Flipped Gumbel versus Gumbel Copula in pairs trading using the NASDAQ 100 and the Russell 1000 Technology Indices

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Abstract

In this paper, we implement a comparative study of pairs trading strategies on the NASDAQ 100 and the Russell 1000 Technology indices using a Gumbel and flipped (also termed survival) Gumbel copula. We divided 15 years (2008-2022) of data into two sections: 10 years of training (2008-2017) and 5 years of testing (2018-2022). We implemented the pairs trading strategy using the two copula approaches on the testing data. To grade the margins of the return data gathered from the two equity indices, we used the Friemer-Mudholkar-Kollia-Lin Generalized Lambda Distribution. A backtesting of the pairs strategy was executed on the testing data which showed that the flipped Gumbel copula outperforms the Gumbel copula. Our results showed that the flipped Gumbel copula approach for the pairs trading strategy traded with higher frequency and generated consistent alpha across all tested years while the Gumbel approach had a large negative alpha in the years 2019 and 2021. This implies that the indices exhibit greater correlations in their lower tails and are thus modeled better using the flipped Gumbel copula.

1 Introduction

We implement the Gumbel and the flipped (survival) Gumbel copula to produce a comparative study in performance between the two approaches in our pairs trading example. The Gumbel is an asymmetric Archimedean copula and is often used in modeling applications since it has the property of being exchangeable, meaning that it has the same structural form even if its dimensions are switched around. As shown in Figure 1, it introduces a greater correlation between two assets' returns at either end of the tails but more so on the positive, upper tails. We approach this assumption with some skepticism since concurrent movements in financial markets are often stronger in periods of downturn. The flipped Gumbel copula models this with greater sufficiency, as shown in Figure 1, with greater correlation in the negative, lower tails.

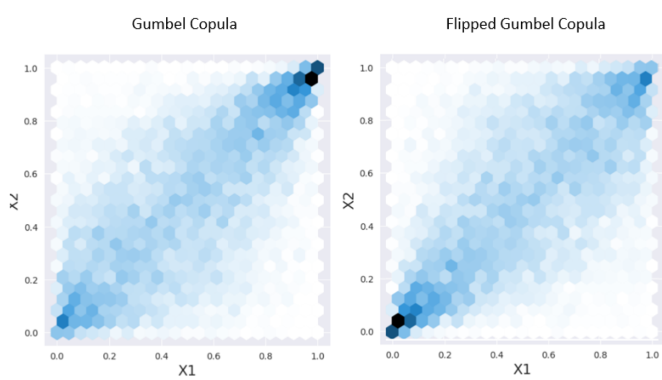

Figure 1:
Left: Example of Gumbel Copula.
Right: Example of Flipped Gumbel Copula.

We chose the Russell 1000 Technology Index and the NASDAQ 100 Index as our asset pair for our pairs trading example. Both are broad-market indices covering similar sectors, and thus exhibit correlated price movements. The Russell 1000 Tech Index, with its overall lower trading volume and large coverage of mid-cap equities, is a less liquid asset relative to the NASDAQ 100 Index with its coverage of higher market cap equities. This liquidity difference can create occasional lags in correlated movements, particularly during market stress, presenting an opportunity for pairs traders to profit from any divergence from the historical spread between the two.

We fit the Generalized Lambda Distribution (GLD) on the daily returns of both indices as the underlying distribution to our training dataset of 10 years (2008-2017). The GLD allows us to parameterize all four moments of the distribution - mean, variance, skew, and kurtosis; and thus offers great flexibility. All four parameters for our distribution are re-estimated based on volatility levels to better cope with heteroskedasticity in the returns. Returns are classified into low, medium, and high volatility regimes based on observed closing VIX prices on each day. Each regime within the training dataset thus has a unique distribution fitted, and CDF values for returns in our testing dataset (2018 - 2022) are calculated based on the corresponding volatility regimes observed on each date.

Using the copulas, we determine the conditional probability of one asset's daily return given the daily return of the other for both our assets. We identify extreme short-term divergence in the two assets if the conditional probabilities are observed outside the 90% confidence interval, and trigger a simultaneous long/short position in the pair. The position is closed once both assets return to their historical spread (with the respective probabilities crossing 0.5), and directly reversed if a new entry signal indicates an inverse long/short position. Net exposure in our positions is always zero, i.e. our size of a long position always corresponds to an equivalently sized short position. We are interested in capturing conditions under which either copulas may significantly under-perform, and thus do not implement any drawdown limits or other forms of stop loss in our strategy.

