Quasicrystals and the cut and project method

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Crystals

- periodic structure
- tiling: set of geometric shapes that fill a plane
- simplest crystals are constructed using regular polygons

Crystals examples 1

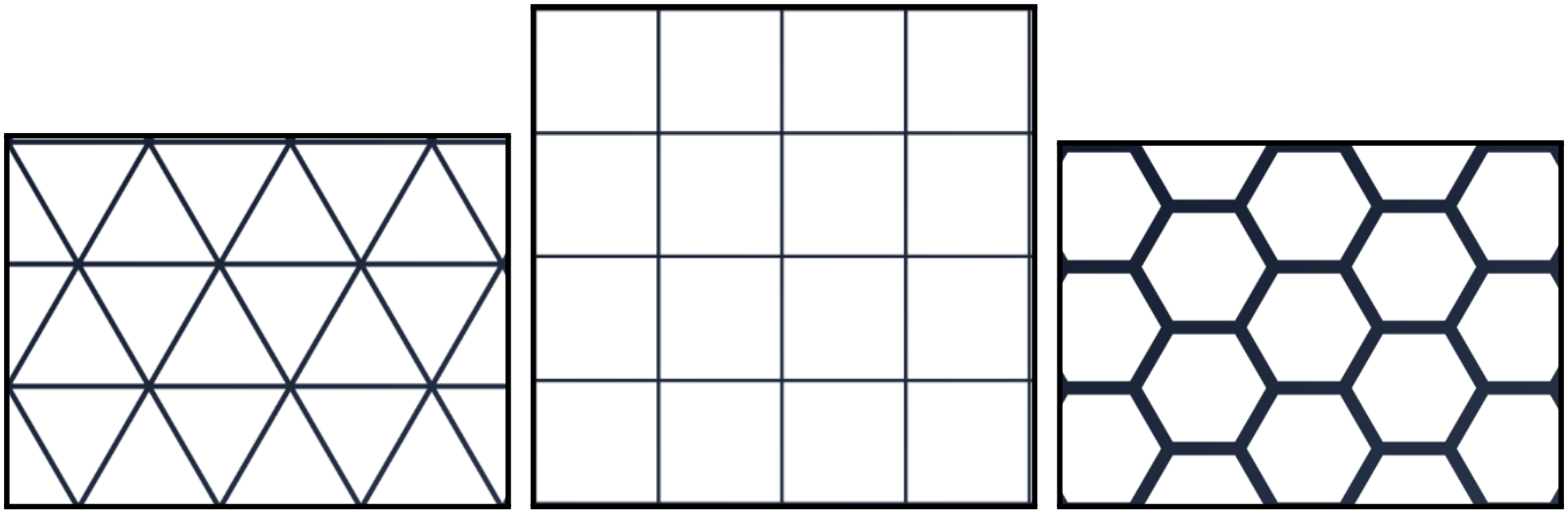


Figure: from regular polygons only triangles, squares and hexagons make a crystal

