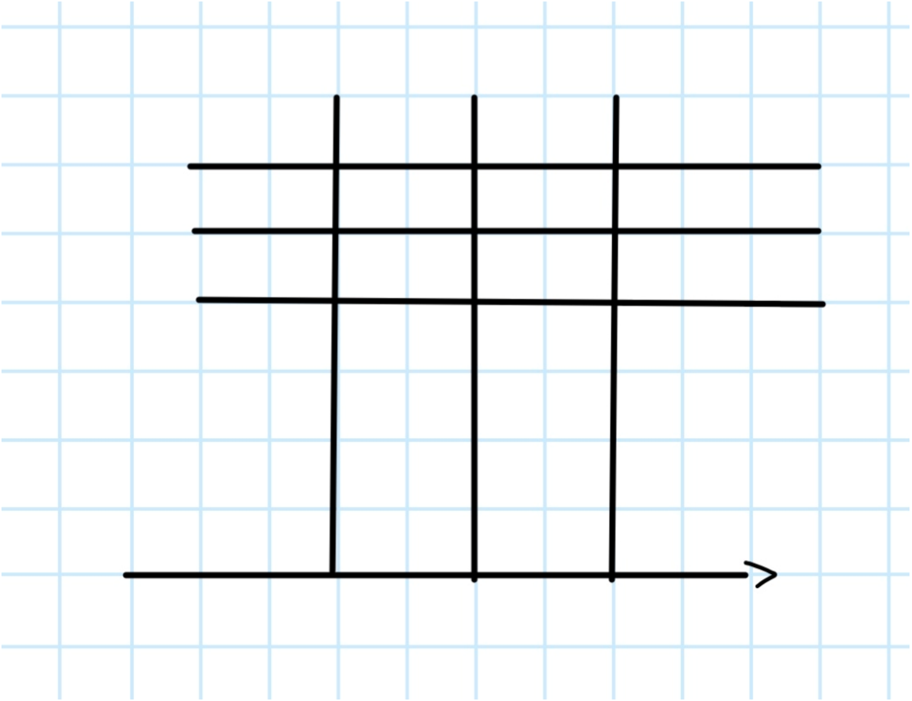
The Poincaré half plane (part II)

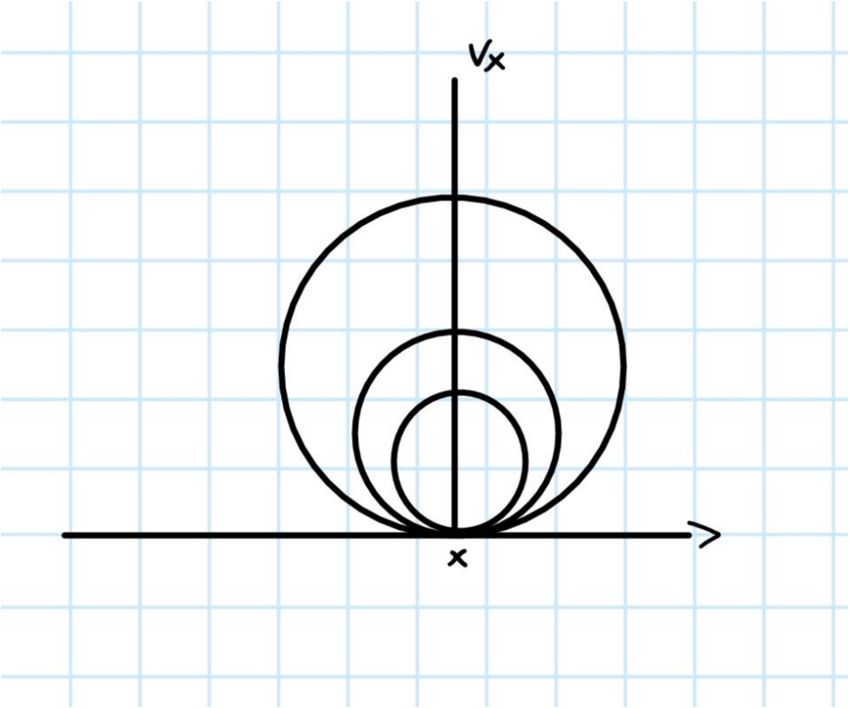
MÖBIUS TRANSFORMATIONS, ISOMETRIES,

HYPERBOLIC TRIANGLES AND HOROCYCLES

Horocycles



Horocycles based at ∞



Horocycles based at $x \in \mathbb{R}$

Möbius Transformations

Definition:

A Möbius map or linear fractional is a map $f\colon\ \mathbb{C}\ \cup\infty \rightarrow \mathbb{C}\ \cup\infty$ given by the formula $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d\ \epsilon\ \mathbb{R}$ and $ad - bc \neq 0$.

