Lecture 5: Strategic Models of Industry Dynamics

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Dynamic Investment Game with Entry/Exit (Ericson and Pakes (1995))

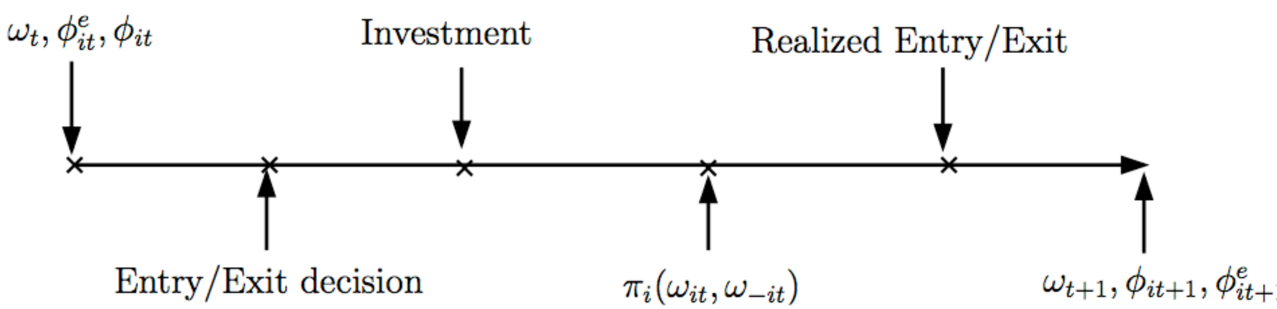
- **Introduction:**
  - Central element: Productivity is stochastically controlled by firms' investments.
  - Investment is strategic: Markov Perfect Equilibrium (MPE).
  - Model is general enough that it can be applied in various contexts:
    1. Quality ladder model (Pakes and McGuire (1994)),
    2. Learning-by-doing (Benkard (2004), Besanko et al. (2010)),
    3. Capacity dynamics (Besanko and Doraszelski (2004)),
    4. Horizontal mergers (Gowrisankaran (1999)),
    5. Advertisement (Doraszelski and Markovich (2005)).

    $*$ See Doraszelski and Pakes (2006) for a review.

Model

- **Actions:**
  1. Static actions: Price/quantity game. Lead to a (unique) static payoff function $\pi_i(\omega_i, \omega_{-i})$.
  2. Dynamic actions: Entry/Exit, Investment $x_i(\omega_i, \omega_{-i})$.
- **Assumptions:** $\pi_i(\omega_i, \omega_{-i})$ is continuous, bounded, and increasing in ω_i.
- **State space:**
  - Firm i's productivity: $\omega_i \in \Omega = \{1, 2, ..., \bar{\omega}\}$ (public information).
    * ω_i can represents a quality index, a marginal cost shifter, capacity, etc.
  - Entry cost: $\phi_i^e \sim F^e(.)$ (private information).
  - Scrap value: $\phi_i \sim F(.)$ (private information).
  - Industry state: $\omega \in S = \{\omega_1, ..., \omega_n | \omega_i \in \Omega, n \leq \bar{n}\}$
  - **Note:** $|S|$ grows exponentially in $\bar{n}$ (curse of dimensionality).

- **Assumptions:** Payoffs and strategies are **symmetric** and **anonymous**.
  - Symmetry: $f_i(\omega_i, \omega_{-i}) = f_j(\omega_i, \omega_{-i})$.
  - Anonymous: $f(\omega_i, \omega_{-i}) = f(\omega_i, \omega_{\mu(-i)})$.
  - Where $\mu(-i)$ is any reordering of indexes $-i$.
- The symmetry and anonymity assumptions reduce dramatically the industry state space. Two ways of writing the industry state:
  1. $\omega \in S^0 = \{\omega_1, ..., \omega_n | \omega_i \in \Omega, \omega_1 > \omega_2 > ... > \omega_n, n \leq \bar{n}\} \subset S$
  2. $\omega \in S^0 = \{s_1, ..., s_{\bar{\omega}} | \omega_i \in \Omega, \sum_\omega s_\omega \leq \bar{n}\} \subset S$
- **Timing:**



- **Investment:**
  - Firms invest x_i to raise the probability of getting a productivity gain ν next period:

$$\Pr(\omega_i' = \omega_i + \nu_i | \omega_i, x_i) = P_\nu(\nu_i | x, \omega_i)$$

    where $\nu_i \in \{0, 1\}$ and $p_\nu(\nu_i = 1 | x_i = 0, \omega_i) = 0$.
  - Assumption: $p_\nu(\nu = 1 | x, \omega)$ is strictly increasing in x_i.
  - Example (Pakes and McGuire (1994)):

$$p_\nu(\nu = 1 | x, \omega) = p_\nu(\nu = 1 | x) = \frac{\alpha x}{1 + \alpha x}$$

- **Exit:**
  - At the beginning of the period incumbents observe scrap value of equipments ϕ_i and decide to exit at the end of the period (i.e. all incumbents receive $\pi(\omega_i, \omega_{-i})$).
  - Since ϕ_i is private information, we can characterize the exit strategy either by an exit cutoff $\bar{\phi}(\omega_i, \omega_{-i})$, or by the continuation probability $r(\omega_i, \omega_{-i}) = \Pr(\phi < \bar{\phi}(\omega_i, \omega_{-i}))$.

- **Entry:**
  - Each period there are $\mathcal{E}$ potential entrants who live **only** one period.
  - Potential entrants privately observe an entry cost ϕ^e and decides to enter or not at the end of the period (one period set-up cost).
  - Entrants choose to invest x^e in order to improve their productivity next period.
  - Entry strategy: Two representations: Optimal cutoff point $\bar{\phi}^e(\omega_i, \omega_{-i})$, or entry probability $r^e = \Pr(\phi < \bar{\phi}^e(\omega_i, \omega_{-i}))$.
- **State-to-state transitions:**
  - Individual state transition:

$$\omega_i' = \omega_i + \underbrace{\nu_i}_{\text{productivity gain}} - \underbrace{\eta}_{\text{aggregate shock}},$$

    where $\eta \in \{0, 1\}$ and $\Pr(\eta = 1) = \delta$.
    Note: The depreciation shock η can either be industry-wide or firm specific.
  - Industry state transition probability matrix: $q(\omega_{-i}'|\omega, \eta)$ is the perceived transition probabilities of competitors' states (i.e. beliefs).

- **Value functions:**
  - Incumbent's problem:

$$V(\omega_i,\omega_{-i},\phi) \;=\; \pi(\omega_i,\omega_{-i}) + \max\Big[-x(\omega_i,\omega_{-i}) + \beta E\big(V(\omega_i',\omega_{-i}',\phi')|\omega_i,\omega_{-i},x_i\big),\phi\Big]$$

Or, prior to observe ϕ:

$$\begin{aligned} E_\phi(V(\omega_i,\omega_{-i},\phi)) \equiv v(\omega_i,\omega_{-i}) &= \pi(\omega_i,\omega_{-i}) + (1-r_i)E[\phi|\beta EW(\omega)-x_i<0] + \\ & \quad r_i\Big[-x_i+\beta\sum_\nu W_i(\nu|\omega)p_\nu(\nu|x_i)\Big] \end{aligned}$$

Where:

* $W_i(\nu|\omega) = \sum_{\omega_{-i}',\eta} v(\omega_i+\nu-\eta,\omega_{-i}')q(\omega_{-i}'|\omega_i,\omega_{-i},\eta)p_\eta(\eta)$,
* $\omega = (\omega_i,\omega_{-i})$.
* r_i is the optimal continuation probability:

$$r_i(\omega,\omega_{-i}) = F\big(\beta\sum_\nu W_i(\nu|\omega)p_\nu(\nu|x_i) - x_i\big)$$

* x_i solves the following KT-FOC:

$$x_i\Big(\beta\sum_\nu W(\nu|\omega)\frac{\partial p_\nu(\nu|x_i)}{\partial x_i} - 1\Big) = 0$$

– Entrant's problem:

$$V^e(\omega, \phi^e) = \max_{\chi^e, x^e} \chi^e(-\phi^e - x_i^e + \beta \sum_\nu W_e(\nu|\omega) p_\nu(\nu|x^e)$$

Where W_e is the continuation value of an entrant (i.e. starting at ω_e), x^e is defined as before. The entry probability is given by:

$$r^e(\omega_i, \omega_{-i}) = F^e\Big(- x^e(\omega) + \beta \sum_\nu W(\nu|\omega) p_\nu(\nu|x^e(\omega)\Big)$$

- **Equilibrium:** A MPE equilibrium is a set of functions $\{x, r, x^e, r^e\}$ such that:

  1. The policy functions solve the incumbents and entrants problems given beliefs $q(.|.)$.
  2. The perceived aggregate transition probabilities $q(\omega'_{-i}|\omega, \eta)$ are consistent with the optimal response of all agents.

