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CHAOS OF LIQUID SURFACE WAVES IN A VESSEL UNDER VERTICAL EXCITATION WITH SLOWLY MODULATED AMPLITUDE

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ABSTRACT: Free surface waves in a cylinder of liquid under vertical excitation with slowly modulated amplitude are investigated in the current paper. It is shown by both theoretical analysis and numerical simulation that chaos may occur even for a single mode with modulation which can be used to explain Gollub and Meyer's experiment. The implied resonant mechanism accounting for this phenomenon is further elucidated.

KEY WORDS: faraday experiment, chaos, resonant interaction

1 INTRODUCTION

Since M. Faraday[1] observed that mercury on a tin plate being vibrated in sunshine gave a very beautiful view by reflection in 1831, much attention has been paid to the formation of surface patterns. Similar experiments can also be conducted in a vessel containing liquid under vertical vibration, which is usually called as the Faraday experiment. When the vessel is forced to vibrate vertically, many typical physical phenomena for surface pattern [2] such as secondary instability, subharmonic bifurcation and solitary wave, can be observed. Recently, Ciliberto and Gollub[3] have experimentally found that competition between two distinct spatial modes may give rise to chaos, which was verified theoretically by Kambe and Umeki[4] later on. To people's surprise, Gollub and Meyer[5,6] assumed that chaos may occur even for a single spatial mode. However, Miles[2,7] indicated that chaotic motion is no longer possible for a single spatial mode instead. In Miles' theory the vertical excitation is modelled as a pure harmonic vibration:

$$z_0 = a_0 \cos(2\omega t) \qquad (\omega^2 |a_0| \ll g)$$

where a_0 is a constant amplitude and ω is the fundamental eigenfrequency of liquid in the vessel. By the average Lagrangian method, we have derived the corresponding governing equations, which turn out to be a nonlinear two-dimensional autonomous system. According to Bendixson' theorem[8], no chaotic motion in this system is possible. As a matter of fact, careful observation shows that there is center displacement and precession occurring in the Gollub and Meyer's experiment and then follows azimuthally modulated chaotic motion in surface pattern. Hence, Mile's argument is reasonable. The controversy motivates us to

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presume that the vertical excitation in the Faraday experiment is slowly modulated, i.e. the amplitude should be described as

$$z_0 = a_0(1 + a\cos\varepsilon\omega t)\cos 2\omega t$$

We assume that $\dfrac{a_0|\omega|^2}{g} = \varepsilon$ and $a = O(\varepsilon)$ is a constant and ε is a small parameter. With this assumption in mind, we reformulate the amplitude equations for a single mode (in Section 2). Theoretical analysis, especially by the Melnikov function method, shows that there do exist chaos for amplitude equations in a certain parametric region (see Section 3). The maximal Liapunov exponent and phase diagrams obtained by numerical computation further verify this result (see Section 4). In this way, both theoretical analysis and numerical simulation convincingly confirm a phenomenon,which is similar to the Gollub and Meyer's observation. (see Section 5)

2 GOVERNING EQUATIONS OF SURFACE WAVES WITH MODULATED AMPLITUDE

Consider an inviscid and irrotational fluid layer of height d with density ρ in a cylinder of the cross-section S. Construct a Cartesian coordinate system moving with the cylinder such that $\boldsymbol{x}$ is in the horizontal plane and z is a vertical upward axis. The free surface displacement of fluid is assumed as $z = \eta(\boldsymbol{x}, t)$, and the bottom $z = -d$ in quiescent states. The cylinder is subjected to vertical excitation with a slowly modulated amplitude.

We expand the free surface displacement in terms of a complete orthonormal set of functions

$$\eta(\boldsymbol{x}, t) = \eta_n(t)\varphi_n(\boldsymbol{x}) \qquad \boldsymbol{x} \in S$$

where repeated dummy subscripts imply a summation over all the participating modes, η_n are generalized coordinates, φ_n are eigenfunctions determined by

$$\begin{aligned}
&(\nabla^2 + k_n)\varphi_n = 0 && (\boldsymbol{x} \in S)\\
&\boldsymbol{n}\cdot\nabla\varphi_n = 0 && (\boldsymbol{x} \in \partial S)\\
&\iint_s \varphi_m\varphi_n \mathrm{d}S = \delta_{mn}S
\end{aligned}$$

where k_n are the corresponding eigenvalues, and δ_{mn}, the Kronecker function.

