Random Processes

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Objectives

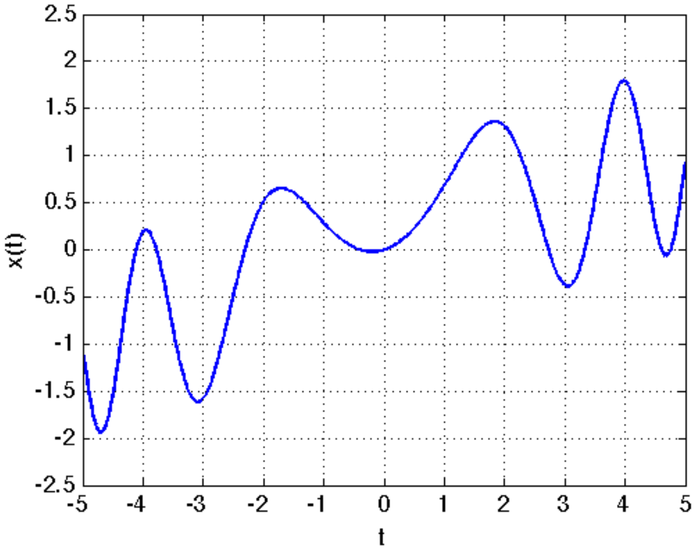
- Extend Discussion to Random Processes
  - Build on foundation of probability and random variables
  - Additionally incorporate “temporal” dependence
  - Often key component of probabilistic models

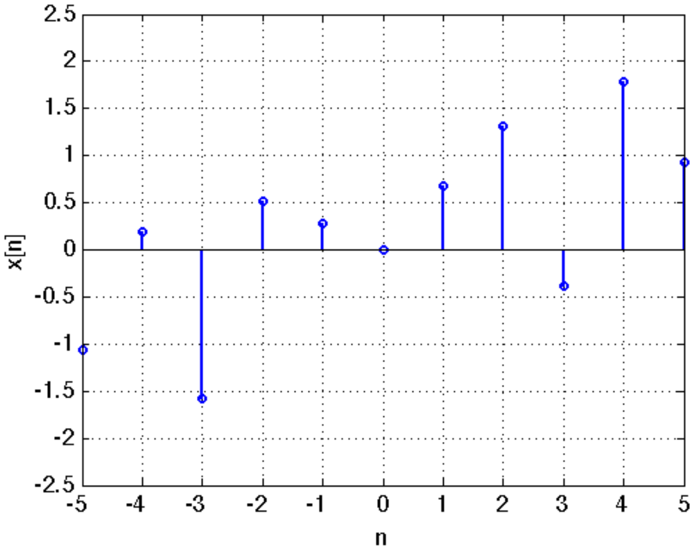
Random Process

- Random signals (functions of “time”)
- Used for:
  - Modeling noise/uncertainty
    - Thermal noise due to random electron movement in a conductor
    - Modeling sources of information: Speech, images, video
- Discrete vs Continuous Time

Discrete vs Continuous "Time" Random Processes

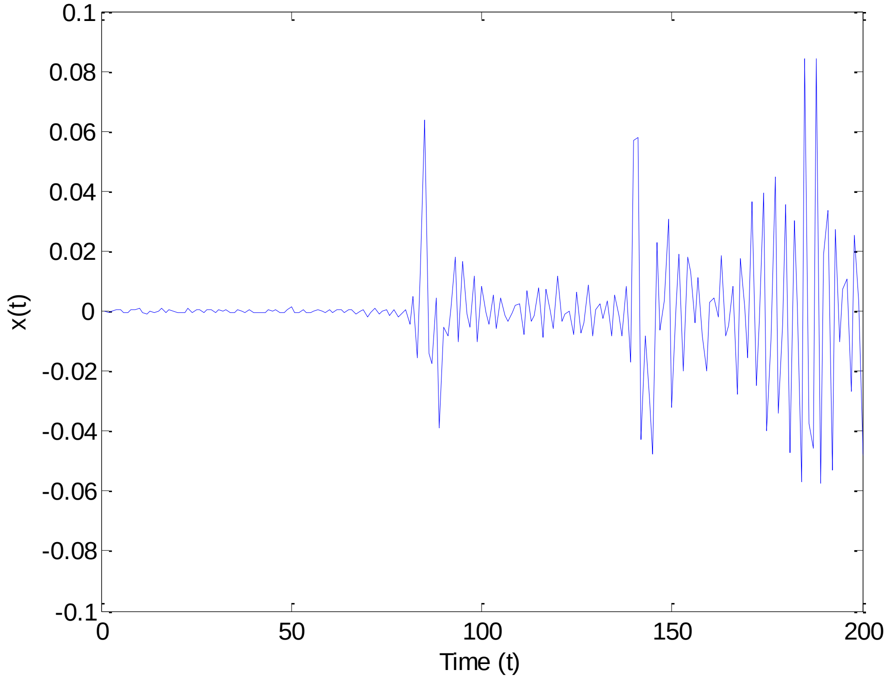
- Continuous time "t" vs discrete time "n"
  - X(t) vs Xn