Properties of the equilibrium

- Existence of a pure strategy equilibrium:
  - Continuity of the scrap value and entry cost distributions ensure the existence of unique entry/exit strategies in probability space (i.e. perfect-bayesian equilibrium, see Doraszelski and Satterthwaite (2005) for the exact conditions of profits and F).
  - As long the $p_\nu(.|x)$ satisfies some regularity conditions such that the continuation value is concave, there exits a unique best-reply investment function (see Ericson and Pakes (1995)).
- Uniqueness: Very difficult to show, apart for some special cases.

- Characterization:
  - There exists a pair $(\bar{n}, \bar{\omega})$ such that with probability one:
    * there will never be more than $\bar{n}$ firms active,
    * we will never observe an active firm with $\omega_i > \bar{\omega}$ (i.e. $x_i(\bar{\omega}, \omega_{-i}) = 0, \forall \omega_{-i}$).

    Therefore, the state space is always **finite** and we can compute an equilibrium.

- The equilibrium defines a finite state Markov chain with transition probability matrix $Q(\omega', \omega)$. Therefore there exists at least one recurrent class of states $R \in S^0$, such that:
  - with probability one each sample path $\{\omega_t\}$ will enter R,
  - and once in R the process will stay within it forever.

  Note: Since R is typically much smaller and S^0, if we can identify it, we can reduce considerably the computation burden (see Pakes and McGuire (2001)).

Computing MPE Nash equilibrium—
Gauss-Jacobi VS Gauss-Seidel algorithms

- **Example:**
  - Static entry game with 2 players and incomplete information.
  - Payoff of entering:

$$\pi_i = \alpha_i - \theta E(N_{-i}) + \epsilon_i = \alpha_i - \theta \tilde{p}_{-i} + \epsilon_i$$

  - If ϵ_i is iid extreme value, the best-response function is given by:

$$p_i = R(\tilde{p}_{-i}) = \frac{\exp(\alpha_i - \theta \tilde{p}_{-i})}{1 + \exp(\alpha_i - \theta \tilde{p}_{-i})}$$

  and a Nash equilibrium is such that $p_i = R(p_{-i})$ for all is.

- **Gauss-Jacobi**:

it 0: Guess p_i^0.

it k: Let $\tilde{p}_j = p_j^{k-1}$, and update strategies using the best-response mapping:

$$p_i^k = R(\tilde{p}_{-i})$$

test: Stop if $||p^k - p^{k-1}|| < \delta$, else update $k+1$.

- **Gauss-Seidel**:

it 0: Guess p_i^0.

it $k.1$: Let $\tilde{p}_{-1} = p_{-1}^{k-1}$, and update strategies using the best-response mapping for player 1:

$$p_1^k = R(\tilde{p}_{-1})$$

...

it $k.j$: Let $\tilde{p}_i = p_i^{k-1}$ for $i > j$, and $\tilde{p}_i = p_i^k$ for $i < j$. Update strategies using the best-response mapping for player i:

$$p_j^k = R(\tilde{p}_{-j})$$

...

test: Stop if $||p^k - p^{k-1}|| < \delta$, else update $k + 1$.

- The Guass-Seidel usually converges faster, but both algorithm can converge to different equilibria (or not converge at all...).

- Alternatives:
  - Damping: $p_i^k = \alpha R(\tilde{p}_{-i}) + (1 - \alpha) p_i^{k-1}$
  - Random updating orders (in the Gauss-Seidel algorithm).

- **Pake-McGuire Algorithm (modified) - Gauss-Jacobi:**
  - Initialization: Set $\underline{\omega}$ and $\bar{\omega}$. How?

    Solve the monopoly problem for arbitrary small $\underline{\omega}$ and large $\bar{\omega}$, and set:

$$\bar{\omega} = \arg\max_{\omega} x^m(\omega) = 0$$
$$\underline{\omega} = \arg\min_{\omega} r^e(\omega) \simeq 0$$

    Also, we need to set a value for $\bar{n}$ "large enough" (see below).

  - Objects in memory at each iteration:

$$M^l(\omega_i, \omega_{-i}) = \{v^l(\omega_i, \omega_{-i}), x^l(\omega_i, \omega_{-i}), x^{e,l}(\omega_i, \omega_{-i}), r^l(\omega_i, \omega_{-i}), r^{e,l}(\omega_i, \omega_{-i})\}$$

– Iteration l: For each $\omega_i \in \Omega$

1. **Compute** $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$. Recall:

$$q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta) = \prod_{j \neq i} p^{l-1}(\omega'_j|\omega_i, \omega_{-i}, \omega) \prod_{k=1}^{E'} p_\nu(\nu_k|x^{e,l-1}) r^{e,l-1}(E', \nu)$$

Where,

$$p^{l-1}(\omega'_j|\omega_i, \omega_{-i}, \eta) = \begin{cases} (1 - r_j^{l-1}) & \text{if } \omega'_j = \emptyset \\ r^{l-1} p_\nu(1|x_j^{l-1}) & \text{if } \omega'_j = \omega_j + 1 - \eta \\ r^{l-1} p_\nu(0|x_j^{l-1}) & \text{if } \omega'_j = \omega_j - \eta \end{cases} \quad (1)$$

$$r^{e,l-1}(E', \omega_i, \omega_{-i}) = \binom{E}{E'} r^{e,l-1}(\omega_i, \omega_{-i})^{E'} (1 - r^{e,l-1}(\omega_i, \omega_{-i}))^{E-E'} \quad (2)$$

In practice, we need to identify the set of "reachable" states $S(\omega_{-i}) \subset S$. How?

∗ For incumbents, $\omega'_j \in \{\emptyset, \omega_j + 1 - \eta, \omega_j - \eta\}$

∗ For entrants, $\omega'_j \in \{\emptyset, \omega^e + 1 - \eta, \omega^e - \eta\}$

If there are n active incumbents and E potential entrants, the dimensionality of $S(\omega_{-i})$ is 3^{n-1+E}.