The rest of the paper follows a literature review of some pertinent existing literature that we have found useful during our research. In section 2, we discuss the data we used for our pairs-trading strategy. Section 3 explains the methods to construct our trading strategy. Section 4 contains a detailed interpretation and discussion of our results. We end the paper with some concluding remarks as well as references to the papers we used during our research.

1.1 Literature Review

It is consistently noted that the use of copulas primarily stems from the need to model financial data that is non-linear and does not sufficiently exhibit normality. Corlu, Metrelliyoz, and Tinic (2016) test such properties of financial data comprehensively across ten different equity index markets around the world. They note that returns from all those markets, unequivocally exhibit distributions that are fatter around the tails, have high kurtosis, and are rarely symmetric around the mean. The assumption of normality to model such data hence becomes futile. They then test alternative distributions for a better fit. Their analysis concludes that the Generalized Lambda Distribution offers the most robust modeling of returns across all markets in the study. The Generalized Lambda Distribution allows us to consider skewness and kurtosis as additional parameters along with location and scale.

Moving from the univariate paradigm to the multivariate space, the idea of non-linearity and tail-heavy correlations between assets are persistent observations in pairs trading strategies. Stander, Marais, and Botha (2012) propose a copula-based approach to capture bivariate dependence structures between two equities and probabilistically identify trading opportunities. They note that the particular flexibility of copulas is that they can model dependence despite different marginal distributions of each asset's returns in the pair. In their example, they explore the N14 copula which produces correlations at both extremes of the joint distribution of a pair, albeit higher in the maxima tails.

Liew and Wu (2013) implement and back-test a similarly constructed pairs trading strategy, but using Gumbel copula to model the correlation. The Gumbel copula, like the N14, captures correlation at both tails but more so on the maxima. However, it allows for various types of dependencies unlike the N14, which only allows for positive dependence. The simple implementation and closed-form properties of Archimedean copulas mean they come into relatively popular use as noted by Mahfoud and Massmann (2012). They further conclude that the Gumbel copula often provides the best fit among its Archimedean peers: Frank (offers symmetric correlation on both tails), and Clayton (offers correlation only on the lower tails).

Besides the Gumbel, the Student-t copula from the elliptical family of copulas is also a prevalent choice as observed in Davallou and Yazdi (2022), Rad et al (2015), and Krauss and Stübinger (2017). The Student-t however, offers a simplistic assumption of symmetric correlations between asset pairs at both tails.

For correlations between assets to be clustered greatly in their upper tails, or even equally clustered in either tail seems skeptical empirically. As Shehadeh, Sadam, Alwadi, and Mohammad (2022) notes, broad-market indices across the globe exhibit greater negative concurrent outliers in their returns compared to positive concurrent outliers. This effect should only amplify between highly correlated asset pairs.

In this paper, we address this gap in the existing literature by implementing a flipped Gumbel copula and conducting a comparative study with the Gumbel copula. The flipped Gumbel copula offers the same properties as any Archimedean copula, but models greater correlation in the lower tails of the pair's returns. If the asset pair in our example does exhibit greater lower tail dependency, then the flipped Gumbel will be an improved model that can generate better trading signals during periods characterized by outlier returns.

2 Data

To implement our pairs trading strategy, we searched for two equity indices with the intent of selecting one index that was less liquid than the other. The rationale behind this approach is that a less liquid market tends to be more volatile, providing more opportunities for statistical arbitrage against a relatively stable market. To identify an equity index that was not very liquid, we examined its market capitalization. Taking this characteristic into account, we selected several equity indices to compare to the S&P 500 and the NASDAQ 100 Index. To choose the best pair for our pairs trading strategy, we calculated the Kendall rank correlation coefficient for each index pair, with a target correlation coefficient greater than 0.7. The Kendall rank calculations led us to select the Russell 1000 Technology Index and the NASDAQ 100 Index for our pairs trading strategy, as they had the highest correlation coefficient of 0.81.

We found that the Russell 1000 Technology Index has a market capitalization of \$10.25 trillion, while the NASDAQ-100 Index has a market capitalization of \$14.33 trillion. To compare the indices, we analyzed the list of companies under each index and divided them into groups based on their market capitalization, as shown in Figure 2.

