DYNAMICS OF SHARES IN A PROOF-OF-STAKE PROTOCOL – ASYMPTOTICS AND PHASE TRANSITIONS

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Abstract.

Key words:

1. Finite population model

Let $K \in \mathbb{N}_+$ be the number of investors, and $N \in \mathbb{R}_+$ be the number of initial coins/tokens. The investors are indexed by $[K] := \{1, \dots, K\}$, and investor k's initial endowment of coins is $n_{k,0}$ with $\sum_{k=1}^K n_{k,0} = N$. We define the investor share as the fraction of coins each investor owns. So the initial investor shares $(\pi_{k,0} : k \in [K])$ are given by

$$\pi_{k,0} := \frac{n_{k,0}}{N}, \quad k \in [K]. \tag{1.1}$$

Similarly, we denote by $n_{k,t}$ the number of coins owned by investor k at time $t \in \mathbb{N}_+$, and the corresponding share is

$$\pi_{k,t} := \frac{n_{k,t}}{N_t}, \quad k \in [K] \quad \text{with } N_t := \sum_{k=1}^K n_{k,t}. \tag{1.2}$$

Here N_t is the total number of coins at time t, and thus $N_0 = N$. Clearly, for each $t \geq 0$ $(\pi_{k,t},\, k \in [K])$ forms a probability distribution on $[K]$.

Now we describe the proof-of-stake (PoS) dynamics, which follows a time-dependent Pólya urn scheme. At time $t \in \mathbb{N}_+$, investor k is selected at random among K investors with probability $\pi_{k,t-1}$. Once selected, the investor receives a deterministic reward of $R_t \in \mathbb{R}_+$ coins. Let $S_{k,t}$ be the random event that investor k is selected at time t. Thus, the number of coins owned by each investor evolve as

$$n_{k,t} = n_{k-1,t} + R_t 1_{S_{k,t}}, \quad k \in [K]. \tag{1.3}$$

Note that the total number of coins satisfy $N_t = N_{t-1} + R_t$. Combining (1.2) and (1.3) yields a recursion of the investor shares:

$$\pi_{k,t} = \frac{N_{t-1}}{N_t} \pi_{k,t-1} + \frac{R_t}{N_t} 1_{S_{k,t}}, \quad k \in [K]. \tag{1.4}$$

Let $\mathcal{F}_t$ be the filtration as the σ-algebra generated by the random events $(S_{k,r} : k \in [K], r \leq t)$. It is easily seen that for each $k \in [K]$, the process of investor k's share $(\pi_{k,t},\, t \geq 0)$ is an $\mathcal{F}_t$-martingale. By the martingale convergence theorem,

$$(\pi_{1,t}, \dots, \pi_{K,t}) \longrightarrow (\pi_{1,\infty}, \dots, \pi_{K,\infty}) \quad \text{as } t \to \infty \text{ with probability 1}, \tag{1.5}$$

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where $(\pi_{1,\infty}, \dots, \pi_{K,\infty})$ is some random probability measure on $[K]$.

As in Roşu and Saleh (2021), we are interested in the evolution of the investor shares $(\pi_{k,t},\ k \in [K])$, as well as its long time limit $(\pi_{k,\infty},\ k \in [K])$. One major problem is to know whether the PoS protocol triggers the 'rich get richer' phenomenon by comparing the investor shares at the initial time and in the long time limit. Roşu and Saleh (2021) took the first step to consider the case of large investors, i.e. $\pi_{k,0} = \Theta(1)$ or equivalently $n_{k,0} = \Theta(N)$. They proved under various reward assumptions that the limiting shares do not deviate much from the initial ones as the initial number of coins N is large:

$$\lim_{N \to \infty} \mathbb{P}(|\pi_{k,\infty} - \pi_{k,0}| > \varepsilon) = 0 \quad \text{for each } k \in [K] \text{ and each fixed } \varepsilon > 0. \tag{1.6}$$

That is to say, the 'rich get richer' phenomenon does not occur when there is a small number $K = \Theta(1)$ of investors each having a large proportion $\pi_{k,0} = \Theta(1)$ of initial coins. However, there may also be intermediate or even small investors whose initial shares $\pi_{k,0} = o(1)$ is relatively small. For instance,

- When the number of investors $K \approx N$, there are intermediate investors with initial endowment of coins $n_{k,0} = f(N)$ such that $f(N) \to \infty$ and $f(N)/N \to 0$, and small investors with initial number of coins $n_{k,0} = \Theta(1)$.
- When the number of investors $K \gg N$, there may also exist tiny investors who own fractional number of coins $n_{k,0} = o(1)$ (this is the case with Robinhood Crypto).

Here we study the evolution of shares for different types of investors, encompassing all the above instances. In particular, we show phase transitions in terms of evolution of shares for different types of investors. Note that if $\pi_{k,0} = o(1)$, for each fixed $\varepsilon > 0$ the probability $\mathbb{P}(|\pi_{k,\infty} - \pi_{k,0}| > \varepsilon)$ always goes to 0 as $N \to \infty$. Thus, we consider the ratios $\pi_{k,t}/\pi_{k,0}$ or $\pi_{k,\infty}/\pi_{k,0}$ instead of the differences.

1.1. **Constant reward.** We start with the case where the reward $R_t \equiv R > 0$. Let $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ be the Gamma function. Recall that the Dirichlet distribution with parameters $(a_1, \dots, a_K)$, which we simply denote $\mathrm{Dir}(a_1, \dots, a_K)$, has support on the standard simplex $\{(x_1, \dots, x_K) \in \mathbb{R}_+^K : \sum_{k=1}^K x_k = 1\}$ and has density

$$f(x_1, \dots, x_K) = \frac{\Gamma\left(\sum_{k=1}^K a_k\right)}{\prod_{k=1}^K \Gamma(a_k)} \prod_{k=1}^K x_k^{a_k - 1}. \tag{1.7}$$

For $K = 2$, the Dirichlet distribution reduces to the beta distribution which is denoted by $\mathrm{Beta}(a_1, a_2)$. It is easily seen that if $(x_1, \dots, x_K) \overset{d}{=} \mathrm{Dir}(a_1, \dots, a_K)$ then for each $k \in [K]$ $x_k \overset{d}{=} \mathrm{Beta}(a_k, \sum_{j \neq k} a_j)$. The following result is concerned with the evolution of shares in a PoS protocol with constant reward.

Theorem 1.1. *Assume that the coin reward is $R_t \equiv R > 0$. Then the investor shares have a limiting distribution*

$$(\pi_{1,\infty}, \dots, \pi_{K,\infty}) \overset{d}{=} \mathrm{Dir}\left(\frac{n_{1,0}}{R}, \dots, \frac{n_{K,0}}{R}\right). \tag{1.8}$$

Moreover,

(i) *For* $n_{k,0} = f(N)$ *such that* $f(N) \to \infty$ *as* $N \to \infty$ *(i.e.* $1/N = o(\pi_{k,0})$*), we have for each* $\varepsilon > 0$ *and each* $t \geq 1$ *or* $t = \infty$*:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \leq \frac{5R}{4\varepsilon^2 f(N)}, \tag{1.9}$$

which converges to 0 *as* $N \to \infty$.

(ii) *For* $n_{k,0} = \Theta(1)$ *(i.e.* $\pi_{k,0} = \Theta(1/N)$*), we have the convergence in distribution:*

$$\frac{\pi_{k,\infty}}{\pi_{k,0}} \xrightarrow{d} \frac{R}{n_{k,0}}\gamma\left(\frac{n_{k,0}}{R}\right) \quad \text{as } N \to \infty, \tag{1.10}$$

where $\gamma\left(\frac{n_{k,0}}{R}\right)$ *is a Gamma random variable with density* $x^{\frac{n_{k,0}}{R}-1}e^{-x}1_{x>0}/\Gamma\left(\frac{n_{k,0}}{R}\right)$.