Theorem 1.27. *The set* $\mathrm{Aut}(\mathbb{H})$ *of analytic bijections* $\mathbb{H} \longrightarrow \mathbb{H}$ *is the subgroup of* $\mathrm{Aut}(\hat{\mathbb{C}})$ *of Möbius maps of the form:*

$$f(z) = \frac{az+b}{cz+d} \quad \textit{with} \quad a,b,c,d \in \mathbb{R}, ad-bc > 0. \tag{1.3}$$

Normalising, this shows that $\mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2,\mathbb{R})/\pm I = \mathrm{PSL}(2,\mathbb{R})$.

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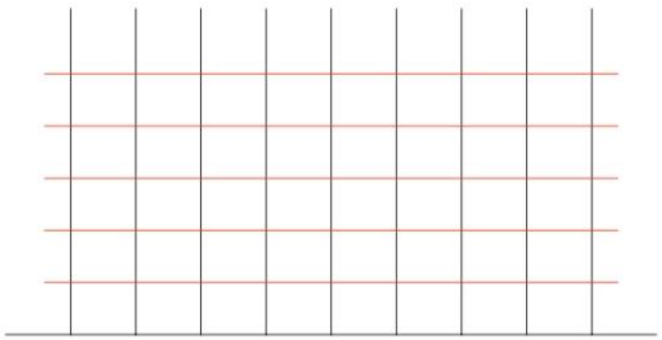
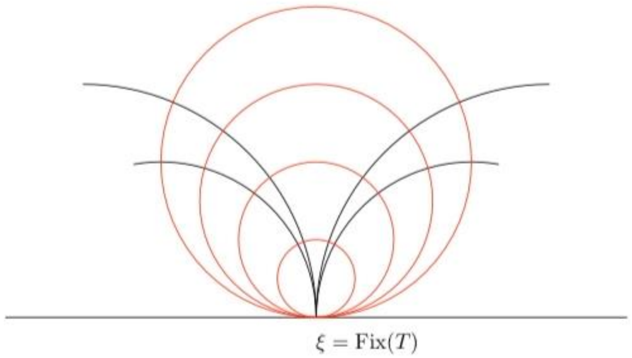
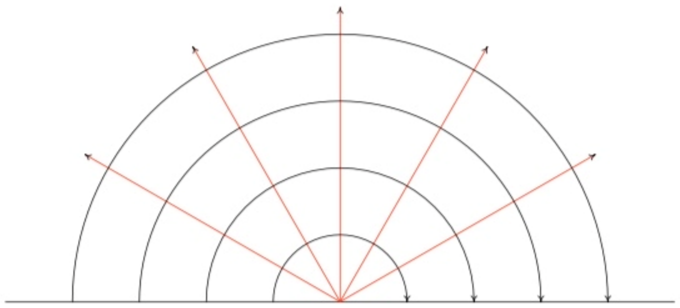
Proposition 2.5. *The groups* $\mathrm{Aut}(\mathbb{H})$ *and* $\mathrm{Aut}(\mathbb{D})$ *act by isometries on* $\mathbb{H}, \mathbb{D}$ *respectively.*