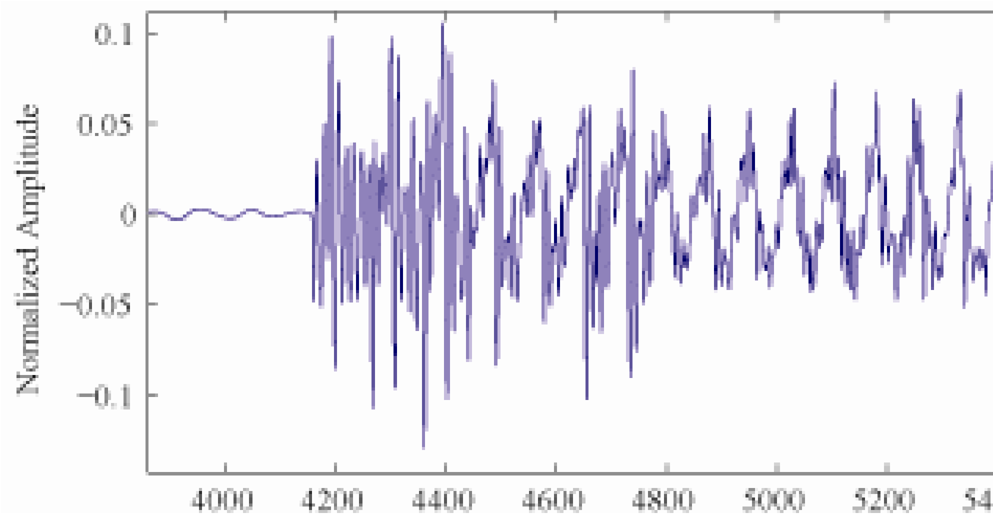




- Examples

Speech Waveform Signal

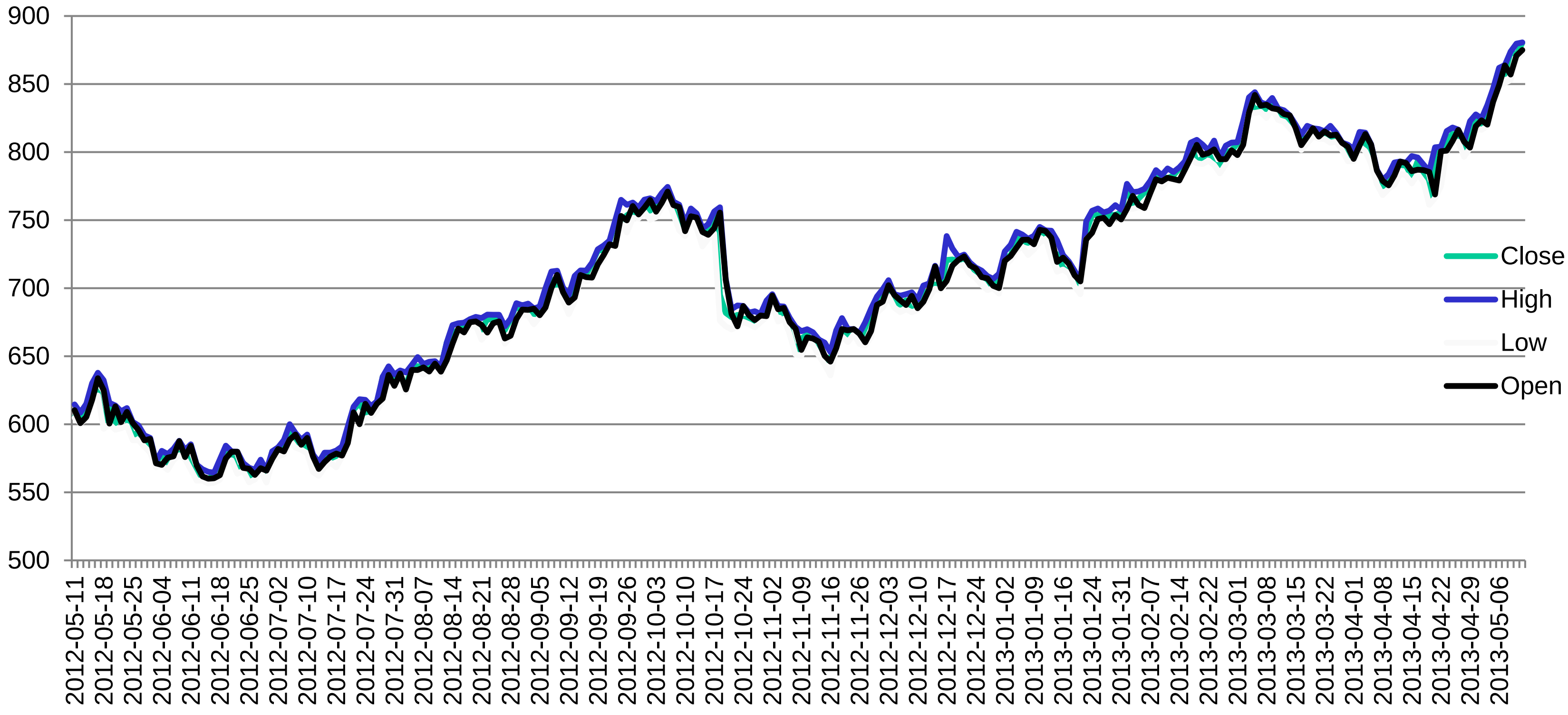




Image

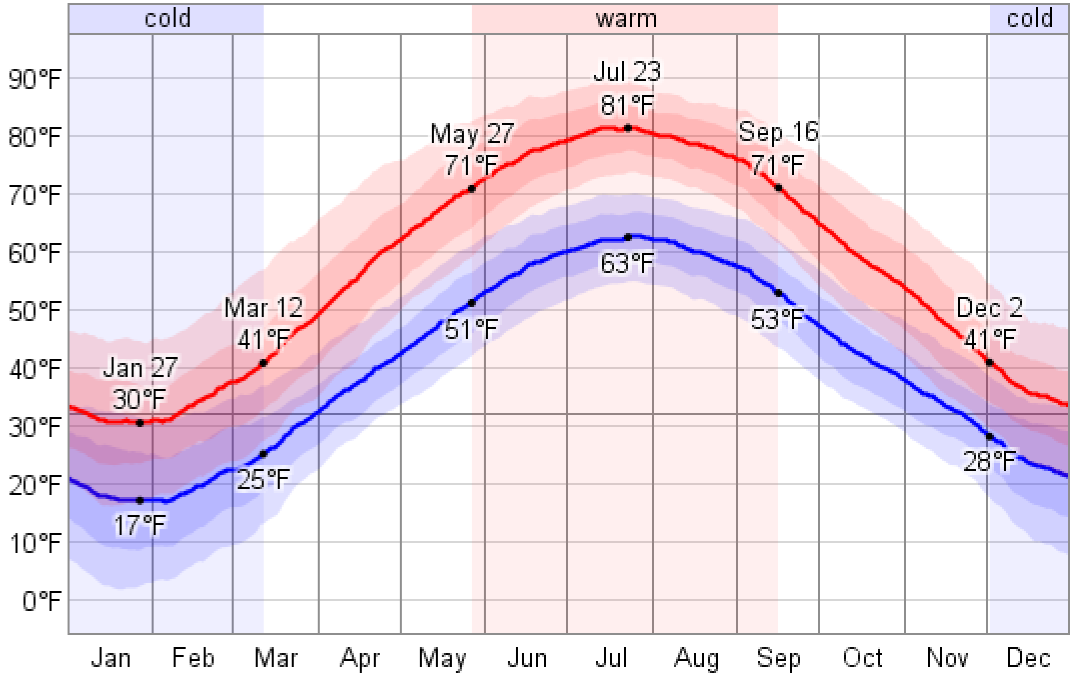
Daily Stock Prices

- Google: Open, Low, High, and Close



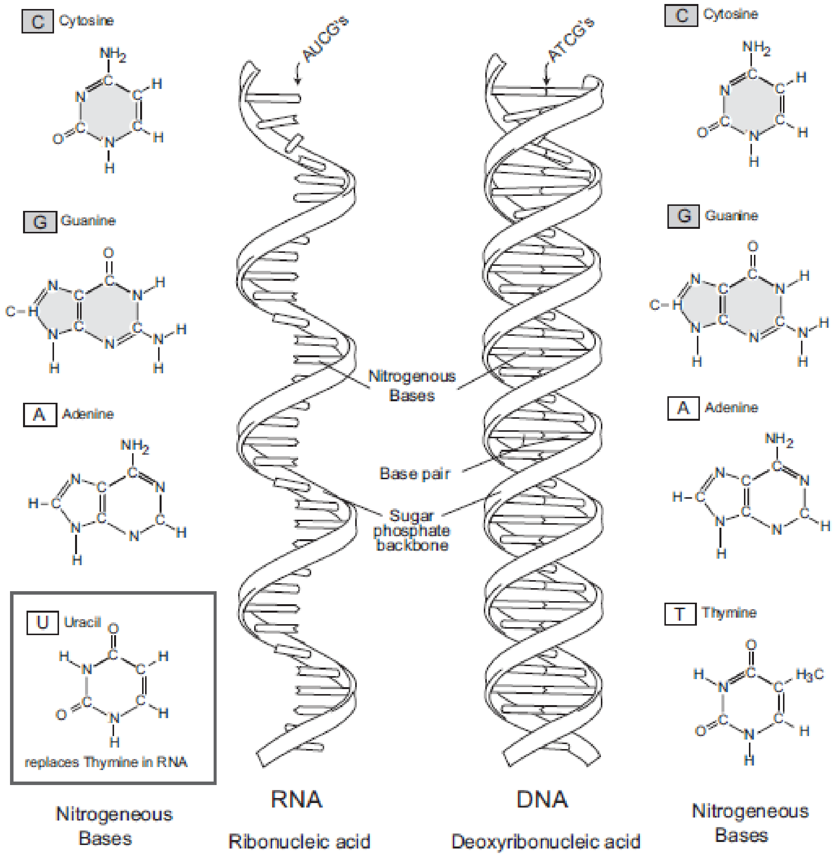
Temperature

- Daily highs and lows in Rochester



http://weatherspark.com/averages/31494/Rochester-New-York-United-States

Signal: Genomic Sequence Data

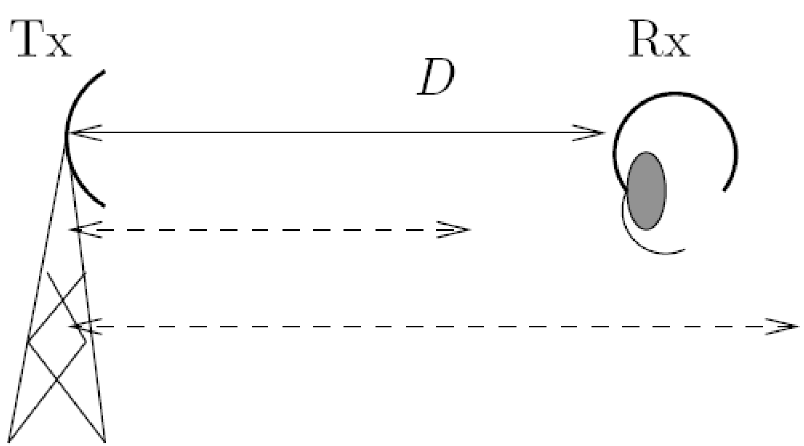