(iii) *For* $n_{k,0} = o(1)$ *(i.e.* $\pi_{k,0} = o(1/N)$*), we have* $\mathrm{Var}(\pi_{k,\infty}/\pi_{k,0}) \to \infty$ *as* $N \to \infty$*. Moreover, for each* $\varepsilon > 0$*:*

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} < \varepsilon\right) \to 1 \quad \text{as } N \to \infty. \tag{1.11}$$

Now we make some comments on Theorem 1.1. The theorem reveals a phase transition of shares in the long run between large, intermediate and small, tiny investors. Part (i) shows that for large and intermediate investors, their shares are stable in the sense that the ratio between the share at a later time t and the initial share, i.e. $\pi_{k,t}/\pi_{k,0}$, converges in probability to 1 as the initial coin offering $N \to \infty$. In particular, for $n_{k,0} = \Theta(N)$ (i.e. $\pi_{k,0} = \Theta(1)$) this is equivalent to the stability of large investor's shares, i.e. (1.6) as proved in Roşu and Saleh (2021). Also note that the concentration bound (1.9) is uniform in time t, implying that the 'rich get richer' phenomenon does not occur at any time. On the other hand, the evolution of shares for small or tiny investors has very different limiting behaviors. Part (iii) shows that small investor's shares are volatile in such a way that the ratio $\pi_{k,\infty}/\pi_{k,0}$ is approximated by a gamma distribution independent of the initial coin offerings, and hence $\mathrm{Var}(\pi_{k,\infty}/\pi_{k,0}) \approx \frac{1}{n_{k,0}}$. For instance, if $n_{k,0} = R = 1$ the limiting distribution of the ratio $\pi_{k,\infty}/\pi_{k,0}$ reduces to the exponential distribution with parameter 1. Thus,

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} > \theta\right) \approx e^{-\theta} \quad \text{as } N \to \infty.$$

So with probability $e^{-2} \approx 0.135$ a small investor's share will double, and with probability $1 - e^{-0.5} \approx 0.393$ this investor's share will be halved. Part (iv) shows that for tiny investors, their shares will be shrinking along the time, and the ratio $\pi_{k,\infty}/\pi_{k,0}$ converges to 0 in probability as $N \to \infty$. Finally, we note that the results in Theorem 1.1 hold jointly for a finite number of investors belonging to the same category:

- For $n_{k_1,0}, \ldots, n_{k_\ell,0}$ with $k_1, \ldots, k_\ell \in [K]$ satisfying the conditions in (i),
$$\left(\frac{\pi_{k_1,t}}{\pi_{k_1,0}}, \ldots, \frac{\pi_{k_\ell,t}}{\pi_{k_\ell,0}}\right) \to 1 \quad \text{in probability as } N \to \infty.$$
- For $n_{k_1,0}, \ldots, n_{k_\ell,0}$ with $k_1, \ldots, k_\ell \in [K]$ satisfying the condition in (ii),
$$\left(\frac{\pi_{k_1,\infty}}{\pi_{k_1,0}}, \ldots, \frac{\pi_{k_\ell,\infty}}{\pi_{k_\ell,0}}\right) \xrightarrow{d} \left(\frac{R}{n_{k_1,0}}\gamma\left(\frac{n_{k_1,0}}{R}\right), \ldots, \frac{R}{n_{k_\ell,0}}\gamma\left(\frac{n_{k_\ell,0}}{R}\right)\right) \quad \text{as } N \to \infty,$$
where $\gamma(n_{k_1,0}/R), \ldots, \gamma(n_{k_\ell,0}/R)$ are independent Gamma random variables.

- For $n_{k_1,0}, \dots, n_{k_\ell,0}$ with $k_1, \dots, k_\ell \in [K]$ satisfying the conditions in (iii),

$$\left(\frac{\pi_{k_1,t}}{\pi_{k_1,0}}, \dots, \frac{\pi_{k_\ell,t}}{\pi_{k_\ell,0}}\right) \to 0 \quad \text{in probability as } N \to \infty.$$

Proof of Theorem 1.1. The fact that the investor shares $(\pi_{1,t}, \dots, \pi_{K,t})$ converges almost surely as $t \to \infty$ to $(\pi_{1,\infty}, \dots, \pi_{K,\infty})$ with the Dirichlet distribution (1.8) is a classical result of Pólya urn, see e.g. Section 6.3 of Johnson and Kotz (1977).

(i) Since $(\pi_{k,t},\, t \ge 0)$ is a martingale, we have $\mathbb{E}(\pi_{k,t}) = \pi_{k,0}$. It follows from Corollary 3.1 of Mahmoud (2009) or Lemma 2.2 of Goldstein and Reinert (2013) that

$$\operatorname{Var}(\pi_{k,t}) = \frac{R^2}{(Rt+N)^2}\left(\frac{R}{N+R}t^2 + \frac{N}{N+R}t\right)\pi_{k,0}(1-\pi_{k,0}). \tag{1.12}$$

Specializing (1.12) to $t = \infty$, we obtain $\operatorname{Var}(\pi_{k,\infty}) = \frac{R}{N+R}\pi_{k,0}(1-\pi_{k,0})$ which recovers Corollary 1 of Roşu and Saleh (2021). By applying Chebyshev's inequality, we get

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) &\le \frac{\operatorname{Var}(\pi_{k,t})}{\varepsilon^2 \pi_{k,0}^2} \\
&= \frac{1-\pi_{k,0}}{\varepsilon^2 \pi_{k,0}}\left(\frac{R^3 t^2}{(Rt+N)^2(N+R)} + \frac{R^2 N t}{(Rt+N)^2(N+R)}\right) \\
&\le \frac{1-\pi_{k,0}}{\varepsilon^2 \pi_{k,0}}\left(\frac{R}{N+R} + \frac{R}{4(N+R)}\right),
\end{aligned} \tag{1.13}$$

which leads to (1.9) by noting that $\pi_{k,0} = \frac{f(N)}{N}$ and $1 - \pi_{k,0} \le 1$.

(ii) Since $(\pi_{1,\infty}, \dots, \pi_{K,\infty}) \stackrel{d}{=} \operatorname{Dir}\left(\frac{n_{1,0}}{R}, \dots, \frac{n_{K,0}}{R}\right)$, we have $\pi_{k,\infty} \stackrel{d}{=} \operatorname{Beta}(\frac{n_{k,0}}{R}.\frac{N-n_{k,0}}{R})$. Now $n_{k,0} = \Theta(1)$, by standard limit theorem we get $N\pi_{k,\infty}/R \xrightarrow{d} \gamma\left(\frac{n_{k,0}}{R}\right)$. This implies the convergence in distribution (1.10).

(iii) By (ii) we have $\frac{\pi_{k,\infty}}{\pi_{k,0}} \stackrel{d}{=} \frac{R}{n_{k,0}}\gamma\left(\frac{n_{k,0}}{R}\right)$. Let $a = \frac{n_{k,0}}{R}$ so $a \to 0$ as $N \to \infty$ by hypothesis. Then

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} < \varepsilon\right) = \mathbb{P}\left(\frac{\gamma(a)}{a} \le \varepsilon\right) = \frac{1}{\Gamma(a)}\int_0^{a\varepsilon} x^{a-1}e^{-x}dx.$$

Note that $\int_0^{a\varepsilon} x^{a-1}e^{-x}dx \ge e^{-a\varepsilon}\frac{(a\varepsilon)^a}{a}$, and $a\Gamma(a) \to 1$ as $a \to 0$. As a result, there exists a constant $C > 0$ (independent of a) such that

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} < \varepsilon\right) \ge Ce^{-a\varepsilon}(a\varepsilon)^a = C\exp\left(-a\varepsilon + a\log a + a\log\varepsilon\right),$$

which converges to 1 as $a \to 0$. This yields the convergence (1.11). □

1.2. **Decreasing reward.**

Theorem 1.2. *Assume that the coin reward is R_t with $R_t \ge R_{t+1}$ for each $t \ge 0$.*

(1) If R_t is bounded away from 0, i.e. $\lim_{t\ge 0} R_t = \underline{R} > 0$, then

(i) For $n_{k,0} = f(N)$ such that $f(N) \to \infty$ as $N \to \infty$ (i.e. $1/N = o(\pi_{k,0})$), we have for each $\varepsilon > 0$ and each $t \ge 1$ or $t = \infty$:

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \le \frac{R_1}{\varepsilon^2 f(N)}, \tag{1.14}$$

which converges to 0 *as* $N \to \infty$.

(ii) For $n_{k,0} = \Theta(1)$ *(i.e.* $\pi_{k,0} = \Theta(1/N)$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \Theta(1)$. *Moreover, there exists* $c > 0$ *independent of* N *such that for* $\varepsilon > 0$ *sufficiently small:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \geq c. \tag{1.15}$$

(iii) For $n_{k,0} = o(1)$ *(i.e.* $\pi_{k,0} = o(1/N)$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$ *as* $N \to \infty$. *Moreover, there exists* $c > 0$ *independent of* N *such that for* $\varepsilon > 0$ *sufficiently small:*

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} > \frac{\varepsilon}{\sqrt{n_{k,0}}}\right) \geq c n_{k,0}. \tag{1.16}$$

(2) If $R_t = \Theta(t^{-\alpha})$ *for* $\alpha > \frac{1}{2}$*, then*

(i) For $n_{k,0} > 0$ *such that* $N n_{k,0} \to \infty$ *as* $N \to \infty$ *(i.e.* $N^2 \pi_{k,0} \to 0$*), we have for each* $\varepsilon > 0$ *and each* $t \geq 1$ *or* $t = \infty$*:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \leq \frac{\sum_{t\geq 1} R_t^2}{N n_{k,0}}, \tag{1.17}$$

which converges to 0 *as* $N \to \infty$.

(ii) For $n_{k,0} = \Theta(1/N)$ *(i.e.* $\pi_{k,0} = \Theta(1/N^2)$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \Theta(1)$.

(iii) For $n_{k,0} = o(1/N)$ *(i.e.* $\pi_{k,0} = o(1/N^2)$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$ *as* $N \to \infty$.

