MATHS

CONTENTS

- LINEAR ALGEBRA
- CALCULUS
- DIFFERENTIAL EQUATIONS
- PROBALITY & STATISTICS
- NUMERICAL METHODS
- COMPLEX VARIABLES
- LAPLACE & FOURIER

CE, ME, CH

GATE	ESE	ISRO/Tech
** 2-3 Q	1Q	1-2Q
*** 3-4Q	2-3Q	2-3Q
* 2Q	1-2Q	2Q
*** 2-4Q	2-3Q	2-3Q
1-2Q	X	1Q
X	2Q	X
X	1Q	1Q
7-10Q	5-15Q	8-10Q
13-15M	10-30M	24-30M

LINEAR ALGEBRA

- ALGEBRA OF MATRICES
- RANK OF MATRICES
- SYSTEM OF EQUATIONS
- EIGEN VALUES & VECTORS

MATRIX

A set of mn objects/elements (real/complex) arranged in rectangular array of

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & \cdots\ n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{2} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ \vdots & & & \\ & & & \end{array}\right]_{m \times n} \end{array}$$

Element of Matrix $= [a_{ij}]$ $\qquad \underset{\text{Rows}}{m} \times \underset{\text{Columns}}{n}$

$i \rightarrow$ Row $\{1 \text{ to } m\}$

$j \rightarrow$ Column $\{1 \text{ to } n\}$

TYPES OF MATRICES

1) ROW MATRIX :- A matrix having one row & any number of columns. Ex :- $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \end{bmatrix}_{1 \times n}$

$m \rightarrow 1 \; ; \; n$

2) COLUMN MATRIX :- A matrix having one column & any numbers of rows. Ex :- $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \end{bmatrix}_{m \times 1}$

$m \; ; \; n \rightarrow 1$

3) NULL MATRIX / ZERO MATRIX :- Any matrix in which all elements are 0. Ex :- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

TYPES OF MATRICES

4) SQUARE MATRIX :- A matrix in which no. of rows & no. of columns are equal.

Ex :- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \\ 6 & 7 & 1 \end{bmatrix}_{3\times 3}$ Trace = 2

TRACE OF MATRIX :- Sum of principal diagonal elements / main diagonal / leading diagonal elements

5) DIAGONAL MATRIX :- A sq. matrix is called diagonal matrix if all its non-diagonal elements are zero.

TYPES OF MATRICES

Ex :- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3 \times 3}$

Diagonal $\neq 0$

Non-diagonal $= 0$

This is a diagonal matrix of order 3.

6) SCALAR MATRIX :- If all elements in a square diagonal matrix are equal.

Ex :- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5 I_3$

TYPES OF MATRICES

7) UNIT MATRIX :- A sq. matrix in which diagonal elements are 1.

→ IDENTITY MATRIX

Ex :- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8) UPPER TRIANGULAR MATRIX :- A sq. matrix is K/a U.T.M. if all elements below principal diagonal are zero.

Ex :- $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ → Above diag. $\neq 0$ → Below diag $= 0$

TYPES OF MATRICES

9) LOWER TRIANGULAR MATRIX :- A sq. matrix is K/a L.T.M. if all elements above principal diagonal is zero.

Ex :- $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -3 & 3 \end{bmatrix}$ → Above diag. = 0 ; Below diag. ≠ 0

10) SUB MATRIX :- A matrix obtained from given matrix by deleting some rows or columns.

$$A = \begin{bmatrix} 0 & 1 & 2 & 4 \\ 2 & -1 & 3 & 5 \\ -1 & 9 & 7 & 6 \\ -5 & 2 & 8 & 0 \end{bmatrix}_{4\times 4} \Rightarrow A_1 = \begin{bmatrix} 2 & -1 & 5 \\ -1 & 9 & 6 \\ -5 & 2 & 0 \end{bmatrix}_{3\times 3} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}_{2\times 2}$$