This process is then repeated for each $\omega_{-i} \in \Omega^{\bar{n}-1}$. **Computing** $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$ **is the main source of the curse of dimensionality problem.**

2. Compute expected continuation values:

$$W^l(\nu|\omega_i,\omega_{-i}) \;=\; \sum_{\omega'_{-i},\eta'} v^{l-1}(\omega_i+\nu_i-\eta')q(\omega'_{-i}|\omega_i,\omega_{-i},\eta')p_\eta(\eta') \tag{3}$$

$$W^{e,l}(\nu|\omega) \;=\; \sum_{\omega',\eta'} v^{l-1}(\omega^e+\nu_i-\eta')q(\omega'|\omega,\eta')p_\eta(\eta') \tag{4}$$

3. Update policy functions (i.e. best-responses):

(a) Investment decisions:

$$x^l(\omega_i,\omega_{-i})\Big(\beta(W^l(1)-W^l(0))\frac{\partial p_\nu(1|x^l)}{\partial x^l}-1\Big)=0$$

$$x^{e,l}(\omega_i,\omega_{-i})\Big(\beta(W^{e,l}(1)-W^{e,l}(0))\frac{\partial p_\nu(1|x^{e,l})}{\partial x^{e,l}}-1\Big)=0$$

(b) Entry/Continuation probabilities:

$$r^l(\omega_i,\omega_{-i})=F\big(\beta\sum_\nu W^l\nu|\omega_i,\omega_{-i})p_\nu(\nu|x_i)-x^l\big)$$

$$r^{e,l}(\omega)=F^e\big(\beta\sum_\nu W^{e,l}(\nu|\omega)p_\nu(\nu|x_i)-x^{el}\big)$$

4. Update value function:

$$\begin{aligned} v^l(\omega_i, \omega_{-i}) &= \pi(\omega_i, \omega_{-i}) + (1 - r^l)E[\phi|\beta W^l(\omega_i, \omega_{-i}) - x^l < 0] + \\ & \quad r^l\Big[- x^l + \beta \sum_{\nu} W^l(\nu|\omega_i, \omega_{-i}) p_\nu(\nu|x^l)\Big] \end{aligned}$$

– Stop if $||v^l(\omega_i, \omega_{-i}) - v^{l-1}(\omega_i, \omega_{-i})|| < \delta$.

- Endogenous $\bar{n}$?
  - Solve the model with $\bar{n}^k$
  - If $\min_\omega r^e(\omega|n(\omega) = \bar{n}) > \epsilon$, recompute the equilibrium with $\bar{n}^{k+1} = \bar{n}^k + 1$.

- **Computation burden:**
  - Number of states $\#S$: Raise the cost of the outer-loop over states (ω_i, ω_{-i}).
  - Transition probabilities $q(\omega'_{-i}|\omega_i, \omega_{-i}, \eta)$ (inside loop to find transient states)
    * $\#S$ grows exponentially in $\bar{n}$.
    * $\#S^0$ grows at polynomial rate in $\bar{n}$.
    * $\#S$ and $\#S^0$ grow exponentially in $\bar{\omega}$: Binding in models with multi-product or multi-state firms (e.g. differentiated products or horizontal mergers).
    * Even when the transient states $S(\omega_{-i})$ can be "pre-computed", the transition probabilities must be re-computed for each new iteration of the policy functions.
- **Alleviating the computational burden:**
  - Stochastic algorithm: Pakes and McGuire (2001).
  - Continuous-time representation: Doraszelski and Judd (2004).
  - Parallel representation: Benkard (2004)
  - Other methods: Approximation methods (e.g. Keane and Wolpin (1997))

Besanko and Doraszelski (2004) Capacity dynamics and endogenous asymmetries in firm size

- **Stylized facts**
  - Market structure is persistent
  - Most industries exhibit important asymmetries: Dominant firm(s) competing with fringes.
  - How heterogeneous initial conditions emerge in the first place?
  - Cournot vs Bertrand: Equivalent market-structure?

- **Model set-up**
  - Duopoly
  - Homogenous products: $D(P) = a - bP$.
  - Heterogeneous (discrete) capacities:

$$0 \leq q_i \leq \bar{q}_i$$

    and industry structure: $\omega_t = \{\bar{q}_{1t}, \bar{q}_{2t}\}$.
  - Two models of competition:
    1. Quantity: Cournot
    2. Price competition: Edgeworth-Bertrand (see for instance Deneckere & Kovenock)
  - Investment: Investing x affects the probability of capacity improvement

$$\Pr(\Delta\bar{q}_{1t}|\bar{q}_{1t}, x_{1t}) = \begin{cases} \frac{(1-\delta)\alpha x_{1t}}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = 1, \\ \frac{(1-\delta)}{1+\alpha x_{1t}} + \frac{\delta \alpha x_{1t}}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = 0, \\ \frac{\delta}{1+\alpha x_{1t}} & \text{If } \Delta q_{1t} = -1 \end{cases}$$

    This is in the interior of the capacity grid. Boundary states: $\bar{q} = M$ and $\bar{q} = 0$.

- **Investment decision:**
  - Bellman equation:
$$V_1^{\tilde{x}}(\omega) = \pi_1(\omega) + \max_{x_1 \geq 0} \left\{ -x_1 + \beta \sum_{\bar{q}_1'} W_1^{\tilde{x}}(\bar{q}_1'|\omega) \Pr(\bar{q}_1'|\bar{q}_1, x_1) \right\}$$
    where $W_1^{\tilde{x}}(\bar{q}_1) = \sum_{\bar{q}_2'} V_1^{\tilde{x}}(\bar{q}_1, \bar{q}_2) \Pr(\bar{q}_2'|\bar{q}_2, \tilde{x}_2)$.
  - Kuhn-Tucker condition:
$$-1 + \beta \sum_{\bar{q}_1'} W_1^{\tilde{x}}(\bar{q}_1'|\omega) \frac{\partial \Pr(\bar{q}_1'|\bar{q}_1, x_1)}{\partial x_1} \geq 0$$
  - Closed-form best-response function:
$$x_1(\omega) = R_1(\tilde{x}|\omega) = \max \left\{ 0, \frac{-1 + \sqrt{\beta\alpha\left((1-\delta)\left(W_1^{\tilde{x}}(\bar{q}_1+1) - W_1^{\tilde{x}}(\bar{q}_1)\right) + \delta\left(W_1^{\tilde{x}}(\bar{q}_1) - W_1^{\tilde{x}}(\bar{q}_1-1)\right)\right)}}{\alpha} \right\}$$
- **MPE Nash equilibrium:**
  - $x_i(\omega) = R_1(x_j(\omega)|\omega)$ for all $i, j = \{1, 2\}$.
  - $V_i^x(\omega) = \pi_i(\omega) + -x_i + \beta \sum_{\bar{q}_1'} W_i^x(\bar{q}_1'|\omega) \Pr(\bar{q}_i'|\bar{q}_i, x_i)$ for all $i, j = \{1, 2\}$.

- **Algorithm:**
  - Guess: $\tilde{x}(\omega)$ and $\tilde{V}(\omega$ for all ω
  - Iteration k:
    - Update $x_i(\omega)$ using $\tilde{x}$ and $\tilde{V}$
    - Update $V_i(\omega)$ using x and $\tilde{V}$
  - Stopping rule: $x_i(\omega) = \tilde{x}_i(\omega)$ and $V_i(\omega) = \tilde{V}_i(\omega)$.
- **Simulations:**
  - Vary the depreciation rate δ, holding fixed the other parameters. Why?
  - δ measures the degree of investment irreversibility. Study the role of "credibility" on long-run industry distribution.