In terms of the generalized coordinates, the corresponding Lagrangian can be represented as

$$\begin{aligned}
L \equiv\ & (\rho S)^{-1}(T - V)\\
=\ & \frac{1}{2}a_{mn}\dot{\eta}_m\dot{\eta}_n - \frac{1}{2}(g + \ddot{z}_0)\eta_n\eta_n\\
=\ & \frac{1}{2}a_n(\dot{\eta}_n^2 - \omega_n^2\eta_n^2) - \frac{1}{2}\ddot{z}_0\eta_n\eta_n + \frac{1}{2}a_{lmn}\eta_l\dot{\eta}_m\dot{\eta}_n + \frac{1}{4}a_{jlmn}\eta_j\eta_l\dot{\eta}_m\dot{\eta}_n
\end{aligned}$$

where $a_n = k_n^{-1}\coth(k_n d) \equiv g/\omega_n^2$, and the other coefficients a_{lmn}, a_{jlmn} are constants dependent on a_n, k_n and the integrals of product of Ψ_n over S.

It is presumed that the primary mode is only resonantly excited , that is, $\omega_1 \approx \omega$, and other modes is detuned from the excitation frequency. Then, we have

$$\eta_n(t) = \delta_{1n}l\{p(\tau)\cos\omega t + q(\tau)\sin\omega t\} + \frac{l^2}{a_1}\{A_n(\tau)\cos 2\omega t + B_n(\tau)\sin 2\omega t + C_n(\tau)\}$$

Choose a length scale $l = 2(\frac{\varepsilon}{|A|})^{\frac{1}{2}} k_1^{-1} \tanh(k_1 d) = O(\varepsilon^{\frac{1}{2}} a_1)$, where $A = \frac{1}{2}\{a_1 a_{1111} + a_{n11} a_{n11} - \frac{1}{2} a_1 (4a_n - a_1)^{-1} (4a_n n - a_{n11})^2\} \tanh^4(k_1 d)$ is a constant coefficient, $\varepsilon = \frac{a_0}{a_1} = \omega_1^2 a_0 / g$ is a small parameter as previously defined, $\tau = \varepsilon\omega t$ is a slowly varying time. In addition, the detuning parameters

$$\beta = \frac{\omega^2 - \omega_1^2}{2\varepsilon\omega_1^2} \approx \frac{\omega - \omega_1}{\varepsilon\omega_1}$$

$$\Omega_n = 4(\frac{\omega_n}{\omega_1})^2 - 1 \approx \frac{4a_n - a_1}{a_1}$$

are introduced for the sake of convenience. Substituting the above representations into L and integrating the expression over one period, we can obtain the averaged Lagrangian function as follows

$$\langle L \rangle = \varepsilon g l^2 \{\frac{1}{2}(\dot{p}q - p\dot{q}) + H(p,q)\}[1 + O(\varepsilon)]$$

where $H = \frac{1}{2}(\beta+1+a\cos\tau)p^2 + \frac{1}{2}(\beta-1-a\cos\tau)q^2 + \frac{1}{4}(p^2+q^2)^2 \mathrm{sgn}(A)$ is a Hamiltanian, $a = O(\varepsilon)$ is a small dimensionless parameter, $p(\tau), q(\tau), A_n(\tau), B_n(\tau), C_n(\tau)$ are slowly varying dimensionless amplitudes, and secondary amplitudes are regarded as $A_n(\tau), B_n(\tau), C_n(\tau)$ are regarded as quasi-steady.

For the time being, we assume $A > 0$. The similar discussion can be carried out for $A < 0$. Requiring $\langle L \rangle$ to be stationary with respect to independent variation of p and q leads to the following differential equations

$$\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -(\beta - 1 - a\cos\tau + p^2 + q^2)q \\ \dot{q} &= -\frac{\partial H}{\partial p} = (\beta + 1 + a\cos\tau + p^2 + q^2)p \end{aligned}\right\} \tag{2.1}$$

Please notice that the symbol dot here denotes the derivatives with respect to the slowly varying variable τ. Apparently, H is a Hamiltonian depending on the slowly varying time τ as well as a couple of canonical conjugate variables (p,q). $(\dot{p},\dot{q})$ represent the differentiation of (p,q) with respect to τ. Therefore, amplitude equations of free-surface displacement in this case constitute a periodically nonautonomous system of two dimensions.