TYPES OF MATRICES

11) HORIZONTAL MATRIX :- $m < n$

Ex :- $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 6 & 7 & 8 & -1 \end{bmatrix}_{3 \times 4}$

12) VERTICAL MATRIX :- $m > n$

Ex :- $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ -1 & 0 & 5 \\ 5 & 6 & 8 \end{bmatrix}_{4 \times 3}$

TYPES OF MATRICES

Properties of Trace :-

1) $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

2) $\mathrm{Tr}(A) = \mathrm{Tr}(A^T)$

3) $\mathrm{Tr}(\{A+B\}^T) = \mathrm{Tr}(A^T) + \mathrm{Tr}(B^T)$

PRODUCT BY A SCALAR

If a matrix is multiplied by scalar, then it is multiplied with each element.

$$A = [a_{ij}]$$

$$KA = [Ka_{ij}]$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -5 & 2 & 4 \end{bmatrix} \qquad 4A = \begin{bmatrix} 4 & 0 & 12 \\ -20 & 8 & 16 \end{bmatrix}$$

ADDITION & SUBTRACTION OF MATRICES

The order of two matrices should be same.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}_{2\times 2}$$

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$$

$$A + (-B)$$

ADDITION & SUBTRACTION OF MATRICES

Properties of Matrix Addition :-

1) Matrix addition is commutative.

$$A + B = B + A$$

2) Matrix addition is associative.

$$(A + B) + C = A + (B + C)$$

3) Cancellation law for matrix addition :-

$$\cancel{A} + B = \cancel{A} + C$$ holds only if $B = C$.

TYPES OF MATRICES

MULTIPLICATION OF MATRICES

The product AB of any two matrices is possible if & only if matrices are conformable.

Matrices are conformable :- $A_{m \times n} \times B_{n \times p}$

No. of columns (A) = No. of rows (B)

Ex :- $\underset{(A)}{\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 5 \end{bmatrix}_{2\times3}} \underset{(B)}{\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 4 \end{bmatrix}_{3\times2}} = \begin{bmatrix} 1+0+9 & 0+0+12 \\ -1-2+15 & 0+4+20 \end{bmatrix}$

$= \begin{bmatrix} 10 & 12 \\ 12 & 24 \end{bmatrix}_{2\times2}$

Total multiplications $= 2 \times 3 \times 2 = 12$

Total additions $= 2 \times (3-1) \times 2 = 8$

MULTIPLICATION OF MATRICES

- Order of $(AB) = m \times p$
- No. of multiplications to obtain each element in AB $= (n)$
- No. of additions to obtain each element in AB $= (n-1)$
- Total no. of multiplications $= mnp$
- Total no. of additions $= m(n-1)p$

MULTIPLICATION OF MATRICES

Properties of Matrix Multiplication :-

1) Matrix multiplication may or may not be commutative.

$$AB \neq BA$$

Ex :- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3\times 3}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}_{3\times 2}$. Will BA exist ?

Matrices multiplication do not commute.

2) Matrix multiplication is associative.

$$(AB)C = A(BC)$$

MULTIPLICATION OF MATRICES

3) Matrix multiplication is distributive w.r.t. addition.

$$A(B+C) = AB + AC$$

" " is not distributive w.r.t. multiplication.

$$A(B \times C) \neq AB \times AC$$

4) Positive integral power of square matrix :-

$$A^m \cdot A^n = A^{m+n} \qquad A \rightarrow \text{square matrix.}$$

5) Zero divisor :- $AB = 0$

- If either $A = 0$ or $B = 0$; then $AB = 0$
- If both $A \neq 0$ & $B \neq 0$; then $AB = 0$ is possible

MULTIPLICATION OF MATRICES

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A \neq 0$ $\quad B \neq 0$ $\quad AB = 0$

6) Multiplication with Identity matrix :-

$A \rightarrow$ Sq. matrix of order n.