- **Simulation results**

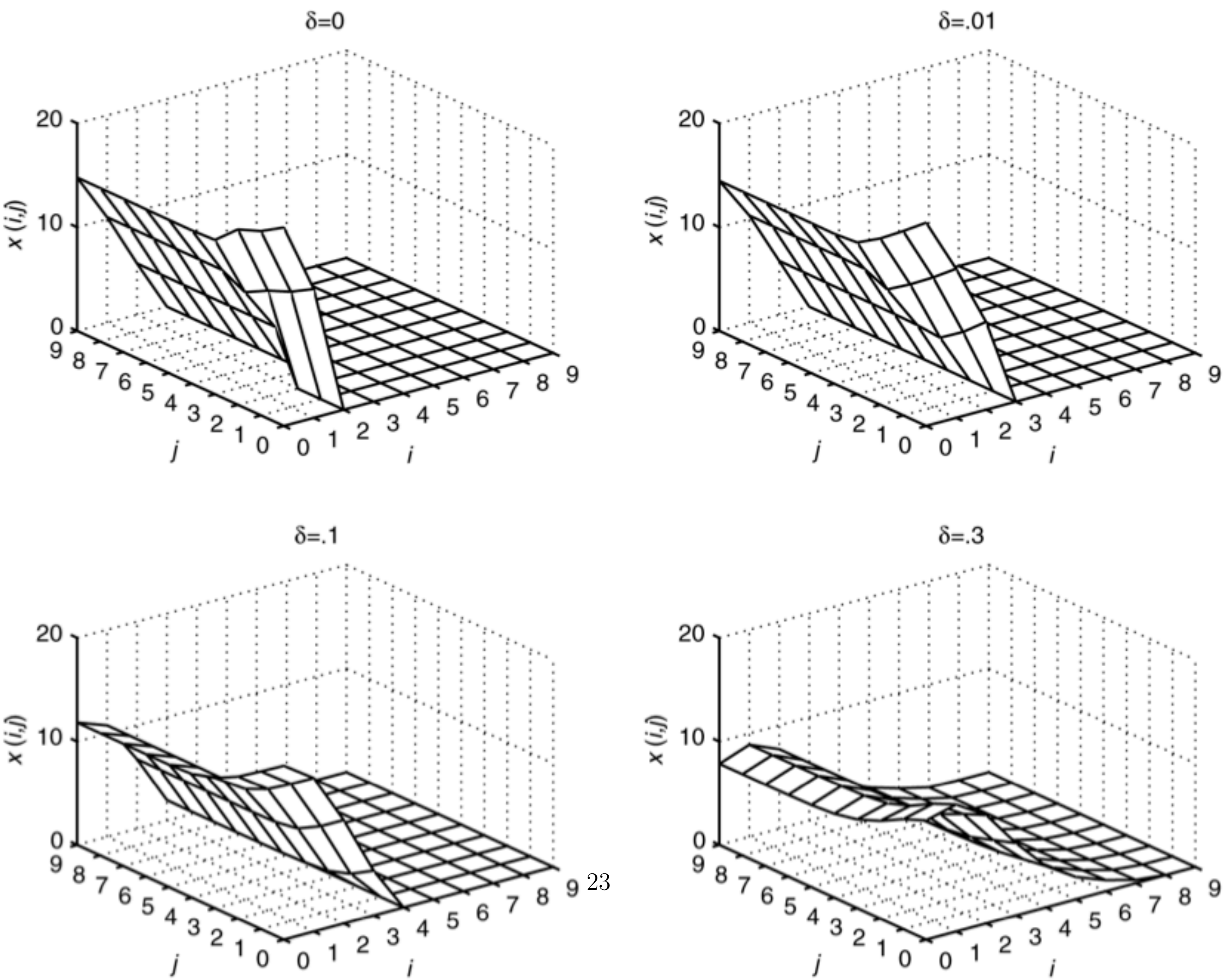
  1. Quantity game

     - If $\delta = 0$: Firms invest at constant rate until they reach Cournot: $x_1 = 0$ if $\bar{q}_1 > 3$.
     - If $\delta > 0$: Firms over-invest (again +/- constant) to maintain the Cournot quantity in the long-run. As a result, firms hold idle capacity, the average long-run capacity levels are increasing in δ, and the industry structure tend to be symmetric.

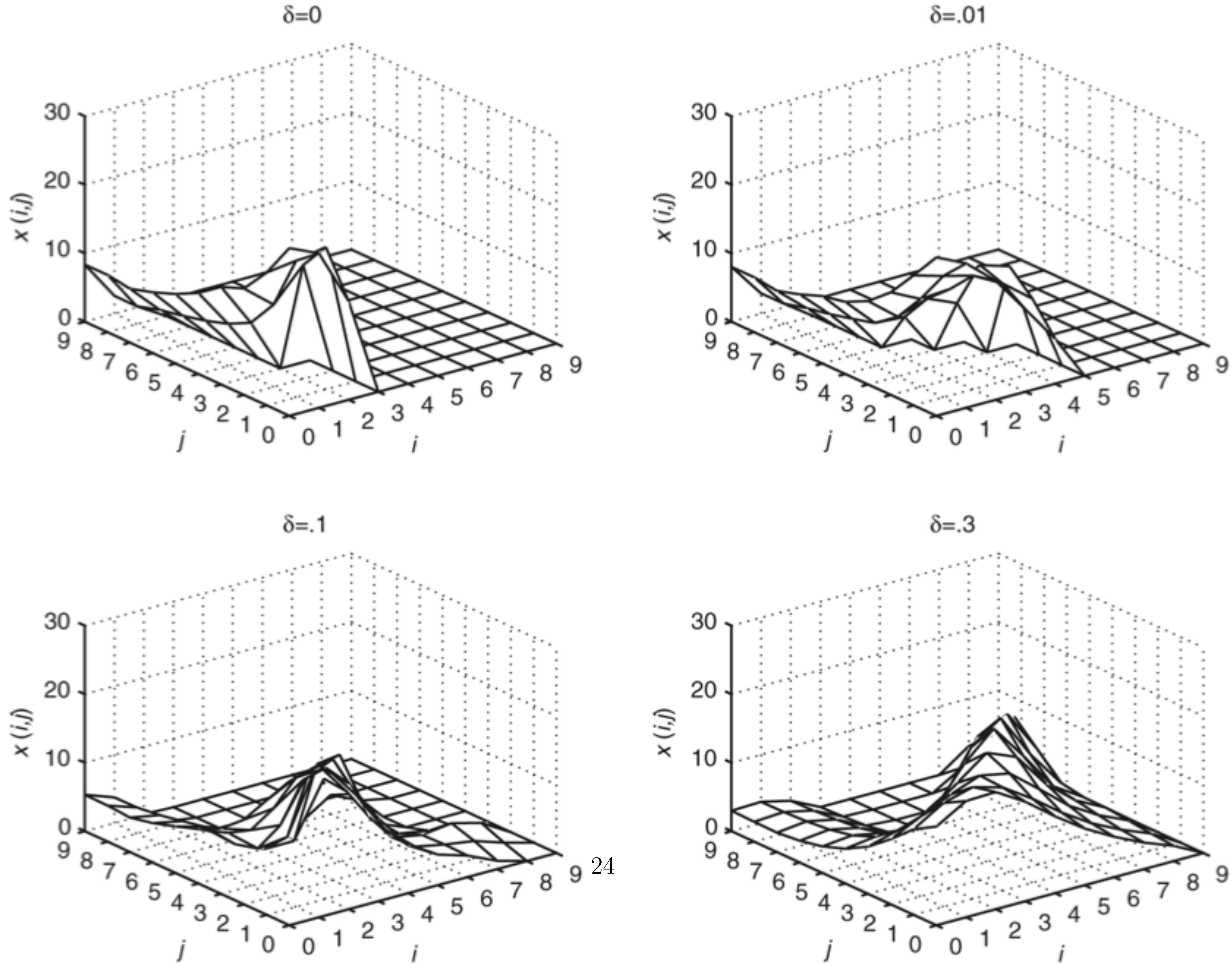
  2. Pricing game

     - Unlike in the Cournot model, investment functions depend crucially on competitors' capacity levels: Firms "give-up" quickly when opponent's level is larger than 3.
     - When firms are symmetric, their investment levels increase: Firm engage in a "pre-emption" race to establish a higher capacity level.
     - Investments are increasing in the depreciation fractor.
     - Why? With price competition, small capacity firms are easily shut-down, and large capacity firms earn monopoly power on the residual demand segment.