3 ANALYSIS OF AMPLITUDE EQUATIONS

In this section, the behaviors of the unperturbed systems ($a = 0$) and the perturbed systems ($a \neq 0$) for Eq.(2.1) are investigated, where $a = O(\varepsilon)$ is a small perturbation parameter.

When $a = 0$, Eq.(2.1) is an autonomous system of two dimensions. Based on the Bendixson theorem[9], the fixed points or closed loops are possible solutions for this system (see Fig.1) but the chaotic motion is absolutely impossible:

(1) For $\beta > 1$, $p = q = 0$ is a center with a nested set of closed loops.

(2) For $-1 < \beta < +1$, $p = 0$, $q = \pm(1-\beta)^{1/2}$ are centers, each of which is enclosed by a set of closed curves; But $p = q = 0$ is a saddle, through which the separatrix is a homoclinic trajectory. And there exist periodic loops outer of the homoclinic trajectory.

(3) For $\beta < -1$, $p = q = 0$ and $p = 0$, $q = \pm(1-\beta)^{1/2}$ are three centers, each of which is enclosed by a nested set of loops. The separatrixes through two saddles $p = \pm(-1-\beta)^{1/2}$, $q = 0$ are homoclinic trajectories surrounded by a family of closed curves.

When $a \neq 0$, Eq.(2.1) is a nonautonomous system of two dimensions. By using the subharmonic bifurcation theory and the Melnikov function method, it follows that

(1) For $\beta > 1$, the solutions of Eq.(2.1) are subharmonic responses, the frequency of which depends on the ratio of $\sqrt{\beta^2 - 1}$ to ω.

(2) For $|\beta| < 1$, we employ the method of the Melnikov function to investigate its homoclinic chaos.

Making a transformation: $p = \sqrt{2r}\cos\theta$, $q = \sqrt{2r}\sin\theta$ and substituting it into Eq.(2.1), we obtain

$$\dot{r} = 2r\sin\theta(1 + a\cos\tau)$$
$$\dot{\theta} = (\beta + 2r) + (1 + a\cos\tau)\cos 2\theta$$

and the corresponding total Hamiltonian

$$H = r[\beta + (1 + a\cos\tau)\cos 2\theta] + r^2$$

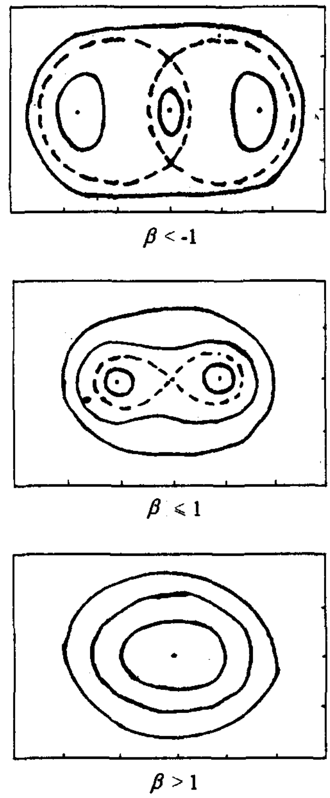

Fig.1 Phase Diagram

Evidently, $H = 0$ with $a = 0$ represents a homoclinic trajectory through the saddle

$$\beta + \cos 2\theta + r = 0 \tag{3.1}$$

The unperturbed system is as follows

$$\dot{r} = 2r\sin 2\theta \qquad \dot{\theta} = \beta + 2r + \cos 2\theta \tag{3.2}$$

Consequently, the Melnikov function is derived by the following integral

$$M(\tau_0, \phi_0) = \int_{-\infty}^{+\infty} [(\beta + 2r + \cos 2\theta)(a \cdot 2r\sin 2\theta\cos(\tau + \tau_0 + \phi_0)) +$$
$$(-2r\sin 2\theta) \cdot a\cos 2\theta\cos(\tau + \tau_0 + \phi_0)]\mathrm{d}\tau$$

Substitute the first one of Eq.(3.2) into the above Melnikov function and integrate it partially

$$M(\tau_0, \phi_0) = a\int_{-\infty}^{+\infty} (2r + \beta)\dot{r}\cos(\tau + \tau_0 + \phi_0)\mathrm{d}\tau =$$
$$-a\int_{-\infty}^{+\infty} \cos(\tau + \tau_0 + \phi_0)\mathrm{d}(\beta r + r^2)\mathrm{d}\tau =$$
$$a\int_{-\infty}^{+\infty} (r + \beta)r\sin(\tau + \tau_0 + \phi_0)\mathrm{d}\tau$$

