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Travelling Waves In One-dimensional Coupled Map Lattices (I) [1]

Guowei HE
(Laboratory for Nonlinear Mechanics, Institute of Mechanics,
Chinese Academy of Sciences, Beijing, 100080, China)
E-mail: he@ariane.saclay.cea.fr

Abstract: We find an exact solution of travelling waves in one-dimensional coupled map lattices (CML) and verify its stability in the moving perturbations . The allowance discrete wave velocities are obtained and the selection principle of wave velocities is presented. By this exact solution, a simple model is suggested to predict the temporal behaviors of travelling waves in CML.
Key Words: travelling waves, CML

Introduction

Wave dynamics play a fundamental role in the spatially extended systems , since wavelike patterns have often been observed thoughout nature, ranging from hydrodynamics to solid physics. Moreover, It is reasonably believed that more complex space-time patterns result from the nonlinear interaction of different waves. For example, spatiotemporal chaos is considered to be due to the interaction of three resonant waves.

Wavelike patterns are defined to be the "almost" spatially periodic structures even though their temporal evolutions are fixed, periodic or chaotic. They may be described in two different frames: the fixed frame and the moving frame. The standing waves are described conveniently in the fixed frame. However, the travelling waves are formulated more suitably in the moving frame. More strictly, the wavelike patterns x_j^t in the moving frame can be written in the form of $x_j^t = x(j+vt,t)$, and the travelling waves are just the solution of the form of $x_j^t = x(j+vt)$. In the model presented in this paper, we can find the wavelike patterns in the moving frame, but cannot find the travelling waves. With this difference in mind, we do not make the strict distinction between them so that they all are called as the travelling waves.

The travelling waves are a novel universality class in one-dimensional coupled map lattices (CML), which is a dynamical system with discrete time ("map"), discrete space ("lattice"), and a continuous state. They exhibit the coexistence of travelling-wave attractors

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with different wave velocities and wavenumbers. The selection mechanism of wave velocities is attributed to the the existence of a phase slip. In this paper, this kind of travelling wave will be investigated further. We find an exact solution for the travelling waves with discrete wave velocities (Sec II), and then discuss the selection of discrete wave velocities (Sec III). In section IV, the stabilities of the travelling waves are studied. Numerical simulation shows that the above results are not restricted only to some special parameter regions but also hold for the more general cases.

1. An exact solution of the travelling waves

Our working model is the diffusive coupled map lattice in which the coupling is between the nearest neighbours:

$$\begin{cases} x_j^{t+1} = (1-\epsilon)f(x_j^t) + \frac{\epsilon}{2}\{f(x_{j-1}^t) + f(x_{j+1}^t)\} \\ x_0^t = x_N^t, \qquad x_1^t = x_{N+1}^t, \qquad j = 1, 2, \cdots, N \end{cases} \tag{1.1}$$

where ϵ is the coupling parameter with $0 \le \epsilon \le 1$, t a discrete time step and j the jth lattice point. f denotes the local map, which is taken as the Logistic form:

$$f(x) = 1 - ax^2. \quad (0 \le a \le 2)$$

We assume the travelling wave solution of (1.1) to be the following:

$$x_j^t = A_t + B_t\cos(j\omega + \sigma t) + C_t\sin(j\omega + \sigma t) \tag{1.2}$$

where $\omega = \frac{2\pi}{N}q$ is a wavenumber, σ a frequency and q an integer to be determined later. Substituting (1.2) into (1.1), we obtain the left-side of (1.1) to be

$$A_{t+1} + (B_{t+1}\cos\sigma + C_{t+1}\sin\sigma)\cos(j\omega + \sigma t) + (-B_{t+1}\sin\sigma + C_{t+1}\cos\sigma)\sin(j\omega + \sigma t),$$

and the right-side of (1.1) as:

$$\{f(A_t) - \frac{1}{2}a[B_t^2 + C_t^2]\} + \{\alpha f'(A_t)B_t\}\cos(j\omega + \sigma t) + \{\alpha f'(A_t)C_t\}\sin(j\omega + \sigma t),$$

in which $\alpha = 1 - \frac{1}{2}\epsilon\sin^2\frac{\omega}{2}$, and the coefficient $\beta = 1 - \frac{1}{2}\epsilon\sin^2\omega$ of the double harmonic term has been assumed to be zero, that is, $\beta = 0$, so that it vanishes.