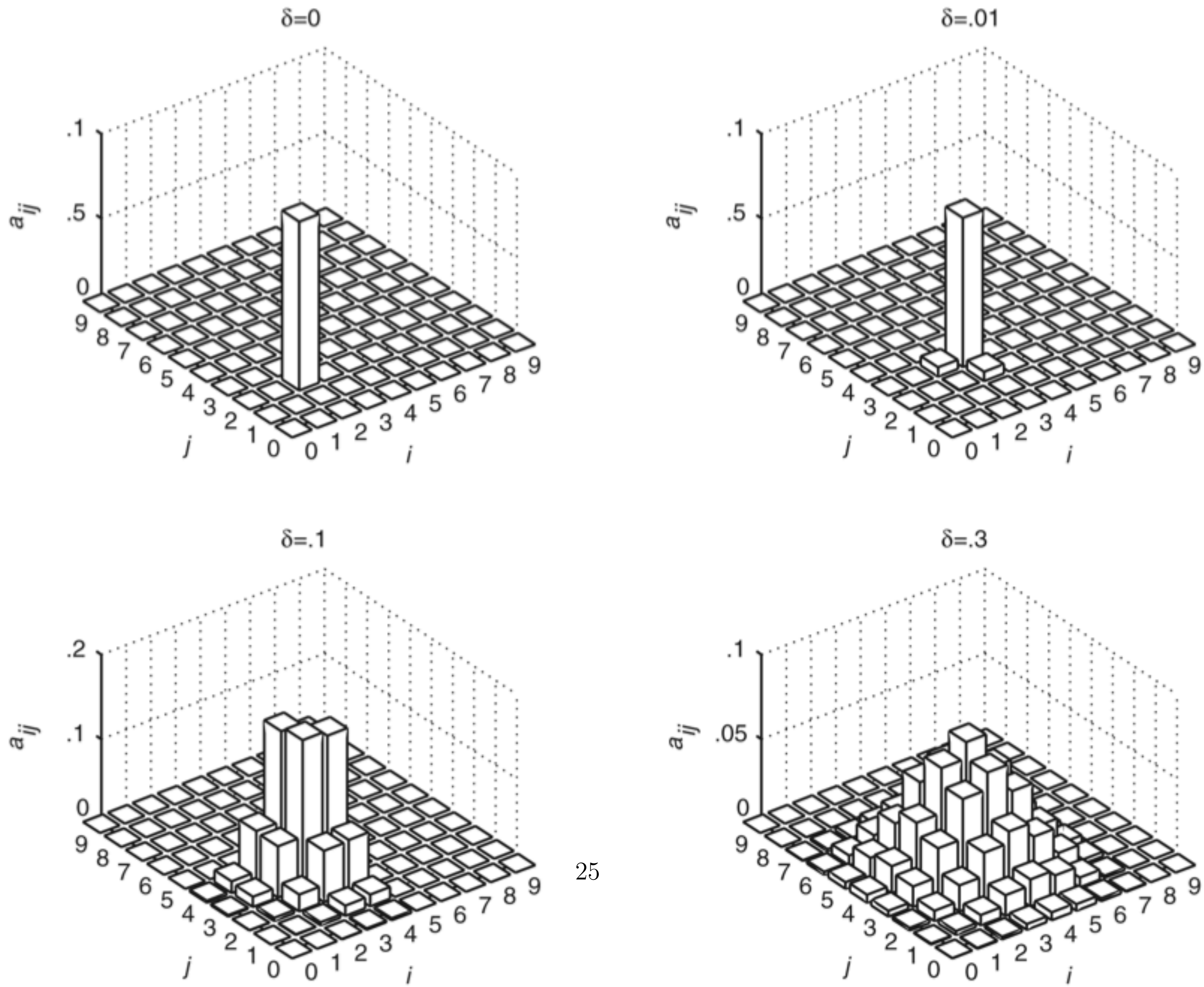
POLICY FUNCTION $x(i, j)$: QUANTITY COMPETITION



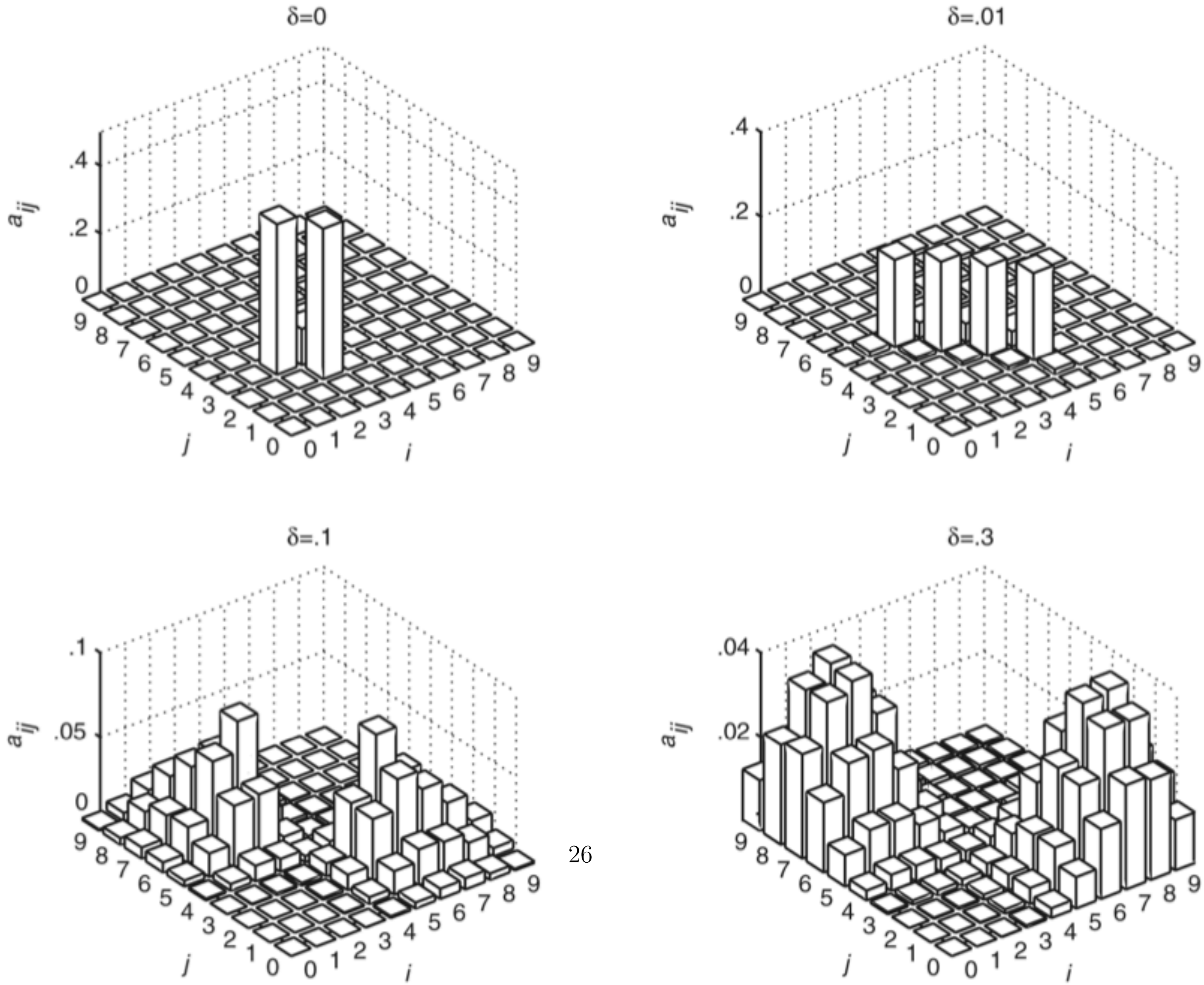
POLICY FUNCTION $x(i, j)$: PRICE COMPETITION



TRANSIENT DISTRIBUTION AFTER $T = 25$ PERIODS WITH INITIAL STATE $i_0 = j_0 = 1$: QUANTITY COMPETITION



TRANSIENT DISTRIBUTION AFTER $T = 25$ PERIODS WITH INITIAL STATE $i_0 = j_0 = 1$: PRICE COMPETITION



Learning-by-doing and organizational forgetting (Besanko et al. (2010))

- **Starting point:**
  - Cabral & Riordan (1994): Learning-by-doing promotes market dominance
  - From there, it might seem like adding forgetting should have a pro-competitive effect
  - However, the paper shows that the opposite is more likely true: Incorporating forgetting induces larger asymmetries and more concentration.
    - "Forgetting" opens the possibility of weakening your opponent by cutting prices
    - This combination of an "offensive" and "defensive" strategy is more pronounced with organizational forgetting
  - Forgetting is a source of equilibrium multiplicity
    - Need to develop a method to trace the equilibrium correspondence.