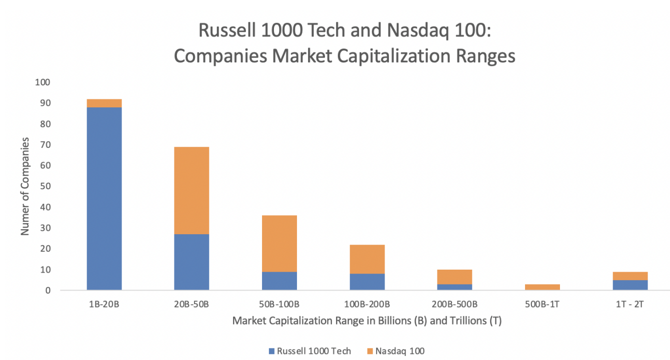

Figure 2: Market Capitalization Ranges for companies listed on the Russell 1000 Technology Index and NASDAQ 100 Index

As seen in Figure 2, the majority of companies listed on the Russell 1000 Technology Index have a market capitalization within the range of \$1 billion to \$20 billion, while the companies listed on the NASDAQ 100 Index are more spread across a market capitalization of $20 billion to $100 billion.

Further analysis revealed that the 140 companies listed on the Russell 1000 Technology Index have an average market capitalization of $84 billion, and the 101 companies listed on the NASDAQ 100 Index have an average market capitalization of $161 billion. Excluding the 34 companies that are in both indices, the average market capitalization of the Russell 1000 Technology Index and the NASDAQ 100 Index were $18 billion and $96 billion, respectively. This shows that the Russell 1000 Technology Index covers some significantly lower market cap equities relative to the NASDAQ 100 Index.

Our analysis shows that 78.6% of the average market capitalization of the Russell 1000 Technology Index comes from companies that are also listed on the NASDAQ 100 Index, which provides strong evidence of the two indices' high correlation.

We utilized data downloaded from the Bloomberg Terminal, specifically, 15 years of data for each index between 2008 to 2022 (Feb. 2008 - Feb. 2023), to calculate the percentage returns for each index for our research. The cumulative returns are shown in Figure 3.

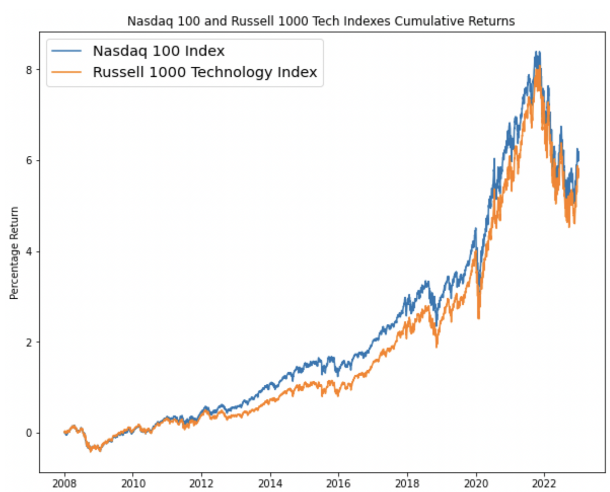

Figure 3: Percentage Return for Russell 1000 Technology and NASDAQ 100

3 Methodology

Our methodology can be summarised as follows:

1. Divide the dataset into training and testing subsets.
2. Fit a distribution to the training set returns data for both indices and generate CDFs for each return in the testing set.
3. Construct a copula to derive a joint relationship between the daily returns of both indices in the testing set and generate conditional probabilities.
4. Derive trading signals to long or short the spread based on the conditional probabilities implying a simultaneous overvalued/undervalued pair.
5. Back-test the trades executed on the testing set.

3.1 Dividing data into training and test sets

Our dataset consists of 15 years of returns from the NASDAQ 100 Index and the Russell 1000 Technology Index. We use the first 10 years (2008 - 2017) of the returns data for training, and the remaining 5 years (2018 - 2022) for testing.