5’ AAUGCAACCCGGU…. 3’

http://www.genome.gov

Random Process Example

- Mobile Receiver (Rx)
  - Variable distance to transmitter D (an RV!)
- Tx Signal: s(t), Rx Signal: r(t)
  - r(t) = signal sent D/c time ago = s(t-D/c)
    - c speed of propagation of light in air
  - Assumptions: Single path, no noise



Random Process Example (contd.)

- Tx Signal Cosine: $s(t) = a_c \cos(2\pi f_c t)$

- Rx signal: $X(t) \;=\; a \cos\left(2\pi f_c t - \dfrac{2\pi D}{\lambda}\right)$

- Random Phase Shift:

$$X(t) \;=\; a \cos\left(2\pi f_c t + \Theta\right)$$

$$\Theta = -\left(\tfrac{2\pi D}{\lambda}\right) \text{ modulo } 2\pi$$

Random Process Example (contd.)

- Typical values: $c = 3 \times 10^8 m/s$

$$f_c = 1GHz = 10^9 Hz$$

$$\lambda = \frac{c}{f_c} = 0.3m = 30cm$$

- Mobile uses move far more than λ
  - Phase shift of 2π for shift of λ
  - Phase “completely random”

$$X(t) \quad = \quad a\cos\left(2\pi f_c t + \Theta\right)$$

Θ uniform [0,2π]