- **Model:**
  - Duopoly without entry/exit
  - Discrete experience level: $e_n \in \{1, 2, ..., M\}$
  - Each period, one client comes to the market
  - Accumulation: $e'_n = e_n + q_n - f_n$. Where:
    * $q_n \in \{0, 1\}$ indicates sales/no-sales for firm n with probability $D_n(\mathbf{p})$.
    * $f_n \in \{0, 1\}$ indicates depreciation (i.e. forgetting).
  - Forgetting probability:
$$\Delta(e_n) = 1 - (1 - \delta)^{e_n}$$
  - Learning curve:
$$c(e_n) = \kappa f(e_n)^{\eta}$$
    where $f(e_n) = m$ if $e_n > m$, where m denotes bottom of the learning curve.
  - Demand/sales probability: $D_n(\mathbf{p}) = \frac{1}{1+\exp\left(\frac{p_n - p_{-n}}{\sigma}\right)}$

- Bellman equation:

$$V_n(\mathbf{e}) = \max_{p_n} D_n(p_n, p_{-n}(\mathbf{e}))(p_n - c(e_n)) + \beta \sum_{k=1}^{2} D_k(p_n, p_{-n}(\mathbf{e}))\bar{V}_{nk}(\mathbf{e})$$

where $\bar{V}_{nk}(\mathbf{e}) = E\left(V_n(\mathbf{e}')|\mathbf{e}, q_k = 1\right)$.

- First-order condition:

$$D_n(\mathbf{e}) + \frac{\partial D_n}{\partial p_n}(p_n - c(e_n)) + \beta \sum_k \frac{\partial \bar{V}_{nk}(\mathbf{e})}{\partial p_n}\bar{V}_{nk}(\mathbf{e}) = 0$$

$$D_n(\mathbf{e})\left(\sigma - (p_n - c(e_n)) + V_n(\mathbf{e}) - \beta\bar{V}_{nn}(\mathbf{e})\right) = 0$$

- Symmetric MPE (arbitrary for firm 1):

$$F^1(e|p^*, V^*) = -V_1^*(\mathbf{e}) + D_1^*(\mathbf{e})(p_1^*(\mathbf{e}) - c(e_1)) + \beta \sum_k D_k^*(\mathbf{e})\bar{V}_{1k}^*(\mathbf{e}) = 0$$

$$F^2(e|p^*, V^*) = D_1^*(\mathbf{e})\left(\sigma - (p_1^*(\mathbf{e}) - c(e_n)) + V_1^*(\mathbf{e}) - \beta\bar{V}_{11}^*(\mathbf{e})\right) = 0$$

for all states. In matrix form:

$$F(p^*, V^*) = \begin{pmatrix} F^1(p^*, V^*) \\ F^2(p^*, V^*) \end{pmatrix} = 0$$

is $2M^2$ system of non-linear equations.

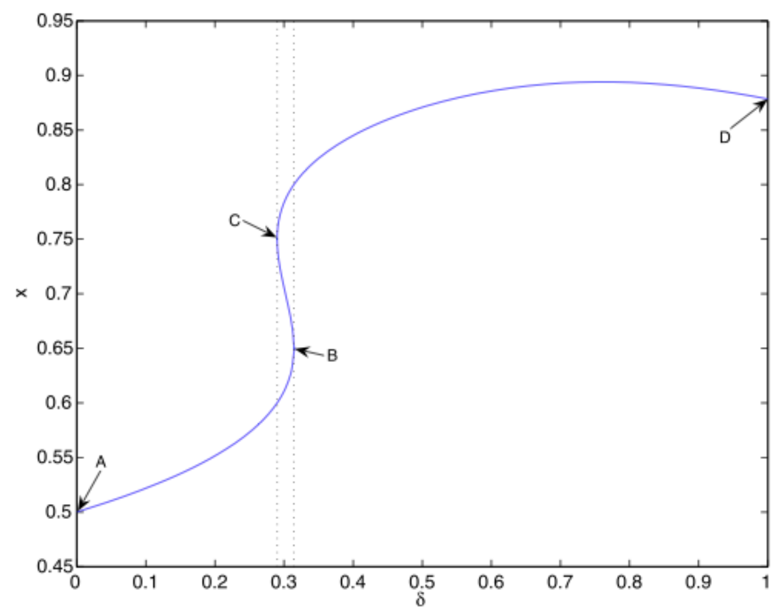
- **Computation: Homotopy method**
  - **Goal:** For each ρ, need to explore the equilibrium correspondence:

$$F^{-1}(\rho) = \{V^*, p^*, \delta | F(V^*, p^*; \delta, \rho) = 0, \delta \in (0,1)\}$$

  - Example:

$$F(x, \delta) = -15.289 - \frac{\delta}{1+\delta^4} + 67.50x - 96.923x^2 + 46.154x^3 = 0$$

How to trace the correspondance?



– The algorithm iterates on a differential equation:

* Lets index each point of the correspondence by s: $(x(s), \delta(s))$
* For any s the following total differential equation applies:

$$\frac{\partial F(x(s),\delta(s))}{\partial x}x'(s) + \frac{\partial F(x(s),\delta(s))}{\partial \delta}\delta'(s) = 0$$

* This differential equation is solved by setting:

$$x'(s) = \frac{\partial F(x(s),\delta(s))}{\partial \delta}$$

$$\delta'(s) = -\frac{\partial F(x(s),\delta(s))}{\partial x}$$

* The idea of the algorithm is to construct a path from $s = 0$ that traces the correspondence of all pairs $(x(s), \delta(s))$ until $\delta = 1$. All we need is a starting point.
* In this example, the starting point can be $\delta(0) = 0$ and $x(0) = 0.5$
* Then, for $s' = s + ds$, we can follow the path traced by the differential equation:

$$x(s') = x(s) + x'(s) = x(s) + \frac{\partial F(x(s),\delta(s))}{\partial \delta}$$

$$\delta(s') = \delta(s) + \delta'(s) = \delta(s) - \frac{\partial F(x(s),\delta(s))}{\partial x}$$

– How does it apply to MPEs?

  * Need to start the homotopy at a starting point (i.e. one arbitrary MPE).
  * How? Follow P-M algorithm for alternative starting values.
  * For a given starting point, iterate on the following multi-dimention differential equation:

$$y_i'(s) = (-1)^{i+1} \det \left(\left(\frac{\partial F(y(s);\rho)}{\partial y} \right)_{-i} \right)$$

    where $y(s) = (V(s), p(s), \delta(s))$, and $i = 1, ..., 2M^2 + 1$.
  * In this particular example, Proposition 2 shows that the MPE is unique for $\delta = 0$ or $\delta = 1$ (i.e. no or full forgetting). This is a natural starting point to trace the "main" correspondence.
  * Finding all the equilibria is not guaranteed however, since some MPEs can be disconnected from the main path. Need to start the algorithm at arbitrary points in the interior.

