The Poincaré Disk $\mathbb{D}$

HEGL Seminar

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Important Definitions

Definition 1 (Set of the Poincaré disk)**.**

$$\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$$

Definition 2 (Line element)**.** Let $z \in \mathbb{C}$ with $z := x + iy$ then the line element is defined as

$$ds^2_{\mathbb{D}} := \frac{(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}$$

Definition 3 (Length of a curve)**.** Let $\gamma : [a, b] \to \mathbb{D}$ be a curve with $\gamma(t) = x(t) + iy(t) \in \mathbb{D}$

$$\int_a^b \frac{2}{1 - x^2 - y^2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition 4 (Distance)**.** Given two points $z_0, z_1 \in \mathbb{D}$ we can calculate the distance $dist_{\mathbb{D}}(z_0, z_1)$ using one of these fomulas[1]:

$$dist_{\mathbb{D}}(z_0, z_1) = arcosh\left(+\frac{2|z_0 - z_1|^2}{(1 - |z_0|^2)(1 - |z_1|^2)}\right) \tag{1}$$

$$dist_{\mathbb{D}}(z_0, z_1) = \frac{1}{2}|log(u : v : z_0 : z_1)| \tag{2}$$

We consider the unique circle orthogonal to the unit disk $\mathbb{D}$ that contains z_0 and z_1. This circle meets the unit circle in two points u and v.

Definition 5 (Cayley transformation)**.** The transformation

$$C : \mathbb{H} \to \mathbb{D}$$
$$z \mapsto \frac{z - i}{z + i}$$

with it's inverse

$$C^{-1} : \mathbb{D} \to \mathbb{H}$$
$$w \mapsto i\left(\frac{1 + w}{1 - w}\right)$$

is called Cayley Transformation and it is an isometry between the two models of the hyperbolic plane $\mathbb{H}$ and $\mathbb{D}$.

[1] We are using the cross ratio $[x : y : z : t] = \frac{z-x}{z-y} \cdot \frac{t-y}{t-x}$

Geodesics

In general the geodesics of the Poincaré disk are the intersections of $\mathbb{D}$ with circles in $\hat{\mathbb{R}}^2$ that are orthogonal to $\partial\mathbb{D}$.

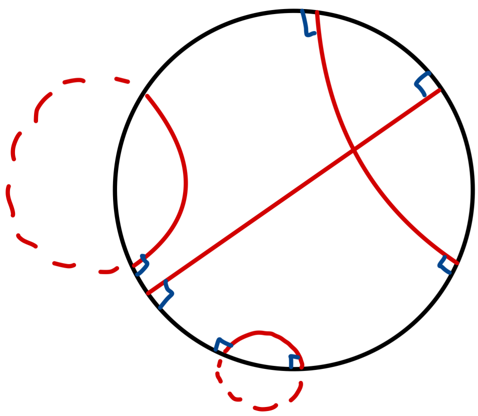

Figure 1: Geodesics in $\mathbb{D}$

Triangles

A triangle on $\mathbb{D}$ consist of three geodesics meeting each other in three different points. An Ideal triangle consist of three geodesics meeting in three points on $\partial\mathbb{D}$ and hence has all angles equal to zero.

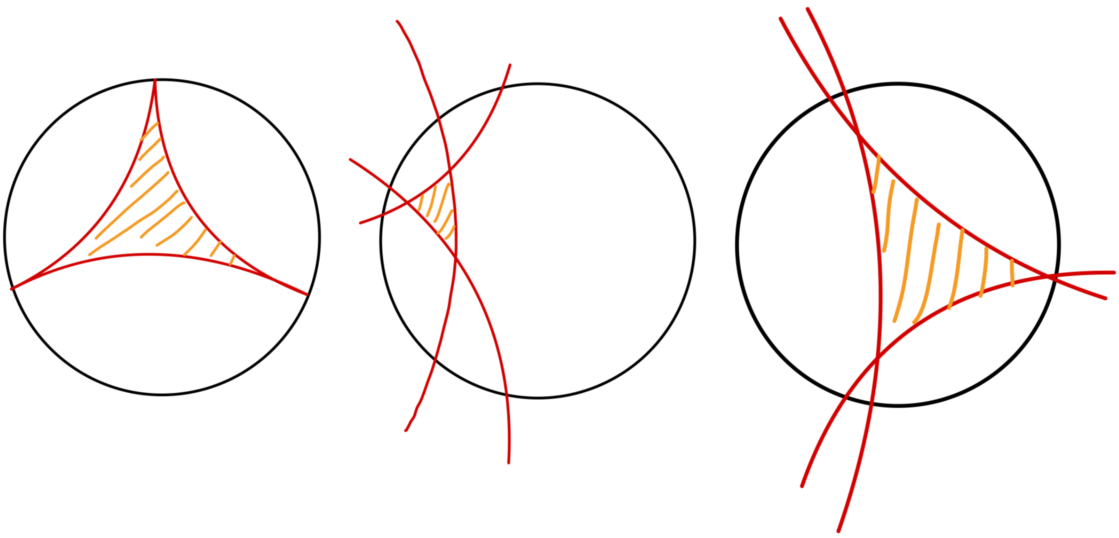

Figure 2: different triangles in $\mathbb{D}$

Angles

Angles on $\mathbb{D}$ are the euclidean angles between the tangents of two geodesics in the point they meet.

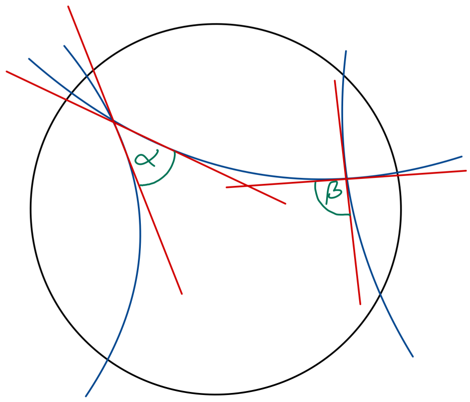

Figure 3: Different angles between the tangents of two geodesics

Horocycles

In the Pioncaré disk model horocycles are represented by circles tangent to $\mathbb{D}$, the centre of the horocycle is the ideal point where the horocycle touches $\partial\mathbb{D}$. Since the ideal point isn't inside $\mathbb{D}$ it's not part of the horocycle. In general a Euclidean Circle that is inside the disk and touches $\partial\mathbb{D}$ is a horocycle.

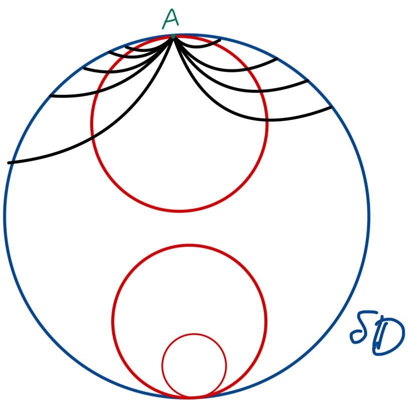

Figure 4: Example of Horocycles and an ideal point A