David's problem: squared root rule

Model 1: Assume that there are K investors/people, and at time 0 their coins/population are $n_{1,0}, \dots, n_{K,0}$ so that $\sum_{k=1}^K n_{k,0} = N_0 := N$, and their shares are $\pi_{k,0} := n_{k,0}/N$. Similar notations $n_{k,t}$ and $\pi_{k,t}$ for coins/population and shares at time t. For simplicity, we will consider the case that K is fixed and $N \to \infty$ so $\pi_{k,0}$'s are of constant order (i.e. large investors/population).

Here is the squared root rule: at time t, the investor/population k is selected with probability $n_{k,t}/N_t^2$, and will be granted a coin/voting power. This way, $N_{t+1} = N_t + 1$ so $N_t = N + t$, and for each $1 \le k \le K$,

$$n_{k,t+1} = \begin{cases} n_{k,t} & \text{with probability} \quad 1 - \frac{n_{k,t}}{N_t^2}, \\ n_{k,t} + 1 & \text{with probability} \quad \frac{n_{k,t}}{N_t^2}. \end{cases} \tag{1}$$

We have then for $1 \le k \le K$,

$$\mathbb{E}(n_{k,t+1} \,|\, n_{k,t}) = n_{k,t}\left(1 + \frac{1}{(N+t)^2}\right), \tag{2}$$

which yields

$$\mathbb{E} n_{k,t} = n_{k,0} \prod_{j=N}^{N+t-1} \left(1 + \frac{1}{j^2}\right). \tag{3}$$

Let $f(n) := \prod_{j=1}^n (1 + j^{-2})$ so that $\mathbb{E} n_{k,t} = n_{k,0} f(N+t-1)/f(N)$. It is well known that

$$f(\infty) := \prod_{j=1}^{\infty} \left(1 + \frac{1}{j^2}\right) = \frac{\sinh \pi}{\pi}, \tag{4}$$

so

$$\mathbb{E} n_{k,\infty} = n_{k,0} \frac{\sinh \pi}{\pi} \frac{1}{f(N)}. \tag{5}$$

It is also easily seen that $f(N) = f(\infty)\left(1 - \frac{1}{N} + o(\frac{1}{N})\right)$. As a result,

$$\mathbb{E} n_{k,\infty} = n_{k,0} \left(1 + \frac{1}{N} + o(\frac{1}{N})\right). \tag{6}$$

There are two implications:

1. $\mathbb{E}\left(\sum_{k=1}^K n_{k,\infty}\right) = N + 1 + o(1) \ll N + \infty$ (here ∞ refers to $t \to \infty$). That is, $\sum_{k=1}^K n_{k,\infty}/N_\infty \to 0$ (the majority of rewards are not handed out).
2. If we only look at the rewards which are handed out, i.e. the small proposition $\sum_{k=1}^K n_{k,t}/N_t \to 0$, formula (6) yields the "relative" stability, i.e. $n_{k,\infty}/n_{k,0} \approx 1$.

Model 2: Now we assume that $N_t = \sum_{k=0}^K n_{k,t}$ and the voting-selection dynamics is the same as (1). Now a first important task is to understand the asymptotic behavior of N_t as $t \to \infty$. This is important as $(N_t,\, t \ge 0)$ represents the economic volume (e.g.

Ethereum has a more reasonable volume while Dogecoin is way too crazy). It is easily seen that

$$N_{t+1} = \begin{cases} N_t & \text{with probability } \; 1 - \frac{1}{N_t}, \\ N_t + 1 & \text{with probability } \; \frac{1}{N_t}. \end{cases} \tag{7}$$

So $(N_t,\, t \ge 0)$ is a nondecreasing time-homogeneous (state-dependent) Markov chain on $\{N, N+1, \ldots\}$ The study of the asymptotic behavior of such a Markov chain is generally referred to as the *Lamperti problem.*

Disgression on the Lamperti problem: In early 60s, John Lamperti considered the problem of recurrence/transition of a general (time-homogenous) Markov chain $(X_n,\, n \ge 0)$ based on

$$m_1(x) = \mathbb{E}(X_{n+1} - X_n \,|\, X_n = x) \quad \text{and} \quad m_2(x) = \mathbb{E}((X_{n+1} - X_n)^2 \,|\, X_n = x), \tag{8}$$

as $x \to \infty$. It was proved that if $m_1(x) \sim \frac{a}{x}$ and $m_2(x) \to b$ (as $x \to \infty$) then

- $2a + b \le 0$ (or $2xm_1(x) + m_2(x) \le 0$ as $x \to \infty$) implies X is recurrent,
- $2a + b > 0$ (or $2xm_1(x) + m_2(x) > 0$ as $x \to \infty$) implies X is transient.

The reason why the above holds is due to the fact that the continuous analogue of the above chain is the Markov process $(X_s,\, s \ge 0)$:

$$dX_s = \frac{a}{X_s} ds + bB_s,\; s \ge 0, \tag{9}$$

which is known as the Bessel process of index $1+2a/b$. Then the recurrence/transience, as well as the asymptotic behavior of the above chain can be basically read from that of the Bessel process. The regime that $m_1(x) = o(1)$ is called the Markov chain with asymptotic zero drift, and features very active research (with many applications in physics model as random walk in random environment).

Now by looking at the dynamics of (7), we have

$$m_1(x) = \frac{1}{x} \quad \text{and} \quad m_2(x) = \frac{1}{x}. \tag{10}$$

Interestingly, there seem to be few result in the case where both $m_1(x)$ and $m_2(x) \to 0$ as $x \to \infty$ except that $(N_t,\, t \ge 0)$ is transient. But this is easy in our case as N is nondecreasing in t. To obtain the asymptotics, one naive idea would be looking at the diffusion:

$$dX_s = \frac{1}{X_s} ds + \frac{1}{\sqrt{X_s}} dB_s, \tag{11}$$

which is of its own interest. But our problem is different as N is nondecreasing in t, and 'upward' variance is larger than the 'downward' variance. Thus, the diffusion approximation (11) **does not** work. This way, our problem is a degenerate version of the general Lamperti problem.