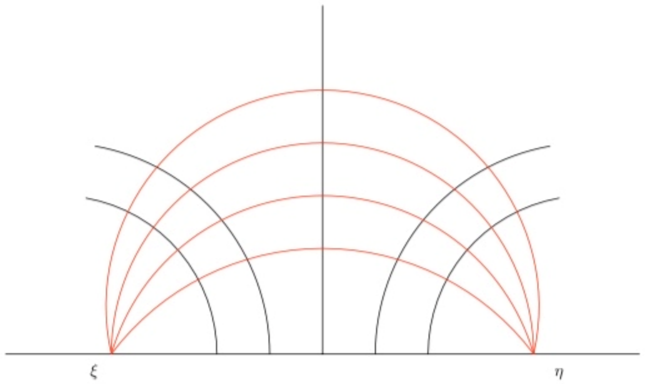
Parabolic transformation



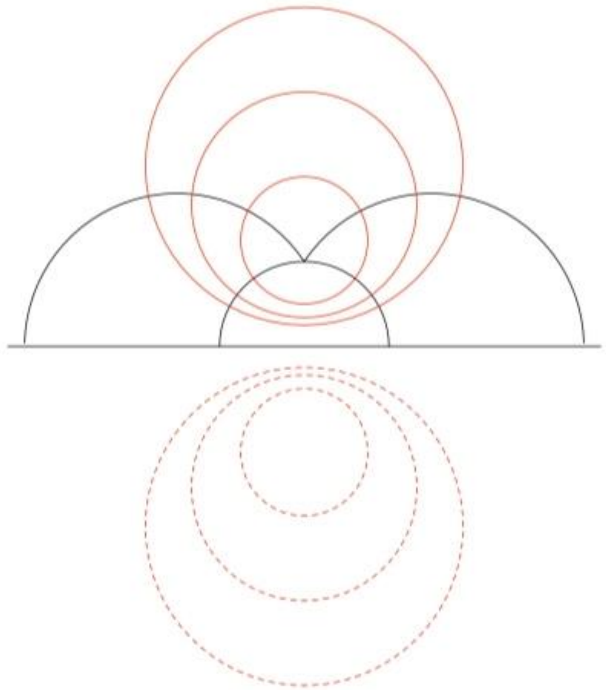


Hyperbolic transformation





Elliptic transformation

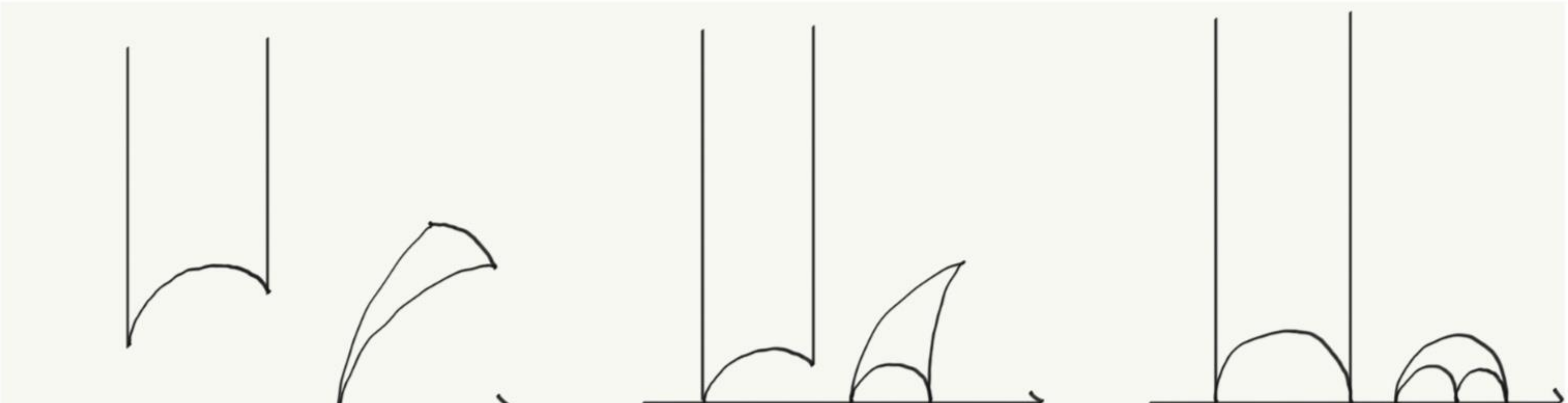


Hyperbolic triangles

Definition 11.1.3

By allowing vertices to be on $\partial\mathbb{H}$, we get *triangles with ideal points*. In particular, a triangle with three ideal points is called an *ideal triangle*.