$$\boxed{A I_n = I_n A = A}$$

MULTIPLICATION OF MATRICES

Minimum number of multiplications & additions :-

$$A_{3\times 2} \qquad B_{2\times 5} \qquad C_{5\times 3}$$

I) $(AB)C \rightarrow (AB)_{3\times 5}\, C_{5\times 3} \rightarrow (ABC)_{3\times 3}$

→ No. of multiplications $\underbrace{(3\times 2\times 5)}_{AB} + \underbrace{(3\times 5\times 3)}_{(AB)C} = 75$

→ No. of additions $\underbrace{\{3\times(2-1)\times 5\}}_{AB} + \underbrace{\{3\times(5-1)\times 3\}}_{(AB)C} = 51$

II) $(BC)A \rightarrow (BC)_{2\times 3}\, A_{3\times 2} \rightarrow (BCA)_{2\times 2}$

No. of multiplication $2\times 5\times 3 + 2\times 3\times 2 = 42$ ✓

No. of additions ; $2\times(5-1)\times 3 + 2\times(3-1)\times 2 = 32$ ✓

MINORS $[M_{ij}]$

The determinant value of square matrix obtained from original matrix of any order by deleting its row & column.

Ex1:-

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & -1 \\ 7 & 8 & 1 & 2 \end{bmatrix}_{3\times 4}$$

Minors of A

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{vmatrix}_{3\times 3} \quad \begin{vmatrix} 5 & 6 \\ 8 & 1 \end{vmatrix}_{2\times 2} \quad \begin{vmatrix} 2 & 0 \\ 8 & 2 \end{vmatrix}_{2\times 2}$$

Ex2:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = 5 \quad M_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 10 \quad M_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} 0 & 0 \\ 2 & 5 \end{vmatrix} = 0 \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1 \quad M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2 \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

COFACTORS

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

$$3\times3 \rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$A_{11} = (-1)^{1+1} M_{11}$$

$$A_{12} = (-1)^{1+2} M_{12}$$

$$A_{13} = (-1)^{1+3} M_{13}$$

$$2\times2 \rightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

DETERMINANT

- only defined for square matrix.
- Determinant (Δ) = Product of any row/column elements * their Cofactors

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix} = +2\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 6\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$= 2(1) - 4(2) + 6(-1)$$

$$= 2 - 8 - 6 = -12$$

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

DETERMINANT & ITS PROPERTIES

TRANSPOSE OF MATRIX :- $A' = A^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Interchanging Rows $\rightleftarrows$ Columns

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

1) $|A| = |A^T|$

2) If we interchange any two rows & columns then the sign of determinant will change.

DETERMINANT & ITS PROPERTIES

$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & -1 & 4 \end{vmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{vmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ -1 & 1 & 4 \end{vmatrix}$$

$\Delta = 6$ $\quad\quad$ $\Delta = -6$ $\quad\quad$ $\Delta = 6$

NOTE :- Δ of U.T.M. / L.T.M. / Diagonal / Scalar matrix

Δ = Product of diagonal elements

3. If any two rows or columns are identical (or proportional) ; then the value of $\Delta = 0$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 5 & 6 \end{vmatrix} \quad\quad\quad \begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{vmatrix}$$

Identical $\quad\quad\quad$ Proportional

DETERMINANT & ITS PROPERTIES

4. Multiplying a determinant by scalar K; then it means multiplying K in any row or column.

$$\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} \xrightarrow{\times 5} 5\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 10 \\ 3 & 8 \end{vmatrix}$$

$\Delta = 2$

$$\Delta = 5 \times 2 = 10 \Leftrightarrow \Delta = 10$$

$$= \begin{vmatrix} 1 & 2 \\ 15 & 40 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 15 & 8 \end{vmatrix}$$

5. Elements of a row/column can be expressed as a sum of two or more elements then the given Δ can be expressed as a sum 2 or more determinants.