The idea is to use the method of moments. To be more precise, we compute $\mathbb{E}(N_t^{2j})$ for $j = 1, 2, \ldots$. For $j = 1$, we have by definition

$$\mathbb{E}(N_{t+1}^2 - N_t^2 \,|\, N_t = x) = 2 + \frac{1}{x}. \tag{12}$$

The discussion in the previous paragraphs implies that almost surely $N_t \to \infty$ as $t \to \infty$. As a result, $\mathbb{E}(N_{t+1}^2 - N_t^2) \to 2$ as $t \to \infty$ which yields

$$\mathbb{E}N_t^2 \sim 2t. \tag{13}$$

Next for $j = 2$,

$$\mathbb{E}(N_{t+1}^4 - N_t^4 \,|\, N_t = x) = 4x^2 + o(x) \quad \text{as } x \to \infty. \tag{14}$$

Thus, $\mathbb{E}(N_{t+1}^4 - N_t^4) = (4 + o(1))\mathbb{E}N_t^2 \sim 4 \cdot 2t$ by (13). So $\mathbb{E}N_t^4 \sim 4 \cdot 2(\frac{t^2}{2}) = 4t^2$. By induction, it is not hard to see that

$$\mathbb{E}N_t^{2j} \sim 2^j t^j \quad \text{for all } j \geq 1. \tag{15}$$

Consequently,

$$\frac{N_t^2}{t} \xrightarrow{d} X \quad \text{as } t \to \infty, \tag{16}$$

where X is a random variable such that $\mathbb{E}X^j = 2^j$ for $j \geq 1$. It is easily seen that the law of X is uniquely determined by the moment sequence $2^j j!$, $j \geq 1$ (e.g. by Carleman's condition), and $X \equiv 2$. Therefore, $N_t/\sqrt{t}$ converges in distribution, and thus in probability, to $\sqrt{2}$. Indeed, the Brownian motion term in (11) can be ignored resulting in a "fluid limit" X_t as follows:

$$dX_t = \frac{1}{X_t} dt. \tag{17}$$

From the above, we have $X_t^2 = 2t$; and inductively, from $X_t^{2j} = (2t)^j$, multiplying both sides of (17) by X_t^{2j+1}, we have

$$X_t^{2j+1} dX_t = X_t^{2j} dt = (2t)^j dt \quad \Rightarrow \quad X_t^{2j+2} = (2t)^{j+1},$$

producing the same result as above, i.e., $\mathbb{E}N_t^{2j} \sim (2t)^j$.

Therefore, the square-root economy ($N_t \approx \sqrt{t}$) is more conservative than the standard setting where $N_t \sim t$ as $t \to \infty$.

Model 2 Continued

Let $n_{k,t}$ denote the number of shares an individual k owns at time t, with $k = 1, \ldots, K$, and $t = 0, 1, \ldots$. Denote $N_t := \sum_k n_{k,t}$ and $\pi_{k,t} := n_{k,t}/N_t$, the latter being the proportion of k's share relative to the total at time t.

In each period t, one additional share will be distributed as follows: each individual k will receive the share w.p. $\theta_{k,t} := n_{k,t}/N_t^{1+\alpha}$, where $\alpha > 0$; and w.p. $1 - \theta_{k,t}$ will receive nothing. Note that similar to what's pointed out in the last section, w.p. $1 - \frac{1}{N_t^\alpha}$ the additional share will not be awarded to any individual, in which case $\pi_{k,t}$ will remain the same, for every k, and so will the total N_t.

Clearly, setting $\alpha = 1$ reduces to the model of the last section. For a general $\alpha > 0$, the equation in (17) becomes

$$dX_t = \frac{1}{X_t^\alpha} dt, \tag{18}$$

with $X_t^{1+\alpha} = (1+\alpha)t$, and analogously $\mathsf{E}(N_t^{1+\alpha}) = (1+\alpha)t$. Furthermore, the limit in (16) becomes

$$\frac{N_t^{1+\alpha}}{t} \xrightarrow{d} X \equiv 1+\alpha \quad \text{as } t \to \infty, \tag{19}$$

Let $\mathcal{F}_t$ denote the filtration associated with the above share distribution outcomes up to time t. Then,

$$\mathsf{E}(\pi_{k,t+1}|\mathcal{F}_t) = \frac{n_{k,t}}{N_t}\left(1 - \frac{1}{N_t^\alpha}\right) + \frac{n_{k,t}}{N_t+1} \cdot \frac{N_t - n_{k,t}}{N_t^{1+\alpha}} + \frac{n_{k,t}+1}{N_t+1} \cdot \frac{n_{k,t}}{N_t^{1+\alpha}}. \tag{20}$$

Recognizing the first term on the RHS, $\frac{n_{k,t}}{N_t} = \pi_{k,t}$, while all other terms sum up to zero, we conclude that $\{\pi_{k,t}\}$ is a martingale, for every k. By the martingale convergence theorem,

$$(\pi_{1,t}, \ldots, \pi_{K,t}) \to (\pi_{1,\infty}, \ldots, \pi_{K,\infty}) \quad \text{almost surely,}$$

where $(\pi_{1,\infty}, \ldots, \pi_{K,\infty})$ is some random probability distribution on $[K]$.

Similarly, we can derive

$$\begin{aligned} \mathsf{E}(\pi_{k,t+1}^2|\mathcal{F}_t) &= \pi_{k,t}^2\left(1 - \frac{1}{N_t^\alpha}\right) + \left(\frac{N_t\pi_{k,t}}{N_t+1}\right)^2 \frac{1-\pi_{k,t}}{N_t^\alpha} + \left(\frac{N_t\pi_{k,t}+1}{N_t+1}\right)^2 \frac{\pi_{k,t}}{N_t^\alpha} \\ &= \pi_{k,t}^2 + \frac{\pi_{k,t}(1-\pi_{k,t})}{N_t^\alpha(N_t+1)^2}. \end{aligned} \tag{21}$$

Hence, we have

$$\mathsf{Var}(\pi_{k,t+1}) = \mathsf{Var}(\pi_{k,t}) + \mathsf{E}\left[\frac{\pi_{k,t}(1-\pi_{k,t})}{N_t^\alpha(N_t+1)^2}\right]. \tag{22}$$