When the vertex is idea, as a convention, we say that the interior angles is 0 at this vertex.



Hyperbolic trigonometry

Hyperbolic sine: $\sinh(x) = \frac{e^x - e^{-x}}{2}$

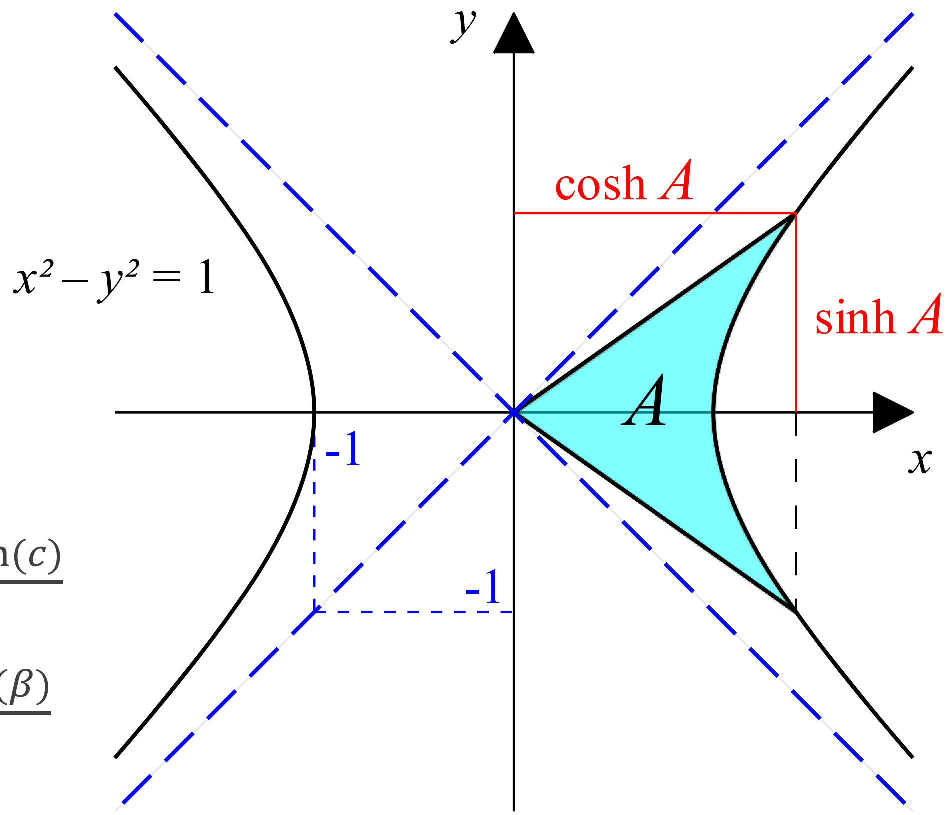
Hyperbolic cosine: $\cosh(x) = \frac{e^x + e^{-x}}{2}$

Hyperbolic tangens: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{2x} - 1}{e^{2x} + 1}$

Hyperbolic sin law: $\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}$

First hyperbolic cosine law: $\cos(\gamma) = \frac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(\mathrm{a})\sinh(b)}$

Second hyperbolic cosine law: $\cos(c) = \frac{\cos(\gamma) + \cos(\alpha)\cos(\beta)}{\sin(\alpha)\sin(\beta)}$



Congurent triangles

Definition 2.30. *Two triangles ABC and $A'B'C'$ are* **congruent** *if there exists a $T \in \mathrm{Isom}(\mathbb{H})$ such that $T(A) = A', T(B) = B'$ and $T(C) = C'$. (Notice that in this definition, T may or may not preserve orientation.)*

Reference list

- Binbin Xu (2021); Introduction to hyperbolic geometry
- Bonahon F. (2009); Low-Dimensional Geometry
- Series C. (2010); Hyperbolic Geometry
- Chang A.; (2010); Isometries of the hyperbolic plane

Example Pictures:

- Series C. (2010); Hyperbolic Geometry
- https://de.wikipedia.org/wiki/Sinus_hyperbolicus_und_Kosinus_hyperbolicus