In the case of (ii) and (iii), there exists $c > 0$ *independent of* N *such that for* $\varepsilon > 0$ *sufficiently small:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \frac{\varepsilon}{\sqrt{N n_{k,0}}}\right) \geq c \pi_{k,0}. \tag{1.18}$$

(3) If $R_t = \Theta(t^{-\alpha})$ *for* $\alpha < \frac{1}{2}$*, then*

(i) For $n_{k,0} > 0$ *such that* $N^{\frac{\alpha}{1-\alpha}} n_{k,0} \to \infty$ *as* $N \to \infty$ *(i.e.* $N^{\frac{1}{1-\alpha}} \pi_{k,0} \to 0$*), there exists* $C > 0$ *independent of* t *and* N *such that for each* $\varepsilon > 0$ *and each* $t \geq 1$ *or* $t = \infty$*:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \leq \frac{C}{N^{\frac{\alpha}{1-\alpha}} n_{k,0}}, \tag{1.19}$$

which converges to 0 *as* $N \to \infty$.

(ii) For $n_{k,0} = \Theta(N^{-\frac{\alpha}{1-\alpha}})$ *(i.e.* $\pi_{k,0} = \Theta(N^{-\frac{1}{1-\alpha}})$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \Theta(1)$. *Moreover, there exists* $c > 0$ *independent of* N *such that for* $\varepsilon > 0$ *sufficiently small:*

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \geq c. \tag{1.20}$$

(iii) For $n_{k,0} = o(N^{-\frac{\alpha}{1-\alpha}})$ *(i.e.* $\pi_{k,0} = o(N^{-\frac{1}{1-\alpha}})$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$ *as* $N \to \infty$. *Moreover, there exists* $c > 0$ *independent of* N *such that for* $\varepsilon > 0$

sufficiently small:

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \frac{\varepsilon}{\sqrt{N^{\frac{\alpha}{1-\alpha}} n_{k,0}}}\right) \geq c N^{\frac{\alpha}{1-\alpha}} n_{k,0}. \tag{1.21}$$

To prove Theorem 1.2, we need a series of lemmas.

Lemma 1.3. *The variance of investor k's share is*

$$\mathrm{Var}(\pi_{k,t}) = a_t \pi_{k,0}(1 - \pi_{k,0}), \tag{1.22}$$

where

$$a_1 = \left(\frac{R_1}{N_1}\right)^2, \quad a_{t+1} = a_t + \left(\frac{R_{t+1}}{N_{t+1}}\right)^2 (1 - a_t). \tag{1.23}$$

The third central moment of investor k's share satisfy the relation

$$\begin{aligned}\mu_3(\pi_{k,t+1}) = \mu_3(\pi_{k,t}) &+ \frac{3R_{t+1}^2}{N_{t+1}^2}\mathbb{E}[(\pi_{k,t} - \pi_{k,0})\pi_{k,t}(1 - \pi_{k,t})] \\ &+ \frac{R_{t+1}^3}{N_{t+1}^3}\mathbb{E}[\pi_{k,t}(1 - \pi_{k,t})(1 - 2\pi_{k,t})],\end{aligned} \tag{1.24}$$

and the fourth central moment of investor k's share satisfy the relation

$$\begin{aligned}\mu_4(\pi_{k,t+1}) = \mu_4(\pi_{k,t}) &+ \frac{6R_{t+1}^2}{N_{t+1}^2}\mathbb{E}[(\pi_{k,t} - \pi_{k,0})^2\pi_{k,t}(1 - \pi_{k,t})] \\ &+ \frac{4R_{t+1}^3}{N_{t+1}^3}\mathbb{E}[(\pi_{k,t} - \pi_{k,0})\pi_{k,t}(1 - \pi_{k,t})(1 - 2\pi_{k,t})] \\ &+ \frac{R_{t+1}^4}{N_{t+1}^4}\mathbb{E}[\pi_{k,t}(1 - \pi_{k,t})(1 - 3\pi_{k,t} + 3\pi_t^2)].\end{aligned} \tag{1.25}$$

Proof. The formula (1.22)–(1.23) for the variance of investor's share is read from Lemma A.2 of Roşu and Saleh (2021). It is easily seen from (1.4) that

$$\pi_{k,t+1} - \pi_{k,0} = (\pi_{k,t} - \pi_{k,0}) + \frac{R_{t+1}}{N_{t+1}}(1_{S_{k,t+1}} - \pi_{k,t}). \tag{1.26}$$

Taking the third and the fourth moment on both sides of (1.26), and using the binomial expansion yields the formulas (1.24)–(1.25). □

Lemma 1.4. *Assume that the reward R_t is decreasing, i.e. $R_t \geq R_{t+1}$ for each $t \geq 0$, and that $\lim_{t\to\infty} R_t = \underline{R} > 0$.*

(1) Let a_t be defined by (1.23). We have

$$\frac{(N - R_1)\underline{R}^2 t}{N(N + R_1)(N + R_1(1 + t))} \leq a_t \leq \frac{R_1}{N}, \quad \textit{for each } t \geq 1. \tag{1.27}$$

(2) Let $\mu_3(\pi_{k,t})$ and $\mu_4(\pi_{k,t})$ be the third and the fourth central moment of investor k's share satisfying (1.24), (1.25) respectively. If $\pi_{k,0} = \mathcal{O}(1/N)$, there exist $C_3, C_4 > 0$ independent of t and N such that

$$\mu_3(\pi_{k,t}) \leq \frac{C_3 \pi_{k,0}}{N^2}, \quad \mu_4(\pi_{k,t}) \leq \frac{C_4 \pi_{k,0}}{N^3} \quad \textit{for each } t \geq 1. \tag{1.28}$$

Proof. (1) The upper bound $a_t \le \frac{R_1}{N}$ follows from Lemma A.4 of Roşu and Saleh (2021). By (1.23), we get $a_{t+1} \ge a_t + \left(1 - \frac{R_1}{N}\right)\left(\frac{R_{t+1}}{N_{t+1}}\right)^2$. This implies that for $t \ge 1$,

$$\begin{aligned} a_t &\ge \left(1 - \frac{R_1}{N}\right)\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^2 \\ &\ge \left(1 - \frac{R_1}{N}\right)\sum_{n=1}^{t}\left(\frac{R_n}{N + nR_1}\right)^2, \end{aligned} \tag{1.29}$$

where the second inequality is due to the fact that R_t is decreasing. By abuse of language, let $(R_s,\, s \in [1,\infty))$ be the linear interpolation of $(R_t,\, t = 1, 2, \ldots)$. Since $s \to \frac{R_s}{N+sR_1}$ is decreasing, we deduce from (1.29) that

$$a_t \ge \left(1 - \frac{R_1}{N}\right)\int_1^{t+1}\frac{R_s^2}{(N+sR_1)^2}ds \ge \left(1 - \frac{R_1}{N}\right)\int_1^{t+1}\frac{\underline{R}^2}{(N+sR_1)^2}ds,$$

which yields the lower bound in (1.27).
(2) Note that $(\pi_{k,t} - \pi_{k,0})\pi_{k,t}(1-\pi_{k,t}) \le \pi_{k,t}^2(1-\pi_{k,t}) \le \pi_{k,t}^2$. Then for each $t \ge 1$,

$$\begin{aligned} \mathbb{E}[(\pi_{k,t} - \pi_{k,0})\pi_{k,t}(1-\pi_{k,t})] \le \mathbb{E}(\pi_{k,t}^2) &= \pi_{k,0}^2 + \mathrm{Var}(\pi_{k,t}) \\ &\le \pi_{k,0}^2 + \frac{R_1\pi_{k,0}}{N} \\ &\le \frac{C\pi_{k,0}}{N} \quad \text{for some } C > 0, \end{aligned} \tag{1.30}$$

where the second inequality follows from the upper bound in (1.27), and the last inequality is due to the fact that $\pi_{k,0} = \mathcal{O}(1/N)$. Also for each $t \ge 1$,

$$\mathbb{E}[\pi_{k,t}(1-\pi_{k,t})(1-2\pi_{k,t})] \le \mathbb{E}(\pi_{k,t}) = \pi_{k,0}. \tag{1.31}$$

Combining (1.24), (1.30) and (1.31), we get

$$\mu_3(\pi_{k,t}) \le \frac{3C\pi_{k,0}}{N}\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^2 + \pi_{k,0}\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^3. \tag{1.32}$$

Using the sum-integral trick as in Lemma A.4 of Roşu and Saleh (2021) we deduce that $\sum_{n=1}^{t}\left(\frac{R_n}{N+\sum_{k=1}^{n} R_k}\right)^k \le (R_1/N)^{k-1}$ for $k = 1, 2, \ldots$ So the bound (1.32) leads to

$$\mu_3(\pi_{k,t}) \le 3CR_1\frac{\pi_{k,0}}{N^2} + R_1^2\frac{\pi_{k,0}}{N^2} \le \frac{C_3\pi_{k,0}}{N^2} \quad \text{for some } C_3 > 0.$$