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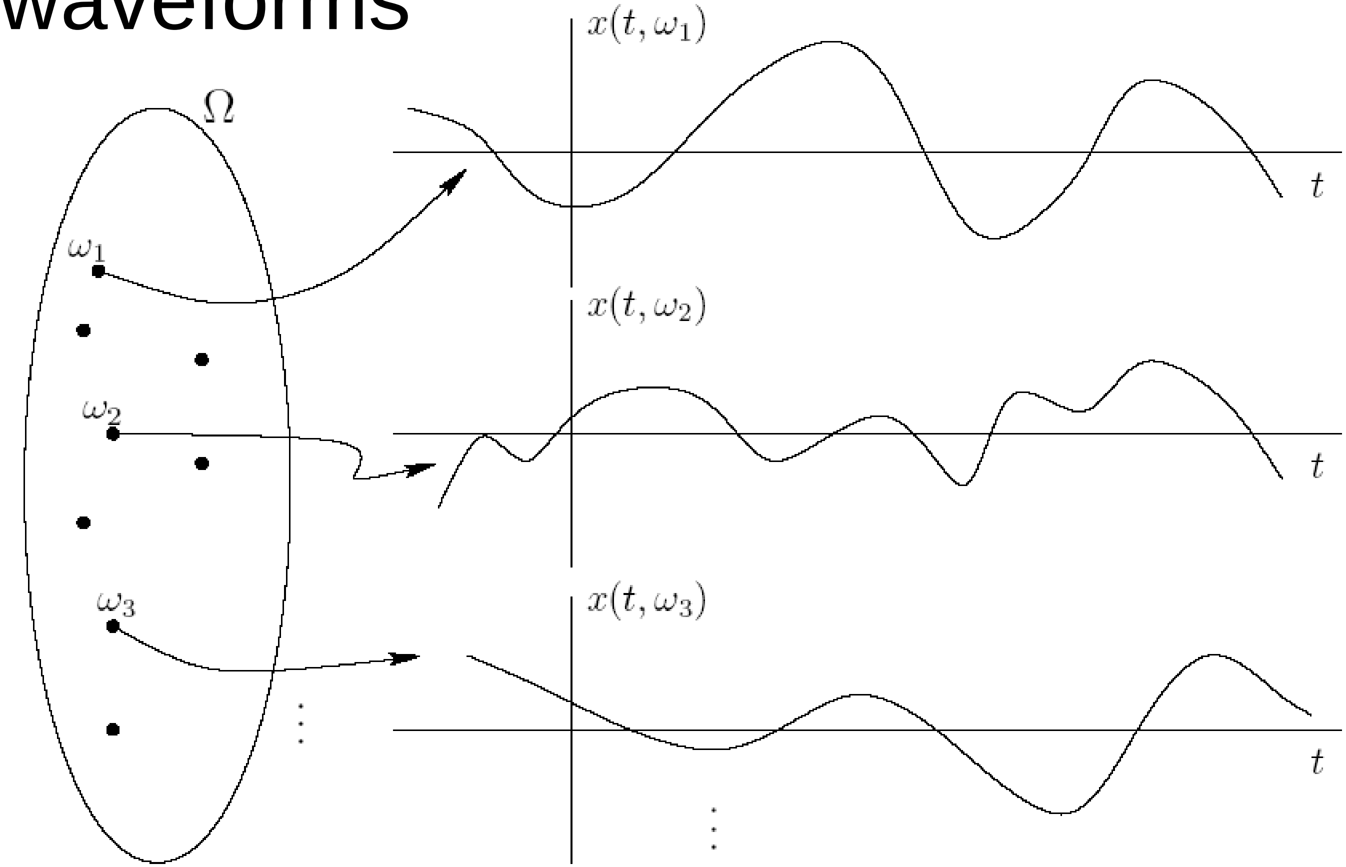
Random Processes: Two Views

- Signal Theory Extension
  - “Deterministic” to random
  - Ensemble View
- Random Variable Extension
  - A random variable for each point in time
    - Time: continuous/discrete
  - “Time” indexed collection of random variables