Next, applying the same derivation as in (20) but to $\theta_{k,t}$ instead, we have

$$\mathsf{E}(\theta_{k,t+1}|\mathcal{F}_t) = \frac{\pi_{k,t}}{N_t^\alpha}\left(1 - \frac{1}{N_t^\alpha}\right) + \frac{N_t\pi_{k,t}}{(N_t+1)^{1+\alpha}} \cdot \frac{1-\pi_{k,t}}{N_t^\alpha} + \frac{N_t\pi_{k,t}+1}{(N_t+1)^{1+\alpha}} \cdot \frac{\pi_{k,t}}{N_t^\alpha}. \tag{23}$$

The last two terms add up to $\frac{\pi_{k,t}}{N_t^\alpha(N_t+1)^\alpha} = \frac{\theta_{k,t}}{(N_t+1)^\alpha}$. Hence, we have

$$\mathsf{E}(\theta_{k,t+1}|\mathcal{F}_t) = \theta_{k,t}\left[1 - \frac{1}{N_t^\alpha} - \frac{1}{(N_t+1)^\alpha}\right] \le \theta_{k,t}. \tag{24}$$

That is, $\{\theta_{k,t}\}$ is a super-martingale, for every k.

Now, let's consider $\alpha = 1$. The second term on the RHS of (22) can be bounded as follows:

$$\mathsf{E}\left[\frac{\pi_{k,t}(1-\pi_{k,t})}{N_t(N_t+1)^2}\right] \le \mathsf{E}\left(\frac{\pi_{k,t}}{N_t^3}\right) \sim \frac{\pi_{k,0}}{(2t+N_0^2)^{3/2}}, \tag{25}$$

where for large t, we write $N_t = (2t+N_0^2)^{1/2}$, with $N_0 := N$, the total number of shares at $t = 0$. Then, along with (22), we have,

$$\mathsf{Var}(\pi_{k,t+1}) \le \frac{\pi_{k,0}}{N_0}. \tag{26}$$

In fact, by (22) and (25),

$$\mathsf{Var}(\pi_{k,t+1}) - \mathsf{Var}(\pi_{k,t}) \le \frac{\pi_{k,0}}{(2t+N_0^2)^{3/2}}, \tag{27}$$

Further by the sum-integral trick, we have

$$\begin{aligned}\mathsf{Var}(\pi_{k,t+1}) &\le \pi_{k,0} \sum_{k=0}^{t} \frac{1}{(2k+N_0^2)^{3/2}} \\ &\le \pi_{k,0} \int_{-1}^{t} \frac{ds}{(2s+N_0^2)^{3/2}} = \pi_{k,0} \left(\frac{1}{\sqrt{N_0^2-2}} - \frac{1}{\sqrt{N_0^2+2t}} \right).\end{aligned} \tag{28}$$

So we always have the upper bound $\mathsf{Var}(\pi_{k,t+1}) \le \frac{\pi_{k,0}}{\sqrt{N_0^2-2}} \sim \frac{\pi_{k,0}}{N_0}$. There are two cases:

1. If t and N_0^2 are of the same order, or $t \gg N_0^2$, then $\frac{1}{\sqrt{N_0^2-2}} - \frac{1}{\sqrt{N_0^2+2t}}$ is of order $1/N_0$.
2. If $t \ll N_0^2$, we can do the Taylor expansion to get

$$\frac{1}{\sqrt{N_0^2-2}} - \frac{1}{\sqrt{N_0^2+2t}} \sim \frac{t+1}{N_0^2}. \tag{29}$$

   In this case, we have an even smaller bound $\pi_{k,0}\frac{t}{N_0^2}$ for $\mathsf{Var}(\pi_{k,t+1})$.

Since we are mostly concerned with large t (in particular, $t \to \infty$), the case 1 is more relevant, and the bound $\pi_{k,0}/N_0$ is sharp. Now by Chebyshev's inequality,

$$\mathsf{P}\left(\left| \frac{\pi_{k,\infty}}{\pi_{k,0}} - 1 \right| > \varepsilon \right) \le \frac{1}{\varepsilon^2 \pi_{k,0} N_0} = \frac{1}{\varepsilon^2 n_{k,0}}. \tag{30}$$

This implies that when $n_{k,0} = f(N_0)$ for $f(N_0) \to \infty$ as $N_0 \to \infty$, then $\frac{\pi_{k,\infty}}{\pi_{k,0}} \to 1$ as $N_0 \to \infty$: this is the concentration property. The same argument shows that this concentration holds for all $\alpha > 0$ – this is the "universal" concentration property.

Now we consider the case where $n_{k,0} = \mathcal{O}(1)$ (and thus $\pi_{k,0} = \mathcal{O}(N_0^{-1})$). In this case,

$$\mathsf{E}\pi_{k,t}^2 = \pi_{k,0}^2 + \mathsf{Var}(\pi_{k,t}) \overset{(26)}{\le} \pi_{k,0}^2 + \frac{\pi_{k,0}}{N_0} \lesssim \frac{\pi_{k,0}}{N_0} \ll \pi_{k,0}. \tag{31}$$

Further in view of (25), we get $\mathsf{Var}(\pi_{k,t+1}) \sim \frac{\pi_{k,0}}{N_0}$ (both lower and upper bound asymptotics). As a result,

- if $n_{k,0} = \Theta(1)$ or $\pi_{k,0} = \Theta(N_0^{-1})$, then $\mathsf{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \Theta(1)$;
- if $n_{k,0} = o(1)$ or $\pi_{k,0} = o(N_0^{-1})$, then $\mathsf{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$.

