Otherwise.} \end{cases}$$

- If $\{\epsilon(0), \epsilon(1)\}$ are distributed according to a T1EV distribution:

$$\Pr(a_{it} = 1 | x_{it}) = \exp(\tilde{v}(x_{it})/(1 + \exp(\tilde{v}(x_{it})))$$

$$\begin{aligned} \bar{V}(x_{it}) &= E_{\epsilon'}\left(V(x', \epsilon')\right) \\ &= E\left(\max_{a_{it}} v(a_{it}, x_{it}) + \epsilon_{it}(a_{it})\right) \\ &= \ln\left(\sum_{a=0,1} \exp(v(a, x_{it}))\right) + \gamma \end{aligned}$$

Solution to the dynamic-programming (DP) problem

- Assumptions (1) and (2) imply that we only need numerically find a fixed-point to the "Emax" function $\bar{V}(x)$ (M elements):

$$\begin{aligned}\bar{V}(x) &= E_\epsilon\left(\max_a \quad v(a,x)+\epsilon(a)\right) \\ &= E_\epsilon\left(\max_a \quad \Pi(a,x)+\beta\sum_{x'}\bar{V}(x')F(x'|x,a)+\epsilon(a)\right) \\ &= \Gamma(x|\bar{V})\end{aligned}$$

where $\Pi(a,x) = Y((1-a)\cdot x) - a\cdot RC(x)$, and $\Gamma(x|\bar{V})$ is a contraction mapping.

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&= \Gamma(x|\bar{V})
\end{aligned}
$$

  where $\Pi(a,x)=Y((1-a)\cdot x)-a\cdot RC(x)$, and $\Gamma(x|\bar{V})$ is a contraction mapping.

- Matrix form representation using the T1EV distribution assumption:

$$
\begin{aligned}
\bar{\mathrm{V}} &= \ln\left(\exp\left(\Pi(0)+\beta F(0)\bar{\mathrm{V}}\right)+\exp\left(\Pi(1)+\beta F(1)\bar{\mathrm{V}}\right)\right)+\gamma \\
&= \Gamma(\bar{\mathrm{V}})
\end{aligned}
$$

  where γ is the Euler constant, $F(0)$ and $F(1)$ are two $M\times M$ conditional transition probability matrix.

Algorithm 1: Value Function Iteration

- Fixed objects (discretization):
  - Payoffs ($M \times 1$):

$$\Pi(a) = \{\theta_0 + \theta_x(1-a)x_i - a \cdot RC(x)\}_{i=1,\dots,M} \text{ for } a \in \{0,1\}$$

  - Conditional transition probability ($M \times M$): $F(a)$ for $a \in \{0,1\}$

$$F_{j,k}(a) = F(x_{t+1} = x_k | x_t = x_j, a_t = a)$$

  - Stopping rule: $\eta \approx 10^{-14}$.

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  - Stopping rule: $\eta \approx 10^{-14}$.
- Value function iteration algorithm:
  1. Guess initial value for $\bar{V}^0(x)$. Example: Static value function

$$\bar{\mathrm{V}}^0(x) = \ln\left(\exp\left(\Pi(0)\right) + \exp\left(\Pi(1)\right)\right) + \gamma$$

  2. Update value function iteration k:

$$\bar{\mathrm{V}}^k = \ln\left(\exp\left(\Pi(0) + \beta F(0)\bar{\mathrm{V}}^{k-1}\right) + \exp\left(\Pi(1) + \beta F(1)\bar{\mathrm{V}}^{k-1}\right)\right) + \gamma$$

  3. Stop if $||\bar{\mathrm{V}}^k - \bar{\mathrm{V}}^{k-1}|| < \eta$. Otherwise, repeat steps (2)-(3).

Policy Function Representation

- Define conditional choice-probability (CCP) mapping:

$$\begin{aligned} P(x) &= \Pr\left(\begin{array}{c} \Pi(1,x) + \beta \sum_{x'} \bar{V}(x')F(x'|x,1) + \epsilon(1) \\ \geq \Pi(0,x) + \beta \sum_{x'} \bar{V}(x')F(x'|x,0) + \epsilon(0) \end{array} \right) \\ &= \exp(\tilde{v}(x)/\left(1 + \exp(\tilde{v}(x))\right) = \left(1 + \exp(-\tilde{v}(x))\right)^{-1} \end{aligned}$$

  Where, $\tilde{v}(x) = v(1,x) - v(0,x)$.

- At the "optimal" CCP, we can write the Emax function as follows:

$$\begin{aligned} \bar{V}^P(x) &= (1 - P(x))\left[\Pi(0,x) + e(0,x) + \beta \sum_{x'} \bar{V}^P(x')F(x'|x,0)\right] \\ &\quad + P(x)\left[\Pi(1,x) + e(1,x) + \beta \sum_{x'} \bar{V}^P(x')F(x'|x,1)\right] \end{aligned}$$

  where $e(a,x) = E(\epsilon(a)|a^* = a, x)$ is the conditional expectation $\epsilon(a)$.

Policy Function Representation (continued)

- If $\epsilon(a)$ is T1EV distributed, we can write this expectation analytically:

$$e(a,x) = \gamma - \ln P(a|x).$$

- This implicitely defines the value function in terms of the CCP vector:

$$\bar{\mathsf{V}}^P = \left(I - \beta \mathsf{F}^P\right)^{-1} \begin{bmatrix} (1-\mathsf{P}) * (\Pi(0) + \mathsf{e}(0)) \\ +\mathsf{P} * (\Pi(1) + \mathsf{e}(1)) \end{bmatrix} \tag{1}$$

  where $\mathsf{F}^P = (1-\mathsf{P}) * F(0) + \mathsf{P} * F(1)$ and $*$ is the element-by-element multiplication operator.

- Equations 1 and 1 define a fixed-point in P:

$$\mathsf{P}^* = \Psi(\mathsf{P}^*)$$

  where $\Psi(\cdot)$ is a contraction mapping.

Algorithm 2: Policy Function Iteration

1. Guess initial value for the CCP. Example: Static choice-probability

$$P(x) = (1 + \exp(-(\Pi(x|1) - \Pi(x|0))))^{-1}$$

2. Calculate expected value function:

$$\bar{\mathrm{V}}^{k-1} = \left(I - \beta \mathrm{F}^{k-1}\right)^{-1} \begin{bmatrix} (1 - \mathrm{P}^{\mathrm{k}-1}) * \left(\Pi(0) + \mathrm{e}^{k-1}(0)\right) \\ +\mathrm{P}^{\mathrm{k}-1} * \left(\Pi(1) + \mathrm{e}^{k-1}(1)\right) \end{bmatrix}$$

3. Update CCP: Logit

$$P^k(x) = \Psi(P^{k-1}(x)) = \left(1 + \exp(-\tilde{v}(x)^{k-1})\right)^{-1}$$

where $\tilde{\mathrm{v}}^{k-1} = \left(\Pi(1) + \beta F(1)\bar{\mathrm{V}}^{k-1}\right) - \left(\Pi(0) + \beta F(0)\bar{\mathrm{V}}^{k-1}\right)$.

4. Stop if $||\mathrm{P}^k - \mathrm{P}^{k-1}|| < \eta$. Otherwise, repeat steps (2)-(4)

Value-function *versus* Policy-function Algorithms

- Both algorithms are guaranteed to converge if $\beta \in (0, 1)$
- Policy-function iteration algorithms converges in fewer steps than value-function iteration.
- However, each step of the policy-function algorithm is **slower** due to the matrix inversion. M is typically very large (in the millions).
- If M is very large, it can be faster and more accurate to find $\bar{\mathrm{V}}$ using linear programing tools (e.g. linsolve in Matlab):

$$
\begin{aligned}
\left(I - \beta \mathrm{F}^{k-1}\right) \bar{\mathrm{V}}^{k-1} &= \begin{array}{c} (1 - \mathrm{P}^{\mathrm{k}-1}) * \left(\Pi(0) + \mathrm{e}^{k-1}(0)\right) \\ + \mathrm{P}^{\mathrm{k}-1} * \left(\Pi(1) + \mathrm{e}^{k-1}(1)\right) \end{array} \\
\Leftrightarrow \mathrm{Ay} &= \mathrm{b}
\end{aligned}
$$