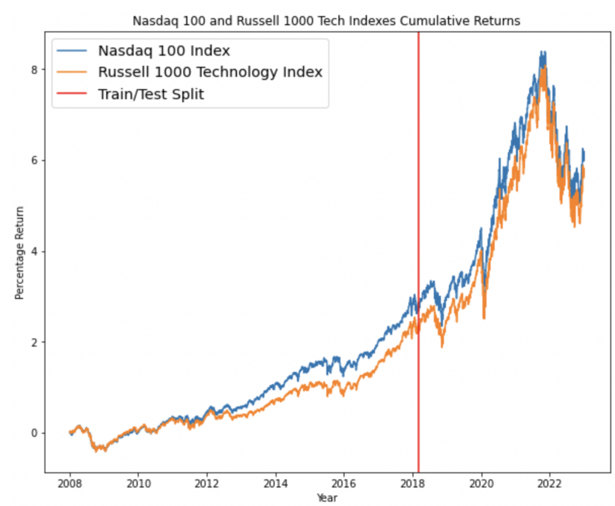

Figure 4: Split of returns data into training and testing sections.

3.2 Marginal distribution fitting

Corlu, Metrelliyoz, and Tinic (2016) explain how assumptions of normality are often inaccurate for financial data and demonstrate how the Generalized Lambda Distribution (GLD) provides the most robust fit for indices across the world. We have incorporated the Friemer-Mudholkar-Kollia-Lin Generalized Lambda Distribution (FMKL GLD) in fitting a distribution to our testing dataset. The parameterization of the GLD is as follows:

$$F^{-1}(u; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)$$

Where λ_1 is the location parameter, λ_2 is the scale parameter, and λ_3 and λ_4 are skewness and kurtosis, respectively. The GLD allows for the parametrization of all four moments of the distribution and thus offers great flexibility.

It is apparent from Figure 4 that daily returns for both indices do not exhibit constant standard deviation, i.e. they are heteroskedastic. Fitting a single parametric distribution across the entire span of our training period would require us to assume constant standard deviation. This assumption is erroneous. Within the existing literature, a common workaround is to not make any parametric assumptions by using an empirical distribution. However, using an empirical distribution does involve assumptions about the smoothness of the distribution to interpolate between the training values when generating distribution values for the testing set. Moreover, additional assumptions are made for extrapolating outside the training data range. These assumptions become increasingly inaccurate when the sample size is smaller.

In our approach, we incorporate our parametric GLD and accommodate for heteroskedasticity by dividing our training period into different volatility regimes. We then fit a unique GLD to returns from each regime. Since both the NASDAQ 100 and Russell 1000 Technology indices are broad market indices, we can use the CBOE Volatility Index (VIX) at each date to determine whether returns on that date exist in a low, medium, or high volatility regime.

Periods in which VIX levels were above the 75th percentile of values observed in the training set, are classified under a high volatility regime. Medium volatility regime correspond to VIX levels between the 25th and 75th percentile of values observed in the training set, whereas VIX values below the 25th percentile correspond to a low volatility regime.These levels and their corresponding regimes are shown in Figure 5.

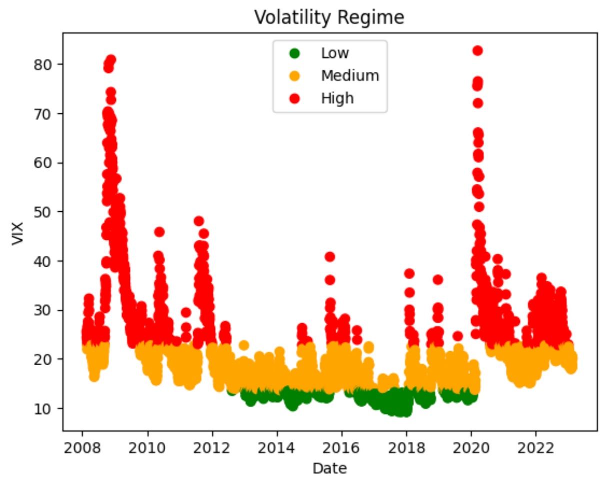

Figure 5: VIX Regime Model

The frequency of each regime's occurrence annually is shown in Figure 6, where we can see 2009, 2020, and 2022 are years with particularly high volatility.