Crystals examples 2

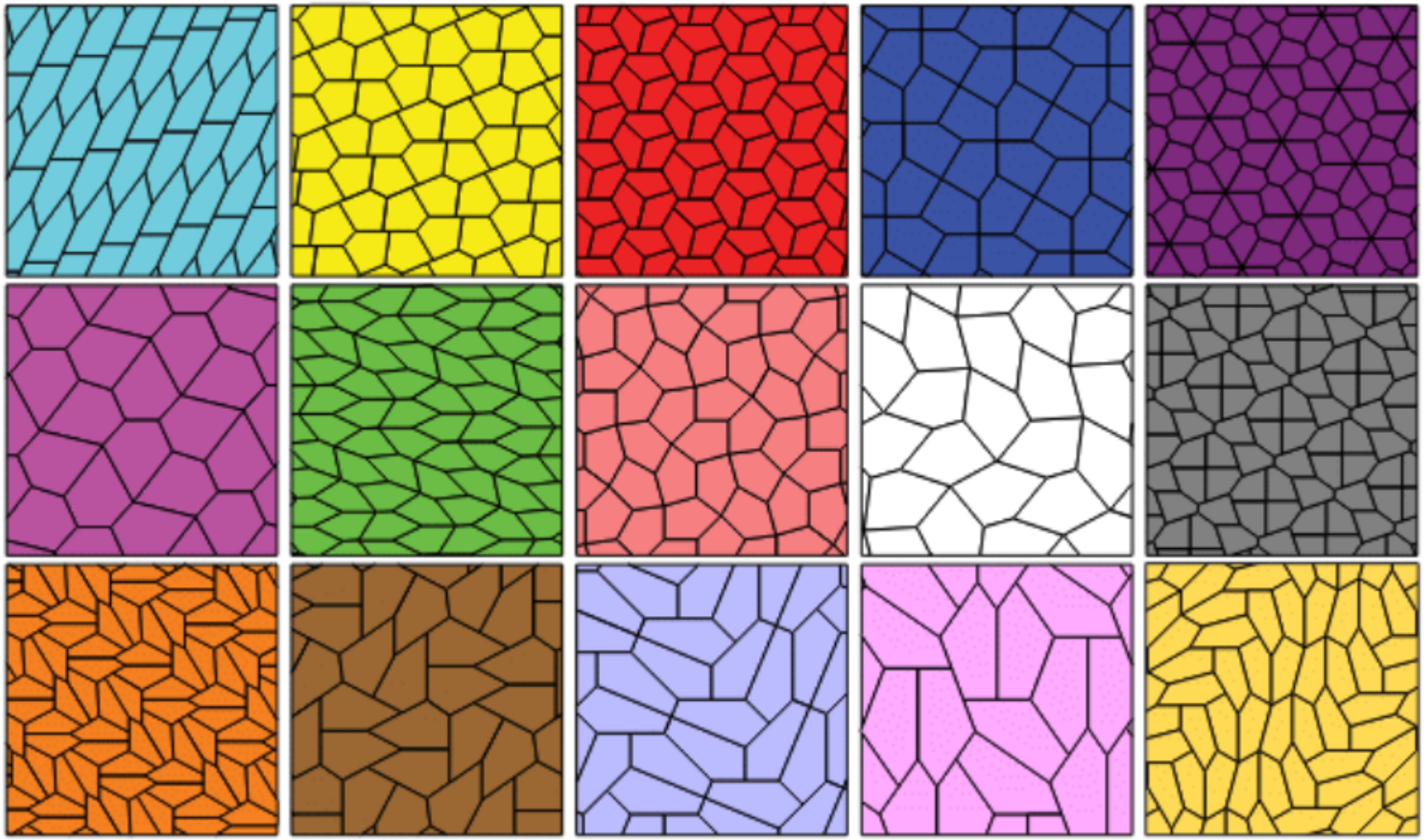


Figure: periodic tilings made of unregular pentagons. Source [2] http://community.wolfram.com/groups/-/m/t/550169

Quasicrystals

- quasi-periodic structure
- also appear in nature
- examples are fibonacci & penrose tilings (coming up)

Cut and project method

- most versatile method to generate quasicrystals
- general working:
  1. starting with an $n \geq 2$ dimensional lattice $\Lambda \in \mathbb{R}^n$
  2. take a affin-linear subspace E of dimension $m < n$ and **cut** $\mathbb{R}^n$
  3. **project** all points of Λ onto $E^{\perp}$ and check if they fall into a certain cut window
  4. take the accepted points and project them onto E

...see example in next slide

Example 1: Fibonacci Tiling

- start with 2D grid (periodic tiling)
- E is a slope with angle Θ
- all points in the cutwindow are projected orthogonal down
- we get a 1D Quasicrystal if Θ is irrational:
- If we choose $\Theta = \tan^{-1}(\frac{1}{\tau})$ we obtain the Fibonacci tiling (τ is the *golden ratio*)

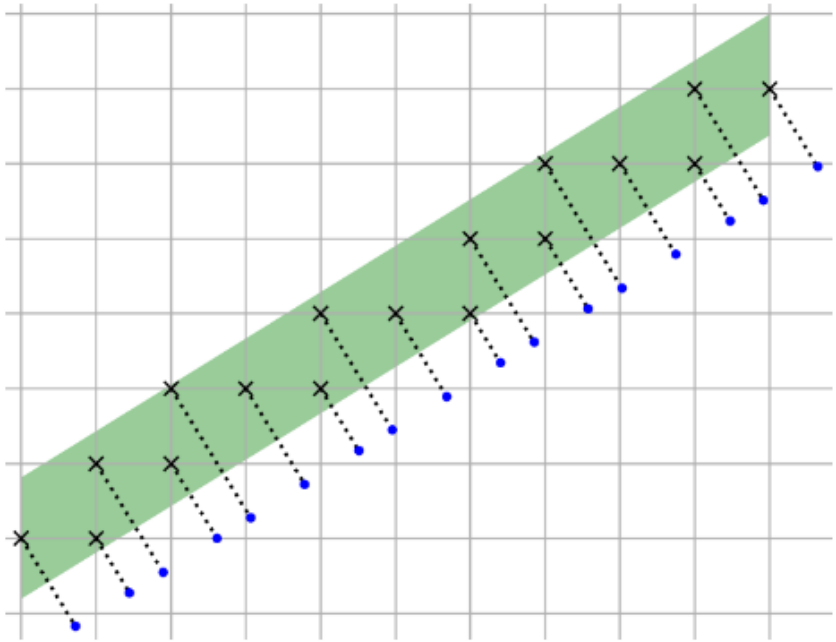
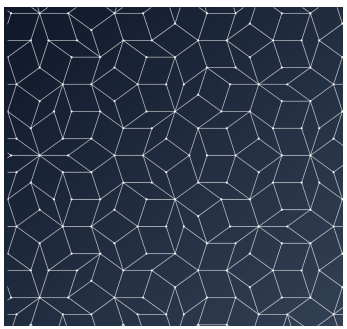


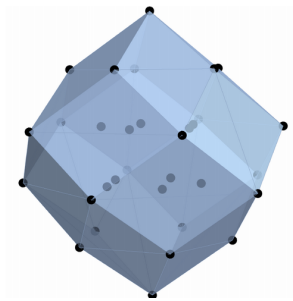
Figure: fibonacci cut&project with $\Theta = \tan^{-1}(\frac{1}{\tau})$ and $\Delta = \sin(\Theta) + \cos(\Theta)$

Example 2: Penrose Tiling

- most famous Quasicrystal
- produced by choosing $\Lambda = \mathbb{Z}^5$ and E as a certain 2 dimensional subspace of $\mathbb{R}^5$



figureA figure



figureAnother figure

Our work progress

- Done:
  - implemented Fibonacci tiling and Penrose tiling
  - website with interactive tools and informative text
- Next steps:
  - add more interactive tools
  - optimize penrose generation using digital geometry techniques
  - implement further tilings such as Wang tilings

Sources

Content and Figures from

- [1] 2021_Daniel_Gouldsbrough_BSc_Thesis.pdf
- [2] http://community.wolfram.com/groups/-/m/t/550169