Similarly, we get from (1.25) that

$$\begin{aligned} \mu_4(\pi_{k,t}) \le \frac{C\pi_{k,0}}{N^2}\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^2 &+ \frac{C'\pi_{k,0}}{N}\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^3 \\ &+ C''\pi_{k,0}\sum_{n=1}^{t}\left(\frac{R_n}{N + \sum_{k=1}^{n} R_k}\right)^4 \\ \le CR_1\frac{\pi_{k,0}}{N^3} + C'R_1^2\frac{\pi_{k,0}}{N^3} + C''R_1^3\frac{\pi_{k,0}}{N^3} &\le \frac{C_4\pi_{k,0}}{N^3} \quad \text{for some } C_4 > 0. \end{aligned}$$

$\square$

Lemma 1.5. *Assume that the reward R_t is decreasing, i.e. $R_t \ge R_{t+1}$ for each $t \ge 0$, and that $R_t = \Theta(t^{-\alpha})$ for $\alpha > \frac{1}{2}$.*

(1) Let a_t be defined by (1.23). *We have*

$$\frac{R_1^2}{(N+R_1)^2} \le a_t \le \frac{\sum_{t\ge 1} R_t^2}{N^2}, \quad \textit{for each } t \ge 1. \tag{1.33}$$

(2) Let $\mu_3(\pi_{k,t})$ and $\mu_4(\pi_{k,t})$ be the third and the fourth central moment of investor k's share satisfying (1.24), (1.25) *respectively. If $\pi_{k,0} = \mathcal{O}(1/N^2)$, there exist $C_3, C_4 > 0$ independent of t and N such that*

$$\mu_3(\pi_{k,t}) \le \frac{C_3 \pi_{k,0}}{N^3}, \quad \mu_4(\pi_{k,t}) \le \frac{C_4 \pi_{k,0}}{N^4} \quad \textit{for each } t \ge 1. \tag{1.34}$$

Proof. (1) By Lemma A.3 of Roşu and Saleh (2021), the sequence $(a_t,\, t \ge 1)$ is increasing. So the lower bound in (1.33) follows from the fact that $a_t \ge a_1$. Further by (1.23), we get

$$a_t \le \sum_{n=1}^t \left(\frac{R_n}{N + \sum_{k=1}^n R_k} \right)^2 \le \sum_{n=1}^t \frac{R_n^2}{N^2} \le \frac{\sum_{t \ge 1} R_t^2}{N^2}, \tag{1.35}$$

where the last inequality is due to the fact that $R_t = \Theta(t^{-\alpha})$ for $\alpha > \frac{1}{2}$ so $\sum_{t\ge 1} R_t^2 < \infty$.

(2) Note that $\sum_{n=1}^t \left(\frac{R_n}{N+\sum_{k=1}^n R_k} \right)^k \le \sum_{t \ge 1} R_t^k / N^k$ for $k = 2, 3, \ldots$ Using the same argument as in Lemma 1.4 and the fact that $\pi_{k,0} = \mathcal{O}(1/N^2)$, we get

$$\begin{aligned} \mu_3(\pi_{k,t}) &\le \frac{C\pi_{k,0}}{N^2}\frac{1}{N^2} + C'\pi_{k,0}\frac{1}{N^3}, \\ \mu_4(\pi_{k,t}) &\le \frac{C\pi_{k,0}}{N^3}\frac{1}{N^2} + \frac{C'\pi_{k,0}}{N^2}\frac{1}{N^3} + C''\pi_{k,0}\frac{1}{N^4}, \end{aligned}$$

for some $C, C', C'' > 0$ independent of t and N. This clearly yields the bounds (1.34). $\square$

Lemma 1.6. *Assume that the reward R_t is decreasing, i.e. $R_t \ge R_{t+1}$ for each $t \ge 0$, and that $R_t = \Theta(t^{-\alpha})$ for $\alpha < \frac{1}{2}$.*

(1) Let a_t be defined by (1.23). *There exist $C > c > 0$ independent of t and N such that*

$$cN^{-\frac{1}{1-\alpha}} \le a_t \le CN^{-\frac{1}{1-\alpha}}, \quad \textit{for each } t \ge N^{-\frac{1}{1-\alpha}}. \tag{1.36}$$

(2) Let $\mu_3(\pi_{k,t})$ and $\mu_4(\pi_{k,t})$ be the third and the fourth central moment of investor k's share satisfying (1.24), (1.25) *respectively. If $\pi_{k,0} = \mathcal{O}(N^{-\frac{1}{1-\alpha}})$, there exist $C_3, C_4 > 0$ independent of t and N such that*

$$\mu_3(\pi_{k,t}) \le C_3 \pi_{k,0} N^{-\frac{2}{1-\alpha}}, \quad \mu_4(\pi_{k,t}) \le C_4 \pi_{k,0} N^{-\frac{3}{1-\alpha}} \quad \textit{for each } t \ge 1. \tag{1.37}$$

Proof. (1) It follows from (1.29) and (1.35) that

$$\frac{a_t}{\sum_{n=1}^t \left(\frac{R_n}{N + \sum_{k=1}^n R_k} \right)^2} \to 1 \quad \text{as } N \to \infty.$$

Thus, we need to study the asymptotic behavior of $\sum_{n=1}^t \left(\frac{R_n}{N+\sum_{k=1}^n R_k}\right)^2$ as $N \to \infty$. Since $R_t = \Theta(t^{-\alpha})$ for $\alpha < \frac{1}{2}$, we have:

$$c\sum_{n=1}^t \frac{1}{(Nn^\alpha + n)^2}. \le \sum_{n=1}^t \left(\frac{R_n}{N+\sum_{k=1}^n R_k}\right)^2 \le C\sum_{n=1}^t \frac{1}{(Nn^\alpha + n)^2}. \tag{1.38}$$

for some $C > c > 0$ independent of t and N. Again by the sum-integral trick, we get

$$\int_1^{t+1} \frac{ds}{(Ns^\alpha + s)^2} \le \sum_{n=1}^t \frac{1}{(Nn^\alpha + n)^2} \le \frac{1}{N^2} + \int_1^\infty \frac{ds}{(Ns^\alpha + s)^2}. \tag{1.39}$$

Now for $t \ge N^{\frac{1}{1-\alpha}}$, we have

$$\begin{aligned}\int_1^t \frac{ds}{(Ns^\alpha + s)^2} &= \int_1^{N^{\frac{1}{1-\alpha}}} \frac{ds}{(Ns^\alpha + s)^2} + \int_{N^{\frac{1}{1-\alpha}}}^t \frac{ds}{(Ns^\alpha + s)^2} \\ &\le \frac{1}{N^2}\int_1^{N^{\frac{1}{1-\alpha}}} \frac{ds}{s^{2\alpha}} + \int_{N^{\frac{1}{1-\alpha}}}^t \frac{ds}{s^2} \le C'N^{-\frac{1}{1-\alpha}},\end{aligned} \tag{1.40}$$

and

$$\begin{aligned}\int_1^t \frac{ds}{(Ns^\alpha + s)^2} &= \int_1^{N^{\frac{1}{1-\alpha}}} \frac{ds}{(Ns^\alpha + s)^2} + \int_{N^{\frac{1}{1-\alpha}}}^t \frac{ds}{(Ns^\alpha + s)^2} \\ &\ge \frac{1}{4N^2}\int_1^{N^{\frac{1}{1-\alpha}}} \frac{ds}{s^{2\alpha}} + \int_{N^{\frac{1}{1-\alpha}}}^t \frac{ds}{4s^2} \ge C''N^{-\frac{1}{1-\alpha}},\end{aligned} \tag{1.41}$$

Combining (1.38), (1.39), (1.40) and (1.41) yields

$$cC'N^{-\frac{1}{1-\alpha}} \le \sum_{n=1}^t \left(\frac{R_n}{N+\sum_{k=1}^n R_k}\right)^2 \le CC'N^{-\frac{1}{1-\alpha}} \quad \text{for } t \ge N^{-\frac{1}{1-\alpha}}.$$

This leads to the bounds (1.36).

(2) Similar to (1.38)–(1.40), we have $\sum_{n=1}^t \left(\frac{R_n}{N+\sum_{k=1}^n R_k}\right)^k \le CN^{-\frac{k-1}{1-\alpha}}$ for $k = 2, 3, \ldots$ Using the same argument as in Lemma 1.4 and the fact that $\pi_{k,0} = \mathcal{O}(N^{-\frac{1}{1-\alpha}})$, we get

$$\begin{aligned}&\mu_3(\pi_{k,t}) \le C\pi_{k,0}N^{-\frac{1}{1-\alpha}}N^{-\frac{1}{1-\alpha}} + C'\pi_{k,0}N^{-\frac{2}{1-\alpha}}, \\ &\mu_4(\pi_{k,t}) \le C\pi_{k,0}N^{-\frac{2}{1-\alpha}}N^{-\frac{1}{1-\alpha}} + C'\pi_{k,0}N^{-\frac{1}{1-\alpha}}N^{-\frac{2}{1-\alpha}} + C''\pi_{k,0}N^{-\frac{3}{1-\alpha}},\end{aligned}$$

for some $C, C', C'' > 0$ independent of t and N. This gives the bounds (1.37). □

Proof of Theorem 1.2. (1) (*i*) By Chebyshev' inequality and the upper bound in (1.27), we get

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \le \frac{a_t(1-\pi_{k,0})}{\pi_{k,0}\varepsilon^2} \le \frac{R_1}{N\pi_{k,0}\varepsilon^2} = \frac{R_1}{n_{k,0}\varepsilon^2}.$$

This proves (1.14): the ratio $\pi_{k,t}/\pi_{k,0}$ converges in probability to 1 as $n_{k,0} = f(N) \to \infty$.