DETERMINANT & ITS PROPERTIES

$$\begin{vmatrix} a+b & c+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & c \\ e & f \end{vmatrix} + \begin{vmatrix} b & d \\ e & f \end{vmatrix}$$

$$\begin{vmatrix} a+b+c & d+e+f \\ g & h \end{vmatrix} = \begin{vmatrix} a+b & d+e \\ g & h \end{vmatrix} + \begin{vmatrix} c & f \\ g & h \end{vmatrix}$$

$$= \begin{vmatrix} a & d \\ g & h \end{vmatrix} + \begin{vmatrix} b & e \\ g & h \end{vmatrix} + \begin{vmatrix} c & f \\ g & h \end{vmatrix}$$

6. $\boxed{\sum \text{Any row/column elements} \times \text{Corresponding cofactors} = \Delta}$

DETERMINANT & ITS PROPERTIES

7. $$\boxed{\sum \text{Any row/column elements} \times \text{Non-corresponding cofactors} = 0}$$

Ex:-
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \Delta$
- $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$
- $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

*** 8. Operations like $\left\{ \begin{matrix} R_i \rightarrow R_i + KR_j \\ C_i \rightarrow C_i + KC_j \end{matrix} \right\}$, then the value of determinant will not change.

$K \rightarrow +, -, 0,$ fraction

DETERMINANT & ITS PROPERTIES

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 2 & 1 \end{vmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\Delta \qquad\qquad \Delta$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & -4 & -8 \end{vmatrix} \xrightarrow{R_3 \to R_3 + 4R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -32 \end{vmatrix}$$

$$\Delta = -32$$

- $R_1 \to R_1 + 5R_2$
  $\Delta \qquad \Delta$
- $C_2 \to C_2 - 2.5C_3$
  $\Delta \qquad \Delta$
- $R_1 \to 5R_1 + 6R_2$
  $\Delta \qquad 5\Delta$
- $C_2 \to 3C_2 - 7C_1$
  $\Delta \qquad 3\Delta$

$$\Delta = 7$$

$$\begin{vmatrix} 5 & 0 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 5 \end{vmatrix} \xrightarrow[R_3 \to R_3 - \frac{2R_1}{5}]{R_2 \to R_2 - \frac{3R_1}{5}} \begin{vmatrix} 5 & 0 & 1 \\ 0 & 0 & 7/5 \\ 0 & 1 & 23/5 \end{vmatrix} \longrightarrow \begin{vmatrix} 5 & 0 & 1 \\ 0 & 1 & 23/5 \\ 0 & 0 & 7/5 \end{vmatrix}$$

$$\Delta = -7$$

DETERMINANT & ITS PROPERTIES

9. $$|ABCD| = |A||B||C||D|$$

$$|A^n| = |A|.|A|\ldots n = |A|^n$$

Ex:-

$$\begin{vmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 2 & -3 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$\boxed{\Delta = -15}$ $\quad \Delta = 15 \quad \Delta = -15$

ADJOINT OF MATRIX :-

$$\boxed{A.(Adj.A) = (Adj\,A)A = |A|I_n}$$

$$Adj\,A = (\text{Cofactor matrix})^T$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad Adj\,A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}}_{adj.\,A} = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

INVERSE OF A MATRIX :-

$$\boxed{A^{-1} = \frac{(\text{adj}.A)}{|A|}}$$

$$A.(\text{Adj}\,A) = |A|\,I_n$$

$$A^{-1}A(\text{Adj}\,A) = A^{-1}|A|\,I_n$$

$$I(\text{Adj}\,A)$$

$$\text{Adj}\,A = |A|\,A^{-1}$$

$$A^{-1} = \frac{\text{Adj}\,A}{|A|}$$

Properties of inverse & Adjoint :-

1. $(AB)^{-1} = B^{-1}A^{-1}$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

2. $A(\operatorname{Adj} A) = |A| I_n$

3. $\operatorname{Adj}.(\operatorname{Adj} A) = |A|^{n-2} A$

$n \rightarrow$ Order of sq. matrix.