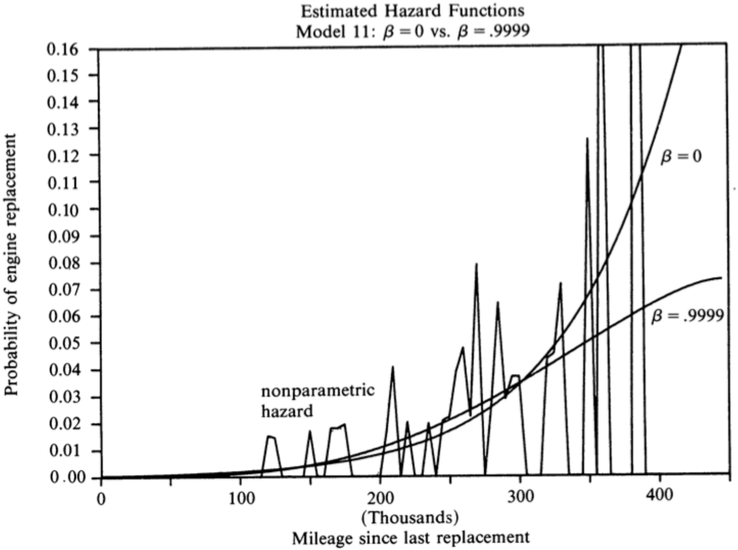
- Suggested algorithm:
  - Start with value-function iteration if $\bar{\mathrm{V}}^{k}(x) - \bar{\mathrm{V}}^{k-1}(x) > \eta^1$
  - Switch to policy-function iteration when $\bar{\mathrm{V}}^{k}(x) - \bar{\mathrm{V}}^{k-1}(x) < \eta^1$
  - Where $\eta^1 < \eta$ (e.g. $\eta^1 = 10^{-2}$)

Rust (1987) Estimation results

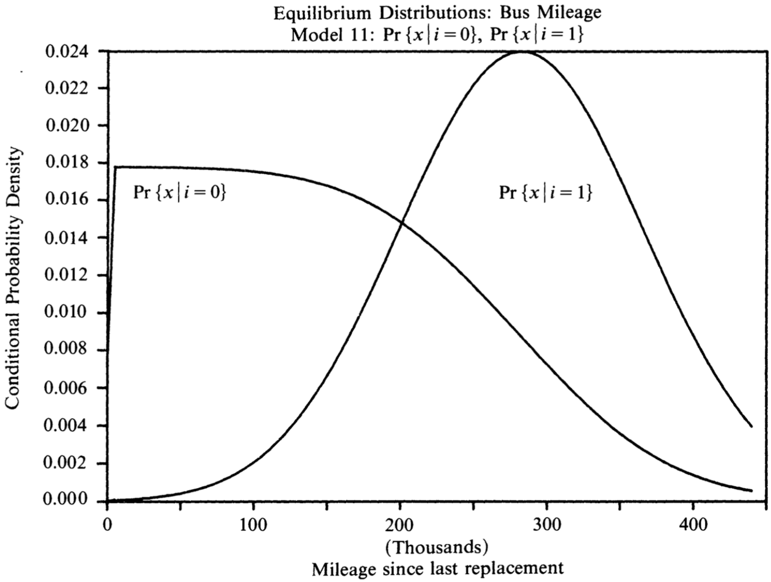
TABLE IX
STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$
FIXED POINT DIMENSION = 90
(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates/ Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic ($df = 4$)	Marginal Significance Level
$\beta = .9999$	RC	11.7270 (2.602)	10.0750 (1.582)	9.7558 (1.227)	85.46	1.2E−17
	θ_{11}	4.8259 (1.792)	2.2930 (0.639)	2.6275 (0.618)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3489 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2708.366	−3304.155	−6055.250		
$\beta = 0$	RC	8.2985 (1.0417)	7.6358 (0.7197)	7.3055 (0.5067)	89.73	1.5E−18
	θ_{11}	109.9031 (26.163)	71.5133 (13.778)	70.2769 (10.750)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3488 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2710.746	−3306.028	−6061.641		
Myopia test:	LR Statistic ($df = 1$)	4.760	3.746	12.782		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0292	0.0529	0.0035		

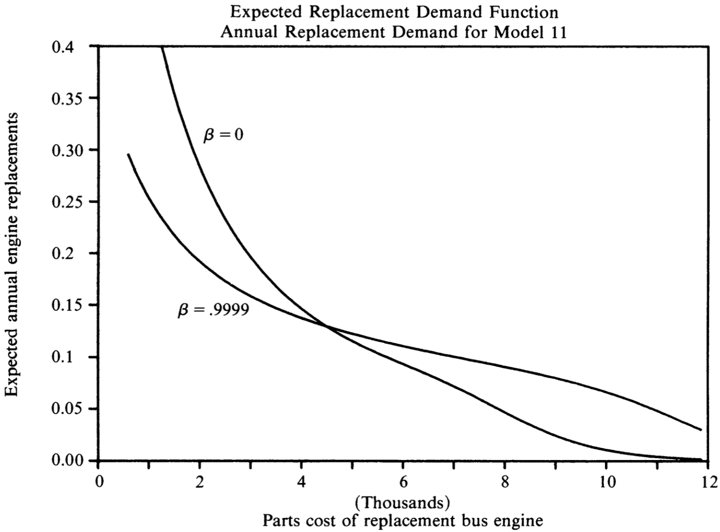
Replacement hazard: Myopic vs Forward looking



Implied distribution of bus mileage and replacement



Aggregate demand for bus engines (Price = RC)



Extension: Finite horizon problems

- **Assumption:** Buses are scrapped after T period

$$V_t(a_{it}|x_{it},\epsilon_{it}) = E\left(\sum_{\tau=0}^{T-t}\beta^\tau \pi_{it+\tau}\middle| x_{it},\epsilon_{it},a_{it}\right)$$

x_{it} is the engine age, and t is the age of the bus.

- **Bellman equation:**

$$\begin{aligned} V_t(a|x,\epsilon) &= Y_t((1-a)\cdot x) - RC(a\cdot x) + \epsilon(a) \\ &\quad +\beta\sum_{x'} E_{\epsilon'}\left(V_{t+1}(x',\epsilon')\right) F_t(x'|x,a) \\ &= v_t(a,x) + \epsilon(a) \end{aligned}$$

where $V_t(x,\epsilon) \equiv \max_{a\in\{0,1\}} V_t(a|x,\epsilon)$.

- Optimal replacement decision:

$$a^* = \begin{cases} 1 & \text{If } v_t(1,x) - v_t(0,x) = \tilde{v}_t(x) > \epsilon(0) - \epsilon(1) = \tilde{\epsilon} \\ 0 & \text{Otherwise.} \end{cases}$$

Solution: Backward induction

- **Last period:** Static problem

$$V_T(x,\epsilon) = \max_{a\in\{0,1\}} Y_T((1-a)\cdot x) - RC(a\cdot x) + \epsilon(a)$$

$$\rightarrow \bar{V}_T(x) = \ln\left(\exp(Y_T(0) - RC(x)) + \exp(Y_T(x))\right) + \gamma$$

$$\rightarrow p_T(x) = \frac{\exp(Y_T(x))}{\exp(Y_T(0) - RC(x)) + \exp(Y_T(x))}$$

Solution: Backward induction

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$$\rightarrow p_T(x) = \frac{\exp(Y_T(x))}{\exp(Y_T(0) - RC(x)) + \exp(Y_T(x))}$$

- **Period** $T-1$**:**

$$\begin{aligned}
\bar{V}_T(x) &= \ln\Big(\sum_a \exp(Y_T((1-a)\cdot x) - RC(a\cdot x) \\
&\qquad +\beta\sum_{x'}\bar{V}_T(x')f_{T-1}(x'|x,a)\Big) + \gamma \\
&= \ln\Big(\sum_a \exp(v_{T-1}(a|x))\Big) + \gamma
\end{aligned}$$

$$\rightarrow p_{T-1}(x) = \frac{\exp(v_{T-1}(1|x))}{\exp(v_{T-1}(0|x)) + \exp(v_{T-1}(1|x))}$$

Solution: Backward induction

- **Summary:** Solution is a collection of choice-probabilities for each state/period combination

$$\{p_t(1), \ldots, p_t(M)\}_{t=1,\ldots,T}$$

  where M is the number of grid points.
- Infinite *vs* Finite horizon?
  - Infinite model requires iterative algorithm ($\beta \to 1$: can be slow)
  - Predictions from finite horizon models can be sensitive to choice of T
  - And have larger state-space: $M \times T$
- Alternative approach: Stochastic aging (equiv. to infinite horizon)

$$\text{State} : x_t = \{\text{age}_t, x_t\}$$

$$\text{Transition: } \Pr(\text{age}_{t+1} = \text{age}_t + 1 | s_t) = \rho(\text{age}_t)$$

  and $\text{age} \in \{0, \ldots, T\}$.