Hence, we have

$$|M(\tau_0,\phi_0)| \le |a| \int_{-\infty}^{+\infty} |(r+\beta)r||\sin(\tau+\tau_0+\phi_0)|d\tau \equiv N(\tau_0,\phi_0)$$

According to the first mean value theorem of a definite integrals, there necessarily exists a $\xi \in R$ such that

$$N(\tau_0,\phi_0) = a|\beta + r(\xi+\tau_0+\phi_0)| \int_{-\infty}^{+\infty} |r\sin(\tau+\tau_0+\phi_0)|\mathrm{d}\tau$$

On the other hand, due to Eq.(3.1) we have

$$\sin 2\theta = \pm\sqrt{1-(r+\beta)^2}$$

Substitute it into the first equation of Eq.(3.2), we produce

$$\dot{r} = 2r\cdot[\pm\sqrt{1-(\beta+r)^2}\,]$$

which leads to

$$-\frac{1}{\sqrt{1-\beta^2}}\ln\left|\frac{\sqrt{1-(r+\beta)^2}+\sqrt{1-\beta^2}}{r} - \frac{\beta}{\sqrt{1-\beta^2}}\right| = 2(\tau+\tau_0+\phi_0)$$

Let $\tau = \xi$ and choose an appropriate pair of (τ_0,ϕ_0) such that

$$\ln\Big(\frac{\sqrt{1-\beta^2}+1}{\beta\sqrt{1-\beta^2}}\Big) = -2\sqrt{1-\beta^2}(\xi+\tau_0+\phi_0)$$

Combining the above two equations, we have $\beta + r(\xi+\tau_0+\phi_0) = 0$, which means $N(\tau_0,\phi_0) = 0$, or

$$M(\tau_0,\phi_0) = 0$$

Similarly, $\dfrac{\partial M}{\partial \tau_0} = 0$ at (τ_0,ϕ_0) can be shown. Therefore, the stable and unstable manifolds intersect transversally such that Eq.(2.1) exhibits chaos in the sense of Smale horseshoe.

(3) For $\beta < -1$, there exist multi-attractors. When initial values are nearby the center like $(0,0)$, the responses are subharmonic solutions similar to case (1). As initial values leave from the center to approach the homoclinic trajectory, they are attracted to the chaotic attractors, which is similar to case (2).

Therefore, when the initial perturbations are restricted to the neighbor of the origin, that is, initial free surface is almost in equilibrium, $|\beta| > 1$ corresponds to subharmonic solutions and $|\beta| < 1$ to chaotic motions.

4 LIAPUNOV EXPONENT AND PHASE DIAGRAMS

In this section, we are concerned with the maximal Liapunov exponent and phase diagrams of Eq.(2.1) to further verify the theoretical conclusions of the preceding section.

Let us begin to discuss the maximal Liapunov exponent. If we exhaust the range of the detuning parameter β from -1 and $+1$ by a step 0.05, where initial conditions are chosen as those approaching to zero, a curve of the maximal Liapunov exponent versus detuning parameter β can be provided in Fig.2. It is found from Fig.2 that the curve for the maximal Liapunov exponents are positive with lump-like peaks in the middle and are zero at both left and right sides, which implies that $|\beta| < 1$ corresponds to chaos and $|\beta| > 1$ to subharmonic branchings. Noting that the detuning parameter β represents the relative difference between the natural and excitation frequencies, we can conclude that nonlinear resonant interaction for smaller β leads to chaos while locking up to subharmonic responses for larger β.

Finally we plot the phase diagrams for unperturbed systems with $a = 0$ and perturbed systems with $a \neq 0$ by numerical integrals respectively as $\beta = \pm 0.5$. Evidently, the phase diagrams of the unperturbed system consists of closed trajectories. In contrast, those of the perturbed system look entirely chaotic. It is an alternative interesting finding that their skeleton is very similar because the latter is the perturbation of the former (See Fig.3).