Next we consider higher moments (to get some anti-concentration result). To this end, write

$$\mathsf{E}(\pi_{k,t+1} \,|\, \mathcal{F}_t) = \frac{n_{k,t}}{N_t} 1_{S_1} + \frac{n_{k,t}}{1+N_t} 1_{S_2} + \frac{1+n_{k,t}}{1+N_t} 1_{S_3}, \tag{32}$$

where S_1, S_2, S_3 are mutually exclusive events with probabilities $\mathsf{P}(S_1) = 1 - (N_t)^{-1}$, $\mathsf{P}(S_2) = \frac{1-\pi_{k,t}}{N_t}$ and $\mathsf{P}(S_3) = \frac{\pi_{k,t}}{N_t}$. Noting that $1_{S_1} = 1 - 1_{S_2} - 1_{S_3}$, we have

$$\mathsf{E}(\pi_{k,t+1} \,|\, \mathcal{F}_t) = \pi_{k,t} - \frac{\pi_{k,t}}{1+N_t} 1_{S_2} + \frac{1-\pi_{k,t}}{1+N_t} 1_{S_3}. \tag{33}$$

We consider the third central moment ($\mu_3(\pi_{k,t+1}) := \mathsf{E}(\pi_{k,t} - \pi_{k,0})^3$)

$$\begin{aligned}\mu_3(\pi_{k,t+1}) = \mu_3(\pi_{k,t}) &\underbrace{-\mathsf{E}\frac{\pi_{k,t}^3(1-\pi_{k,t})}{(1+N_t)^3 N_t}}_{(a)} + \underbrace{\mathsf{E}\frac{\pi_{k,t}(1-\pi_{k,t})^3}{(1+N_t)^3 N_t}}_{(b)} \\ &\underbrace{-3\mathsf{E}(\pi_{k,t} - \pi_{k,0})^2 \frac{\pi_{k,t}(1-\pi_{k,t})}{(1+N_t)N_t}}_{(c)} \underbrace{+3\mathsf{E}(\pi_{k,t} - \pi_{k,0})^2 \frac{\pi_{k,t}(1-\pi_{k,t})}{(1+N_t)N_t}}_{(d)} \\ &\underbrace{+3\mathsf{E}(\pi_{k,t} - \pi_{k,0}) \frac{\pi_{k,t}^2(1-\pi_{k,t})}{(1+N_t)^2 N_t}}_{(e)} + \underbrace{3\mathsf{E}(\pi_{k,t} - \pi_{k,0}) \frac{\pi_{k,t}(1-\pi_{k,t})^2}{(1+N_t)^2 N_t}}_{(f)}. \end{aligned} \tag{34}$$

The terms (c) and (d) cancel out. Furthermore,

$$\begin{aligned}(e) + (f) &= 3\mathsf{E}(\pi_{k,t} - \pi_{k,0}) \frac{\pi_{k,t}(1-\pi_{k,t})}{(1+N_t)^2 N_t} \\ &\lesssim \frac{\mathsf{E}\pi_{k,t}^2}{N_t^3} \overset{(31)}{\lesssim} \frac{\pi_{k,0}}{N_0(2t + N_0^2)^{3/2}}, \end{aligned} \tag{35}$$

and

$$\begin{aligned}(a) + (b) &= 3\mathsf{E}\frac{\pi_{k,t}(1-\pi_{k,t})(1-2\pi_{k,t})}{(1+N_t)^3 N_t} \\ &\lesssim \frac{\pi_{k,0}}{N_t^4} = \frac{\pi_{k,0}}{(2t + N_0^2)^2}. \end{aligned} \tag{36}$$

Combining (34)–(36) and using the sum-integral trick again yield

$$\mu_3(\pi_{k,t+1}) \lesssim \frac{\pi_{k,0}}{N_0^2} \quad \text{and} \quad \mathsf{E}\pi_{k,t+1}^3 \lesssim \frac{\pi_{k,0}}{N_0^2} + \frac{\pi_{k,0}^2}{N_0} + \pi_{k,0}^3 \lesssim \frac{\pi_{k,0}}{N_0^2}, \tag{37}$$

as $\pi_{k,0} = \mathcal{O}(N_0^{-1})$. Further we consider the fourth central moment

$$\begin{aligned}\mu_4(\pi_{k,t+1}) = \mu_4(\pi_{k,t}) &\underbrace{+\mathsf{E}\frac{\pi_{k,t}^4(1-\pi_{k,t})}{(1+N_t)^4 N_t}}_{(a')} + \underbrace{\mathsf{E}\frac{\pi_{k,t}(1-\pi_{k,t})^4}{(1+N_t)^4 N_t}}_{(b')} \\ &\underbrace{-4\mathsf{E}(\pi_{k,t} - \pi_{k,0})^3 \frac{\pi_{k,t}(1-\pi_{k,t})}{(1+N_t)N_t}}_{(c')} + \underbrace{4\mathsf{E}(\pi_{k,t} - \pi_{k,0})^3 \frac{\pi_{k,t}(1-\pi_{k,t})}{(1+N_t)N_t}}_{(d')} \\ &\underbrace{+6\mathsf{E}(\pi_{k,t} - \pi_{k,0})^2 \frac{\pi_{k,t}^2(1-\pi_{k,t})}{(1+N_t)^2 N_t}}_{(e')} + \underbrace{6\mathsf{E}(\pi_{k,t} - \pi_{k,0})^2 \frac{\pi_{k,t}(1-\pi_{k,t})^2}{(1+N_t)^2 N_t}}_{(f')} \\ &\underbrace{-4\mathsf{E}(\pi_{k,t} - \pi_{k,0}) \frac{\pi_{k,t}^3(1-\pi_{k,t})}{(1+N_t)^3 N_t}}_{(g')} + \underbrace{4\mathsf{E}(\pi_{k,t} - \pi_{k,0}) \frac{\pi_{k,t}(1-\pi_{k,t})^3}{(1+N_t)^3 N_t}}_{(h')}. \end{aligned} \tag{38}$$

The terms (c') and (d') cancel out. Furthermore,

$$(e') + (f') \lesssim \mathsf{E}\frac{\pi_{k,t}^3}{N_t^3} \overset{(37)}{\lesssim} \frac{\pi_{k,0}}{N_0^2(2t+N_0^2)^{3/2}}, \tag{39}$$

$$(g') + (h') \lesssim \mathsf{E}\frac{\pi_{k,t}^2}{N_t^4} \overset{(31)}{\lesssim} \frac{\pi_{k,0}}{N_0(2t+N_0^2)^{2}}, \tag{40}$$

$$(a') + (b') \lesssim \mathsf{E}\frac{\pi_{k,t}}{N_t^5} \overset{(31)}{\lesssim} \frac{\pi_{k,0}}{(2t+N_0^2)^{5/2}}. \tag{41}$$

Combining (38)–(41) and using the sum-integral trick, we have

$$\mu_4(\pi_{k,t+1}) \lesssim \frac{\pi_{k,0}}{N_0^3}. \tag{42}$$

Since $\pi_{k,0} = \mathcal{O}(N_0^{-1})$, we get $\mu_4(\pi_{k,t}) = \mathcal{O}(N_0^{-4})$. Now by using the same argument as (1.43) on p.10 of the writeup, we get the anti-concentration in the case of $n_{k,0} = \Theta(1)$: there exists $c > 0$ independent of N_0 such that for $\varepsilon > 0$ sufficiently small,

$$\mathsf{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right) > c. \tag{43}$$

Interesting thing is that in this model the expected reward $\mathbb{E}(n_t^{-1}) \to 0$ (decreasing), while the phase transition is the same as Theorem 1.2 (i) with a deterministic decreasing reward $R_t \downarrow \underline{R} > 0$.