Estimation: Nested fixed-point MLE

- **Data:** Panel of choices a_{it} and observed states x_{it}
- **Parameters:** Technology parameters $\theta = \{\theta_{Y_0}, \theta_{Y_1}, \theta_{R_0}, \theta_{R_1}\}$, discount factor β, and distribution of mileage shocks $f_x(\xi_{it})$.
- **Initial step:** If the panel is long-enough, we can estimate $f_x(\xi)$ from the data. The estimated process can then be *discretized* to construct $\hat{F}(1)$ and $\hat{F}(0)$.
- Maximum likelihood problem:

$$\begin{aligned} \max_{\theta,\beta} \quad & \sum_i \sum_t a_{it} \ln P(x_{it}) + (1 - a_{it}) \ln(1 - P(x_{it})) \\ \text{s.t.} \quad & P(x_{it}) = \Psi(x_{it}) \quad \forall x_{it} \end{aligned}$$

- In practice, we need two functions:
  - *Likelihood:* Evaluate $L(\theta, \beta)$ given $P(x_{it})$.
  - *Fixed-point:* Routine that solves $P(x_{it})$ for every guess of θ, β.

Incorporating Unobserved Heterogeneity

- **Why?** Relax the conditional independence assumption.
- **Example:** Buses have heterogeneous replacement costs (K types)
  - This increases the number of parameters by $K(K-1)$: $\{\theta^1_{R_0}, ..., \theta^K_{R_0}\} + \{\omega_1, .., \omega_{K-1}\}$ (probability weights).
  - E.g.: discretize a parametric distribution: $\ln \theta^i_{R_0} \sim N(\mu, \sigma^2)$
  - This changes the MLE problem:

$$\max_{\theta,\beta,\omega} \quad \sum_i \ln \left[\sum_k g(k|x_{i1}) \prod_t P_k(x_{it})^{a_{it}} (1 - P_k(x_{it}))^{1-a_{it}} \right]$$
$$\text{s.t.} \quad P_k(x_{it}) = \Psi_k(x_{it}) \quad \forall x_{it} \text{ and type } k$$

    Where $g(k|x_{i1})$ is the probability that bus i is type k conditional on initial milage x_{i1} (i.e. initial condition problem).
  - How to calculate $g(k|x_{i1})$?

The initial condition problem

- Unobserved heterogeneity creates a correlation between the initial state (i.e. x_{i1} mileage) and types (Heckman 1981).
- Two solutions:
  - **New buses:** Exogenous initial assignment $g(k|x_{i1}) = \omega_k$.
  - **Limiting distribution:** The bus engine replacement creates a *finite-state Markov chain* defined by

$$F_k(x'|x) = \sum_a P_k(a|x)F(x'|x,a) \text{ for each type } k$$

Under fairly general assumptions, this process generates a unique limiting distribution:

$$\pi_k(x) = \sum_{i=1}^{M} F_k(x_{t+1} = x|x_t = x_i)\pi_k(x_i) \leftrightarrow \pi_k = F_k^T \pi_k$$

We can use the limiting distribution to calculate the type probability conditional on initial mileage:

$$g(k|x_{i1}) = \frac{\omega_k \pi_k(x_{i1})}{\sum_{k'} \omega_{k'} \pi_{k'}(x_{i1})}$$

Identification: Unobserved heterogeneity

- **Example:** Retirement decision
  - Payoff:

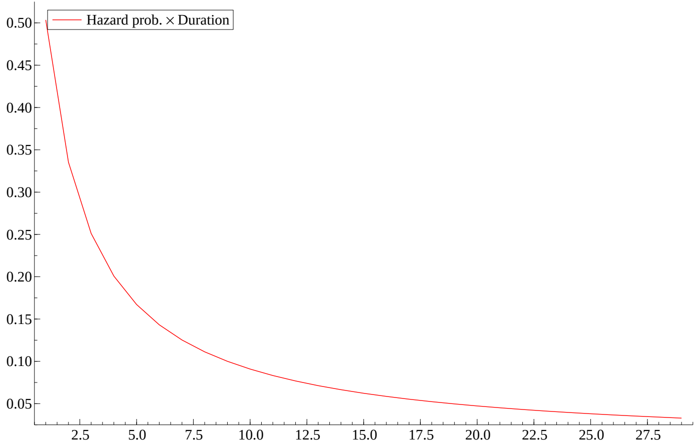
$$V_{it}^{work} = \alpha_i + X_i\beta + \epsilon_{it}$$
$$V_{it}^{retire} = 0$$

    where $\epsilon_{it} \sim N(0,1)$ (iid).
  - Conditional on α_i, the likelihood of observing i retiring after T_i years is:

$$\Pr(T_i|X_i, \alpha_i) = \left(1 - \Phi(-\alpha_i - X_i\beta)\right)^{T_i - 1} \Phi(-\alpha_i - X_i\beta)$$

  - Key prediction: Declining hazard rate
    - ★ Why? Persistence in α_i implies that individual who "survive" longer are less likely to retire (selection effect)

Numerical example: Retirement probability and selection



- **Implication:** Slope of the hazard function identifies the importance of unobserved heterogeneity

Identification: Discount Factor

- The data is summarized by the empirical hazard function:

$$h(x) = \Pr(\text{replacement}_t | \text{miles}_t = x)$$

- This corresponds to the reduced form of the model:

$$\begin{aligned} h(x) &= F_{\tilde{\epsilon}}(\tilde{v}(x)) \\ &= F_{\tilde{\epsilon}}\left(\begin{array}{c} (\Pi(1,x) - \Pi(0,x)) \\ - \quad \beta \sum_{x'} V(x')(F(x'|x,1) - F(x'|x,0)) \end{array} \right) \end{aligned}$$

- **Claim:** β is not identified, unless we parametrize payoffs: Y and RC.
  - If $\Pi(x)$ is linear in x, then non-linearity in the **observed** hazard function identifies β.
  - If $\Pi(x)$ is a non-parametric function, we cannot distinguish between a non-linear myopic model ($\beta = 0$), and a forward-looking model ($\beta > 0$).

Identification: Discount Factor

- What would identify β?
  - **Exclusion restriction:** The model includes a state variable z that only enters the Markov transition function (i.e. $F(x'|x,z,a)$), and not the static payoff function.

$$
\begin{aligned}
h(x,z) &= F_{\tilde{\epsilon}}(\tilde{v}(x,z)) \\
&= F_{\tilde{\epsilon}}\left(- \frac{(\Pi(1,x) - \Pi(0,x))}{\beta \sum_{x'} V(x')(F(x'|x,z,1) - F(x'|x,z,0))} \right)
\end{aligned}
$$

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\end{aligned}
$$

- **Example:** Donut hole (Dalton, Gowrisankaran and Town, 2019)
  - Non-linear insurance contract: Reduced subsidies in a coverage gap leads to (expected) cost increase (x = cumulative spending)
  - Forward-looking agent: Gap is expected, and should not lead to abrupt change in claims (smoothing)
  - Myopic agent: Gap is unexpected, and spending drops abruptly
  - Evidence: "Enrollees have flat spending in a period before the doughnut hole and a large spending drop in the gap"

Sequential estimators of DDC models

- **Key references:**
  - Hotz and Miller (1993)
  - Hotz, Miller, Sanders, and Smith (1994)
  - Aguirregabiria and Mira (2002)
  - Identification: Magnac and Thesmar (2002), Kasahara and Shimotsu (2009)
- Consider the following dynamic discrete choice model with additively separable (AS) and conditional independent (CI) errors.
  - A discrete actions.
  - Payoff function: $u(x|a)$
  - State space: (x, ϵ).
  - Where x is a discrete state vector, and ϵ is an A-dimensions continuous vector.
  - Distribution functions:
    - ⋆ $\Pr(x_{t+1} = x'|x_t, a) = f(x'|x, a)$
    - ⋆ $g(\epsilon)$ is a type-1 EV density with unit variance.