(ii) Note that $\operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \frac{Na_\infty(1-\pi_{k,0})}{n_{k,0}}$. It follows from (1.27) that

$$\frac{R^2}{R_1} \le \liminf_{N\to\infty} Na_\infty \le \limsup_{N\to\infty} Na_\infty \le R_1. \tag{1.42}$$

Since $n_{k,0} = \Theta(1)$, we deduce from the lower bound of (1.42) that there exists $c > 0$ independent of N such that $\operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \ge c$. For $\varepsilon > 0$, we have

$$\begin{aligned} c \le \operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) &= \mathbb{E}\left(\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right)^2 1_{\left\{\left|\frac{\pi_{k,\infty}}{\pi_{k,0}}-1\right|>\varepsilon\right\}}\right) + \mathbb{E}\left(\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right)^2 1_{\left\{\left|\frac{\pi_{k,\infty}}{\pi_{k,0}}-1\right|\le\varepsilon\right\}}\right) \\ &\le \sqrt{\mu_4\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right)\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right)} + \varepsilon^2. \end{aligned} \tag{1.43}$$

By the upper bound (1.28), we get $\mu_4\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \le \frac{C_4}{n_{k,0}^3} := C$. As a result, for $\varepsilon < \sqrt{c}$,

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \ge \frac{(c-\varepsilon^2)^2}{C},$$

which leads to the anti-concentration bound (1.20).
(iii) Again from (1.42) we obtain

$$\frac{R^2}{R_1} \le \liminf_{N\to\infty} n_{k,0} \operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \le \limsup_{N\to\infty} n_{k,0} \operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \le R_1,$$

which implies $\operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$ since $n_{k,0} = o(1)$. Further by substituiting c with $c/n_{k,0}$, and ε with $\varepsilon/\sqrt{n_{k,0}}$ in (1.43), we derive the tail bound (1.16).
(2) and (3) follow from Lemma 1.5 and 1.6 exactly in the same way as (1) is derived from Lemma 1.4. ☐

1.3. **Increasing reward.**

Theorem 1.7. *Assume that the reward $R_t = \rho N_{t-1}^\gamma$ for some $\rho > 0$ and $\gamma > 0$.*

(1) If $\gamma > 1$, then $\pi_{k,\infty} \in \{0,1\}$ almost surely with

$$\mathbb{P}(\pi_{k,\infty} = 1) = \pi_{k,0}, \quad \mathbb{P}(\pi_{k,\infty} = 0) = 1 - \pi_{k,0} \tag{1.44}$$

(2) If $\gamma < 1$, then

(i) For $n_{k,0} = f(N)$ such that $f(N)/N^\gamma \to \infty$ as $N \to \infty$ (i.e. $\pi_{k,0})/N^{\gamma-1} \to \infty$), we have for each $\varepsilon > 0$ and each $t \ge 1$ or $t = \infty$:

$$\mathbb{P}\left(\left|\frac{\pi_{k,t}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \le \frac{\rho N^\gamma}{(1-\gamma)n_{k,0}\varepsilon^2}, \tag{1.45}$$

which converges to 0 as $N \to \infty$.

(ii) For $n_{k,0} = \Theta(N^\gamma)$ (i.e. $\pi_{k,0} = \Theta(N^{\gamma-1})$), we have $\operatorname{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = \Theta(1)$. Moreover, there exists $c > 0$ independent of N such that for $\varepsilon > 0$ sufficiently small:

$$\mathbb{P}\left(\left|\frac{\pi_{k,\infty}}{\pi_{k,0}} - 1\right| > \varepsilon\right) \ge c. \tag{1.46}$$

(iii) For $n_{k,0} = o(N^\gamma)$ *(i.e.* $\pi_{k,0} = o(N^{\gamma-1})$*), we have* $\mathrm{Var}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) \to \infty$ *as* $N \to \infty$*. Moreover, there exists* $c > 0$ *independent of* N *such that for* $\varepsilon > 0$ *sufficiently small:*

$$\mathbb{P}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}} > \varepsilon\sqrt{\frac{N^\gamma}{n_{k,0}}}\right) \ge c\frac{n_{k,0}}{N^\gamma}. \tag{1.47}$$

To prove Theorem 1.7, we need the following lemma.

Lemma 1.8. *Assume that the reward* $R_t = \rho N_{t-1}^\gamma$ *for some* $\rho > 0$ *and* $\gamma \in [0,1)$*.*

(1) Let a_t *be defined by* (1.23)*. There exist* $c > 0$ *independent of* t *and* N *such that*

$$cN^{\gamma-1} \le a_t \le \frac{\rho}{1-\gamma}N^{\gamma-1}, \quad \textit{for } t \textit{ sufficiently large.} \tag{1.48}$$

(2) Let $\mu_3(\pi_{k,t})$ *and* $\mu_4(\pi_{k,t})$ *be the third and the fourth central moment of investor k's share satisfying* (1.24), (1.25) *respectively. If* $\pi_{k,0} = \mathcal{O}(N^{\gamma-1})$*, there exist* $C_3, C_4 > 0$ *independent of* t *and* N *such that*

$$\mu_3(\pi_{k,t}) \le C_3\pi_{k,0}N^{2\gamma-2}, \quad \mu_4(\pi_{k,t}) \le C_4\pi_{k,0}N^{3\gamma-3} \quad \textit{for each} t \ge 1. \tag{1.49}$$

Proof. (1) The upper bound $a_t \le \frac{\rho}{1-\gamma}N^{\gamma-1}$ follows from Lemma A.5 of Roşu and Saleh (2021). Recall that $N_{t+1} = N_t + \rho N_t^\gamma$, and thus N_t behaves asymptotically as $t^{\frac{1}{1-\gamma}}$. As a result, there exists $c > 0$ independent of t and N such that $N_t \ge cN_{t+1}$ for t sufficiently large. This also implies that $R_t \ge c^\gamma R_{t+1}$ for t sufficiently large. By (1.29), we have

$$\begin{aligned} a_t &\ge \left(1 - \frac{\rho}{1-\gamma}N^{\gamma-1}\right)\sum_{n=1}^t\left(\frac{R_n}{N_n}\right)^2 \\ &\ge c'\sum_{n=1}^t\left(\frac{R_{n+1}}{N_n}\right)^2 = c'\sum_{n=1}^t\frac{N_{n+1}-N_n}{N_n^{2-\gamma}}, \end{aligned} \tag{1.50}$$

where the last equality is due to the fact that $R_{n+1}^2 = \rho N_n^\gamma(N_{n+1} - N_n)$. Using the sum-integral trick, we get

$$\sum_{n=1}^t\frac{N_{n+1}-N_n}{N_n^{2-\gamma}} \ge c''\int_N^{N_t}\frac{ds}{s^{2-\gamma}} \ge c'''N^{\gamma-1}, \tag{1.51}$$

for t sufficiently large. Combining (1.50) and (1.51) yields the lower bound in (1.48).

(2) Similar to the proof of the upper bound in (1.48), we can show that $\sum_{n=1}^t\left(\frac{R_n}{N_n}\right)^2 \le \rho^{k-1}N^{(k-1)(\gamma-1)}$ for $k = 2, 3, \ldots$ Using the same argument as in Lemma 1.4 and the fact that $\pi_{k,0} = \mathcal{O}(N^{\gamma-1})$, we obtain:

$$\begin{aligned} &\mu_3(\pi_{k,t}) \le C\pi_{k,0}N^{\gamma-1}N^{\gamma-1} + C'\pi_{k,0}N^{2(\gamma-1)}, \\ &\mu_4(\pi_{k,t}) \le C\pi_{k,0}N^{2(\gamma-1)}N^{\gamma-1} + C'\pi_{k,0}N^{\gamma-1}N^{2(\gamma-1)} + C''\pi_{k,0}N^{3(\gamma-1)}, \end{aligned}$$

for $C, C', C'' > 0$ independent of t and N. This leads to the bounds (1.49). □

Proof of Theorem 1.7. (1) It follows from the proof of Proposition 4 of Roşu and Saleh (2021) that for $R_t = \rho N_{t-1}^{\gamma}$ with $\gamma > 1$, the sequence a_t increases to the limit $a_\infty = 1$. Consequently,

$$\mathbb{E}(\pi_{k,t}^2) \to \mathbb{E}(\pi_{k,\infty}^2) = \pi_{k,0}^2 \quad \text{as } t \to \infty. \tag{1.52}$$

We conclude by noting that $\pi_{k,\infty} \in [0,1]$ and $\mathbb{E}(\pi_{k,\infty}) = \pi_{k,0}$, so $\mathbb{E}(\pi_{k,\infty}^2) \le \pi_{k,0}^2$ with equality if and only if $\pi_{k,\infty} \in \{0,1\}$ and (1.44) holds.
(2) follows from Lemma 1.8 exactly in the same way as Theorem 1.1 follows from Lemmas 1.4–1.6. □

2. Infinite population model

2.1. **Discrete infinite population.** In the previous section, we consider a finite number of investors, i.e. $K < \infty$, and study the limiting behavior as the initial coin offering $N \to \infty$. Here we work directly with $K = \infty$ investors, under Pólya urn dynamics.