4. $|A^{-1}| = 1/|A|$

5. $|\operatorname{Adj}. A| = |A|^{n-1}$

6. $|\operatorname{Adj}.(\operatorname{Adj} A)| = |A|^{(n-1)^2}$

7. $|\operatorname{Adj}(\operatorname{Adj}.(\operatorname{Adj} A))| = |A|^{(n-1)^3}$

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}^T$$

i) Adj A

$$\boxed{\text{Adj}.A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}}$$

ii) $A^{-1} = (\text{Adj } A)/|A|$

$$\boxed{A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}}$$

iii) $|A^{-1}| = 1/|A| = -\frac{1}{2}$

iv) $|\text{Adj} A| = |A|^{n-1} = (-2)^{3-1} = 4$

v) $|\text{Adj}.(\text{Adj}A)| = |A|^{(n-1)^2}$

$= (-2)^{(3-1)^2}$

$= (-2)^4 = 16$

$$|A| = -1 + 0 + (-1)$$
$$= -2$$

vi) $|A^2| = |A|^2 = 4$

vii) $|A^3| = |A|^3 = -8$

Inverse of 2X2 matrix :-

Ex :-

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Find A^{-1}

$$A^{-1} = \frac{Adj\ A}{|A|}$$

$$= \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

TRICK :-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex:- Find A^{-1}

$$A = \begin{bmatrix} 5 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\rightarrow \quad A^{-1} = \frac{1}{-5-2} \begin{bmatrix} -1 & -2 \\ -1 & 5 \end{bmatrix}$$