There is a missing piece in the above argument: we replace N_t with $\sqrt{2t}$ in expectations. This approximation becomes subtle when t is large. Here we need some concentration/deviation bound for N_t as t is large. To this end, we will further expand the moments $\mathsf{E}N_t^{2j}$, $j \geq 1$. Recall that $\mathsf{E}N_t^{2j} \sim (2t)^j$ for $j \geq 1$. This implies that $\mathbb{E}N_t \lesssim \sqrt{2t}$. Together with $N_t/\sqrt{t} \to \sqrt{2}$ in probability, we get $(1-\varepsilon)\sqrt{2t} \leq \mathsf{E}N_t \leq (1+\varepsilon)\sqrt{2t}$. From now on, we will write $\mathsf{E}N_t \approx \sqrt{2t}$ to indicate the equality up to a factor of ε. Note that

$$\mathsf{E}N_t = \mathsf{E}N_0 + \sum_{k<t} \mathsf{E}N_k^{-1}, \tag{44}$$

which implies that $\sum_{k<t} \mathsf{E}N_k^{-1} \approx \sqrt{2t}$. Let $M_t := N_t^2 - 2t$ so

$$M_{t+1} = \begin{cases} M_t - 2 & \text{with probability } \ 1 - \frac{1}{N_t}, \\ M_t + 2N_t - 1 & \text{with probability } \ \frac{1}{N_t}. \end{cases} \tag{45}$$

We have then

$$\mathsf{E}\left(M_{t+1}|\mathcal{F}_t\right) = M_t + \frac{1}{N_t}, \tag{46}$$

and hence $\mathsf{E}M_t \sim \sqrt{2t}$. Furthermore,

$$\mathsf{E}(M_{t+1}^2 \,|\, \mathcal{F}_t) = M_t^2 + \frac{2M_t}{N_t} + 4N_t - \frac{3}{N_t}. \tag{47}$$

As a result, $\mathsf{E}(M_{t+1}^2 - M_t^2) = 6\mathsf{E}N_t - 2t\mathsf{E}N_t^{-1} - 3\mathsf{E}N_t^{-1} \lesssim 6\sqrt{2t}$, and hence $\mathsf{E}M_t^2 \lesssim Ct^{\frac{3}{2}}$ and $\mathsf{Var}(M_t) \lesssim Ct^{\frac{3}{2}}$. By Chebyshev's inequality,

$$\mathsf{P}\left(|N_t^2 - 2t| > \delta t\right) = \mathsf{P}(|M_t| > \delta t) \lesssim Ct^{-\frac{1}{2}}. \tag{48}$$

Now let $\Omega_t := \{|N_t^2 - 2t| \le \delta t\} = \{|M_t| \le \delta t\}$ so that $\mathsf{P}(\Omega_t) \ge 1 - C\delta^{-2}t^{-\frac{1}{2}}$. We have then

$$\begin{aligned}
\mathsf{E}\left(\frac{M_t}{N_t}\right) &= \mathsf{E}\left(\frac{M_t}{N_t}1_{\Omega_t}\right) + \mathsf{E}\left(\frac{M_t}{N_t}1_{\Omega_t^c}\right) \\
&\le \frac{\delta t}{\sqrt{(2-\delta)t}}\,\mathsf{P}(\Omega_t) + \mathsf{E}(N_t 1_{\Omega_t^c}) \\
&\lesssim C\delta\sqrt{t} + \sqrt{\mathsf{E}N_t^2 \cdot \mathsf{P}(\Omega_t^c)} \\
&\lesssim C\delta\sqrt{t} + Ct^{\frac{1}{4}} \lesssim C\delta t^{\frac{1}{2}}. \qquad (49)
\end{aligned}$$

Combining (47) and (49), we have $\mathsf{E}(M_{t+1}^2 - M_t^2) \lesssim 4\sqrt{2t}$, and hence $\mathsf{E}M_t^2 \lesssim \frac{8\sqrt{2}}{3}t^{\frac{3}{2}}$. This is slightly better (up to a constant factor) than the estimates after (47), but this idea will be useful in upper bounding higher moments.

Next we consider $k \ge 3$ and k even. Observe that the leading terms of $\mathsf{E}(M_{t+1}^k - M_t^k)$ for $k \ge 3$ are

$$\underbrace{k\mathsf{E}\left(\frac{M_t^{k-1}}{N_t}\right)}_{(a)} + \underbrace{2^k\mathsf{E}\left(N_t^{k-1}\right)}_{(b)} + \underbrace{2^{k-1}k\mathsf{E}\left(M_t N_t^{k-2}\right)}_{(c)}. \qquad (50)$$

All other terms are smaller than these three terms in t. Note that $(b) \sim 2^k(2t)^{\frac{k-1}{2}} = 2^{\frac{3k-1}{2}}t^{\frac{k-1}{2}}$. Now we deal with the term (c). Note that

$$\mathsf{E}N_t^2 \approx 2t + \sum_{k<t}\mathsf{E}N_k^{-1} \approx 2t + \sqrt{2t}. \qquad (51)$$