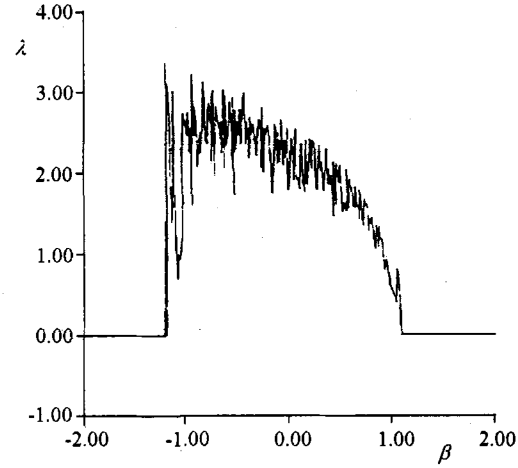

Fig.2 Liapunov exponent when $a = 0.1$

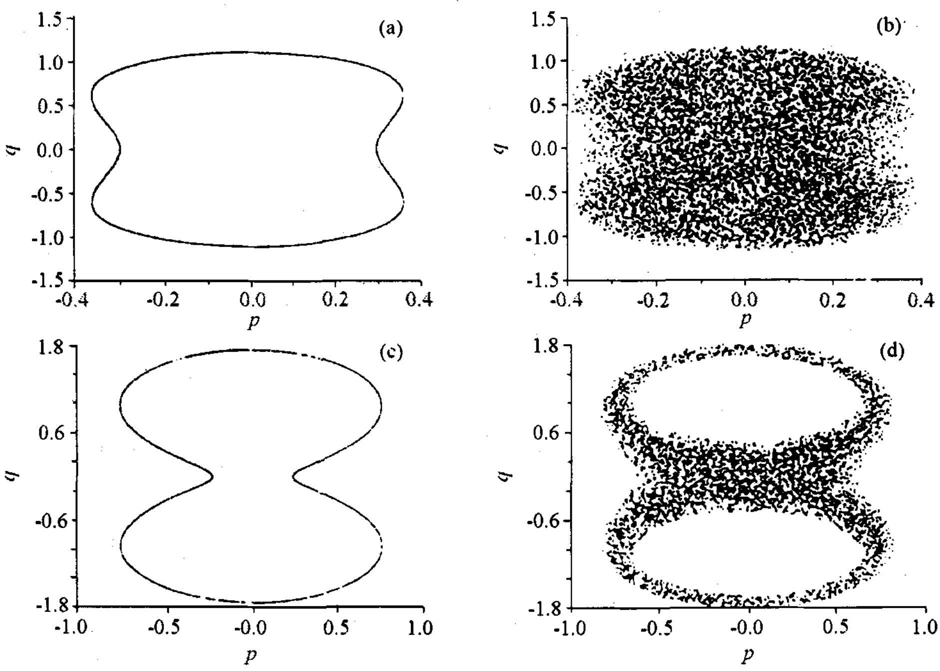

Fig.3 Phase diagram

(a) $a = 0$, $\beta = 0.5$; (b) $a = 0.1$, $\beta = 0.5$; (c) $a = 0$, $\beta = -0.5$; (d) $a = 0.1$, $\beta = -0.5$

5 CONCLUDING REMARKS

Based on the average Lagrangian method, the periodically parameter-excited equations are derived for modeling the liquid surface waves in a vessel under vertical excitation with a slowly modulated amplitude. By theoretical analysis and numerical simulation, especially the method of the Melnikov function and the Liapunov exponents, we find that (i) for larger detuning parameters, the patterns on free surfaces represent standing waves with subharmonic modulation. (ii) for smaller detuning parameters, that is, the exciting frequency nearly equals natural one, the surface patterns display chaotic modulation. In other words, resonance will lead to slowly modulated chaos in the pattern of surface waves.

Our results seem to explain Gollub and Meyer's experiment in which chaos may occur even for a single mode. Actually, the excitation amplitude regarded as unchanged in Miles' theory is reassumed as periodically slow varying in the present model. Recently, Gollub has also indicated that resonant interaction of two different frequencies, generally belonging to distinct spatial modes, is a necessary condition for chaotic phenomena in the Faraday experiments[7]. As a matter of fact, for a system with slowly modulated amplitude, there actually exist two frequencies, one from a spatial mode and another due to slow modulation of amplitude. Their resonant interaction by parametric excitation results in chaos of the Faraday experiment, which also is the very mechanism responsible for Gollub and Meyer's experiment due to azimuthal modulation. Furthermore we can investigate spatiotemporal patterns, such as competition, intermittency, chaos and spatiotemporal chaos in the Faraday experiment under vertical excitation with slowly modulated amplitude in the near future.

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