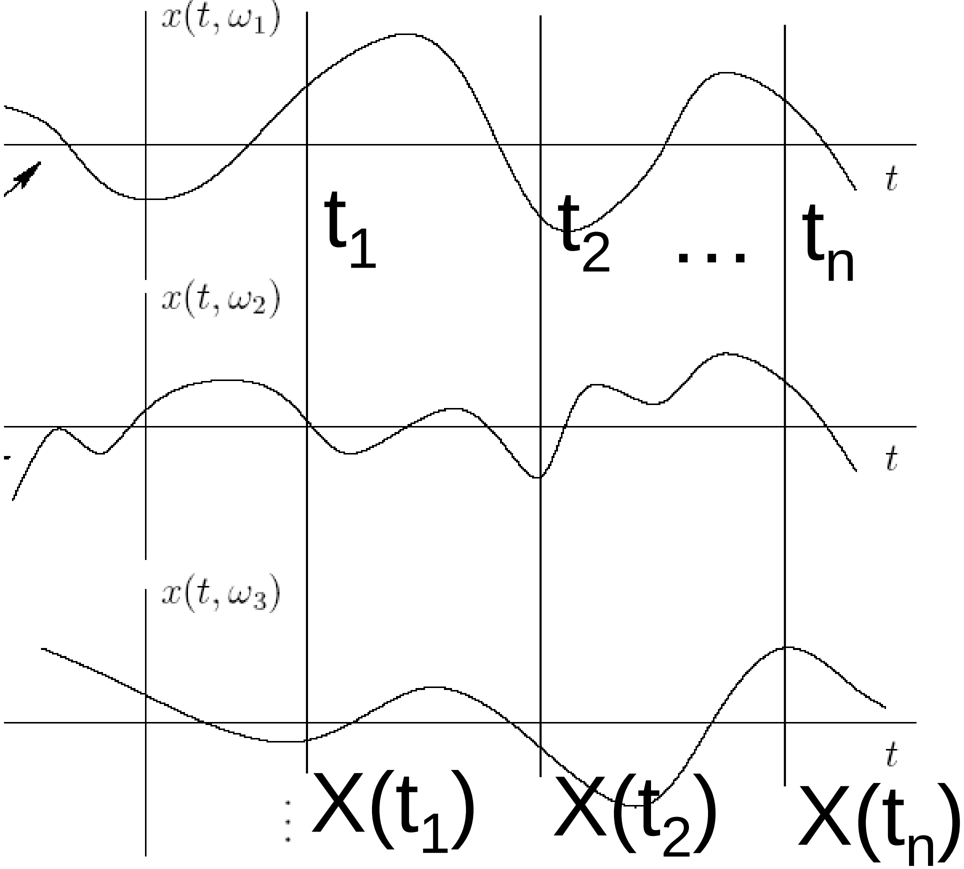
Ensemble View

- Selection from an “ensemble” of waveforms



Indexed RV Collection View

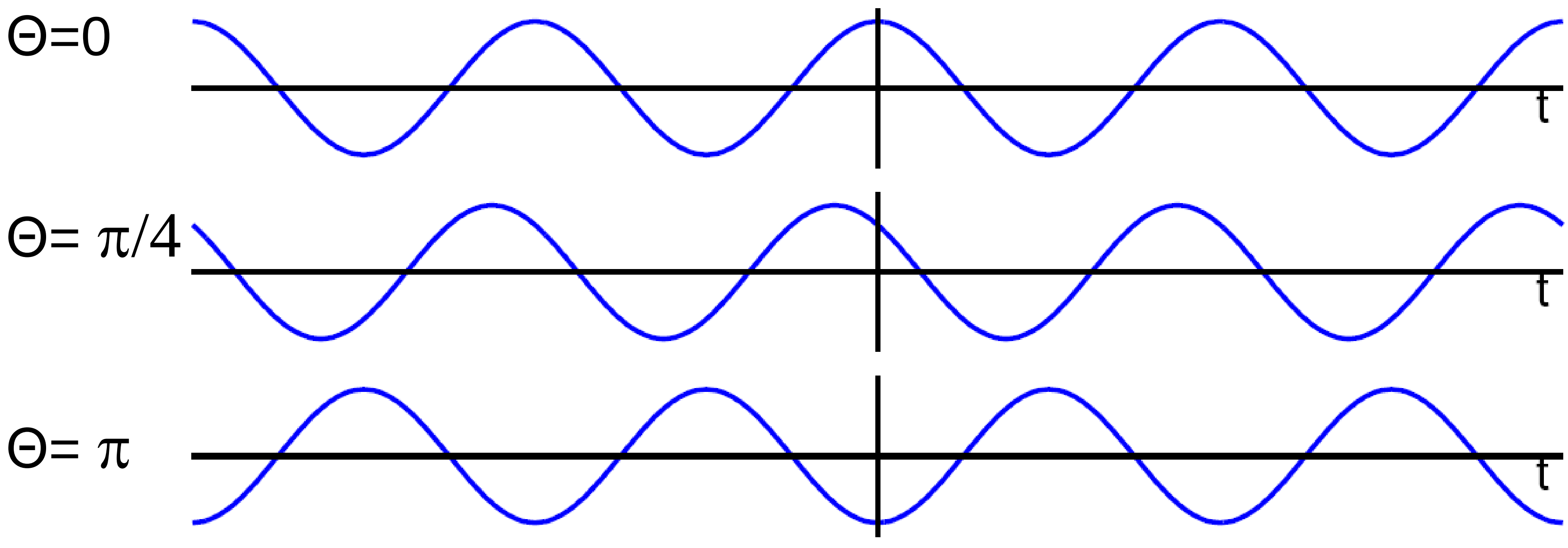
- X(t) – random Process
  - – $X(t_1)$, $X(t_2)$, …. , $X(t_n)$ random variables



Example: Delayed Cosine

$$X(t) \quad = \quad a\cos\left(2\pi f_c t + \Theta\right)$$

Θ uniform [0,2π]



X(0) an RV

Random Process (RP) Characterization

- Ensemble View: $X(t) = f(t, \boldsymbol{\Theta})$
  - $\boldsymbol{\Theta} = [\Theta_1, \Theta_2, \ldots, \Theta_n]$, RP parameters
  - Random Process characterization
    - Joint distribution of random vector $\boldsymbol{\Theta}$

- Indexed RV collection view
  - Characterize by joint pdf (multiple RVs)

$$p_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots, x_n)$$

  - for every choice of n, and t_1, t_2, …. t_n

Statistical Averages for a Random Process

- Mean:

$$m_X(t) \stackrel{\text{def}}{=} E\left[X(t)\right] = \int_{-\infty}^{\infty} x p_{X(t)}(x) dx$$

- Auto-correlation function:

$$R_X(t_1, t_2) \stackrel{\text{def}}{=} E\left[X(t_1) X(t_2)\right]$$

- Expectation operator = pdf weighted integral

Example: RP Statistical Averages

$$X(t) = a\cos(2\pi f_c t + \Theta)$$

Θ uniform $[0,2\pi]$

$$p_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

- Mean:

$$m_X(t) = \frac{1}{2\pi}\int_0^{2\pi} a\cos(2\pi f_c t + \theta)\, d\theta = 0$$