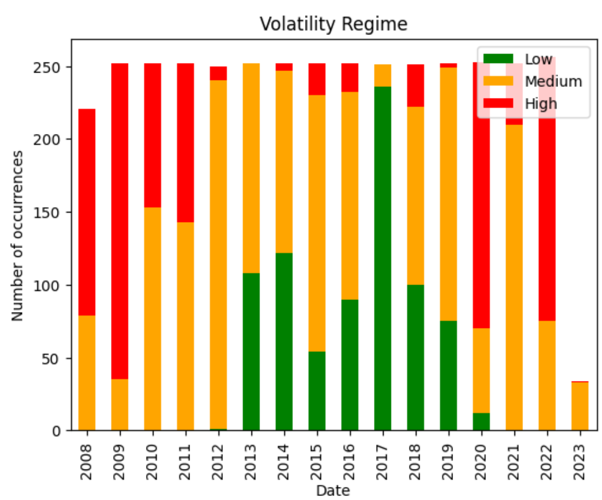

Figure 6: Occurrence of High, Medium, Low Regimes Grouped By Year

The GLD parameters are calculated separately and a unique distribution is fitted for each of the three regimes using our training dataset. We can then generate CDFs for daily returns in our testing dataset based on observed volatility levels on their respective dates. Our findings have been presented in detail (Table 1) and (Table 2) for the NASDAQ 100 Index and Russell 1000 Technology Index, respectively.

	Mean	Std	Skew	Kurtosis
Low	0.002165	0.005958	0.008853	1.561915
Medium	0.000848	0.009878	0.084762	0.696611
High	-0.001352	0.022531	0.246001	3.380982

Table 1: Parameters used to calculate the NDX CDF for each Regime.

	Mean	Std	Skew	Kurtosis
Low	0.002204	0.006645	-0.288609	2.225049
Medium	0.000694	0.010196	0.061139	0.555966
High	-0.001292	0.022595	0.282797	2.731529

Table 2: Parameters used to calculate the RGUSTL CDF for each Regime.

3.3 Constructing Copulas

We implement both the Gumbel and the flipped Gumbel copula and generate a comparison of their performance on our test dataset.

While the Gumbel copula introduces greater correlation in the upper tails of the assets' returns, the flipped Gumbel does so on lower tails.

The construction of the Gumbel copula is as follows:

$$C(u, v) = e^{-(-\ln(u)^{\theta} - \ln(v)^{\theta})^{\frac{1}{\theta}}}$$

Where u and v are the CDF generated in the previous step for the daily returns of the NASDAQ 100 and Russell 1000 Technology indices. Theta is a parameter derived from Kendall's Tau (ρ):

$$\theta = \frac{1}{1 - \rho}$$

The flipped Gumbel copula can then be constructed:

$$C_{\text{flipped}}(u, v) = u + v - 1 + C(u, v)$$

Both copulas should perform similarly if returns in the test set are mostly in their normal range, i.e. respective CDFs are consistently near the center and away from either tails. However, we can expect large discrepancies during periods that exhibit frequent outlier returns because that is where the two copulas fundamentally differ. For the majority of our testing period (2018-2022), it seems returns are frequently outliers and we expect significant discrepancies in performance between the two copula approaches.

3.4 Trading Strategy

Once we compute our copulas, we partially differentiate them and determine conditional probabilities of one asset's daily return given the daily return of the other in the pair (e.g. $P(u|v) = \frac{\partial C}{\partial v}$). These conditional probabilities form the basis of our entry and exit conditions.

The entry conditions are as follows:

$$P(u|v) > 0.95 \,\&\, P(v|u) < 0.05 \tag{1}$$

$$P(v|u) > 0.95 \,\&\, P(u|v) < 0.05 \tag{2}$$

Where u and v are CDFs of the two assets in the pair, respectively. (1) Refers to shorting the spread (short asset u and long asset v), whereas (2) refers to longing the spread (long asset u and short asset v).

Exit conditions can be defined when both assets revert to their historical relationship:

$$P(u|v) < 0.5 \,\&\, P(v|u) > 0.5 \tag{3}$$

$$P(v|u) < 0.5 \,\&\, P(u|v) > 0.5 \tag{4}$$

(3) Refers to the exit signal corresponding to a short position, whereas (4) refers to the exit signal to close a long position.

When examining our sequence of signals, we found that there were instances where signals jumped directly between a long and short before an exit condition could be achieved. In such instances, we simultaneously closed our long (or short) position and opened a new short (or long) position on the same day.

4 Results

Results from backtesting the pairs trading strategy in the training period between February 2018 and February 2023 show that the flipped Gumbel copula approach significantly outperforms the simple Gumbel approach. Figure 7 shows consistent positive alpha across all years for the flipped Gumbel approach, with particularly high returns in 2020 and 2022, which were also years with the highest volatility in the testing period as implied by Figure 6.