Comparing the above two expressions, we have

$$\begin{cases} A_{t+1} = f(A_t) - \frac{1}{2}a(B_t^2 + C_t^2) \\ B_{t+1} = \alpha f'(A_t)(B_t\cos\sigma - C_t\sin\sigma) \\ C_{t+1} = \alpha f'(A_t)(B_t\sin\sigma + C_t\cos\sigma) \end{cases} \tag{1.3}$$

If $\beta = 0$, (1.2) is just an exact travelling wave solution of (1.1). The amplitudes of the travelling waves are governed by (1.3). The wavenumbers of the travelling waves are determined by the expression $\beta = 0$, that is,

$$\omega = \arcsin\frac{1}{\sqrt{2\epsilon}}.$$

In order for the above expression to hold, we have to require $\epsilon \ge 0.5$. In other words, the travelling waves exsit only if ϵ is larger than 0.5, which agrees with the result of numerical simulations .

2. Selection of the discrete wave velocities

Now we solve the amplitude equations (1.3), and derive the wave velocities of the solution (1.2). Here, it is only the periodic solutions of (1.3) which are addressed since amplitude chaos is impossible (for details, see the next section).

Let $(A_1, B_1, C_1), \cdots, (A_T, B_T, C_T)$ be the solution of (1.2) with the period T. Substituting them into the 2nd and 3rd equations of (1.3), we obtain the following expressions:

$$\begin{pmatrix} B_1 \\ C_1 \end{pmatrix} = \lambda_T \begin{pmatrix} \cos(\sigma T) & -\sin(\sigma T) \\ \sin(\sigma T) & \cos(\sigma T) \end{pmatrix} \begin{pmatrix} B_1 \\ C_1 \end{pmatrix} \tag{2.1}$$

where $\lambda_T = \alpha^T f'(A_1) \cdots f'(A_T)$. It is easily seen that the eigenvalues of the coefficient matrix in (2.1) are $\lambda_T \exp(\pm i\sigma T)$. In order that (2.1) has a nontrivial solution, its coefficient matrix must have at least one real eigenvalue equal to unity. Thus, $\sin(\sigma T) = 0$, or

$$\sigma = \frac{m\pi}{T}, \quad (m = 0, \pm 1, \cdots, \pm(T-1)).$$

The corresponding wave velocities are immediately found:

$$v = \frac{\sigma}{\omega} = \frac{m\pi}{T}[\arcsin(\frac{1}{\sqrt{2\epsilon}})]^{-1} \tag{2.2}$$

To simplify the travelling wave solutions, we introduce the polar coordinates:

$$A_t = r_t \cos\phi$$

$$B_t = r_t \sin\phi,$$

where ϕ is a phase angle independent of time t. Then, (1.2) can be rewritten as

$$x_j^t = A_t + r_t \cos(j\omega + \sigma t - \phi), \tag{2.3}$$

where A_t and r_t, by (1.3), satisfy the following equations:

$$\begin{cases} A_{t+1} = f(A_t) - \frac{1}{2} a r_t^2 \\ r_{t+1} = \alpha f'(A_t) r_t, \end{cases} \tag{2.4}$$

which are the same as the amplitude equations of the standing waves in (1.1) .

If $T = 1$, then $\sigma = 0$ or $v = 0$, and the fixed solution (1.2) cannot be a travelling wave. If $T > 1$, (1.2) may be either a standing wave ($\sigma = 0$) or travelling waves ($\sigma \neq 0$). For $T \geq 4$, the real travelling waves may be observed.

The travelling waves (1.2) with the period T may have T different wave velocities. In other words, there coexist T travelling wave attractors with the different wave velocities. The allowance wave velocities are determined by both wavenumber and frequency. According to (2.2) and (2.4), they are dependent on the structural parameters - the diffusive coefficient ϵ and nonlinearity parameter a - of the CML (1.1). Therefore, the allowance wave velocities are determined by the structural parameters of (1.1), and the selected wave velocity depends on its attractor basin.

It seems that the above conclusion is restricted to the special condition $\beta = 0$. In fact, this condition is not very necessary. By the implicit function theorem , we can relax this condition to the travelling wavelike solution, which is just like what we have done for the standing waves . The numerical simulation also supports this conslusion. When $\beta \neq 0$, the simple model (1.3) can exhibit the same temporal behaviors as the travelling waves in CML. Therefore, we may use (1.3) to predict the behaviors of the whole lattice chain. For example, (2.2) indicates the center values of the narrow bands for the dicscrete wave velocities.

3. Remarks on stabilities of travelling waves

In stability analysis, we shall be interested in the following perturbations:
(1) Perturbations relevant to the moving frame: They move with the same velocity v as the solutions (1.2), in such a way that they are travelling waves.
(2) The perturbation is a uniformly small deformation: i.e. one which is both uniformly small with respect to spatial lattices and a deformation of the travelling wave solution (1.1).