- Auto-correlation (use trig identity):

$$R_X(t_1, t_2) = \frac{a^2}{2}\cos(2\pi f_c(t_1 - t_2))$$

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Stationary Random Process: Motivation

- What's in a time origin ?
  - Is our choice of t=0 special
- Example:
  - Case I: Rand amplitude cosine

$$X_1(t) = A\cos(2\pi f_c t)$$, A uniform [-1,1]

  - Case II: Rand phase cosine

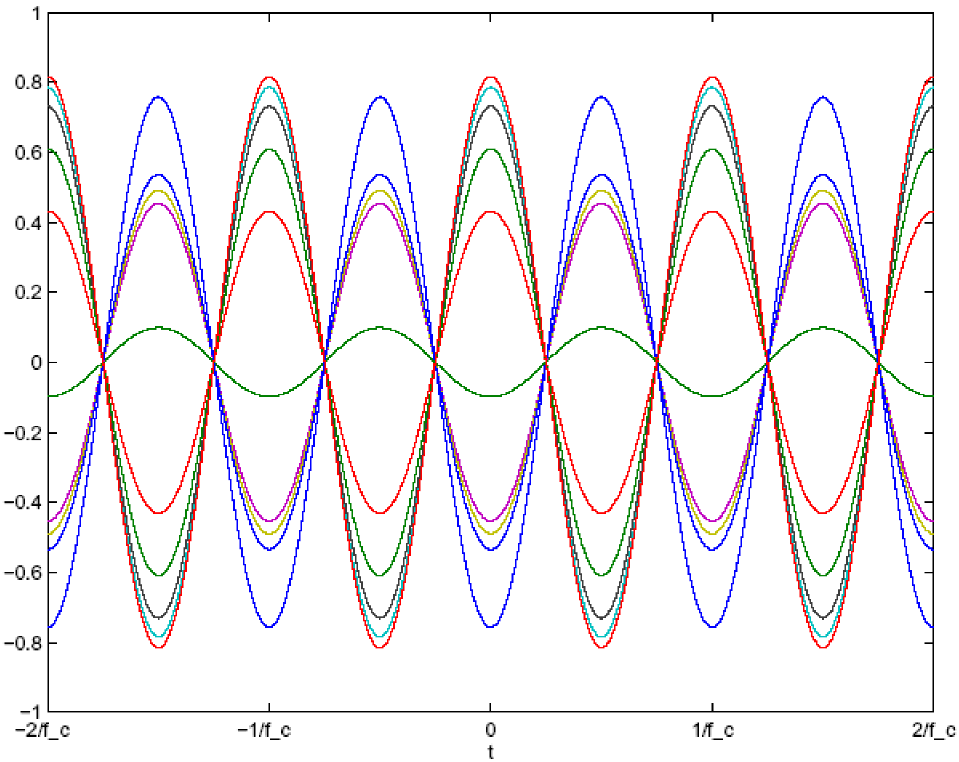
$$X(t) = a\cos(2\pi f_c t + \Theta)$$

Θ uniform $[0,2\pi]$

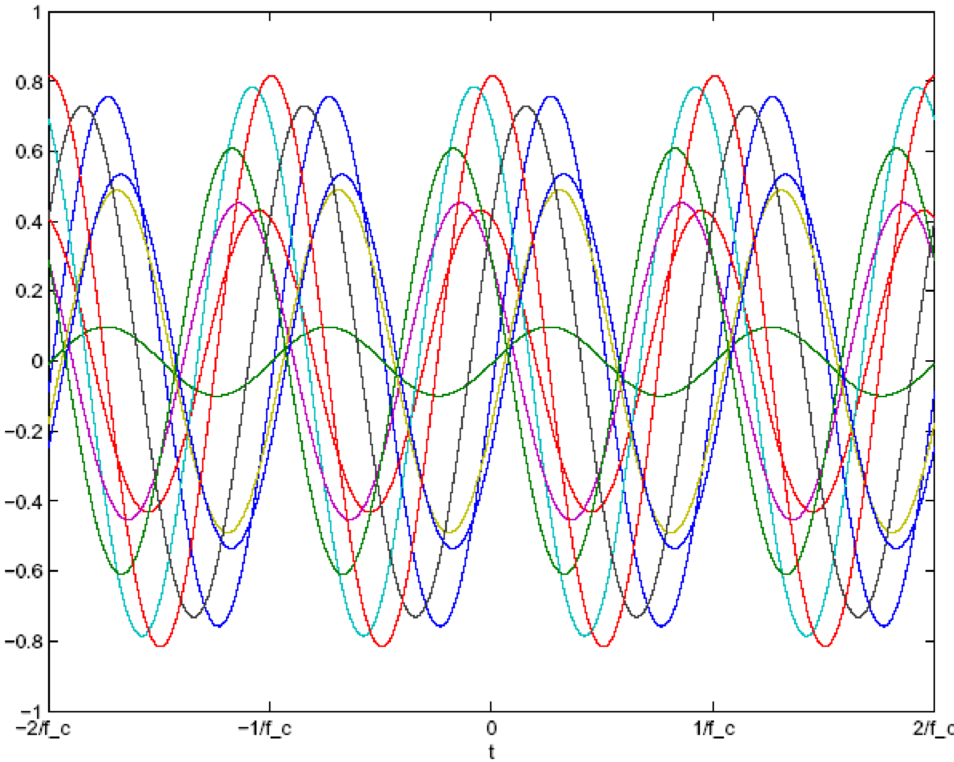
Stationary Random Process: Motivation (contd.)