Bellman Operator

- Bellman equation:

$$
\begin{aligned}
V(x) &= \int \max_{a\in A} \left\{ u(x|a) + \epsilon(a) + \beta \sum_{x'} V(x') f(x'|x,a) \right\} g(\epsilon) d\epsilon \\
&= \int \max_{a\in A} \left\{ v(x|a) + \epsilon(a) \right\} g(\epsilon) d\epsilon \\
&= \ln \left(\sum_{a} \exp(v(x|a)) \right) + \gamma \\
&= \Gamma\big(V(x)\big)
\end{aligned}
$$

CCP Operator

- Express $V(x)$ as a function of $P(a|x)$.

$$V(x) = \sum_a P(a|x) * \left\{ u(x|a) + E(\epsilon(a)|x,a) + \beta \sum_{x'} V(x') f(x'|x,a) \right\}$$

Where,

$$\begin{aligned} & E(\epsilon(a)|x,a) \\ & \quad = \frac{1}{P(a|x)} \int 1\Big(v(x|a) + \epsilon(a) > v(x|a') + \epsilon(a'), a' \neq a \Big) g(\epsilon) d\epsilon \\ & \quad e(a, P(a|x)) = \gamma - \ln P(a|x) \end{aligned}$$

CCP Operator (continued)

- In Matrix form:

$$V = \sum_a P(a) * \left[u(a) + e(a,P) + \beta F(a) V\right]$$

$$\left[I - \beta \sum_a P(a) * F(a)\right] V = \sum_a P(a) * \left[u(a) + e(a,P)\right]$$

$$V(P) = \left[I - \beta \sum_a P(a) * F(a)\right]^{-1} \left[\sum_a P(a) * \left(u(a) + e(a,P)\right)\right]$$

where $F(a)$ is $|X| \times |X|$ and V is $|X| \times 1$.

CCP Operator (continued)

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where $F(a)$ is $|X| \times |X|$ and V is $|X| \times 1$.

- The CCP contraction mapping is:

$$\begin{aligned} P(a|x) &= \Pr\left(v(x|a, P) + \epsilon(a) > v(x|a', P) + \epsilon(a'), a' \neq a\right) \\ &= \frac{\exp\left(\tilde{v}(x|a, P)\right)}{1 + \sum_{a'>1} \exp\left(\tilde{v}(x|a, P)\right)} \\ &= \Psi(a|x, P) \end{aligned}$$

where $\tilde{v}(x|a, P) = v(x|a, P) - v(x|1, P)$.

Two Special Cases

1. **Linear payoff:** If $u(x|a,\theta) = x\theta$, the value function is also linear in θ.

$$
\begin{aligned}
V(P) &= Z(P)\theta + \lambda(P) \\
\text{Where} \qquad & Z(P) = \left[I - \beta \sum_a P(a) * F(a)\right]^{-1} \left[\sum_a P(a) * X\right] \\
& \lambda(P) = \left[I - \beta \sum_a P(a) * F(a)\right]^{-1} \left[\sum_a P(a) * e(a,P)\right]
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2. **Absorbing state:** $v(x|0) = 0$ (e.g. Exit or retirement). This change the value function:

$$V(x,\varepsilon) = \max \left\{ u(x) + \varepsilon(1) + \beta \sum_{x'} \underbrace{E_{\varepsilon'}[V(x',\varepsilon')]}_{=\bar{V}(x')} F(x'|x), \varepsilon(0) \right\}$$

As before, the expected continuation value is:

$$\begin{aligned} \bar{V}(x) &= \log\left(\exp(0) + \exp\left(u(x) + \beta \sum_{x'} \bar{V}(x')F(x'|x)\right)\right) + \gamma \\ &= \log\left(1 + \exp(v(x))\right) + \gamma \end{aligned}$$

Two Special Cases (continued)

- The choice probability is given by:

$$\Pr(a = 1|x) = P(x) = \frac{\exp(v(x))}{1 + \exp(v(x))}$$

Note that the log of the "odds-ratio" is equal to the choice-specific value function:

$$\log\left(\frac{P(x)}{1 - P(x)}\right) = v(x)$$

- Therefore, the expected continuation value can be expressed as a function of $P(x)$:

$$\begin{aligned}
\bar{V}^p(x) &= \log\left(1 + \exp(v(x))\right) + \gamma = \log\left(1 + \frac{P(s)}{1 - P(x)}\right) + \gamma \\
&= -\log\left(1 - P(x)\right) + \gamma
\end{aligned}$$

- **Implication:** With an absorbing state, we don't need to invert $\left[I - \beta \sum_a P(a) * F(a)\right]$ to apply the CCP mapping.

Two-Step Estimator

- The objective is to estimate the structural parameters θ without repeatedly solving the DP problem
- **Initial step:** Reduced form of the model
  - Markov transition process: $\hat{f}(x'|x, a)$
  - Policy function: $\hat{P}(a|x)$
  - **Constraint:** Need to estimate both functions at EVERY state point x.

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- How? Ideally $\hat{P}(a|x)$ is estimated non-parametrically to avoid imposing a particular functional form on the policy function (i.e. no theory involved at this stage). This would correspond to a frequency estimator:

$$\hat{P}(a|x) = \frac{1}{n(x)} \sum_{i \in n(x)} 1(a_i = a)$$

- For finite samples, we need to impose smooth the policy function and interpolate between states are not visited (or infrequently). Kernels or local-polynomial techniques can be used.

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- **Second-step:** Structural parameters conditional on $(\hat{P}, \hat{f})$

Pseudo-likelihood estimators (PML)

- **Source:** Aguirregabiria and Mira (2002)
- **Data:** Panel of n individuals of T periods:

$$(A, X) = \{a_{it}, x_{it}\}_{i=1,\ldots,n; t=1,\ldots,T}$$

- **2-Step estimator:**
  1. Obtain a flexible estimator of CCPs $\hat{P}^1(a|x)$
  2. Feasible PML estimator:

$$Q^{2S}(A, X) = \max_{\theta} \sum_{t} \sum_{i} \Psi(a_{it}|x_{it}, \hat{P}^1, \hat{F}, \theta)$$

  If $V(P)$ is linear, the second step is a linear probit/logit model.

- This process can be extended to K-steps to improve the performance of the estimator (very useful!)

NFXP *vs* Two-step Methods

- Two-steps methods require:
  - A lot of care to estimate the first-stage (reduced-form)
  - Ideally semi-parametric/flexible functional forms
  - Very simple "second-stage": Linear Logit estimation ($\equiv$ static)
  - Cannot (easily) handled unobserved heterogeneity

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- NFXP method require:
  - Efficient computing of the fixed-point
  - Less data
  - High-performance computing
- **Suggestion:** Do not use two-step methods when the sample-size/state-points ratio is small
  - Non-stationary models are difficult to estimate with two-step methods (e.g. aggregate shock, common policy change, etc)
  - When the "first-stage" CCP is too noisy $\rightarrow$ Inconsistent estimates (i.e. non-linear measurement error)

Extension: Models with Large State Space

- Most problems have multiple continuous state variables:
  - Human capital accumulation: ability
  - Demand for durable goods: quality
  - Learning with experimentation: beliefs
  - Investment: serially correlated productivity
- **Curse of dimensionality problem:** The state space points increase exponentially with the dimensionality of the state space.

Example 1: Crawford and Shum (2005) “Uncertainty and Learning in Pharmaceutical Demand”

TABLE II

SWITCHING PROBABILITIES OVER THE COURSE OF TREATMENT[a]

Prescription Number	Total Treatment Length					
	5	6	7	8	9	10
2	14.3	13.6	10.9	10.0	7.8	9.2
3	11.6	11.6	6.3	8.8	7.8	6.6
4	8.9	5.6	5.4	3.1	7.8	3.9
5	13.4	7.9	10.0	8.8	4.9	5.3
6		11.3	6.3	5.7	2.9	5.3
7			9.5	10.0	7.8	11.8
8				8.1	4.9	11.8
9					7.8	5.3
10						11.8

[a]The (i, j)th entry is the percentage of treatment sequences of length j in which a switch was observed during the ith $(i \le j)$ prescription.

- **Fact:** Marked decreasing trend in the drug switching probability at the very beginning of treatment.
- Two forces:
  - Initial experimentation and risk aversion.
  - Forward looking behavior: Incentive for patients (or doctors) to acquire more information by experimenting.