We give a formal description of the model. The PoS scheme of infinite population is just as described in Section 1 but for $K = \infty$. To be more precise, there are a countably infinite number of investors indexed by $\mathbb{N}$. Let $n_{k,t}$, $k \in \mathbb{N}$ be investor k's endowment of coins at time t with $\sum_{k=1}^{\infty} n_{k,t} = N_t$. At time $t+1$, investor k is selected at random among all investors with probability $\pi_{k,t} := n_{k,t}/N_t$. Once selected, the investor receives a deterministic reward of $R_t \ge 0$. This way, equations (1.1)–(1.4) hold for $K = \infty$, and there are infinitely many tiny investors whose initial number of coins $n_{k,0} = o(1)$ (i.e. $\pi_{k,0} = o(1/N)$). The investor shares are given by a vector of infinite length $(\pi_{1,t}, \pi_{2,t}, \dots)$. Again by the martingale convergence theorem,

$$(\pi_{1,t}, \pi_{2,t}, \dots) \to (\pi_{1,\infty}, \pi_{2,\infty}, \dots) \quad \text{as } t \to \infty, \tag{2.1}$$

where $(\pi_{1,\infty}, \pi_{2,\infty}, \dots)$ is random probability measure on $\mathbb{N}$.

We have seen in Theorem 1.1 that for a finite number of K investors, if the reward R_t is constant, the limiting investor shares $(\pi_{1,\infty}, \dots, \pi_{K,\infty})$ have the Dirichlet distribution. For the infinite population model, one important problem is to understand the limiting shares $(\pi_{1,\infty}, \pi_{2,\infty}, \dots)$ with constant or more general rewards. To this end, we recall the following definition of the Dirichlet-Ferguson measure, or simply Dirichlet measure which was introduced by Ferguson (1973); Blackwell and MacQueen (1973) in the context of nonparametric Bayesian analysis.

Definition 2.1. *Let S be a Polish space with Borel σ-field $\mathcal{S}$, and let μ be a positive measure on $(S, \mathcal{S})$ with $0 < \mu(S) < \infty$. We say that F has $\mathrm{Dir}(\mu)$ distribution if F is a random distribution on S such that for every measurable partition $B_1, \dots, B_k$ of S, the random vector $(F(B_1), \dots, F(B_k))$ has $\mathrm{Dir}(\mu(B_1), \dots, \mu(B_k))$ distribution.*

The following theorem is concerned with the evolution of shares in a PoS protocol of infinite population.

Theorem 2.2. *Assume that the coin reward $R_t \equiv R > 0$. Then the investor shares have a limiting distribution*

$$(\pi_{1,\infty}, \pi_{2,\infty}, \dots) \overset{d}{=} \mathrm{Dir}(\mu), \tag{2.2}$$

where μ is a positive measure on $\mathbb{N}$ with $\mu(\{k\}) = \frac{n_{k,0}}{R}$, $k \in \mathbb{N}$. Moreover, for each investor or a finite number of investors in the same category, the results in Theorems 1.1, 1.2 and 1.7 hold under the corresponding reward assumptions.

In particular, if there are a finite number of K investors, the measure μ is then supported on $[K]$ which recovers the identity in distribution (1.8) in Theorem 1.1. To prove Theorem 2.2, we need the following result of the Blackwell-MacQueen scheme which generalizes the Pólya urn scheme.

Lemma 2.3 (Blackwell and MacQueen (1973))**.** *Let μ be a positive and finite measure on a Polish space $(S, \mathcal{S})$. Define a sequence $(X_t,\, t = 1, 2, \dots)$ as follows: X_1 is distributed as $\mu(\cdot)/\mu(S)$, and for $t \ge 1$,*

$$\mathbb{P}(X_{t+1} \in \cdot | X_1, \dots, X_t) = \frac{\mu(\cdot) + \sum_{n=1}^t \delta_{X_n}(\cdot)}{\mu(S) + t} := F_t(\cdot), \tag{2.3}$$

where $\delta_X(\cdot)$ is the Dirac mass at point X. Then

- *F_t converges in total variation (and thus in distribution) almost surely to a random discrete distribution F, which has $\mathrm{Dir}(\mu)$ distribution.*
- *Conditional given F, $X_1, X_2, \dots$ are independent and identically distributed as F.*

Proof of Theorem 2.2. First assume that $R_t \equiv R$, and let μ be a positive measure on $S = \mathbb{N}$ such that $\mu(\{k\}) = \frac{n_{k,0}}{R}$, so $\mu(S) = \frac{N}{R}$. Let X_t, $t \ge 1$ be the index of the investor who is selected by the PoS protocol at time t. By definition of the PoS scheme, we have

$$\begin{aligned}
&\mathbb{P}(X_1 = k) = \frac{n_{k,0}}{N} = \frac{\mu(\{k\})}{\mu(S)}, \\
&\mathbb{P}(X_{t+1} = k | X_1, \dots, X_t) = \frac{n_{k,0} + R \sum_{n=1}^t 1_{\{X_n = k\}}}{N + Rt} = \frac{\mu(\{k\}) + \sum_{n=1}^t \delta_{X_n}(\{k\})}{\mu(S) + t}.
\end{aligned}$$

Lemma 2.3 then implies the identity in distribution (2.2). By Definition 2.1, we get $\pi_{k,0} \stackrel{d}{=} \mathrm{Beta}(\frac{n_{k,0}}{R}, \frac{N - n_{k,0}}{R})$ for each fixed k, and thus the results in Theorem 1.1 hold. The results in Theorems 1.2 and 1.7 are stated for each investor k, and do not depend on the number K of investors. So these results also hold in the infinite population setting. ☐

2.2. **Infinite population from continuum.** Here we consider a model of the PoS protocol of infinite population which is sampled from a continuous space. The motivation comes from understanding the influence induced by common features among investors in the PoS protocol. The influence of a particular feature is measured by the total shares that investors having this feature own. In many generic cases, features are represented or approximated by elements in a continuous sample space, e.g. geolocation of an investor, market experience of an investor measured in time, index assessing the level of risk aversion of an investor, and so on. To fix the idea, we abstract the feature space as the unit interval $S = [0, 1]$.

The model of the PoS feature scheme is inspired from the Blackwell-MacQueen construction of a Pólya urn on general state spaces as described in Lemma 2.3. At each time $t \ge 1$, an investor with some feature $X_t \in S = [0, 1]$ is selected to receive a deterministic reward $R_t \ge 0$. Now we specify the rule of selection along the time. At time $t = 0$, the initial coin offering is $N_0 = N$ and these coins are distributed among investors with features in $S = [0, 1]$ according to a diffuse probability measure ν on $S = [0, 1]$. That is, the number of coins owned by investors with features in $[x, x + dx]$ is $N\nu(x)dx$. At time $t = 1$, an investor with feature X_1 is selected by the rule

$$\mathbb{P}(X_1 \in \cdot) = \nu(\cdot), \tag{2.4}$$

and then receives a reward R_1. So the total number of coins becomes $N_1 = N + R_1$. At time $t = 2$, a investor with feature X_2 is selected with probability $\mathbb{P}(X_2 \in \cdot) = (R_1 \delta_{X_1}(\cdot) + N\nu(\cdot))/N_1$. More generally, at time $t \geq 2$ an investor with feature X_t is selected by the rule:

$$\mathbb{P}(X_t \in \cdot | X_1, \dots, X_{t-1}) = \frac{\sum_{n=1}^{t-1} R_n \delta_{X_n}(\cdot)}{N + \sum_{n=1}^{t-1} R_n} + \frac{N\nu(\cdot)}{N + \sum_{n=1}^{t-1} R_n}. \tag{2.5}$$

The main difference between the PoS feature scheme (2.4)–(2.5) and the models discussed in previous sections is that there are uncountably many features among investors for selection but there are only a countable number of investors selected at time $t = 1, 2, \dots$ Two natural questions arise:

(1) How to label the features of investors selected by the PoS scheme (2.4)–(2.5)?
(2) What are the limiting shares corresponding to these features?