Further by induction, we have

$$\mathsf{E}N_t^{2j} \approx (2t)^j + a_j t^{j-\frac{1}{2}} \quad \text{for } j \ge 1, \qquad (52)$$

with $a_1 = \sqrt{2}$ and

$$a_{j+1} = \frac{2(j+1)}{j+\frac{1}{2}}a_j + 2^{j+\frac{1}{2}}(j+1) \quad \text{for } j \ge 1, \qquad (53)$$

where the previous recursion comes from the equation $\mathsf{E}N_t^{2j+2} \approx (2j+2)\sum_{k<t}\mathsf{E}N_t^{2j} + (k+1)(2k+1)\sum_{k<t}\mathsf{E}N_t^{2k-1}$. So $\mathsf{E}N_t^k = (2t)^{\frac{k}{2}} + \mathcal{O}\left(t^{\frac{k-1}{2}}\right)$ for $k \ge 2$ and k even. As a result,

$$\begin{aligned}
\mathsf{E}\left(M_t N_t^{k-2}\right) &= \mathsf{E}N_t^k - 2t\,\mathsf{E}N_t^{k-2} \\
&= (2t)^{\frac{k}{2}} - 2t(2t)^{\frac{k-2}{2}} + \mathcal{O}(t^{\frac{k-1}{2}}) = \mathcal{O}(t^{\frac{k-1}{2}}). \qquad (54)
\end{aligned}$$

Thus, the term (c) is of order $\mathcal{O}(t^{\frac{k-1}{2}})$.

The nasty case is (a). Let $\Omega_{t,\alpha} = \{|N_t^2 - 2t| \le t^\alpha\} = \{|M_t| \le t^\alpha\}$ so that $\mathsf{P}(\Omega_{t,\alpha}) \le Ct^{\frac{3}{2}-2\alpha}$ (we would require $3/4 < \alpha < 1$). Following the same idea as in (49), we have

$$\begin{aligned}
\mathsf{E}\left(\frac{M_t^{k-1}}{N_t}\right) &= \mathsf{E}\left(\frac{M_t^{k-1}}{N_t}1_{\Omega_{t,\alpha}}\right) + \mathsf{E}\left(\frac{M_t^{k-1}}{N_t}1_{\Omega_{t,\alpha}^c}\right) \\
&\lesssim t^{\alpha(k-1)-\frac{1}{2}} + \mathsf{E}(N_t^{2k-3}1_{\Omega_{t,\alpha}^c}) \\
&\lesssim t^{\alpha(k-1)-\frac{1}{2}} + \mathsf{E}(N_t^{2k-3}p)^{\frac{1}{p}}\mathsf{P}(1_{\Omega_{t,\alpha}^c})^{\frac{1}{q}}, \qquad (55)
\end{aligned}$$

with $1/p + 1/q = 1$. By taking $p \to \infty$ and $q \to 1$, we get the above estimate as

$$t^{\alpha(k-1)-\frac{1}{2}} + t^{k-2\alpha}. \tag{56}$$

By equating the two terms, we obtain

$$\alpha = \frac{k+\frac{1}{2}}{k+1} \quad \text{and} \quad \mathsf{E}\left(\frac{M_t^{k-1}}{N_t}\right) \lesssim t^{k-2+\frac{1}{k+1}}. \tag{57}$$

Consequently, $\mathsf{E}M_t^k \lesssim t^{k-1+\frac{1}{1+k}}$. As k (even) becomes large, we basically have $\mathsf{E}M_t^k \lesssim t^{k-1+\varepsilon}$ for any $\varepsilon > 0$. By Markov's inequality, we get an improved bound

$$\mathsf{P}\left(|N_t^2 - 2t| > \delta t\right) = \mathsf{P}(M_t^k > \delta^k t^k) \lesssim t^{-1+\varepsilon}. \tag{58}$$

Now let us go back to (25). Set now $\Omega_t = \{|N_t^2 - N_0^2 - 2t| > \delta t\}$. We have

$$\begin{aligned} \mathsf{E}\left(\frac{\pi_{k,t}}{N_t^3}\right) &= \mathsf{E}\left(\frac{\pi_{k,t}}{N_t^3}1_{\Omega_t}\right) + \mathsf{E}\left(\frac{\pi_{k,t}}{N_t^3}1_{\Omega_t^c}\right) \\ &\lesssim \frac{\pi_{k,0}}{(2t+N_0^2)^{\frac{3}{2}}} + \frac{\pi_{k,0}^{\frac{1}{p}}}{N_0^3}\mathsf{P}(\Omega_t^c)^{\frac{1}{q}}, \end{aligned} \tag{59}$$

where $1/p + 1/q = 1$, and one can take e.g. $p \to \infty$ and $q \to 1$. So if we can show that $\mathsf{P}(\Omega_t^c) = \mathcal{O}(t^{-\alpha})$ for $\alpha > 1$ then we are done... The main problem is the lower tail bound $\Omega_t^- := \{N_t^2 - N_0^2 - 2t < -\delta t\}$...

Actually we can prove a less stronger result than the concentration lower tail bound around $\sqrt{2t}$. All we need is that $\mathsf{P}(N_t < a\sqrt{t})$ is small for some $a > 0$: this is sufficient to make all the approximation valid up to a constant factor. We will rely on combinatorics to prove that for all $\lambda < \lambda_-$ (λ_- is a number less than $\sqrt{2}$ that we will make explicit later),

$$\mathsf{P}(N_t < \lambda\sqrt{t}) \lesssim \exp(-f(\lambda)\sqrt{t}) \quad \text{with } f(\lambda) > 0. \tag{60}$$

First note that $(N_t,\, t \geq 0)$ has increments $\{0, 1\}$ so there are $\binom{t}{k}$ paths ending at $N_t = k$ (you will choose k increments $\uparrow$ out of t steps). It is easy to see that the probability of each path ending at $N_t = k$ is upper bounded by

$$\frac{1}{k!}\left(1 - \frac{1}{k}\right)^{t-k}, \tag{61}$$

since the k $\uparrow$ steps will contribute $1/k!$, and the remaining $\to$ steps will at most have probability $\left(1 - \frac{1}{k}\right)^{t-k}$. Consequently,

$$\mathsf{P}(N_t \leq m) \leq \sum_{k \leq m} \underbrace{\binom{t}{k}\frac{1}{k!}\left(1 - \frac{1}{k}\right)^{t-k}}_{a_k}. \tag{62}$$

As we will see, a_k is nondecreasing up to $k \approx \lambda_-\sqrt{t}$. The idea is to use the Stirling approximation to study the term

$$a_{\lambda\sqrt{t}} = \binom{t}{\lambda\sqrt{t}}\frac{1}{(\lambda\sqrt{t})!}\left(1 - \frac{1}{\lambda\sqrt{t}}\right)^{t-\lambda\sqrt{t}}. \tag{63}$$