- What causes the multiplicity?
  * A **sufficient** condition for uniqueness of MPE is: (i) statewise uniqueness (i.e. stage game), and (ii) the movements through the state space is unidirectional.
  * Without organizational forgetting, firms with probability one end up at (M, M).
  * Working backward establish uniqueness.
- Distribution of the number of computed equilibria:

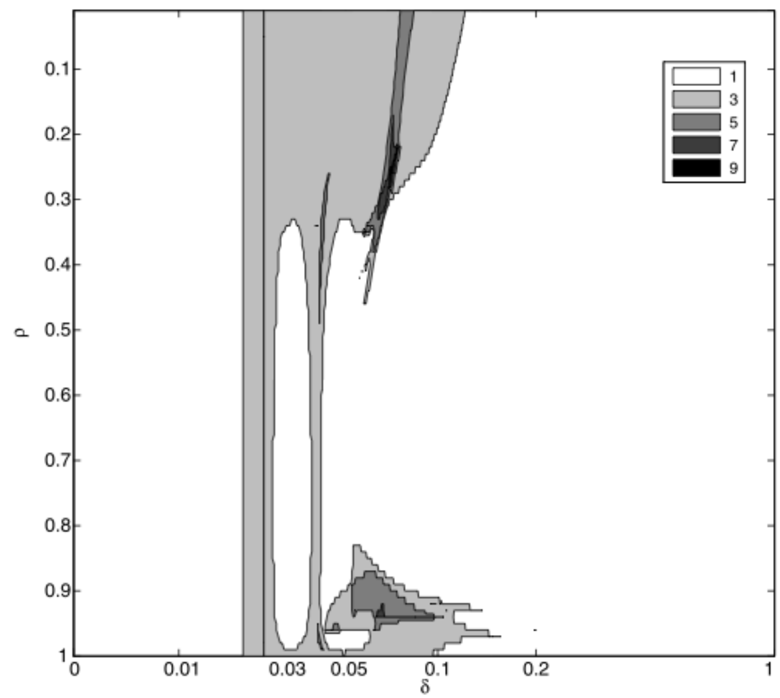


FIGURE 2.—Number of equilibria.

- Types of equilibria:
    1. Flat equilibrium without well (i.e. without forgetting)
    2. Flat equilibrium with well (i.e. small level of forgetting)
    3. Trenchy equilibrium (i.e. small level of forgetting)
    4. Extra trenchy equilibrium (i.e. large level of forgetting)

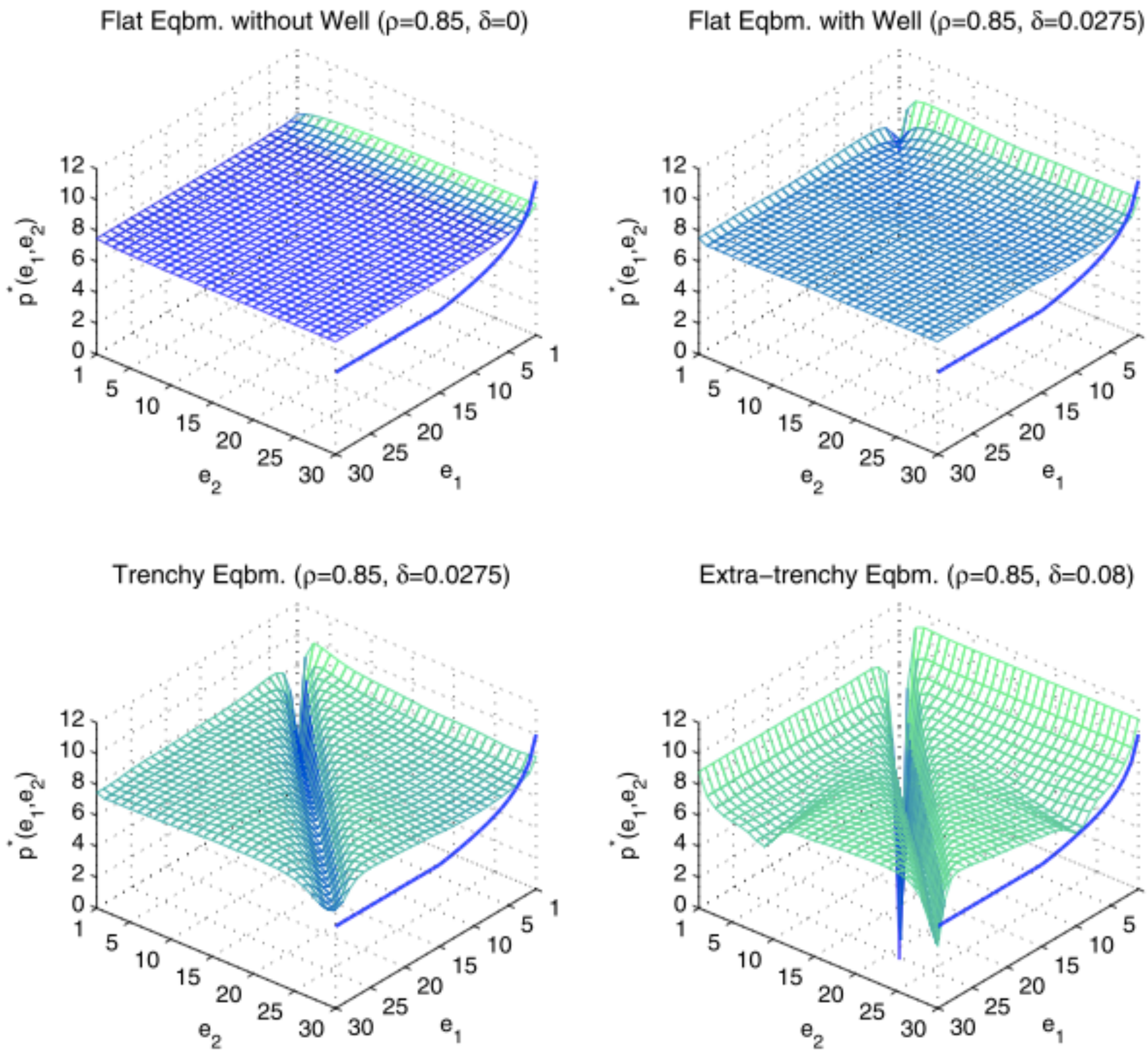


FIGURE 4.—Policy function $p^*(e_1, e_2)$; marginal cost $c(e_1)$ (solid line in $e_2 = 30$ plane).

- Industry dynamics:
  - No forgetting: Symmetric structure
  - Introducing forgetting lead to more asymmetric structures:
    * In "flat" equilibria, asymmetries are mostly temporary, and are associated with initial price wars (i.e. when $e_1 = e_2 = 1$).
    * In trenchy equilibria, asymmetries are persistent
  - The type of equilibria and the level of forgetting therefore determine the degree of concentration in the long-run, and the life cycle of "dominant" firms.