CONJUGATE OF A MATRIX :-

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1.25 \end{bmatrix}$$

Real matrix

$$\bar{A} = \begin{bmatrix} 2 & 3 \\ 0 & 1.25 \end{bmatrix}$$

$$A = \bar{A}$$

$$A = \begin{bmatrix} 1+i & 5i \\ 1-i & 2-i \end{bmatrix}$$

Complex matrix

$$\bar{A} = \begin{bmatrix} 1-i & -5i \\ 1+i & 2+i \end{bmatrix}$$

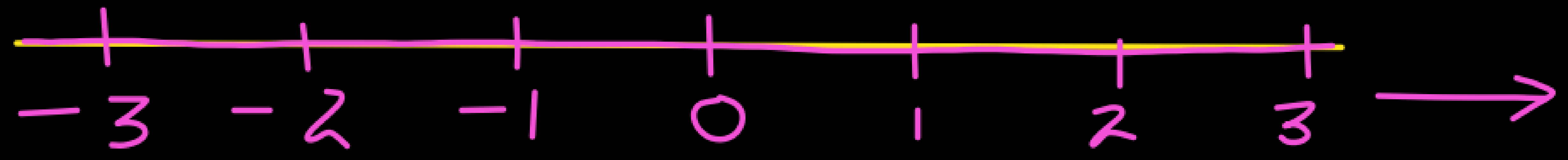
$$A \neq \bar{A}$$

$N \rightarrow 1, 2, 3, 4 \ldots\ldots\ldots$

$W \rightarrow 0, 1, 2, 3 \ldots\ldots\ldots$

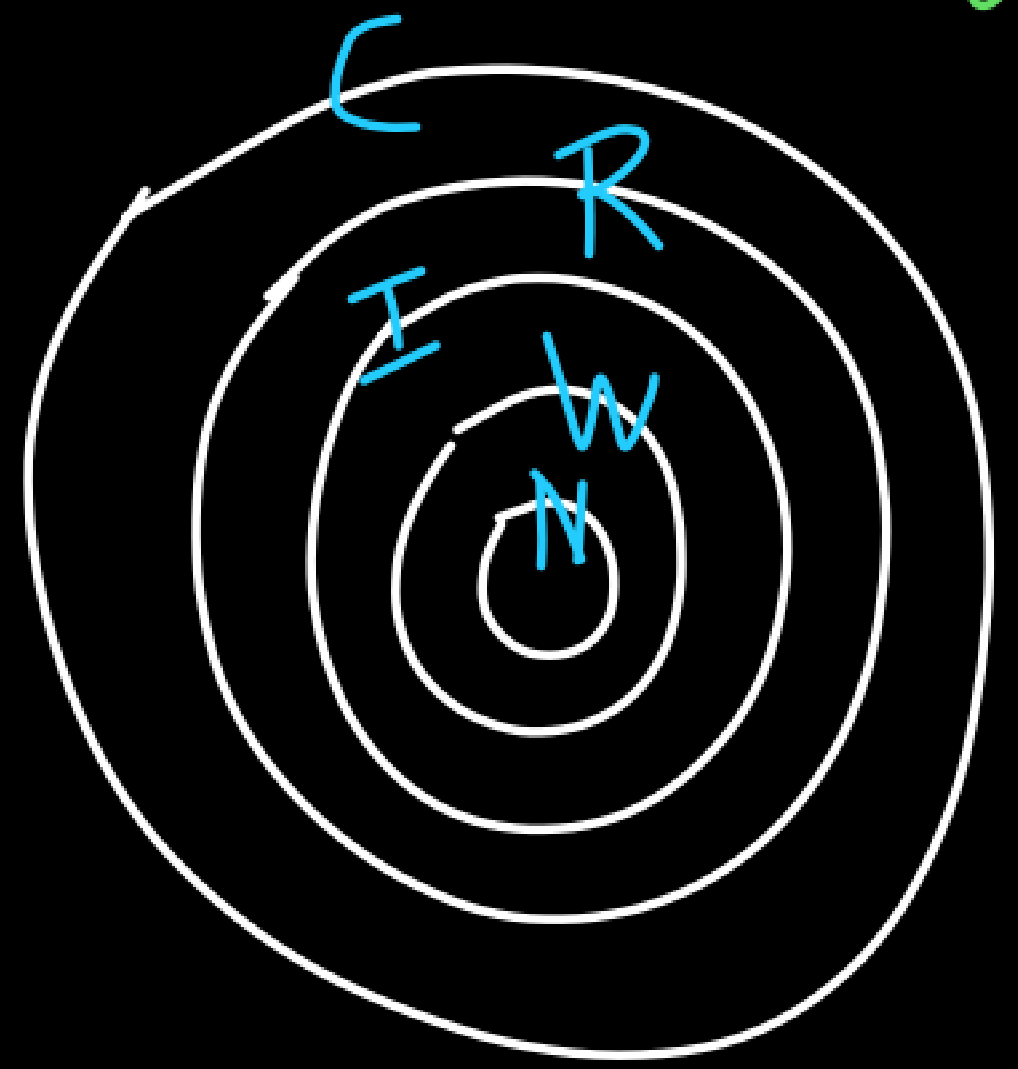
Z or $I \rightarrow \ldots\ldots -3, -2, -1, 0, 1, 2, 3, 4 \ldots$

$R \rightarrow$



Complex → Real + Imaginary

↳ Root of −ve no.



SPECIAL TYPES OF MATRICES

CONJUGATE TRANSPOSE OF MATRIX :-

Transjugate $= A^{*} = A^{\theta} = (\bar{A})^{T} = \overline{(A^{T})}$

Transpose   Conjugate

- $(A+B)^{*} = A^{*} + B^{*}$
- $(AB)^{*} = B^{*}A^{*}$

- $(A^{*})^{*} = A$
- If A is real matrix, $A^{*} = A^{T}$

Ex :- $A = \begin{bmatrix} 5i & 1-i \\ 1+i & 2+3i \end{bmatrix}$

Find $A^{*} = \begin{bmatrix} -5i & 1+i \\ 1-i & 2-3i \end{bmatrix}^{T} = \begin{bmatrix} -5i & 1-i \\ 1+i & 2-3i \end{bmatrix}$

