Minkowski space

when physics made hyperbolic geometry popular

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HEGL Proseminar/Seminar: Visualizations in hyperbolic geometry

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Outline

"Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

—Hermann Minkowski—

Bilinear form

Let V be a vector space over the field $\mathbb{K}$.
Remember what a bilinear form is:

- $b : V \times V \to \mathbb{K}$
- can be symmetric
- $v, u \in V$ are called (b-)**orthogonal** if $b(v, u) := \langle v, u \rangle = 0$.
- **(semi)positive definite** if associated qudratic form takes (semi)positive Values on $V \setminus \{0\}$

Quadratic form

Definition

The associated **qudratic form** $q : V \to \mathbb{K}$ to b is defined by $q(v) = b(v, v)$.

Example

For $x, y \in V = \mathbb{R}^2$ $b(x, y) = x_1 y_2 - x_2 y_1$ is an bilinear form. So the associated quadratic form is $q(x) = x_1 x_2 - x_2 x_1 = 0$ for all $x \in V$.

Theorem (Sylvester's law of inertia)

Let (V, b) be a real vector space with dimenson n, equiped with an symmetric bilinear form.

The dimension of any maximal positive [resp. negative] subspace is the same, it's called **positive index** *[resp.* **negative index***].*
The pair (p,q) is the **signature** *of b.*

Euclidian vector space

- finite-dimensional real vector space
- the negativ index is zero
- positive definite symmetric bilinear form called **inner product** $b = \langle \cdot, \cdot \rangle$
- norm $\|v\| = \sqrt{\langle v, v \rangle}$ provides proper distance

pseudo-Euclidian vector space

- finite-dimensional real vector space
- has mixed signature
- nondegenerate symmetric bilinear form called **pseudo inner product** $b = \langle \cdot, \cdot \rangle$
- pseudo norm $\|v\| = \sqrt{|\langle v, v \rangle|}$ doesn't provide a proper distance

Example

Let $V = \mathbb{R}^n$ and choose $p, q \in \mathbb{N}_0$ so that $p + q = n$. We consider

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \ldots + x_p y_p - x_{p+1} y_{p+1} - \ldots - x_{p+q} y_{p+q}$$

This is a symmetric bilinear form with signature (p, q) and also nondegenerate. The space $(V, \langle\cdot,\cdot\rangle)$ is the **canonical pseudo-Euclidian space of sigature** (p, q), denoted $\mathbb{R}^{p,q}$.

Definition of Minkowski space

Definition (mathematical definition)

A pseudo-Eucildian space with negative index $q = 1$ is called a **Minkowski space**. The canonical Minkowski space of dimension $n + 1$ is $\mathbb{R}^{n,1}$.

A nonzero vector $v \in V \setminus \{0\}$ is called

- **spacelike** if $\langle v, v \rangle > 0$
- **ligthlike** if $\langle v, v \rangle = 0$
- **timelike** if $\langle v, v \rangle < 0$

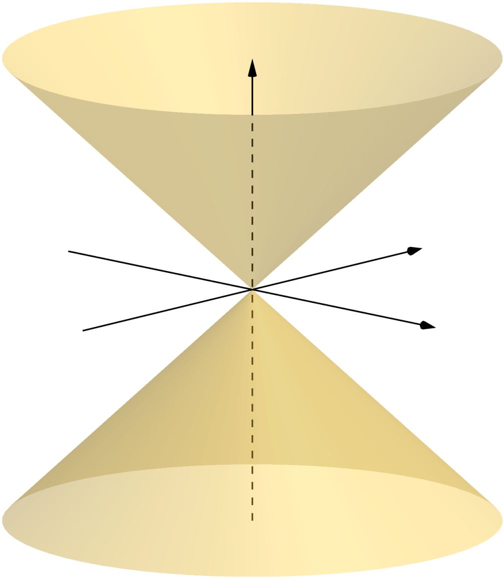
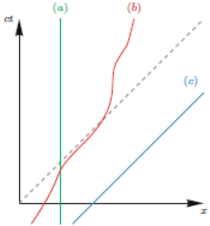


Figure: The lightcone in $\mathbb{R}^{2,1}$.

Definition (physical definition)

The Minkowski space or **Minkowski spacetime** is the fourdimensional vectorspace, which has one time and three spatial coordinates. A point in Minkowski spacetime $x^\mu = (ct, x, y, z)^T$ is called event. An object moving through spacetime usually is given in a Minkowski diagramm by it's so called world line.

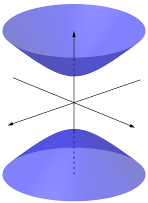


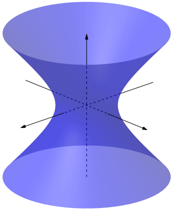
Pseudo euclidian spheres

Definition

For $m \in \mathbb{R}$ the subset $S_m = \{v \in V \mid \langle v, v \rangle = m\}$ is called **(pseudo-) sphere** and m is it's square radius.

Figure: Spheres in $\mathbb{R}^{2,1}$:two-sheeted hyperboloid ($m = -1$), one-sheeted hyperboloid ($m = 1$)





How at all do we measure an angle?

Definition

Let u and v be two vectors in a pseudo-Euclidean vector space V contained in a vector plane $P \subset V$. The angle between u and v is the arc length between the matching points on the unit circle in P, if $\langle u, u \rangle$, $\langle v, v \rangle$ and $\langle u, v \rangle$ are all positive or all negative. So we get:

$$\measuredangle(u, v) = \ell(\gamma) = \int_\gamma ds$$

We essentially defined (one-dimensional) angles in spherical, hyperbolic, and Euclidean geometries

Theorem

Let $u, v \in V$ and $P \subset V$ a vector plane containing them. Assume that u and v are both spacelike or both timelike and let $\epsilon = \pm 1$ indicate their common sign.
If P has mixed signature, then reversed Cauchy–Schwarz holds: $|\langle u, v\rangle| \geq \|u\| \, \|v\|$.
The angle $\measuredangle(u, v)$ is well-defined when the sign of $\langle u, v\rangle$ is ϵ, in which case it is the so called **hyperbolic angle** *and equals the unique real number $\theta \in [0, +\infty)$ such that:*

$$\langle u, v\rangle = \epsilon \|u\| \|v\| cosh(\theta)$$

Linear isometries

Definition

Let V be a pseudo-Euclidean vector space of signature (p, q). The **orthogonal group** of V is the group of linear automorphisms of V that preserve the inner product:

$$O(V) := \{f \in GL(V) \mid \forall u, v \in V \ \langle f(u), f(v) \rangle = \langle u, v \rangle\}$$

For the Matrix representation we find:

$$O(V) := O(p, q) = \{M \in M(n, \mathbb{R}) \mid M^T I_{p,q} M = I_{p,q}\}$$

For $q = 1$ it is called **Lorentz group**.

The Lorentz Group consists of:

- a time inversion (T)
- a space inversion (P)
- a combination of the two above (PT)
- spatial rotations
- hyperbolic rotations (Lorentz boosts)

The last two together form a Lie Algebra and contain the identity.

Example (Lorentz boost in z direction)

This represents a transformation into a refrence frame which moves at a constant speed v into direction z.

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & -\beta\gamma \\ & & -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

With $\beta = tanh(\psi)$ we find why it is called a hyperbolic rotation.

Conclusion

Sources

- Loustau - hyperbolic geometry, (not yet published)
- Bartelmann, Feuerbach et al. - Theoretische Physik, 2015 Springer
- Lee - Geometry: from Isometries to Special Relativity, 2020 Springer