Note that

$$\begin{aligned}\binom{t}{\lambda\sqrt{t}} &= \frac{t!}{(\lambda\sqrt{t})!(t-\lambda\sqrt{t})!}\\ &\approx t^{\frac{1}{4}}\frac{t^t}{(\lambda\sqrt{t})^{\lambda\sqrt{t}}(t-\lambda\sqrt{t})^{t-\lambda\sqrt{t}}}\\ &\approx t^{\frac{1}{4}}\exp\left(\frac{\lambda}{2}\sqrt{t}\log t + (\lambda-\lambda\log\lambda)\sqrt{t}\right), \end{aligned}\tag{64}$$

$$(\lambda\sqrt{t})! \approx t^{\frac{1}{4}}\left(\frac{\lambda\sqrt{t}}{e}\right)^{\lambda\sqrt{t}} \approx t^{\frac{1}{4}}\exp\left(\frac{\lambda}{2}\sqrt{t}\log t + (\lambda\log\lambda-\lambda)\sqrt{t}\right), \tag{65}$$

and

$$\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t-\lambda\sqrt{t}} \approx \exp\left(-\frac{\sqrt{t}}{\lambda}\right). \tag{66}$$

Combining the above estimates yields

$$a_{\lambda\sqrt{t}} \approx \exp\left(-\left(2\lambda\log\lambda - 2\lambda + \frac{1}{\lambda}\right)\sqrt{t}\right). \tag{67}$$

Now set $f(\lambda) := 2\lambda\log\lambda - 2\lambda + \frac{1}{\lambda}$ for $\lambda > 0$. The function has two roots $\lambda_- \approx 0.56$ and $\lambda_+ \approx 2.51$, and $f > 0$ on the interval $(0, \lambda_-)$. See the figure below for a plot of this function.

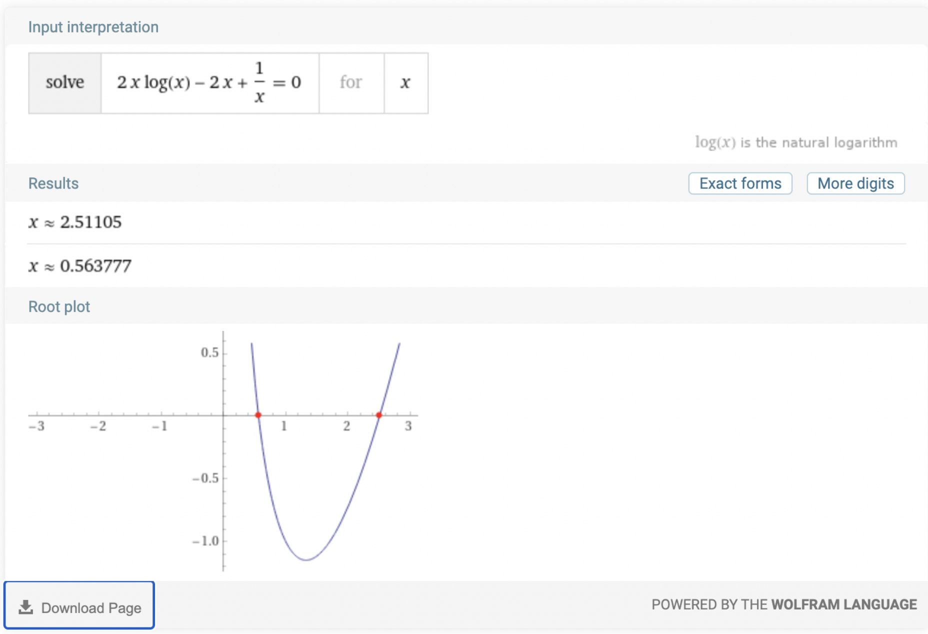


Figure 1: Function f

As a result, for $\lambda < \lambda_-$,

$$\mathsf{P}(N_t \le \lambda\sqrt{t}) \lesssim \sqrt{t}\exp(-f(\lambda)\sqrt{t}) \lesssim \exp\left(-\frac{f(\lambda)}{2}\sqrt{t}\right). \tag{68}$$

We are done.

There is a way to slightly improve the value of λ_- from 0.56 to 0.60. The idea is to split the interval $[0,t]$ to $[0,t/2]$ and $[t/2,t]$. The task is to upper bound $\mathbb{P}(N_t=\lambda\sqrt{t})$ for $\lambda>0$ (we will neglect the polynomial terms and only focus on the exponential terms). Note that

$$\mathsf{P}(N_t=\lambda\sqrt{t})=\sum_{k\le\lambda\sqrt{t}}\binom{t/2}{k}\binom{t/2}{\lambda\sqrt{t}-k}\frac{1}{(\lambda\sqrt{t})!}\underbrace{\left(1-\frac{1}{k}\right)^{t/2-k}\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t/2+k-\lambda\sqrt{t}}}_{a}. \tag{69}$$

Next we will split the range of $k\le\lambda\sqrt{t}$ into $S_1:=\{k\le a\sqrt{t}\}\cup\{k\ge(\lambda-a)\sqrt{t}\}$, and $S_2:=\{a\sqrt{t}<k<(\lambda-a)\sqrt{t}\}$ with $a<\frac{\lambda}{2}$. For $k\in S_1$, we simply bound the term (a) by $\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t/2-\lambda\sqrt{t}}$ while for $k\in S_2$ we bound the term (a) by

$$\left(1-\frac{1}{(\lambda-a)\sqrt{t}}\right)^{t/2-(\lambda-a)\sqrt{t}}\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t/2-a\sqrt{t}}. \tag{70}$$

Consequently,

$$\begin{aligned}\mathsf{P}(N_t=\lambda\sqrt{t})\le&\underbrace{\left(\sum_{k\in S_1}\binom{t/2}{k}\binom{t/2}{\lambda\sqrt{t}-k}\right)\frac{1}{(\lambda\sqrt{t})!}\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t/2-\lambda\sqrt{t}}}_{(b)}\\&+\underbrace{\left(\sum_{k\in S_2}\binom{t/2}{k}\binom{t/2}{\lambda\sqrt{t}-k}\right)\frac{1}{(\lambda\sqrt{t})!}\left(1-\frac{1}{(\lambda-a)\sqrt{t}}\right)^{t/2-(\lambda-a)\sqrt{t}}\left(1-\frac{1}{\lambda\sqrt{t}}\right)^{t/2-a\sqrt{t}}}_{(c)}.\end{aligned} \tag{71}$$