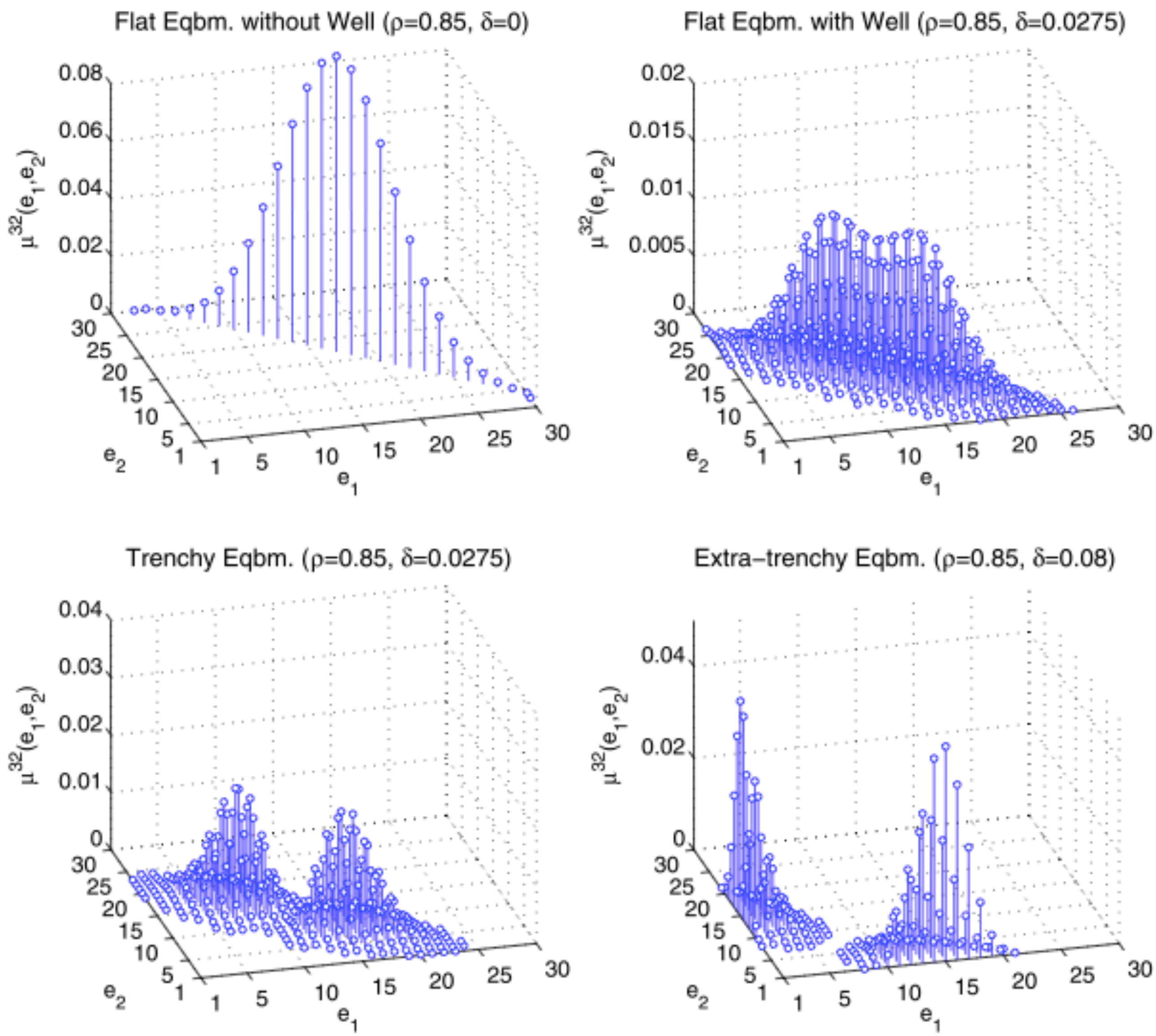


FIGURE 6.—Transient distribution over states in period 32 given initial state (1, 1).

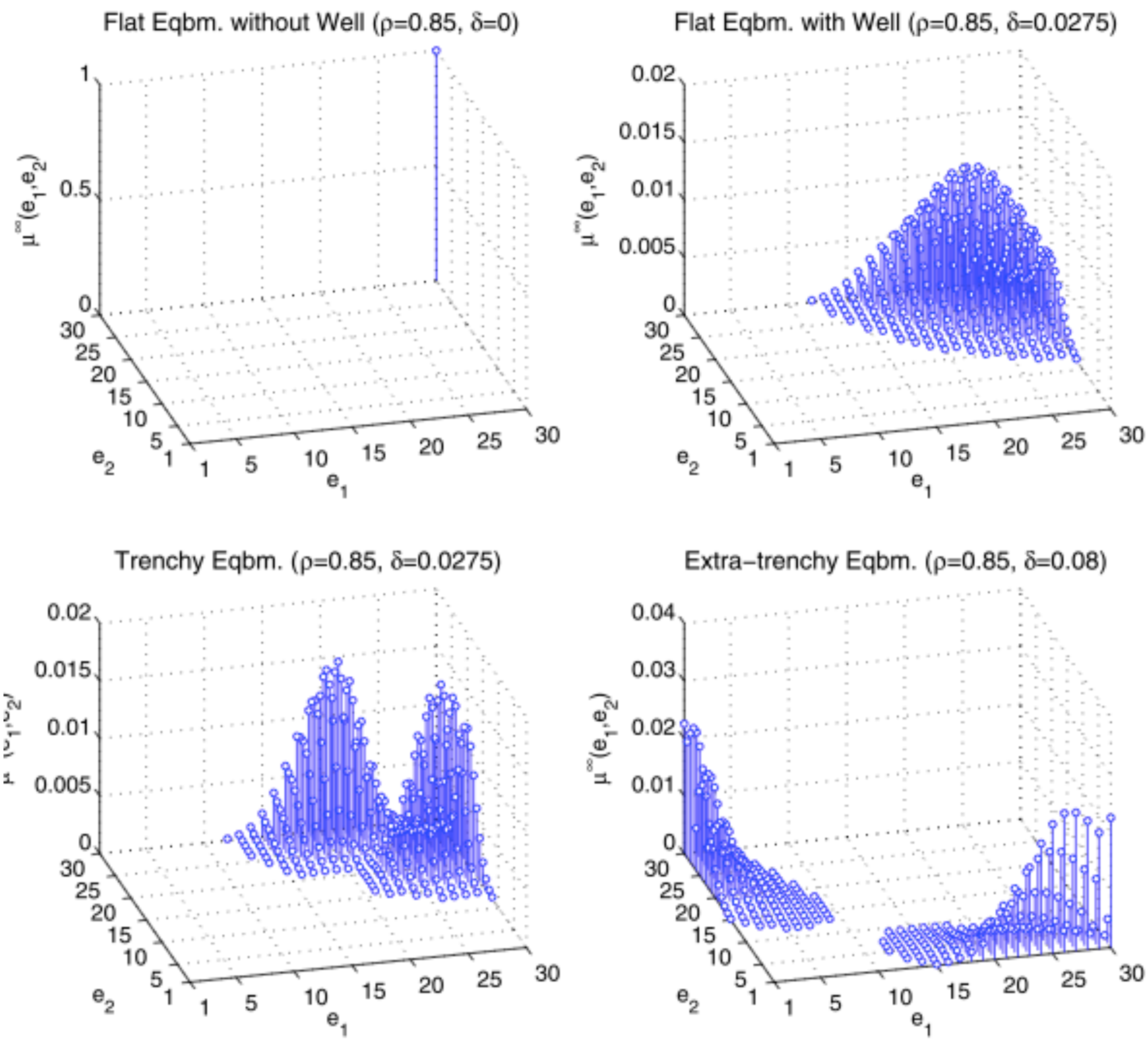


FIGURE 7.—Limiting distribution over states.

- Dynamic pricing and price wars:
  - We can re-write the FOC as follows:

$$p^*(\mathbf{e}) = c^*(\mathbf{e}) + \frac{\sigma}{1 - D_1^*(\mathbf{e})}$$

    where $c^*(\mathbf{e}) = c(e_1) - \beta\left(\bar{V}_{11}(\mathbf{e}) - \bar{V}_{12}(\mathbf{e})\right)$.
  - **Wells VS Trenches:**
    - Both phenomena have an investment dimension:
      - Firms want to price low to acquire experience, and reach a level of dominance.
    - Trenches also have a defensive incentive:
      - Dominant firms try to maintain their dominance position as soon as their opponent gets closer to 1/2 share.
  - **Sideway trenches:**
    - With "extra trenchy" equilibria, price wars occur even with asymmetric market structures.
    - Why? When asymmetries are large, followers with very low experience face a larger probability of loosing experience, than making a sale (i.e. gaining e). Therefore, dominant firms can "invest" in their dominance position by increasing the probability that the followers stays in that region.

Summary of Predictions

PRICING BEHAVIOR AND INDUSTRY DYNAMICS

	Flat Eqbm. Without Well	Flat Eqbm. With Well	Trenchy Eqbm.	Extratrenchy Eqbm.
Leading example	$\rho = 0.85$, $\delta = 0$	$\rho = 0.85$, $\delta = 0.0275$	$\rho = 0.85$, $\delta = 0.0275$	$\rho = 0.85$, $\delta = 0.08$
Preemption battle (well)	no	yes	no	no
Price war triggered by imminent threat (diagonal trench)	no	no	yes	yes
Price war triggered by distant threat (sideways trench)	no	no	no	yes
Short-run market dominance	no	yes	yes	yes
Long-run market dominance	no	no	yes, modest	yes, extreme
Dominance properties	yes	no, mostly	no, mostly	no, mostly

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