With the above assumptions, the perturbed travelling waves can be specified as follows:

$$\begin{aligned}\tilde{x}_t^j &= (A_t + sD_t\cos(j\Omega)) + (B_t + 2sE_t\cos(j\Omega))\cos(j\omega+\sigma t)\\ &\quad +(C_t + 2sF_t\cos(j\Omega))\sin(j\omega+\sigma t)\\ &= A_t + B_t\cos(j\omega+\sigma t) + C_t\sin(j\omega+\sigma t) + s\big(D_t\cos(j\Omega)\\ &\quad +E_t[\cos(j(\omega+\Omega)+\sigma t)+\cos(j(\omega-\Omega)+\sigma t)]\\ &\quad +F_t[\sin(j(\omega+\Omega)+\sigma t)+\sin(j(\omega-\Omega)+\sigma t)]\big),\end{aligned}$$

where $A_t, B_t, C_t, D_t, F_t, E_t$ are of order 1. s is a small parameter and Ω is of the same order as s.

Substituting $\tilde{x}_j^t$ into (1.1), we obtain

$$\begin{aligned}\tilde{x}_{t+1}^j &= A_{t+1} + B_{t+1}\cos(j\omega+\sigma t) + C_{t+1}\sin(j\omega+\sigma t) + s\big(D_{t+1}\cos(j\Omega)\\ &\quad +E_{t+1}[\cos(j(\omega+\Omega)+\sigma t)+\cos(j(\omega-\Omega)+\sigma t)]\\ &\quad +F_{t+1}[\sin(j(\omega+\Omega)+\sigma t)+\sin(j(\omega-\Omega)+\sigma t)]\big) + O(s^2),\end{aligned}$$

in which A_t, B_t, C_t are given by (1.3), and D_t, E_t, F_t are given by

$$\begin{cases} D_{t+1} = f'(A_t)D + f'(B_t)E + f'(C_t)F \\ E_{t+1} = \alpha[\frac{1}{2}(f'(B)\cos\sigma - f'(C)\sin\sigma)D + f'(A)E\cos\sigma - f'(A)F\sin\sigma] \\ F_{t+1} = \alpha[\frac{1}{2}(f'(B)\sin\sigma + f'(C)\cos\sigma)D + f'(A)E\sin\sigma + f'(A)F\cos\sigma] \end{cases} \tag{3.1}$$

The Jacobian matrices of (1.3) and (2.1) are respectively

$$J(A_t,B_t,C_t) = \begin{pmatrix} f'(A_t) & \frac{1}{2}f'(B_t) & \frac{1}{2}f'(C_t) \\ \alpha(f'(B_t)\cos\sigma - f'(C_t)\sin\sigma) & \alpha f'(A_t)\cos\sigma & -\alpha f'(A_t)\sin\sigma \\ \alpha(f'(B_t)\sin\sigma + f'(C_t)\cos\sigma) & \alpha f'(A_t)\sin\sigma & \alpha f'(A_t)\cos\sigma \end{pmatrix}$$

and

$$J(D_t,E_t,F_t) = \begin{pmatrix} f'(A_t) & f'(B_t) & f'(C_t) \\ \frac{\alpha}{2}(f'(B_t)\cos\sigma - f'(C_t)\sin\sigma) & \alpha f'(A_t)\cos\sigma & -\alpha f'(A_t)\sin\sigma \\ \frac{\alpha}{2}(f'(B_t)\sin\sigma + f'(C_t)\cos\sigma) & \alpha f'(A_t)\sin\sigma & \alpha f'(A_t)\cos\sigma \end{pmatrix}$$

It is easily verified that the matrices $J(A_t,B_t,C_t)$ and $J(D_t,E_t,F_t)$ have the same characteristic polynomial. Therefore, they have the same eigenvalues. If all eigenvalues of the matrix $J(A_t,B_t,C_t)$ are located inside the unit circle, the perturbations D_t, E_t, F_t will converge to zero as the time t goes to infinity. In other words, for the perturbation under consideration, the amplitude stability implies that the travelling waves (1.2) are stable. On the contrary, if the matrix $J(A_t, B_t, C_t)$ has at least one eigenvalue larger than 1, the perturbations will grow exponentially. In this case, (1.2) is no longer valid, and the high-order harmonic waves are excited. Noting that the existence of an eigenvalue larger than 1 implies a positive Lyapounov exponent, i.e. chaotic amplitudes are impossible for a single travelling wave (1.2).

5. Conclusion

The aim of this paper was to study the travelling waves in one-dimensional CML. We found an exact solution for the travelling waves, and investigated its stability with respect to the perturbations, which are the uniformal small deformation, moving at the same velocity as the travelling wave. The allowance wave velocities form a discrete set dependent on the parameters a and ϵ of (1.1). The wave velocity selected from this discrete set is determined by the initial condition. This type of travelling wave is a spatiotemporal periodic solution and never demonstrates the amplitude chaos. The amplitute equation (1.3), as a simple (or small) model, can be used to predict the temporal behaviors of travelling waves in CML.

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