SPECIAL TYPES OF MATRICES

1) IDEMPOTENT MATRIX :-

- A sq. matrix is idempotent if $\boxed{A^2 = A}$
- If A is idempotent; then $(I-A)$ is idempotent matrix.
- If A & B are idempotent matrix; then $A+B$ is idempotent when $AB = BA = O$

Ex :- $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

2) INVOLUTARY MATRIX :-

- A sq. matrix is involutary ; if $\boxed{\begin{array}{l} A = A^{-1} \\ A^2 = I \end{array}}$

Ex :- $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

SPECIAL TYPES OF MATRICES

3) NIL POTENT MATRIX :-

$A^K = 0$; A is nilpotent matrix of order K.

Ex:- $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A is nilpotent matrix of order 2.

SPECIAL TYPES OF MATRICES

Square Matrices

Real Matrices

→ Symmetric matrix

$$A = A^T$$

→ Skew-symmetric matrix

$$A = -A^T$$

→ Orthogonal matrix

- $AA^T = I$
- $|A| = \pm 1$

Complex Matrices

→ Hermitian matrix

$$A = A^* = (\bar{A})^T$$

→ Skew-Hermitian matrix

$$A = -A^*$$

→ Unitary matrix

- $AA^* = I$
- $|A| = 1$

SPECIAL TYPES OF MATRICES

A) SYMMETRIC MATRIX :- if $a_{ij} = a_{ji}$

- $A = A^T$

Ex :- $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$

→ Symmetric about leading diagonal.

A is sq. any matrix

→ $A + A^T$ is always symmetric

→ AA^T is always symmetric.

→ Max. unique elements in Symmetric matrix $= \boxed{\frac{n^2 - n}{2}} + \boxed{n} = \frac{n(n+1)}{2}$

Non-diag. Diag.

SPECIAL TYPES OF MATRICES

B) SKEW-SYMMETRIC MATRIX :– if $\{a_{ij} = -a_{ji}\}$

- $A^T = -A$

Ex :– $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

Properties :–

→ Sum of all elements = 0

→ Diagonal elements are always 0.

→ Δ of odd order skew-symmetric matrix = 0

→ $A - A^T$ is always skew-symmetric.

SPECIAL TYPES OF MATRICES

NOTE :- Every square matrix can be expressed as a sum of symmetric & skew-symmetric.

$$\boxed{A = \underbrace{\left(\frac{A + A^T}{2}\right)}_{\text{Sym.}} + \underbrace{\left(\frac{A - A^T}{2}\right)}_{\text{Skew-sym.}}}$$

Ex :-

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}}{2} + \frac{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}}{2}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

SPECIAL TYPES OF MATRICES

c) ORTHOGONAL MATRIX :-

consist of orthogonal row/col. unit vectors.

- $AA^T = I$ or $A^T = A^{-1}$

Ex:-

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} ; A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$$

If X_1 & X_2 are 2 vectors; then they are orthogonal vectors if :-

$$X_1 . X_2^T = 0$$

Properties :-

$\rightarrow R_1 . R_2^T = 0$ $\rightarrow C_1 . C_2^T = 0$

$\rightarrow R_2 R_3^T = 0$ $\rightarrow C_2 . C_3^T = 0$

$\rightarrow R_3 R_1^T = 0$ $\rightarrow C_3 . C_1^T = 0$

$\rightarrow |R_1| = |R_2| = |R_3| = |C_1| = |C_2| = |C_3| = 1$

SPECIAL TYPES OF MATRICES

→ Δ of orthogonal matrix is always ± 1.