Model Description

- Expected Utility (CARA):

$$\begin{aligned}
&\widetilde{EU}(\mu_{ij}(t), \nu_{ij}(t), p_n, \epsilon_{ijt}) \\
&= -\exp\Big(-r\mu_{ij}(t) + 1/2r^2(\sigma_n^2 + V_{ij}(t))\Big) - \alpha p_n + \epsilon_{ijt} \\
&= EU(\mu_{ij}(t), V_{ij}(t), p_n) + \epsilon_{ijt}
\end{aligned}$$

- **State space:**

$$s_{jt} = \Big\{\mu_{ij}(t), \nu_{ij}(t), l_{ij}(t), h_j(t)\Big\}_{j=1...5}$$

Where $l_{ij}(t) = \sum_{s<t} d_{ij}(s)$ is the cumulative number of prescriptions.

- **Value Function:** Infinite horizon problem with absorbing state (i.e. recovery)

$$\begin{aligned}
V(s) &= \int \max_j EU_j(s) + \epsilon_j + \beta E\big[(1-h(s'))V(s')|d_j = 1, s\big]\, dF(\epsilon) \\
&= \log\Big[\sum_j \exp\big(EU_j(s) + \beta E\big[(1-h(s'))V(s')|d_j = 1, s\big]\big)\Big]
\end{aligned}$$

Example 2: Keane and Wolpin (1997) "Career decisions of young men"

- Career choice for age a in state s

$$V(s,a) = \max_{m \in M} V_m(s,a)$$

Where the choice-specific value function is given by:

$$V_m(s,a) = \begin{cases} R_m(s,a) + \beta E[V(s', a+1) | S, d_m(a) = 1] & \text{If } a < A \\ R_m(s,A) & \text{If } a = A \end{cases}$$

- The "return" function is given by:

$$R_m(s,a) = \begin{cases} \underbrace{r_m \exp\left(e_{m0} + e_{m1} g(a) + e_{m2} x_m(a) - e_{m3} x_m(a)^2 + \epsilon_{im}(a)\right)}_{\text{Wage (data)}} & \text{If } m \in 1,2,3 \\ e_4 - tc_1 1[g(a) \geq 12] - tc_2 1[g(a) \geq 16] + \epsilon_{i4}(a) & \text{If Schooling (4)} \\ e_5 + \epsilon_{i5}(a) & \text{If Home (5)} \end{cases}$$

Where $\epsilon_m(a) \sim N(0, \sigma_m^2)$ and e_{m0} (endowment) is distributed according to a finite-mixture with 4 types.

Models with Large State Space

- Consider the following DDC model:

$$V(s) = \max_{a \in A} \underbrace{u(s,a) + \beta \int V(s')f(s'|s,a)ds'}_{v(a,s)}$$

  where $s \in S$ is a d-dimension state vector.

- Two numerical challenges:
  - Numerical integration (esp. when d is large)
  - Obtain $v(a,s)$ for all s's in the data
- We will consider two approaches:
  - Randomization: Rust (1997)
  - Interpolation+simulation: Keane and Wolpin (1994)

Models with Large State Space

- Random Bellman Operator (Rust, 1997):

$$\tilde{\Gamma}_M(V)(s) = \max_a u(a,s) + \frac{\beta}{M}\sum_{k=1}^{M} V(\tilde{s}_k) f(\tilde{s}_k|s,a)$$

where $s \in \{\tilde{s}_1, \dots, \tilde{s}_M\}$ are uniform random grid points in S. For instance, if $S = [0,1]^d$, $\tilde{s}_k$ is a d vector of $U(0,1)$ random-variables.

- Since $\tilde{s}$ is random, the law of large number guarantees that:

$$E\left[\frac{\beta}{M}\sum_{k=1}^{M} V(\tilde{s}_k) f(\tilde{s}_k|s,a)\right] = \int V(\tilde{s}_k) f(k'|s,a) ds'$$

$$E\left[\frac{1}{M}\sum_{k=1}^{M} f(\tilde{s}_k|s,a)\right] = \int f(k'|s,a) ds' = 1$$

Random Grid Approach

Reference: Rust (1997)

- **Implication:** Convergence to the true expected-value function depends on M, and is independent of the dimensionality d (breaks the curse of dimensionality)

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- In "finite" sample, the density must sums to one for $\tilde{\Gamma}(V)$ to be a contraction. Normalization:

$$\bar{f}_M(\tilde{s}_k|s,a) = \frac{f(\tilde{s}_k|s,a)}{\sum_k f(\tilde{s}_k|s,a)}$$

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- **Importantly:** The random operator has a *self-approximating* property

$$V_M(s) = \max_a u(a,s) + \sum_{k=1}^{M} V(\tilde{s}_k)\bar{f}_M(\tilde{s}_k|s,a)$$

- That is, for large M, $V_M(s)$ is a consistent estimate of the "true" value function for any s outside of the random-grid.
  - No need for interpolation or function approximation

Random value-function iteration

Reference: Rust (1997)

1. Define "observed" state points $\{s_i\}_{i=1,\ldots,N}$ (e.g. data)
2. Draw M uniform random state points $\tilde{S}_M = \{\tilde{s}_k\}_{k=1,\ldots,M}$ and compute $\bar{f}_M(\tilde{s}_k|s,a)$
3. Iterate on the random bellman operator for all $\tilde{s} \in \tilde{S}_M$:

$$V^{t+1}(\tilde{s}_j) = \max_a u(\tilde{s}_k, a) + \sum_k \beta V^t(\tilde{s}_k)\bar{f}_M(\tilde{s}_k|\tilde{s}_j, a)$$

   until $||V^{t+1}(\tilde{s}_j) - V^t(\tilde{s}_j)|| < \epsilon$.

4. Evaluate value function over observed states $\{s_i\}_{i=1,\ldots,N}$ (i.e. data)

$$V_M(s_i) = \max_a u(s_i, a) + \sum_k \beta V^*(\tilde{s}_k)\bar{f}_M(\tilde{s}_k|s_i, a)$$

5. Increase the random grid (M) by Δ and repeat steps $2-4$.
6. Stop if $||V_M(s_i) - V_{M+\Delta}(s_i)|| < \eta$ for all s_i.

Random Grid Approach

Reference: Rust (1997)

- Additional comments:
  - The quality of the approximation is likely better if the pseudo-random numbers are replaced by Halton sequences (or other deterministic random number generator) - See Rust and Train for discussion.
  - As with Importance Sampling, the density $f(s'|s, a)$ must have positive mass for all (most?) random points. Works poorly if the support of the transition function is very small.
  - Method is applicable also for finite horizon models (i.e. step 3 is backward induction)
- Application: Gordon (2009), "A Dynamic Model of Consumer Replacement Cycles in the PC Processor Industry"

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- **General Idea:** Solve the value function exactly only at a subset $\tilde{S}$ of the states and interpolate between them using least-squares to compute $EV(s)$ at $s \notin \tilde{S}$.
- As in Rust (1997), $\tilde{S}$ can be a uniform random sample of grid points (e.g. d-dimensions Halton sequences)

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- **General Idea:** Solve the value function exactly only at a subset $\tilde{S}$ of the states and interpolate between them using least-squares to compute $EV(s)$ at $s \notin \tilde{S}$.
- As in Rust (1997), $\tilde{S}$ can be a uniform random sample of grid points (e.g. d-dimensions Halton sequences)
- **Model:** Finite horizon dynamic discrete choice model

$$V_t(s) = \max_{a \in A} u_t(s,a) + \epsilon(a) + \beta \int V_{t+1}(s') f_t(s'|s,a) ds'$$

where $\epsilon(a) \sim T1EV(0,1)$.

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- Backward induction algorithm for a fix grid $\tilde{S}$: Period T
  1. Calculate $EV_T(\tilde{s}_k)$ for all $\tilde{s}_k \in \tilde{S}$:

$$\begin{aligned} EV_T(\tilde{s}_k) &= \int \Big\{ \max_a u_T(a, \tilde{s}_k) + \epsilon_T(a) \Big\} dG(\epsilon) \\ &= \log \left(\sum_a \exp \Big(u_T(a, \tilde{s}_k) \Big) \right) + \gamma \end{aligned}$$

  2. Run the following interpolation regression:

$$EV_T(\tilde{s}_k) = B(\tilde{s}_k)\hat{c}_T + \text{Residual} = \widehat{EV}_T(\tilde{s}_k) + \text{Residual},$$

where $B(\tilde{s}_k)$ is vector containing flexible transformations of the state variables, and b_T is an interpolation parameter for period T

Note: For the approximation to work, the R^2 of the regressions must be very high.