Recall that

$$\binom{\alpha t}{\beta\sqrt{t}}\approx\exp\left(\frac{\beta}{2}\sqrt{t}\log t+(\beta\log\alpha+\beta-\beta\log\beta)\sqrt{t}\right). \tag{72}$$

Thus,

$$\begin{aligned}\sum_{k\in S_1}\binom{t/2}{k}\binom{t/2}{\lambda\sqrt{t}-k}&\approx\exp\left(\frac{\lambda}{2}\sqrt{t}\log t+(-\lambda\log 2+\lambda-a\log a-(\lambda-a)\log(\lambda-a))\sqrt{t}\right),\\\sum_{k\in S_2}\binom{t/2}{k}\binom{t/2}{\lambda\sqrt{t}-k}&\approx\exp\left(\frac{\lambda}{2}\sqrt{t}\log t+(\lambda-\lambda\log\lambda)\sqrt{t}\right).\end{aligned} \tag{73}$$

Thus, we get the exponential bounds from (65), (66) and (73) for the terms (b) and (c):

$$\begin{aligned}(b)&\approx\exp\left((-\lambda\log 2+2\lambda-a\log a-(\lambda-a)\log(\lambda-a)-\lambda\log\lambda-\frac{1}{\lambda})\sqrt{t}\right),\\(c)&\approx\exp\left((2\lambda-2\lambda\log\lambda-\frac{1}{2\lambda}-\frac{1}{2(\lambda-a)})\sqrt{t}\right).\end{aligned} \tag{74}$$

By equating the two coefficients before $\sqrt{t}$ in (74), we get

$$-\lambda\log 2+2\lambda-a\log a-(\lambda-a)\log(\lambda-a)-\lambda\log\lambda-\frac{1}{\lambda})\sqrt{t}=2\lambda-2\lambda\log\lambda-\frac{1}{2\lambda}-\frac{1}{2(\lambda-a)}. \tag{75}$$

By letting $a=\theta\lambda$ with $\theta<\frac{1}{2}$, the above equation yields

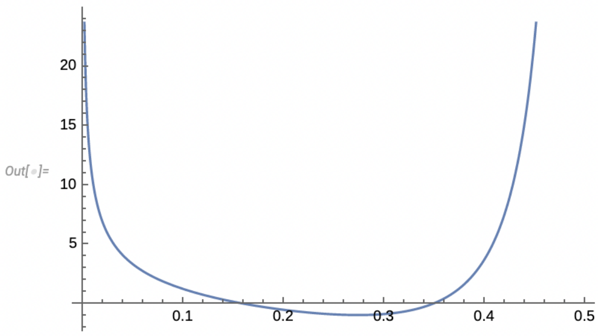
$$\lambda=\sqrt{\frac{\theta}{2(1-\theta)(\log 2+\theta\log\theta+(1-\theta)\log(1-\theta))}}. \tag{76}$$

So the coefficient is

$$f(\lambda)=2\lambda\log\lambda-2\lambda+\frac{1}{2\lambda}+\frac{1}{2(1-\theta)\lambda}, \tag{77}$$

with θ given by (76). One can also inject the expression (76) into (77) so f is also a function of θ. It can be checked that $f(\theta)$ has only one root on $(0,1/2)$ which is approximately 0.1575, and the corresponding value for λ_- is 0.60. See the plot of the function $\theta\to f(\theta)$, and related Mathematica computations.

```
In[]:= f[x_] := Sqrt[x/(2*(1-x)*(Log[2] + x* Log[x] + (1-x)*Log[1-x]))]
       g[x_] := 2*f[x]*Log[f[x]] - 2*f[x] + 1/(2*f[x]) + 1/(2*(1-x)*f[x])
       Plot[g[x], {x, 0, 0.5}]
       FindRoot[g[z], {z, 0.2}]
       f[0.157505]
```




```
Out[]= {z → 0.157505}

Out[]= 0.602343
```

Figure 2: Function f

One can continue this procedure, for instance to split $[0,t]$ into $[0,t/3]$, $[t/3,2t/3]$ and $[2t/3,t]$... to get better and better numerical values of λ_- so that $\lambda_-\sqrt{t}$ is the threshold for the $\exp(-\sqrt{t})$ lower tail bound. But it is not clear whether this approach will get all the way to $\sqrt{2}-$.

For the upper tail bound, write $a_j=2^j b_j$, and we get $b_1=1/\sqrt{2}$,

$$b_{j+1}=\frac{j+1}{j+\frac{1}{2}_j}+\frac{1}{\sqrt{2}}(j+1)\quad\text{for } j\geq 1. \tag{78}$$

It is not hard to prove by induction that $b_j \leq j^2$, and hence, $a_j \leq 2^j j^2$ for $j \geq 1$. Now consider the moment generating function $\mathsf{E}e^{\theta N_t^2}$ for $\theta > 0$. Note that

$$\begin{aligned}
\mathsf{E}e^{\theta N_t^2} &= \sum_{j\geq 0} \frac{\theta^j \mathsf{E} N_t^{2j}}{j!} \\
&\lesssim \sum_{j\geq 0} \frac{(2t\theta)^j}{j!} + \sum_{j\geq 1} \frac{j^2(2t\theta)^j}{j!} t^{-\frac{1}{2}} \\
&\approx e^{2\theta t} + e^{2\theta t}\Theta(t^{\frac{3}{2}}).
\end{aligned} \tag{79}$$

Next by the Chernoff bound,

$$\mathsf{P}\left(N_t^2 - 2t > \delta t\right) \leq e^{-\delta t}\mathsf{E}e^{N_t^2 - 2t} = e^{-\delta t}\Theta(t^{\frac{3}{2}}) \leq e^{-\delta t/2}, \tag{80}$$

for t sufficiently large. Here (82) establishes the upper tail bound for N_t as $t \to \infty$. (More rigorously, one needs to do a truncation in (81) which will entail small polynomial terms in t).