$$|AA^T| = |I|$$

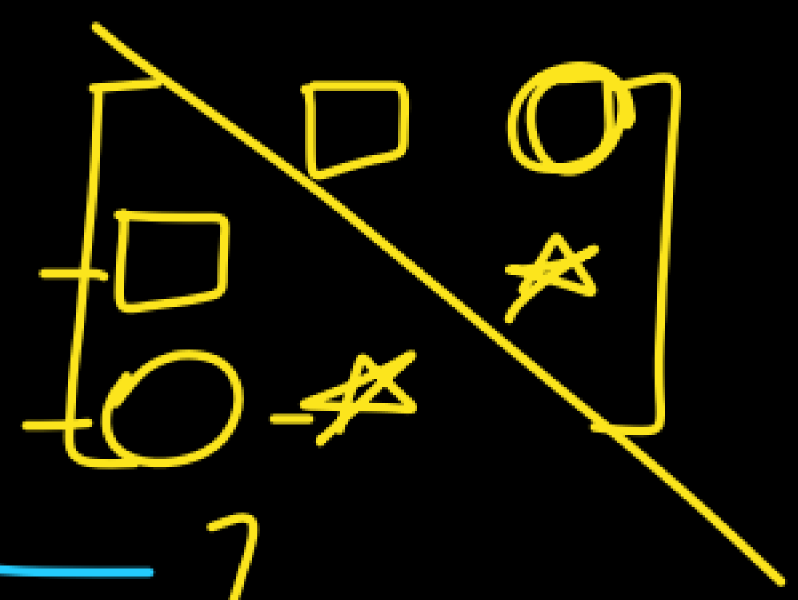
$$|A|\,|A^T| = 1$$

$$|A|\,|A| = 1$$

$$|A|^2 = 1 \Rightarrow |A| = \pm 1$$

→ If A & B are orthogonal; then AB & BA are also orthogonal.

SPECIAL TYPES OF MATRICES



Types of Complex Matrices :-

A) Hermitian Matrix :- if $\{a_{ij} = \overline{a_{ji}}\}$

- $A = A^*$

→ $(A + A^*)/2$ is always Herm.

→ Diagonal elements are always real.

Ex :- $\begin{bmatrix} 5 & 1+i \\ 1-i & 6 \end{bmatrix}$ $\begin{bmatrix} 5 & 1+i \\ 1-i & 6 \end{bmatrix}$

$A \quad = \quad A^*$

B) Skew - Hermitian Matrix :- if $\{a_{ij} = -\overline{a_{ji}}\}$

- $A = -A^*$

→ $(A - A^*)/2$ is always skew-herm.

→ Diagonal elements are always 0 or imagonary

Ex :- $\begin{bmatrix} i & 1+i \\ -1+i & 3i \end{bmatrix}$ $\begin{bmatrix} -i & -1-i \\ 1-i & -3i \end{bmatrix}$

$A \quad = \quad -A^*$

SPECIAL TYPES OF MATRICES

NOTE:- • Every sq. matrix can be expressed as sum of Hermitian & skew-hermitian matrix.

$$A = \boxed{\left(\frac{A+A^*}{2}\right)} + \boxed{\left(\frac{A-A^*}{2}\right)}$$

• Every matrix can be uniquely expressed as $P + iQ$, where P & Q are Hermitian

$$A = \boxed{\left(\frac{A+A^*}{2}\right)} + i\,\boxed{\left(\frac{A-A^*}{2i}\right)}$$

$$P \quad + \quad i \quad Q$$

SPECIAL TYPES OF MATRICES

c) Unitary matrix :-

- $A A^* = I$

Ex :-

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}}_{A} \underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}}_{A^*} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

NOTE:-

$$|A A^*| = |I|$$

$$|A|\,|A^*| = 1$$

$$|A|\,|\bar{A}'| = 1$$

$$|A|\,|\bar{A}| = 1$$

$$|A|\,|A| = 1$$

$$|A|^2 = 1 \Rightarrow |A| = 1.$$