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- Period $T-1$:
  1. Draw M random variables: $s_m \sim f(s'|\tilde{s}_k, a)$ for each $\tilde{s}_k$.
  2. For each state $\tilde{s}_k$ compute the expected value of choosing option a in $T-1$:

$$E\left[\hat{V}_T|\tilde{s}_k, a\right] = \frac{1}{M}\sum_m \widehat{EV}_T(s_m)$$

     Where $\widehat{EV}_T(s_m)$ uses $B(s_m)\hat{c}_T$ (OLS prediction).
  3. For each $\tilde{s}_k$ calculate $EV_{T-1}(\tilde{s}_k))$

$$EV_{T-1}(\tilde{s}_k) \quad = \quad \log\left(\sum_a \exp\left(u_{T-1}(\tilde{s}, a) + \beta E\left[\hat{V}_T|\tilde{s}_k, a\right]\right)\right) + \gamma$$

  4. Run the interpolation regression:

$$EV_{T-1}(\tilde{s}_k) = B(\tilde{s}_k)\hat{c}_{T-1} + \text{Residual} = \widehat{EV}_{T-1}(\tilde{s}_k) + \text{Residual}$$

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- Repeat simulation/interpolation steps for the remaining periods $t = T - 2, ..., 0$.
- **Solution:** Sequence of interpolation parameters $\hat{c}_T, \hat{c}_{T-1}, \ldots, \hat{c}_0$.

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- Repeat simulation/interpolation steps for the remaining periods $t = T-2, ..., 0$.
- **Solution:** Sequence of interpolation parameters $\hat{c}_T, \hat{c}_{T-1}, \ldots, \hat{c}_0$.
- **Likelihood:** Predicted choice-probabilities at observed states s_{it} and observed actions a_{it}
  - For each observation (i, t) calculate the choice-specific value function

$$v_t(s_{it}, a; \theta) = u_t(s_{it}, a; \theta)) + \beta E\left[\hat{V}_{t+1}|s_{it}, a\right]$$

    where $E\left[\hat{V}_{t+1}|s_{it}, a\right] = \frac{1}{M}\sum_m B(s_m)\hat{c}_{t+1}$ and $s_m \sim f(s'|s_{it}, a)$

  - Calculate predicted choice-probabilities:

$$\hat{P}_t(a_{it}|s_{it}, \theta) = \frac{\exp(v_t(s_{it}, a_{it}; \theta))}{\sum_a \exp(v_t(s_{it}, a; \theta))}$$

  - Simulated likelihood function:

$$L(Y, X|\theta) = \sum_i \sum_t \ln\left(\hat{P}_t(a_{it}|s_{it}, \theta)\right)$$

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- It is straightforward to incorporate unobserved heterogeneity

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

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- **Example:** $s_{it} = \{x_{it}, \omega_{it}\}$ where ω_{it} is unobserved (to the econometrician) and evolves over time as

$$\omega_{it} = \rho_0 + \rho_1 \omega_{it-1} + \eta_{it}, \quad \eta_{it} \sim N(0, \sigma_\eta^2)$$

- We can solve the model over a grid as before: $\hat{c}_t$. The choice-probability is obtained by simulating sequences of ω_{it}

$$\hat{P}_t(a_{it}|x_{it}, \omega_{mt}, \theta) = \frac{\exp(v_t(x_{it}, \omega_{mt}, a_{it}; \theta))}{\sum_a \exp(v_t(x_{it}, \omega_{mt}, a; \theta))}$$

$$\hat{P}(a_{i1}, \ldots, a_{iT}|x_{i1}, \ldots, x_{iT}, \theta) = \frac{1}{M} \sum_m \prod_{t=1}^{T} \hat{P}_t(a_{it}|x_{it}, \omega_{mt}, \theta)$$

- Likelihood function:

$$L(Y, X|\theta) = \sum_i \ln \hat{P}(a_{i1}, \ldots, a_{iT}|x_{i1}, \ldots, x_{iT}, \theta)$$

Simulation and Interpolation Approach

Reference: Keane and Wolpin (1994)

- How to choose interpolation function $B(s)$?
- $B(s)$ is called a "basis-function". Challenge: s is potentially a large dimensionality vector, which leads to a curse of dimensionality.
- A common approach is to use polynomials in the "static" expected utility (also called "monomial" basis functions)

$$EU_t(s) = \ln\left(\sum_a \exp(u_t(s,a;\theta)\right) + \gamma$$

$$\rightarrow B_t(s) = \left\{EU_t(s), EU_t(s)^2, \ldots, EU_t(s)^K\right\}$$

Primer on Collocation Methods

Reference: Miranda and Fackler (2002) Chapter 6

- Drawbacks of uniformly spaced polynomial approximation:
  - Poor approximation
  - Near-colinearity poblems
- Solution: Orthogonal polynomials (aka Collocation methods)
- Define approximation function as:

$$\hat{f}(x) = \sum_{j=1}^{n} \phi(x_j) c_j$$

  where $\phi_j(x)$ are independent basis functions, and c_j are basis coefficients

- If $\{x_j\}_{j=1,\ldots,n}$ is a collection of interpolation nodes, c is a solution to a system of linear equations

$$f(x_j) = y_j = \Phi(x_j) c \rightarrow c = (\Phi'\Phi)^{-1}\Phi' y$$

Primer on Collocation Methods

Reference: Miranda and Fackler (2002) Chapter 6

- **Method 1:** Chebychev polynomials over $x \in [a, b]$
- Location of the grid points:

$$x_j = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{n-j+0.5}{n}\pi\right)$$

  Chebychev basis functions are by construction “orthogonal” functions (see Miranda). Additionally: derivatives are polynomial (degree $n-1$)
- Polynomial basis functions (single dimension):

$$\phi(z) = \{\phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_J(z)\}$$

  where

$$\phi_0(z) = 1, \phi_1(z) = z, \phi_2(z) = 2z^2 - 1, \ldots, \phi_j(z) = 2z\phi_{j-1}(z) - \phi_{j-2}(z)$$

  where $z \in (-1, 1)$.

Primer on Collocation Methods

Reference: Miranda and Fackler (2002) Chapter 6

- **Method 2:** Piecewise polynomial basis (or Spline)
  - Define a uniform grid of dimension p: $v_i \in [a, b]$
  - Basis function (or B-Spline) of degree k is constructed recursively:

$$\text{First degree: } \phi_{i,0} = \begin{cases} 1 & v_i \leq x < v_{i+1} \\ 0 & \text{Else.} \end{cases}$$

$$\text{Degree } k\text{: } \phi_{i,k} = \frac{x - \nu_i}{\nu_{i+1} - \nu_i} \phi_{i,k-1}(x) + \frac{\nu_{i+1} - x}{\nu_{i+1} - \nu_i} \phi_{i+1,k-1}(x)$$

  - As before, evaluate the interpolation parameter by evaluating the function at $n = p + k - 1$ points, and solving a system of linear equations
  - In practice, it is common to use Cubic splines because the approximation is (twice) differentiatiable (note the case for linear splines)

Primer on Collocation Methods

Reference: Miranda and Fackler (2002) Chapter 6

- **Method 2:** Piecewise polynomial basis (or Spline)
  - Define a uniform grid of dimension p: $v_i \in [a, b]$
  - Basis function (or B-Spline) of degree k is constructed recursively:

$$\text{First degree: } \phi_{i,0} = \begin{cases} 1 & v_i \leq x < v_{i+1} \\ 0 & \text{Else.} \end{cases}$$

$$\text{Degree } k\text{: } \phi_{i,k} = \frac{x - \nu_i}{\nu_{i+1} - \nu_i}\phi_{i,k-1}(x) + \frac{\nu_{i+1} - x}{\nu_{i+1} - \nu_i}\phi_{i+1,k-1}(x)$$

  - As before, evaluate the interpolation parameter by evaluating the function at $n = p + k - 1$ points, and solving a system of linear equations
  - In practice, it is common to use Cubic splines because the approximation is (twice) differentiatiable (note the case for linear splines)
- Matlab Packages: ComEcon Toolbox

https://pfackler.wordpress.ncsu.edu/compecon/154-2/

Also available for Python and Julia.