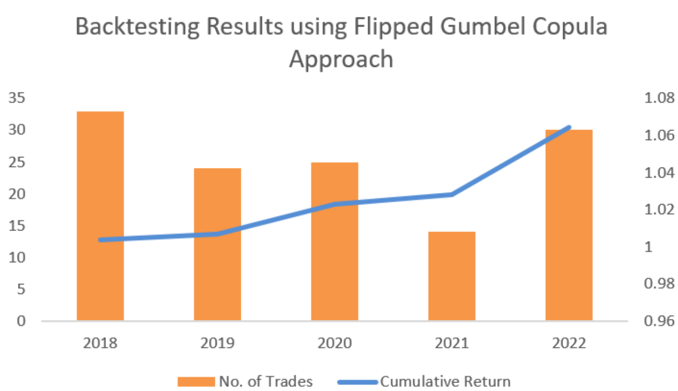

Figure 7: Cumulative Returns across the testing period and number of positions taken every year in the Flipped Gumbel

The simple Gumbel approach suffered greatly in 2021 and 2019, with negative alpha that was large enough to make the cumulative returns for the entire period negative. This is not surprising given the simple Gumbel's relative weakness in modeling left tail movements in the pair. As seen in Figure 4, both 2019 and 2021 were years of significant downside movements for both indices.

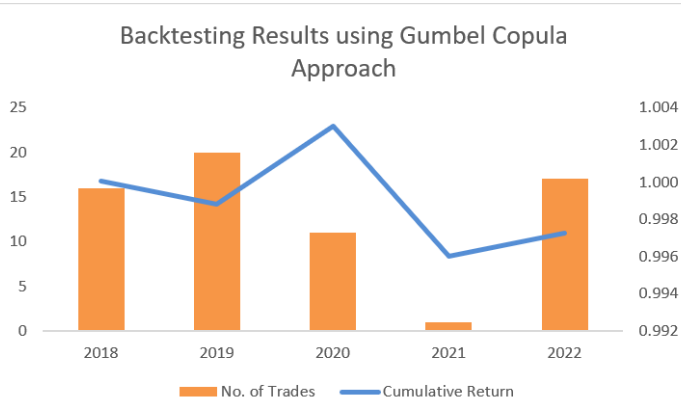

Figure 8: Cumulative Returns across the testing period and number of positions taken every year in the Gumbel

The performance of both models is examined in greater detail in Table 3 and Table 4. We see that across all years, the flipped Gumbel produces trades with higher frequency. We performed a simple volatility adjustment to mean returns for each year (Mean/Std), showing that the flipped Gumbel produces better trades across all years even when adjusted for risk. This implies that while the flipped Gumbel copula models downside co-movements well, it also sufficiently provides for upside co-movements.

Period	No. of Trades	Total Return	Mean Return	Std	Mean/Std
2018-2022	**126**	**6.44%**	**0.05%**	**0.002**	**0.220**
2018	33	0.35%	0.01%	0.002	0.061
2019	24	0.35%	0.01%	0.001	0.097
2020	25	1.58%	0.06%	0.002	0.333
2021	14	0.55%	0.04%	0.002	0.233
2022	30	3.62%	0.12%	0.004	0.334

Table 3: Performance of the Flipped Gumbel Copula

Period	No. of Trades	Total Return	Mean Return	Std	Mean/Std
2018-2022	**65**	**-0.273%**	**-0.004%**	**0.002**	**-0.0196**
2018	16	0.006%	0.001%	0.002	0.003
2019	20	-0.124%	-0.006%	0.001	-0.062
2020	11	0.421%	0.039%	0.003	0.140
2021	1	-0.701%	-0.701%	N/A	N/A
2022	17	0.125%	0.008%	0.002	0.035

Table 4: Performance of the Gumbel Copula

We can further analyze the returns profile from the approaches based on which areas of the copula or joint distribution they come from. Figure 9 shows such a profile for the simple Gumbel copula, where the size of the bubbles implies the size of the returns. We see the biggest returns generated in either tail, with a cluster of smaller returns in the tails too, and the rest of the returns scattered across the diagonal. Returns from the upper tail were the highest but few. This implies that the NASDAQ 100 and Russell 1000 Technology Index seldom exhibited upper tail dependence in our testing period, and thus opportunities to trade using the simple Gumbel copula were fewer. Considering average returns in the simple Gumbel implementation, it is likely that a vast majority of the rest of the bubbles corresponds to negative returns.