These problems are closely related to the problem of species sampling and exchangeable partitions studied in Pitman (1995, 1996). Let us spell out in the PoS setting as follows. For (1) the simplest way to label the features among selected investors is by their order of appearance. For $j \geq 1$, denote $\widetilde{X}_j$ as the j^{th} feature to appear in the sequence of $X_1, X_2, \dots$ Let $M_1 := 1$ and

$$M_j := \inf\{n : n > M_{j-1}, X_n \notin \{X_1, \dots, X_{n-1}\}\} \quad \text{for } j \geq 2, \tag{2.6}$$

with the convention $\inf \emptyset = \infty$. So M_j is the index at which the j^{th} feature appears for the first time, and $\widetilde{X}_j = X_{M_j}$ on the event $M_j < \infty$. For instance, if

$$(X_1, X_2, \dots) = (0.1, 0.1, 0.3, 0.2, 0.2, 0.3, 0.1, 0.4, \dots),$$

then $M_1 = 1$, $M_2 = 3$, $M_3 = 4$, $M_4 = 8, \dots$ and $\widetilde{X}_1 = 0.1$, $\widetilde{X}_2 = 0.3$, $\widetilde{X}_3 = 0.2$, $\widetilde{X}_4 = 0.4, \dots$ For general rewards R_t, it seems challenging to specify the limiting distribution of shares $(\pi_{\widetilde{X}_1,\infty}, \pi_{\widetilde{X}_2,\infty}, \dots)$. One exception is for the constant reward $R_t \equiv R$ which we assume from now on.

Theorem 2.4. *Assume that the coin reward $R_t \equiv R > 0$. Let $\widetilde{X}_1, \widetilde{X}_2, \dots$ be the features appearing in the order of appearance of the PoS feature scheme specified by* (2.4)–(2.5)*. Then there is the stick-breaking representation for the limiting shares:*

$$\pi_{\widetilde{X}_j} = \left[\prod_{i=1}^{j-1} (1 - W_i)\right] W_j \quad \textit{for } j \geq 1, \tag{2.7}$$

where $W_1, W_2, \dots$ are independent and identically distributed as $\text{Beta}(1, N/R)$*. Moreover, let $K_t := \sup\{j : M_j \leq t\}$ be the number of features appeared among the first t selected investors. Then $K_t / \log t \to N/R$ almost surely.*

Proof. The stick-breaking representation (2.7) follows from the Blackwell-MacQueen urn construction (Lemma 2.3), along with various constructions of the Dirichlet measure by Ferguson (1973); McCloskey (1965). See e.g. Section 2.2 of Pitman (1996) for a review of the circle of ideas. The fact that K_t behaves asymptotically as $\frac{N}{R} \log t$ is read from Theorem 2.3 of Korwar and Hollander (1973). $\square$

Theorem 2.4 shows that whatever the initial distribution of features is, the PoS scheme (2.4)–(2.5) with constant reward yields a limiting share distribution on a countable number of features. This distribution, which only depends on the initial coin offering N and the reward R, is known as the Griths-Engen-McCloskey (GEM) distribution (Ewens (1990)).

We may also consider more general PoS feature scheme. One such instance is when the rule of selection relies on the history of features of previously selected investors. For $j \geq 1$, let

$$N_{jt} := \sum_{n=1}^{t} 1(X_n = \widetilde{X}_j, M_j < \infty), \tag{2.8}$$

be the number of times that investors with feature j (in the order of appearance) are selected up to time t, and $\boldsymbol{N}_t := (N_{1t}, N_{2t}, \ldots)$ be the vector of counts of various features of investors until time t. We can also regroup the investors according to their features and rewrite the selection rule (2.5) as

$$\mathbb{P}(X_t \in \cdot | X_1, \ldots, X_{t-1}, K_{t-1} = k) = \sum_{j=1}^{k} \frac{N_{jt-1}}{N/R + t - 1} 1(\widetilde{X}_j \in \cdot) + \frac{N/R}{N/R + t - 1} \nu(\cdot). \tag{2.9}$$

Here we look for general selection rules of form $\mathbb{P}(X_1 \in \cdot) = \nu(\cdot)$ and for $t \geq 2$,

$$\mathbb{P}(X_t \in \cdot | X_1, \ldots, X_{t-1}) = \sum_{j=1}^{k} p_j(\boldsymbol{N}_{t-1}) 1(\widetilde{X}_j \in \cdot) + p_{k+1}(\boldsymbol{N}_{t-1}) \nu(\cdot), \tag{2.10}$$

for some functions p_j, $j = 1, 2, \ldots$ defined on $\cup_{k=1}^{\infty} \mathbb{N}^k$. The meaning of the selection rule (2.10) is as follows: given the histogram $\boldsymbol{N}_{t-1}$ of k features of investors selected from time 1 to time $t-1$, an investor with feature j is selected with probability $p_j(\boldsymbol{N}_{t-1})$ for $1 \leq j \leq k$, and an investor with a new feature $k+1$ is selected with probability $p_{k+1}(\boldsymbol{N}_{t-1})$. It is easily seen the selection rule (2.9) is a special case of the general rule (2.10) with

$$p_j(n_1, \ldots, n_k) = \frac{n_j}{N/R + t - 1} 1(1 \leq j \leq k) + \frac{N/R}{N/R + t - 1} 1(j = k - 1), \tag{2.11}$$

with $\sum_{j=1}^{k} n_j = t - 1$. A closely related selection rule is defined by the functions

$$p_j(n_1, \ldots, n_k) = \frac{n_j - \alpha}{N/R + t - 1} 1(1 \leq j \leq k) + \frac{N/R + k\alpha}{N/R + t - 1} 1(j = k - 1), \tag{2.12}$$

for some $\alpha \geq 0$. In this case, the limiting share distribution also has the stick-breaking representation (2.7) with $W_1, W_2, \ldots$ independent, and W_k distributed as Beta$(1 - \alpha, N/R + k\alpha)$. This is known as the Pitman-Yor distribution (Pitman (1996); Pitman and Yor (1997)).

In general, we need the following condition on p_j to define the selection rule (2.10):

$$p_j(\boldsymbol{n}) \geq 0, \qquad \sum_{j=1}^{|\boldsymbol{n}|_0 + 1} p_j(\boldsymbol{n}) = 1 \quad \text{for } \boldsymbol{n} \in \cup_{k=1}^{\infty} \mathbb{N}^k, \tag{2.13}$$

where $|\boldsymbol{n}|_0$ is the number of nonzero entries in $\boldsymbol{n}$. Pitman (1996) provides an additional condition on p_j in terms of the exchangeable partition probability functions (EPPFs) so that the sequence $X_1, X_2, \ldots$ specified by the selection rule (2.4)–(2.10) is exchangeable, and thus the limiting share distribution is well-defined. Such a sequence $X_1, X_2, \ldots$ is called the species sampling sequence, see also Hansen and Pitman (2000) for related discussions.

2.3. **From finite to infinite population.** In this final section, we propose a PoS protocol through a dynamic approach. We start with a finite number of investors, and then at each time a new investor may come into the market, which evolves to an infinite population. This combines the ideas from previous sections, especially the Blackwell-MacQueen urn scheme.

Now we describe the model. At time $t = 0$ there are K investors indexed by $[K]$. These investors are initial 'capitalists' in the market, so they play a very important role in the PoS scheme. For $k \in [K]$, let $n_{k,0}$ be the initial endowment of coins of investor k, and $\pi_{k,0} := n_{k,0}/N$ with $N = \sum_{k=1}^K n_{k,0}$ be the corresponding shares. It is natural to think of $n_{k,0} = \Theta(n)$ or equivalently $\pi_{k,0} = \Theta(1)$. At time $t = 1$, there are two possibilities: either one of these K initial investors are selected, or a new investor is selected from the population. This is often the case in reality, since many cryptocurrencies or stocks are initially owned by a handful of coin miners or venture capitalists, and then their shares will be diluted by new investors along the time. Since the population is large, we approximate the population space by the unit interval $S = (0, 1)$, and a new investor is selected from $S = (0, 1)$ by a diffuse probability measure ν (as in Section 2.2). We also introduce a dilution parameter $\theta > 0$, which is the weight that a new investor comes to the market. To be more precise,

- For each $k \in [K]$, investor k is selected with probability $\frac{n_{k,0}}{N+\theta}$.
- A new investor with 'index' in $(0, 1)$ is selected with probability $\frac{\theta\nu(\cdot)}{N+\theta}$.

By letting X_1 be the index of the investor selected at time 1, we have

$$\mathbb{P}(X_1 \in \cdot) = \sum_{k=1}^K \frac{n_{k,0}}{N+\theta}\delta_k(\cdot) + \frac{\theta\nu(\cdot)}{N+\theta}, \tag{2.14}$$

and the selected investor receives a deterministic reward $R_1 > 0$. More generally, at time t the selection rule is given by

$$\mathbb{P}(X_t \in \cdot|X_1, \dots, X_{t-1}) = \sum_{k=1}^K \frac{n_{k,t-1}}{N_{t-1}+\theta}\delta_k(\cdot) + \sum_{X_n \in (0,1)} \frac{n_{X_n,t-1}}{N_{t-1}+\theta}\delta_{X_n}(\cdot) + \frac{\theta\nu(\cdot)}{N_{t-1}+\theta}, \tag{2.15}$$

where $n_{k,t}$ is the number of coins that investor k owns at time t, $n_{X_n,t}$ is the number of coins that a new investor with index $X_n \in (0, 1)$ owns at time t, and $N_t = N + \sum_{n=1}^t R_n$ is the total number of coins up to time t. There are three terms on the right side of the selection rule (2.15): the first one comes from the K initial investors, the second term is from new investors previously entering the market, and the third term is the probability that a new investor is introduced.