Example 1: Crawford and Shum (2005)

- Expected Utility (CARA):

$$
\begin{aligned}
&\widetilde{EU}(\mu_{ij}(t), \nu_{ij}(t), p_n, \epsilon_{ijt}) \\
&= -\exp\Big(-r\mu_{ij}(t) + 1/2r^2(\sigma_n^2 + V_{ij}(t))\Big) - \alpha p_n + \epsilon_{ijt} \\
&= EU(\mu_{ij}(t), V_{ij}(t), p_n) + \epsilon_{ijt}
\end{aligned}
$$

- **State space:**

$$
s_{jt} = \Big\{\mu_{ij}(t), \nu_{ij}(t), l_{ij}(t), h_j(t)\Big\}_{j=1...5}
$$

  Where $l_{ij}(t) = \sum_{s<t} d_{ij}(s)$ is the cumulative number of prescriptions.
- **Value Function:** Infinite horizon problem with absorbing state (i.e. recovery)

$$
\begin{aligned}
V(s) &= \int \max_j EU_j(s) + \epsilon_j + \beta E\big[(1 - h(s'))V(s')|d_j = 1, s\big] dF(\epsilon) \\
&= \log\Big[\sum_j \exp\big(EU_j(s) + \beta E\big[(1 - h(s'))V(s')|d_j = 1, s\big]\big)\Big]
\end{aligned}
$$

Solution method: Interpolation and simulation

1. Define a discrete grid $S^* \in S$. For each state $s \in S^*$ make an initial guess at the value function $V^0(s)$.
2. Run regression:
$$V^0(s) = B(s)\theta^0 + u_s$$
3. Draw M random signals $\{x_{jn}^m, y_{jn}^m\}$
4. Compute the expected value of choosing dug n for each $s \in S^*$:
$$E\left[\hat{V}^0(s)|d_h = 1, s\right] = \frac{1}{M}\sum_m (1 - h(s^m))B(s^m)'\theta^0$$
Where s^m is the state associated draw m and drug j being chosen.
5. Update the value function for each $s \in S^*$:
$$V^1(s) = \log\left[\sum_n \exp\left(EU(s) + \beta E\left[\hat{V}^0(s|d_n = 1, s\right]\right)\right] + \gamma$$
6. Repeat step 2-5 until convergence.

Simulated MLE

- Let $P(d_{it}|s_{it}, k)$ denotes the multinomial Logit choice probability given state s_{it}
- The likelihood contribution for an observed *sequence* of choices is:

$$\hat{P}(d_{i1}, \ldots, d_{iT_i}) = \frac{1}{M}\sum_{m}^{M}\sum_{k=1}^{K}\omega_k \left[\prod_t (1 - h_{mt,k}) P(d_{it}|s_{mt}, k)\right] h_{mT_i,k} P(d_{i,T_i}|s_{m,T_i}, k)$$

where M is the number of simulated signal sequences and T_i is the last period for i (cured).

- SML:

$$\max_{\theta} \sum_{i} \ln \hat{P}(d_{i1}, \ldots, d_{i,T_i}|\theta)$$

What is the effect of incomplete information?

- Learning: Incentive to "experiment" when uncertainty is large
- Risk aversion: Switching cost and sticky choices
- Combined result: Over-prescription of "popular" drugs and higher concentration

Baseline Specification[a]	
Avg. discounted utility	−28.7
Avg. treatment length	4.8
Avg. treatment cost	245
Avg. number of different drugs	1.4
Market shares	
Drug 1	60.4
Drug 2	14.1
Drug 3	3.7
Drug 4	2.5
Drug 5	19.3
Herfindahl index	4,242
Counterfactual I: Complete Information[b]	
Avg. discounted utility	−26.4
Avg. treatment length	8.8
Avg. treatment cost	385
Avg. number of different drugs	1.9
Market shares	
Drug 1	22.4
Drug 2	12.9
Drug 3	12.0
Drug 4	10.9
Drug 5	41.8
Herfindahl index	2,676
Counterfactual II: No Experimentation[c]	
Avg. discounted utility	−30.6
Avg. treatment length	4.8
Avg. treatment cost	248

Dynamic Pricing Contracts

- Non-linear pricing contracts:
  - Labor supply and non-linear taxes
  - Financial contracts
  - Health-insurance
  - Incentive pay
  - Energy/telecommunication
- **Key feature:** Price depends on past consumption/productivity
- Firms/governments use these types of contracts to solve moral hazard problems
  - Linear pricing contract leads to sub-optimal effort (labor), or excess consumption (energy/health)
- Two examples:
  - *Usage-based internet plans:* Nevo, Turner, and Williams (2016)
  - *Pay-for-performance teacher contracts:* Duflo, Hanna, and Ryan (2012)

Example 2: Usage-based internet plans

Reference: Nevo, Turner, and Williams (2016)

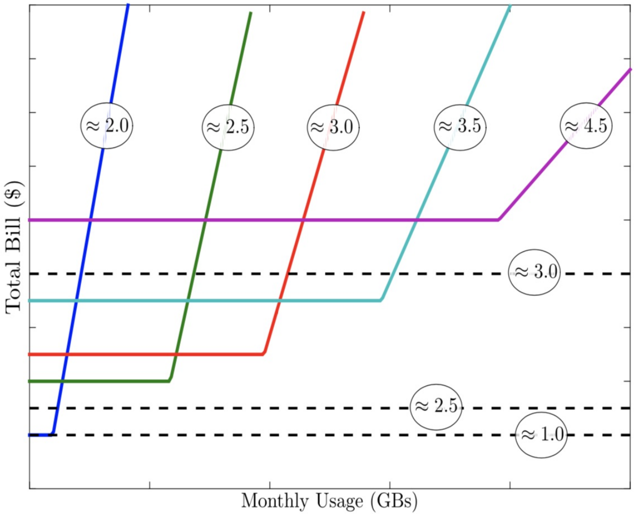
- **Data:**
  - Daily internet consumption from a large ISP over one billing cycle
  - Key variables: (i) menu of (non-linear) contract, (ii) cumulative usage, and (iii) contract choice.
- **Motivating fact:** Changes in patterns of internet consumption

	% of Usage		% Usage Growth
Source	2012	2015	2012 to 2015
Video	34.1	61.1	260.1
Web Browsing	31.9	21.5	36.2
File Sharing	8.3	0.2	−95.2
Gaming	1.3	3.1	357.1
Music	0.4	3.4	1,650.0
Backup	0.2	0.5	400.0
Other	23.7	10.3	−12.4
Total	100.0	100.0	101.5

- **Implication:** Increased network congestion during "prime-time" hours (negative externality)

Non-linear pricing contracts

Allowances, Speed and Cost



Evidence of Forward-Looking Behavior

Daily consumption (log) as function of distance to allowance limits

	$\mathbb{1}[10 \leq t < 15]$	$\mathbb{1}[15 \leq t < 20]$	$\mathbb{1}[20 \leq t < 25]$	$\mathbb{1}[25 \leq t < 31]$
$\mathbb{1}[0 \leq \frac{C_{jk(t-1)}}{\overline{C}_k} < 0.40]$	-0.04^{**}	-0.04^{**}	0.03^{**}	0.08^{**}
	(0.01)	(0.01)	(0.01)	(0.01)
$\mathbb{1}[0.40 \leq \frac{C_{jk(t-1)}}{\overline{C}_k} < 0.60]$	-0.02	-0.12^{**}	-0.12^{**}	-0.04^{**}
	(0.02)	(0.01)	(0.01)	(0.01)
$\mathbb{1}[0.60 \leq \frac{C_{jk(t-1)}}{\overline{C}_k} < 0.80]$	-0.07^{**}	-0.12^{**}	-0.20^{**}	-0.16^{**}
	(0.03)	(0.02)	(0.02)	(0.01)
$\mathbb{1}[0.80 \leq \frac{C_{jk(t-1)}}{\overline{C}_k} < 1.00]$	-0.19^{**}	-0.26^{**}	-0.39^{**}	-0.42^{**}
	(0.05)	(0.03)	(0.02)	(0.02)
$\mathbb{1}[1.00 \leq \frac{C_{jk(t-1)}}{\overline{C}_k}]$	-0.12^{**}	-0.35^{**}	-0.41^{**}	-0.47^{**}
	(0.05)	(0.03)	(0.02)	(0.02)
Adjusted R^2	0.46			

[a]This table presents OLS estimates of Equation (1) using 1,644,030 subscriber-day observations. The dependent variable is natural logarithm of daily usage. Each cell in the table gives the coefficient on the interaction between the indicators in the respective row and column. Controls include a constant, time trend, indicators for the day of the week, and subscriber fixed effects. Asterisks denote statistical significance: **1% level, *5% level.