Ten Sample Realizations

- Case I



- Case II:



Stationary Random Process Definition

- Intuition: Statistics not dependent on choice time origin
  - Invariant to time shifts
- Strict Sense Stationary
  - Joint pdf invariant to time shifts

$$p_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots, x_n) =$$
$$p_{X(t_1+\Delta)X(t_2+\Delta)\cdots X(t_n+\Delta)}(x_1, x_2, \ldots, x_n)$$

  - often joint pdf unavailable

Wide-sense Stationary (WSS) Process

- First two moments, invariant to time shifts
- Mean: not dependent on time

$$m_X(t) = E[X(t)] = m_X$$

- Auto-correlation: function of time difference alone

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

- Stationary means wide-sense stationary

Example: Delayed Cosine Revisited

$$X(t) \quad = \quad a\cos\left(2\pi f_c t + \Theta\right)$$

Θ uniform [0,2π]

- Mean:

$$m_X(t) = \frac{1}{2\pi}\int_0^{2\pi} a\cos\left(2\pi f_c t + \theta\right) d\theta = 0$$

- Auto-correlation:

$$R_X(t_1, t_2) = \quad \frac{a^2}{2}\cos\left(2\pi f_c (t_1 - t_2)\right)$$

- X(t) is wide-sense stationary
- Also strict sense stationary

Autocorrelation Function Properties (WSS Process)

- Auto-correlation function

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

- Even function:

$$R_X(-\tau) = R_X(\tau)$$

- Maxima at origin:

$$|R_X(\tau)| \leq R_X(0)$$

Multiple Random Processes

- X(t), Y(t) two random processes
- Joint Characterization:
  - Joint pdf of

$$X(t_1), X(t_2), \ldots X(t_n), Y(u_1), Y(u_2), \ldots Y(u_m)$$

  - for every choice of n,m and

$$t_1, t_2, \ldots t_n, u_1, u_2, \ldots u_m$$

Independent/Uncorrelated Processes

- Independent: Joint pdf is product of pdfs
  - pdf of samples of X(t)
  - pdf of samples of Y(t)
- Uncorrelated: $X(t_1)$ and $Y(t_2)$ uncorrelated random variables
  - for any t_1 and t_2
  - a LOT simpler than independence
- Independent implies uncorrelated
  - Converse NOT true

Cross-correlation, Joint Stationarity

- Cross-correlation function:

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

  – Time interrelation betw. processes (partial)

- Joint Stationarity: X(t), Y(t) random proc
  – Individually stationary
  – Cross-correlation function of time difference only

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$$

$$R_{XY}(\tau) = E[X(t+\tau)X(t)]$$

Gaussian Processes

- X(t) Gaussian Process
  - Random vector $\mathbf{X} = [X(t_1), X(t_2) \ldots X(t_n)]^T$ jointly Gaussian
  - pdf

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x}-\mathbf{m})}{2}\right)$$

    - Mean $\mathbf{m}_X = E[\mathbf{X}]$
    - covariance $\mathbf{C} = E[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^T]$

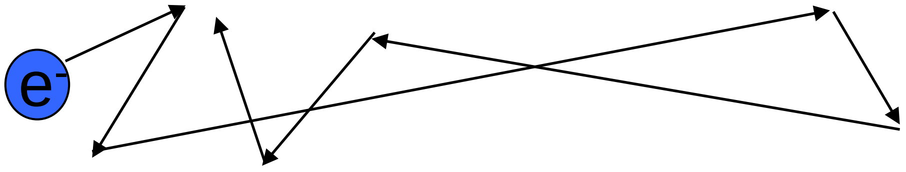
Fully characterized by $m_x(t)$ and $R_X(t_1, t_2)$

Gaussian Process Properties

- Jointly Gaussian Processes that are uncorrelated are independent
  - commonly invoked

Thermal Noise

- Random electron motion in a conductor
  - Thermal motion above T>0 Kelvin
  - Charged electron → random current



  - Contributes unavoidable noise in receivers
  - Noise sum of currents from lot of electrons
    - Gaussian pdf by Central Limit Theorem
  - (Almost) Independent across different time instants

Key Concepts

- Random Processes
  - Models for noise/information
  - Ensemble view/indexed collection of RVs
  - Discrete vs Continuous
- Stationary
  - Desirable: time origin non special
  - Spectral Representation
- Gaussian Processes
  - Full characterization from mean and autocorrelation