Figure 10 shows the returns profile for the flipped Gumbel approach. Unlike in the simple Gumbel case, we have almost all the returns concentrated in the bottom-left quadrant close to the lower tail where the copula models correlation the highest. The clustering of relatively

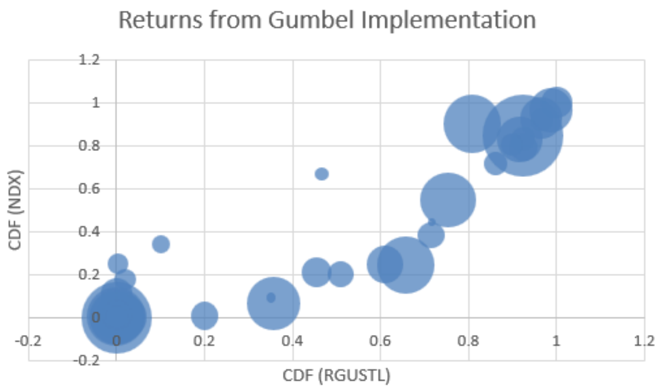

Figure 9: Returns from positions taken in the Gumbel Implementation, against respective CDFs of the indices. (Size of the bubble implies the size of the return)

large bubbles around the extreme lower tail show that frequent and profitable trade opportunities arose during large simultaneous downturns in the two assets. This suggests that high levels of lower tail dependence between the indices were observed during the testing period. The few large returns near the center of the copula are likely from periods that the simple Gumbel characterized as high upper tail dependence in its assumptions. Although fewer, the flipped Gumbel copula does generate profitable trade signals during those periods.

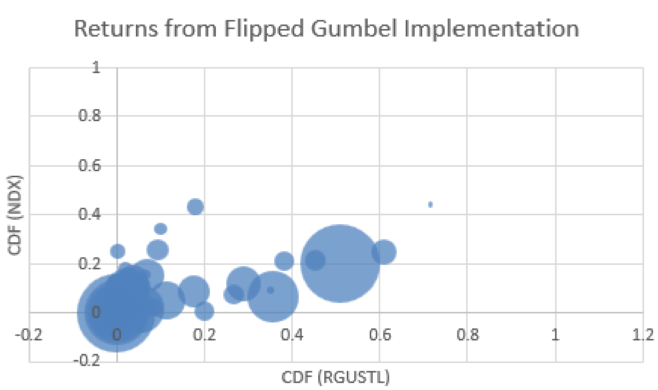

Figure 10: Returns from positions taken in the Gumbel Implementation, against respective CDFs of the indices (Size of the bubble implies the size of the return)

5 Conclusion

The flipped Gumbel copula's superior performance over the simple Gumbel copula implies, at least for the data we examined, that correlation tends to be higher among asset pairs in their left(lower) tails rather than their right(higher) tails. This observation is consistent with the tenets of behavioral finance where significant market downturns are often macroeconomic and simultaneous across a range of assets. For highly correlated assets, this effect is further amplified.

In addition to implementing a copula model that models both upper and lower tail dependence but with greater correlation in the lower tails, our research develops on existing literature by specifically testing performance in a period characterized by historically atypical market movements. This is especially critical to test from a risk management standpoint but is not addressed in most existing literature. The Gumbel copula performs well in several examples in existing literature because most of the time correlated assets exhibit simultaneous positive drift. Furthermore, it is often tested against copulas that also offer questionable assumptions by modeling similar tail dependences, symmetric tail dependences, or containing almost all correlations in only either of the tails.

As a future direction, we could implement the GARCH model to further account for heteroskedasticity in our dataset. However, since the NASDAQ 100 and Russell 1000 Technology indices are broad-market indices, we were able to use the VIX to identify volatility regimes and fit different GLDs accordingly. Therefore, the use of VIX may be a simpler and more practical approach in our case.

Another idea to explore would be running a dynamic pairs trade on a basket of indices. Based on the volatility regime, the model would execute pairs to trade and extract the most alpha. This would look at the statistical relationship between sets of pairs that work well in high volatility versus low volatility.

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