A first question is how the shares of the K initial investors are diluted along the time. This is answered in the following theorem.

Theorem 2.5. *For $k \in [K]$, let $\pi_{k,t}$ be the shares of investor k at time t under the PoS protocol* (2.14)–(2.15)*. Then $(\pi_{k,t}, t \ge 0)$ is a supermartingale. Consequently, the shares of initial investors $(\pi_{1,t}, \dots, \pi_{K,t}) \to (\pi_{1,\infty}, \dots, \pi_{K,\infty})$ almost surely for some random sub-probability distribution $(\pi_{1,\infty}, \dots, \pi_{K,\infty})$.*

(1) Assume that the coin reward R_t is decreasing with $R_t \ge R_{t+1}$ for each $t \ge 0$.

(i) If $\lim_{t\to\infty} R_t = R > 0$ or $R_t = \Theta(t^{-\alpha})$ for $\alpha < 1$, we have $0 < \mathbb{E}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) < 1$, and $\lim_{N\to\infty} \mathbb{E}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = 1$.

(ii) If $R_t = \Theta(t^{-\alpha})$ for $\alpha > 1$, we have $\pi_{k,\infty} = 0$ almost surely.

(2) Assume that the coin reward $R_t = \rho N_{t-1}^{\gamma}$ for $\rho, \gamma > 0$. We have $0 < \mathbb{E}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) < 1$, and $\lim_{N\to\infty} \mathbb{E}\left(\frac{\pi_{k,\infty}}{\pi_{k,0}}\right) = 1$.

(3) Assume that the coin reward $R_t = R > 0$. We have $\pi_{k,\infty} \overset{d}{=} \mathrm{Beta}\left(\frac{n_{k,0}}{R}, \frac{N+\theta-n_{k,0}}{R}\right)$. Consequently, the results in Theorem 1.1 hold.

Proof. Note that $n_{k,t+1} = n_{k,t} + R_t$ with probability $\frac{n_{k,t}}{N_t+\theta}$ and $n_{k,t+1} = n_{k,t}$ with probability $1 - \frac{n_{k,t}}{N_t+\theta}$. As a result,

$$\begin{aligned}\mathbb{E}(\pi_{k,t+1}|\mathcal{F}_t) &= \frac{n_{k,t}+R_t}{N_t+R_t}\frac{n_{k,t}}{N_t+\theta} + \frac{n_{k,t}}{N_t+R_t}\left(1 - \frac{n_{k,t}}{N_t+\theta}\right) \\ &= \frac{n_{k,t}(N_t+\theta+R_t)}{(N_t+R_t)(N_t+\theta)} < \frac{n_{k,t}}{N_t} = \pi_{k,t}.\end{aligned}$$

So $(\pi_{k,t},\, t \ge 0)$ is a supermartingale. By the martingale convergence theorem, $(\pi_{1,t}, \ldots, \pi_{K,t})$ converges almost surely to a random vector $(\pi_{1,\infty}, \ldots, \pi_{K,\infty})$. Observe that $\mathbb{E}(\pi_{k,t+1}) = \frac{N_t(N_{t+1}+\theta)}{N_{t+1}(N_t+\theta)}\mathbb{E}(\pi_{k,t})$ which implies that

$$\mathbb{E}(\pi_{k,\infty}) = \pi_{k,0}\prod_{t=1}^{\infty}\left(1 - \frac{\theta R_{t+1}}{N_{t+1}(N_t+\theta)}\right). \tag{2.16}$$

(1) Assume that R_t is decreasing. If $\lim_{t\to\infty} R_t = R > 0$, we have $\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} = \mathcal{O}(t^{-2})$, and if $R_t = \Theta(t^{-\alpha})$ for $\alpha < 1$, we have $\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} = \mathcal{O}(t^{-2+\alpha})$. In both cases, we get $\sum_{t=1}^{\infty}\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} < \infty$ and thus the infinite product in (2.16) converges to some number in $(0,1)$. If $R_t = \Theta(t^{-\alpha})$ for $\alpha > 1$, we have $\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} \ge Ct^{-1}$ for t sufficiently large. In this case, $\sum_{t=1}^{\infty}\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} = \infty$. Consequently, we get $\mathbb{E}(\pi_{k,\infty}) = 0$ which implies that $\pi_{k,\infty} = 0$ almost surely.

(2) Assume that $R_t = \rho N_{t-1}^{\gamma}$ for $\rho, \gamma > 0$. If $\gamma < 1$, it follows from the proof of Lemma 1.8 that $R_t = \Theta\left(t^{\frac{\gamma}{1-\gamma}}\right)$. We have $\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} = \mathcal{O}\left(t^{-2-\frac{\gamma}{1-\gamma}}\right)$. If $\gamma > 1$, then $\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} = \Theta(N_t^{-1})$ and N_t grows exponentially in t. In both cases, we have $\sum_{t=1}^{\infty}\frac{R_{t+1}}{N_{t+1}(N_t+\theta)} < \infty$ which implies that the infinite product in (2.16) converges to some number in $(0,1)$.

(3) It is easily checked that the PoS scheme (2.14)–(2.15) with constant reward is just the Blackwell-MacQueen urn with $\mu := (\sum_{k=1}^{K} n_{k,0}\delta_k + \theta\nu)/R$. It follows from Lemma 2.3 that selection probability (2.15) converges almost surely to a random discrete distribution $F \overset{d}{=} Dir(\mu)$, and given F the indices of investors selected are independent and identically distributed as F. This implies that the limiting share of investor k is $\mathrm{Beta}\left(\frac{n_{k,0}}{R}, \frac{N+\theta-n_{k,0}}{R}\right)$, and the results in Theorem 1.1 follow. □

Theorem 2.5 shows that for the dynamic PoS model (2.14)–(2.15), the shares of initial investors will decrease along the time. If the coin reward does not decay too fast, i.e. $R_t \gg t^{-1}$, the expectation of the ratio $\pi_{k,\infty}/\pi_{k,0}$ tends to 1 as the initial coin offering N is large. On the contrary, if the coin reward decays very fast, i.e. $R_t \ll t^{-1}$, the initial investors' shares will be eventually diluted to zero. More can be said if the coin reward is constant. As is clear in the proof of Theorem 2.5, the selection rule (2.15) as a probability measure converges almost surely to a random discrete probability distribution $F \stackrel{d}{=} \mathrm{Dir}((\sum_{k=1}^K n_{k,0}\delta_k + \theta\nu)/R)$, and given F the indices of selected investors, i.e. $X_1, X_2, \ldots$ are independent and identically distributed as F. Recall that F has the representation $\sum_{t=1}^\infty P_t \delta_{Z_t}$ where $(P_1, P_2, \ldots)$ has $\mathrm{GEM}(\frac{N+\theta}{R})$ distribution, and $Z_1, Z_2, \ldots$ are independent and identically distributed as $(\sum_{k=1}^K n_{k,0}\delta_k + \theta\nu)/(N+\theta)$ and also independent of $(P_1, P_2, \ldots)$. Therefore, the indices of new investors (in order of their appearance) are just independent and identically distributed as ν, and the limiting expectation of their total shares is $\frac{\theta}{N+\theta}$. So the larger the initial coin offering N is, the less influence of new investors have.

References

D. Blackwell and J. B. MacQueen. Ferguson distributions via Pólya urn schemes. *Ann. Statist.*, 1:353–355, 1973.

W. J. Ewens. Population genetics theory—the past and the future. In *Mathematical and statistical developments of evolutionary theory (Montreal, PQ, 1987)*, volume 299 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 177–227. Kluwer Acad. Publ., 1990.

T. S. Ferguson. A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, 1: 209–230, 1973.

L. Goldstein and G. Reinert. Stein's method for the beta distribution and the Pólya-Eggenberger urn. *J. Appl. Probab.*, 50(4):1187–1205, 2013.

B. Hansen and J. Pitman. Prediction rules for exchangeable sequences related to species sampling. *Statist. Probab. Lett.*, 46(3):251–256, 2000.

N. L. Johnson and S. Kotz. *Urn models and their application: an approach to modern discrete probability theory.* John Wiley & Sons, 1977.

R. M. Korwar and M. Hollander. Contributions to the theory of Dirichlet processes. *Ann. Probability*, 1:705–711, 1973.

H. M. Mahmoud. *Pólya urn models.* Texts in Statistical Science Series. CRC Press, 2009.

J. W. McCloskey. *A model for the distribution of individuals by species in an environment.* 1965. Thesis (Ph.D.)–Michigan State University.

J. Pitman. Exchangeable and partially exchangeable random partitions. *Probab. Theory Related Fields*, 102(2):145–158, 1995.

J. Pitman. Some developments of the Blackwell-MacQueen urn scheme. In *Statistics, Probability and Game theory*, volume 30 of *IMS Lecture Notes Monogr. Ser.*, pages 245–267. Inst. Math. Statist., 1996.

J. Pitman and M. Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, 25(2):855–900, 1997.

I. Roşu and F. Saleh. Evolution of shares in a proof-of-stake cryptocurrency. *Manag. Sci.*, 67(2):661–672, 2021.

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