- **Takeaways:** (i) demand is downward sloping with respect to the "shadow prices" (prob of hitting the allowance limit), and (ii) sensitivity to the limit is monotonic in time and distance (↑ prob of price hike)

Model description

- Assumptions:
  - Forward looking: Consumers have (daily) discount factor equal to one
  - Plans are chosen optimally (rational expectation)
  - Finite horizon: Consumers solve a $T = 30$ dynamic programming problem (no link across billing periods)

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$$u_h(c_t, y_t, v_t; k) = v_t \left(\frac{c_t^{1-\beta_h}}{1-\beta_h} \right)^{\beta_h} - c_t \left(\kappa_{1h} + \frac{\kappa_{2h}}{\ln s_k} \right) + y_t$$

where c_t is internet consumption, s_{kt} is the download speed of plan k, and y_t is consumption of the numeraire good.

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  where c_t is internet consumption, s_{kt} is the download speed of plan k, and y_t is consumption of the numeraire good.
- Budget constraint (end of month):

$$F_k + p_k \max\{C_T - \bar{C}_k, 0\} + Y_T \leq I$$

  where $C_T = \sum_{t=1}^{T} c_t$, and $\{F_k, p_k, \bar{C}_k\}$ are the terms of the contract.

Model description

- Dynamic consumption problem (conditional on k):

$$V_{hk} = \max_{c_1,\ldots,c_T} E\left[\sum_t u_h(c_t, y_t, v_t; k)\right] \text{ s.t. } F_k + p_k \max\{C_T - \bar{C}_k, 0\} + Y_T \leq I$$

- Plan choice problem:

$$\max_k V_{hk}$$

Note: There are no "logit" shocks. Conditional on being of type k, every households choose plan $k(h)$ (deterministic).

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- Uncertainty: Internet marginal utility shocks

$$v_t \sim LN(\mu_h, \sigma_h^2)$$

  where (μ_h, σ_h) are specific to households of type h. Assumption: v_t is IID over time.

- State variables and heterogeneity:
  - State: $s_t = (C_t, v_t)$
  - Household types: $(\mu_h, \sigma_h, \beta_h, \kappa_{1h}, \kappa_{2h})$

Model description

- Dynamic decision:

$$\max_{c_t} v_t \quad \left(\frac{c_t^{1-\beta_h}}{1-\beta_h}\right)^{\beta_h} - c_t\left(\kappa_{1h} + \frac{\kappa_{2h}}{\ln s_k}\right)$$
$$-p_t O_{kt}(C_{t-1}+c_t) + E[V_{hk,t+1}(C_{t-1}+c_t)]$$

where $O_{kt}(C_{t-1}+c_t) = 1$ if $C_{t-1}+c_t > \bar{C}_k$, 0 otherwise.

- Shadow price:

$$\tilde{p}_k(c_t, C_{t-1}) = \begin{cases} p_k & \text{If } O_{kt}(C_{t-1}+c_t) = 1 \\ \frac{\partial E[V_{hk,t+1}(C_{t-1}+c_t)]}{\partial c_t} & \text{Else.} \end{cases}$$

- Optimal consumption:

$$c^*_{hkt} = \left(\frac{v_t}{\kappa_{1h} + \kappa_{2h}/\ln s_k + \tilde{p}_k(c_t, C_{t-1})}\right)^{1/\beta_h}$$

Model Solution: Backward induction

- Random household types: $(\mu_h, \sigma_h, \beta_h, \kappa_{1h}, \kappa_{2h})$
  - Uniform grid: 7 points along each dimension (16,807 total)
  - Integration method: Importance sampling ($\approx$ Rust's random-grid method)
- **Discretization:** $C_t \in \{0, \Delta, 2\Delta, \ldots, C^{\max}\}$ (2000 points)
  - Consumption is a discrete choice
  - Conditional on C_{t-1}, c_t is a choice over $C_{t-1} + \Delta$, $C_{t-1} + 2\Delta$, ...

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- Last period $T = 30$:
  - For each C_{T-1} calculate $E[c^*_{hkT} | C_{T-1}]$ and $E[V_{hkT}(C_{T-1})]$
  - How? Adaptive Simpson quadrature
- Period $t < T$:
  - For each C_t calculate $E[c^*_{hkT} | C_{T-1}]$ and $E[V_{hkT}(C_{T-1})]$
  - Calculate transition probability matrix (driven by v_t): $f(C_t | C_{t-1}, k, h)$

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  - Calculate transition probability matrix (driven by v_t): $f(C_t | C_{t-1}, k, h)$
- Initial period: $t = 0$
  - Identify the plan choice for type h: $k(h) = \max_k E[V_h k1(0)]$

Estimation

- **Goal:** Identify the distribution of types

$$\{\theta_1, \theta_2, \ldots, \theta_{16807}\} \text{ and } \sum_h \theta_h = 1$$

- **Insight:** The solution of the model (over 16K types) is independent of the "econometrics parameters" (Ackerberg (2009)). The solution of the model can be computed "offline".

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- **Moment conditions:**
  - Conditional mean consumption:

$$\underbrace{\bar{c}_{kt}(C_{t-1})}_{\text{data}} - \underbrace{\sum_h E[c^*_{hkt}|C_{t-1}]\gamma_{hkt}(C_{t-1})\theta_h}_{\text{model}}$$

    where $\gamma_{hkt}(C_{t-1})$ is the prob. of reaching state (k, C_{t-1}) for type h.
  - Fraction of subscribers in state (k, C_{t-1})

$$\bar{s}_{kt}(C_{t-1}) - \sum_h \gamma_{hkt}(C_{t-1})\theta_h$$

  - Total: 120,000 moments per plan (8).

Estimation

- The moments are **linear** in the parameters

$$\min_{\theta} \mathbf{m}(\theta)^T V^{-1} \mathbf{m}(\theta)$$
$$\text{s.t. } \sum \theta_h = 1$$

where $m_i(\theta) = \hat{m}_i^{\text{data}} - \hat{m}_i^{\text{model}}\theta$ and $V = \text{Identity}$.

- **Computation:**
  - Abstracting from the constraint, $\hat{\theta}$ can be estimated by OLS!
  - With the integration constraint, this is a well-defined convex optimization problem.
  - Reference: Fox, Kim, Ryan, and Bajari (2011)

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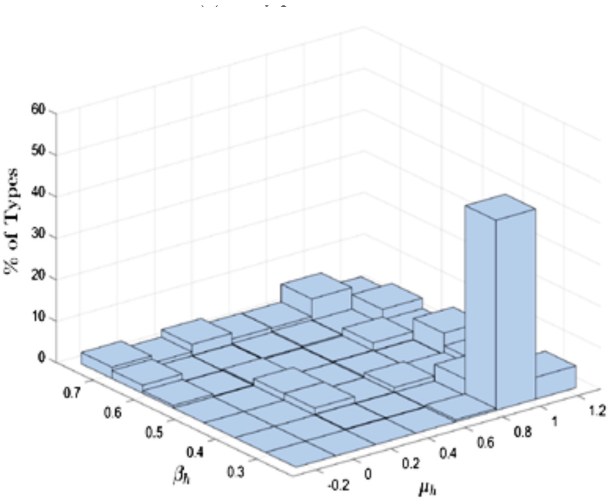
  - With the integration constraint, this is a well-defined convex optimization problem.
  - Reference: Fox, Kim, Ryan, and Bajari (2011)
- **Standard-errors:** Block bootstrap
  - Compute $\hat{\theta}$ from $\hat{m}^{\text{data}}$
  - Sample (with replacement) N consumer *sequences* of consumption
  - Compute $\hat{\theta}^2$ by matching moments $\hat{m}^s$
  - Repeat $1,000$ times: Bootstrap standard-errors over $\hat{\theta}$ and counterfactual simulations (e.g. elasticities)

Results: Distribution of types



Results: Limited *vs* Unlimited plans

	(1)	(2)	(3)	(4)
Scenario Description				
UBP/Unlimited	UBP	Unlim	Unlim	Unlim
Plan Attributes	current	current	typical US	rev-max F_k
Usage and Surplus				
Usage (GBs)	48.2	60.2	62.0	65.4
	(0.203)	(0.261)	(0.264)	(0.322)
Speed (Mb/s)	14.2	10.3	10.8	12.6
	(0.021)	(0.010)	(0.018)	(0.069)
Consumer Surplus ($)	84.7	111.9	113.5	97.1
	(0.810)	(0.791)	(0.789)	(0.810)
Revenue ($)	69.4	42.1	44.8	64.3
	(0.132)	(0.044)	(0.068)	(0.209)

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